Coalition-proof Stackelberg equilibria

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Abstract

Coalition proofness has been discussed in the context of Bertrand and Cournot oligopolies and oligopolistic markets with supply function equilibria. This paper introduces a solution concept on similar lines for Stackelberg games with multiple leaders and multiple followers. The key property of a coalition-proof Stackelberg equilibrium is that it is *self-enforcing*. Coalition-proofness for both Nash and correlated equilibria are studied for such games.

JEL: C62, C72, D43, L13

Keywords: Coalition-proofness, multiple-leader-follower Stackleberg game, Nash equilibrium, Correlated equilibrium

1 Introduction

A sequential move quantity game between firms was first intensively studied by Von Stackelberg (1952) in which firms take prices as given. This is referred to as the Stackelberg game, and a Stackelberg equilibrium is a subgame perfect Nash equilibrium of a such a game. A standard result, discussed for instance in Myerson (1991), is that every perfect information sequential move game has atleast one subgame-perfect Nash equilibrium. Stackelberg games can also have multiple leaders and followers. In such games, firms at each level play a simultaneous move game between themselves and the two groups of firms play a sequential move game. have firms at each level playing their simultaneus move either Nash equilibrium strategies as in Julien (2011) and Tesoriere (2017a,b) or correlated equilibrium strategies as in Von Stengel and Zamir (2010), Černỳ (2016), Castiglioni, Marchesi, and Gatti (2021) and Yu, Xu, and Chen (2022). Comparisons with other oligopolitic models has been found in Amir and Grilo (1999) and Cumbul (2021).

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A coalition-proof Nash equilibrium was first defined by Bernheim, Peleg, and Whinston (1987) and a perfectly coalition-proof Nash equilibrium was discussed in detail in Peleg (1992). Coalition-proofness has been applied to several models, particularly oligopolies. In a companion paper to the one where they define a CPNE, Bernheim and Whinston (1987) apply the equilibrium concept to a Cournot oligopoly. A coalition-proof Bertrand equilibrium is defined in Chowdhury and Sengupta (2004). For an oligopoly with a supply function equilibrium, coalition-proofness has been defined in Delgado and Moreno (2004). The concept of coalition-proofness has been applied to other equilibrium concepts as well apart from Nash equilibrium, for instance to correlated equilibria. CPCE has been defined in Moreno and Wooders (1996), Ray (1996) and Milgrom and Roberts (1996).

The lattice-theoretic approaches used here were initiated by Topkis (1978) and Topkis (1979). A good review is in Topkis (1998). Particular models with strategic complementarities and subtitutes were studied in Vives (1990). Coalition-proofness in a special class of games with strategic complementarities has been studied in Quartieri and Shinohara (2015).

2 Coalition-proof equilibria

Consider the game $\Gamma = (N, S, \pi)$. A coalition is denoted $Z \subseteq N$. A feasible coalitional deviation is a plan for all members of a coalition to play differently from what was agreed by all players. This was the reasoning for the equilibrium concept of a *strong Nash equilibrium* introduced by Aumann (1959). However, this is too general and unstable an equilibrium concept as there could be further sub-coalitions with profitable deviations. For the stability of coalitional deviations, Bernheim, Peleg, and Whinston (1987) introduced the idea of *self-enforcing deviations* when the players have finite strategy sets. A self enforcing deviation is a feasible coalitional deviation from which no sub-coalition can deviate and increase the payoff of each member of the sub-coalition. The inductive definition of a coalition-proof Nash equilibrium is reproduced below.

Definition 1. *(Coalition-proof Nash equilibrium, CPNE)*

1. *A single player game indexed by $i = 1$, Γ has a CPNE if and only if strategy $s^* \in S$ maximizes $u_i(s)$.*

2. *Assume that a CPNE has been defined for the game with $n = 1$. Then for $n > 1$,*

   (a) *A strategy profile $s^* \in S$ is self-enforcing if for a coalition $Z \subset N$ of players the strategy is a CPNE in the game Γ/s^*_{-j}.*

(b) *A strategy $s^* \in S$ is a CPNE if it is self-enforcing and if there does not exist any other self-enforcing strategy profile $s \in S$ such that $u_i(s) > u_i(S^*)$ for all $i \in N$.*

A coalition-proof correlated equilibrium has also been defined. While other authors have studied this for games with finite strategy spaces, the following definitions 1, 2 and 3 from Milgrom and Roberts (1996) are for games with infinite strategy spaces and are reproduced here for ease of exposition.

Definition 2. *(Coalition communication structure) A collection Σ of sequences σ of subsets of N that are in decreasing set-inclusion order is a coalition communication structure.*

Definition 3. *(Initial segment) Given a sequence $\sigma = (S_1, S_2...S_T)$, the sequence $\underline{\sigma} = (S_1, S_2...S_t)$ for some $t < T$ is called an* initial segment *of σ, and $\underline{\sigma}$ is* initial *on Σ.*

Definition 4. *(Self-enforcing deviation) A feasible coalitional deviation ν by a coalition S to a correlated strategy profile μ is self-enforcing if for the game $(\Gamma, \Sigma(\underline{\sigma}, S))$ if (S) is* initial *on $(\underline{\sigma})$ and there is no S' on ξ with ξ a self-enforcing, payoff-improving deviation for S' from ν in $(\Gamma, \Sigma(\underline{\sigma}, S))$.*

Permitting communication between players is a way to characterise coalition-proofness for correlated equilibria as well. This has been done for static games by Ray (1996) and Moreno and Wooders (1996). When the strategy space S is infinite, a pure strategy correlation device is a pair $D = (M, P)$ where $M = (M_i)_{i \in N}$ and $P \in \Delta(M)$. A canonical (or direct) correlation device messages from the players' action sets and is hence represented as $D = (S, P)$ where $S = (S_i)_{i \in N}$.

Definition 5. *(Coalition-proof correlated equilibrium, CPCE) A correlation device D and the correlated strategy P is a CPCE of the game (Γ, Σ) if there exists no coalition S such that (S) is initial in Σ and has a payoff-improving self-enforcing deviation.*

3 Setup

3.1 Stackelberg game with multiple leaders and followers

Consider a Stackelberg (Von Stackelberg, 1952) game with multiple agents The game is denoted $\Gamma_S = (N, M, A^L, A^F, \pi)$. Let $N = \{1...n\}$ represent the set of leaders and $M = \{1...m\}$ represent the set of followers. Leader firm i chooses from its strategy space A_i and a A firm that produces x, has a cost function $c(\cdot)$, competes in a market wherein the other firms produce a total amount y and faces an inverse demand function $p(\cdot)$ has a profit function

$\pi(\cdot, \cdot)$ given by:

$$\pi(x, y) = p(x + y)x - c(x) \tag{1}$$

Leader i's choice variable is q_i, the choices of the other leaders are denoted as q_{-i}. The profit maximisation problem is then:

$$\max_{q_i} \quad \pi_i(q_i, q_{-i}, \widehat{q_i}) = p(q_i + \sum_{k \neq i, k \in N} q_k + \sum_{j \in M} \widehat{q_{ij}})q_i - c(q_i) \tag{2}$$

$$\begin{aligned} s.t. \quad & q_i \in A_i^L \\ & q_{-i} \in A_{-i}^L \\ & \widehat{q_i} \in S(q^L) \end{aligned} \tag{3}$$

where $q^L = (q_i)_{i \in N}$ is the action profile of the leaders, and the set $S(q^L)$ is the solution set of the followers, that is, the set from which the followers choose their strategy profile given that the leaders' strategy vector q^L is known to the followers. Each follower j takes q^L as the parameter when solving for the optimal strategy q_j. Follower j solves the problem:

$$\max_{q_j} \quad \pi_j(q_j, q_{-j}; q^L) = p(q_j + \sum_{l \neq j, l \in M} q_l + \sum_{i \in N} q_i)q_j - c(q_j) \tag{4}$$

$$\begin{aligned} s.t. \quad & q_j \in A_j^F \\ & q_{-j} \in A_{-j}^F \end{aligned}$$

The solution to this problem forms the followers' strategy profile $q^F = (q_j)_{j \in N}$. The solution set $S(q^L)$ consists of all such profiles $q^F \in Y^L$ where $Y^L = \prod_{j \in M} q_j$. When $S(q^L)$ is a singleton for each q^L, the leader's problem permits a unique solution. When $S(q^L)$ is set-valued for each q^L, there can be multiple solutions and the leader can approach the problem in different ways. Two important approaches are those in which the leader solves either for the best or the worst outcomes. The best outcome for the leader is called the *optimistic* outcome and would be the one in which followers choose their strategies to maximise their profits and those of the leaders. The worst outcome for the leader is called the *pessimistic* outcome and would be the one in which the followers would choose their strategies to maximise their profits but to minimise the profits of the leaders.

3.2 Stackelberg equilibrium

Stackelberg equilibria can be Nash equilibria or correlated equilibria. These are defined below.

Definition 6. *(Stackelberg Nash equilibrium, SNE) A Stackelberg equilibrium in the multi-leader-follower game is a subgame perfect Nash equilibrium $(q_i, \lambda(\cdot)) \in X \times \Lambda$ that satisfies the following conditions:*

$$\pi_i(q_i, q_{-i}, \lambda(q_i, q_{-i})) \geq \pi_i(\widetilde{q}_i, q_{-i}, \lambda(\widetilde{q}_i, q_{-i})) \quad \forall i \in N \tag{5}$$

$$\pi_i(q_i, q_{-i}, q) \geq \pi_i(\widetilde{q}_i, q_{-i}, q^L) \quad \forall i \in M \tag{6}$$

Here the first equation is the solution to a leader's profit maximisation problem. This is a hierarchical optimization problem that has as its variables the quantity q_i and the conjectures about the quantities of all followers $\lambda(\cdot)$. The second equation is a follower's profit maximisation problem which takes as parameters the quantities chosen by all the leaders. The concept of a correlated equilibrium has been noted by authors to be more intuitive than a Nash equilibrium. A correlated strategy is one from which a mediator suggests an action to each player.

Definition 7. *(Stackelberg correlated equilibrium, SCE) In a correlated equilibrium, players play the correlated strategy* obediently. *A correlated strategy pair $(\mu, \nu) \in \Delta(A^L) \times \Delta(A^F)$ is a Stackelberg correlated equilibrium if it forms a Stackelberg equilibrium wherein the leaders play a correlated equilibrium μ and the followers play a correlated equilibrium ν. The Stackelberg equilibrium thus satisfies:*

$$\pi_i(\mu(q_i, q_{-i}), \nu(q_i, q_{-i})) \geq \sum_{q^L \in A^L} \mu(q^L) \pi_i(\widetilde{q}_i, q_{-i}, \nu(q_i, q_{-i})) \tag{7}$$

$$\pi_i(\mu(q_i, q_{-i}), \nu(q_i, q_{-i}^F)) \geq \sum_{q^F \in A^F} \nu(q^F) \pi_i(\mu(q_i^L, q_{-i}^L), \widetilde{q}_i, q_{-i}) \tag{8}$$

The correlated equilibrium (μ, ν) is a correlation device. A Stackelberg game Γ_S extended by the correlation device is denoted $\Gamma_{S,\mu,\nu} = (\Gamma_S, (\mu, \nu))$.

3.3 Coalition-proof Stackelberg equilibrium

A coalition of leaders in a Stackelberg game $S^L \subseteq N$ chooses a coalitional strategy $(q_i)_{i \in S^L}$. A coalition of followers $S^F \subseteq M$ chooses a coalitional strategy $(q_i)_{i \in S^F}$. Coalition-proofness can thus be defined for Stackelberg equilibria as well. A Nash equilibrium based solution concept can be defined as follows:

Definition 8. *(Coalition-proof Stackelberg Nash equilibrium, CPSNE) A coalition-proof Stackelberg Nash equilibrium in a multi-leader-follower Stackelberg game has a Stackelberg equilibrium with both the set of leaders and the set of followers playing their own coalition-proof Nash equilibrium strategies.*

A similar definition for a coalition-proof correlated equilibrium in a Stackelberg game would be as follows:

Definition 9. *(Coalition-proof Stackelberg correlated equilibrium, CPSCE) A coalition-proof Stackelberg correlated equilibrium in a multi-leader-follower Stackelberg game is a Stackelberg equilibrium with both the set of leaders and the set of followers playing their own coalition-proof correlated equilibrium strategies.*

Aumann (1987) discusses how the convex hull of the set of Nash equilibria is a subset of the set of correlated equilibria. Thus, every Nash equilibrium is also a correlated equilibrium. However, there may be correlated equilibria that lie outside the convex hull of the set of Nash equilibria. This result can be extended to coalition-proof stackelberg equilibria as well.

Remark 1. *The convex hull of the set of CPSNE is a subset of the set of CPSCE. Every CPSNE is also a CPSCE but there may be CPSCE that are not CPSNE.*

4 Strategic complementarity

The lattice-theoretic approach to studying the existence of equilibria in games with strategic complementarities has several advantages. Amir and Grilo (1999) list them as follows:

1. Profit functions can be non-concave.
2. Pure-strategy Nash equilibria necessarily exist.
3. When there are multiple equilibria, they are ordered and can be ranked by individual firms according to their preferences.

4.1 Games with strategic complementarities

Definition 10. *(Normal-form game with strategic complementarities, Milgrom and Roberts (1996)) A normal-form game $\Gamma = (N, A, \pi)$ with strategic complementarities if:*

1. *each A_i is a compact lattice*

2. *each π_i is upper semi-continuous in x_i and continuous in x_{-i}*

3. *each π_i is quasisupermodular in x_i and supermodular in (x_i, x_{-i})*

An extensive-form game can also have strategic complementarities if the payoff structure is such that pairs of players have marginal payoffs that are increasing in the other player's strategy. In particular, a Stackelberg game with strategic complementarities is one with supermodularities in both the levels.

Definition 11. *(Stackelberg game with strategic complementarities) A Stackelberg game $\Gamma_S = (N, M, A^L, A^F, \pi)$ is a game with strategic complementarities if both the leaders and followers play normal-form games with strategic complementarities.*

Assuming that both the leaders and followers play games that have unique Nash equilibria, the following proposition is a direct result of Theorem 2 in Milgrom and Roberts (1996).

Proposition 1. *A CPSCE exists for every Stackelberg game with strategic complementarities in both the levels.*

Remark 1 implies that proposition 1 leads to the following corollary:

Corollary 1. *A CPSNE exists for every Stackelberg game with strategic complementarities in both the levels.*

5 Conclusion

This note defines coalition-proof Stackelberg equilibria. The solution concept is defined for both Nash equilibria and correlated equilibria. Bundling, rationing and capacity competition are interesting issues that can be studied while considerings coalition-proofness properties in such games.

6 Appendix

6.1 Supermodularity and submodularity

Consider the lattice X. The following definitions and results are standard in the literature on games with supermodularities and submodularities and are relevant to the propositions here.

Definition 12. *(Supermodular (Submodular), Topkis (1998)) A function $f : X \to \mathbb{R}$ is supermodular (submodular) on X if:*

$$f(x') + f(x'') \leq (\geq) f(x' \vee x'') + f(x' \wedge x'')$$

for all x' and x'' in X. It is strictly supermodular (submodular) if the inequality $\leq$ $(\geq)$ holds strictly.

Definition 13. *(Quasisupermodular, Milgrom and Shannon (1994)) A function $f : X \to \mathbb{R}$ is quasisupermodular if for all x' and x'' in X:*

$$f(x') \geq (>) f(x' \wedge x'') \implies f(x' \vee x'') \geq (>) f(x'')$$

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