GRADUAL MATCHING WITH AFFIRMATIVE ACTION[1]

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ABSTRACT. Admissions to technical colleges in India feature a multi-period semi-centralized matching process and are subject to sophisticated affirmative action. At each stage, applicants can decide whether to finalize their assignments or participate in the next stage by updating their submitted *rank-ordered lists*. We propose a sequential matching model where institutions' selection criteria are formulated via choice rules that admit *slot-specific priorities*. We show that the applicants will be (weakly) better off in the subsequent stages when their updated rank-ordered lists *adhere* to the assignments in the previous stages. Moreover, the mechanism that implements the individual-proposing deferred acceptance outcome at each stage is *gradually stable*, a stability notion adapted to the sequential problem. We use our results to analyze the recently reformed admission procedures for engineering colleges in India, where applicants are provided various options to update their rankings at additional stages.

JEL CODES: $D0$, $D9$.
KEYWORDS: Market design, affirmative action, gradual matching, rank order lists

1 Introduction

Recently, there has been a great deal of discussion on multi-period assignment procedures in college admissions in countries such as India, Brazil, China, France, and Germany. Baswana et al. (2019) proposed a semi-centralized, multi-period matching mechanism for engineering college admissions in India. They observe that the welfare of candidates improves in every round of the mechanism.

"*New semi-centralized, multi-period matching mechanism enjoys monotonicity*[4] *across runs. The options available to candidates are only enhanced in going from one period to the next....*"

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[4]By "monotonicity", the authors mean that candidates are assigned to (weakly) better programs with respect to their submitted rank-ordered lists in each period of the sequential procedure.

In this paper, we provide a general *gradual* (or sequential) matching framework to study when this sort of monotonicity happens under complicated diverity and affirmative action considerations. Our framework allows us to prove the claim in the quote above, among our other contributions.

The admission process for engineering colleges in India matches approximately 1.3 million students to 34,000 university positions. The Indian Institute of Technologies (henceforth, IITs) and the non-IIT Centrally Funded Technical Institutes (henceforth, non-IITs) have implemented the mechanism developed by Baswana et al. (2019). Before that, both types of institutions conducted their admission processes separately and independently. When an applicant received an offer from both IITs and non-IITs and chose one, it resulted in a vacancy in the other set of institutions. These vacant seats were either left unfilled or allocated in an ad-hoc and decentralized manner, which caused inefficiency and/or unfairness. Under this new combined seat allocation procedure, students are required to rank all programs (including both IITs and non-IITs) according to their rank order lists (ROLs) and submit a single ranking. Both types of colleges, IITs and non-IITs, independently run the individual-proposing deferred acceptance (DA) mechanism introduced by Gale and Shapley (1962) to find a match. The assignment of students is determined based on their complete ranking. In the subsequent rounds of the admissions process, students are given various options to update their ROLs. These options include "withdraw", "reject", "freeze", "slide", and "float"[5]. Depending on the option chosen, the students may either exit the procedure with or without an assignment or choose to participate in the subsequent periods. After the completion of a fixed number of periods, each student is then assigned to their finalized program.

This process is subject to a comprehensive affirmative action program, which has been implemented via a reservation system. There are two types of reservations: *vertical* and *horizontal*. Each institution reserves a certain percentage of its slots for students from a vertical reserved category– Scheduled Castes (SC), Scheduled Tribes (ST), Other Backward Classes (OBC), and Economically Weaker Sections (EWS). Specifically, 15%, 7.5%, 27%, and 10% of the slots are reserved for SC, ST, OBC, and EWS students, respectively. Applicants who do not belong to any of these vertical reserved categories are referred to as General Category (GC) and positions that are not reserved are referred to as open-category positions. They are available to all applicants, including those

[5]All these options are formally defined in Section 6.

from reserved categories who do not declare their membership. A minimum number of positions within each vertical category are earmarked for women as horizontal reservations. Vertical reservations are implemented as "over-and-above" by filling open-category positions before vertically reserved categories. Horizontal reservations are implemented as minimum guarantees by filling horizontally reserved positions before unreserved positions.

Situations such as the one described earlier can be modeled using *gradual* matching mechanisms framework, introduced in Haeringer and Iehlé (2021) (henceforth, H&I). They introduce a multi-period college admissions problem where individuals are offered repeated opportunities to participate in the mechanism, using updated ROLs. The final matching is constructed *gradually* over several periods. An important assumption made in the H&I model is that each institution has a *responsive*[6] choice rule. However, the underlying assumption of responsiveness does not accommodate affirmative action considerations and hence rule out important real world applications.

Motivated by engineering college admissions in India, we adopt the *gradual matching problem* of H&I. We assume that institutions implement affirmative action policies via the *slot-specific priorities* approach of Kominers and Sönmez (2016)). The latter is a model of the individual-institution matching model where each institution has a set of positions (slots) that can be assigned to different individuals. Positions have their own (potentially independent) rankings for contracts (here individuals). Within each institution, a linear order – referred to as the precedence order – determines the sequence in which positions are filled.

In this paper, we investigate the restrictions on individual's ROLs across different periods that result in *monotone* outcomes, without modifying the priority ranking in a way to guarantee a position to individuals proposed a match in previous period[7]. We refer to a matching outcome as monotone when each individual is matched to an institution that is weakly higher than the match of previous rounds (see Theorem 1). Further, we introduce a "backward-looking" notion of stability for gradual matching mechanisms adjusted for affirmative action constraints. We take into consideration individual rationality, non-wastefulness and justified envy of individuals across dif-

[6] An institution has a *responsive* choice rule if it can be generated by a strict preference order that always selects the $q-$best alternatives whenever available. Here q is the capacity of the institution (see Roth (1985) and Chambers and Yenmez (2018)).

[7] There are several applications where institutions are mandated to to not modify their priority ranking across different rounds of the mechanism (see Section 6).

ferent periods. Theorem 2 establishes a relationship between this notion of stability, we refer to as *gradual stability*, and monotone outcomes. These results generalize the findings presented in H&I. However, we do not rely on the proof of H&I to validate our first result. We apply our theoretical findings to analyze the *multi-run multi-stage DA mechanism* that has been implemented in engineering college admissions in India since 2016. We also relate our findings to the characterization result of Kojima and Manea (2010) for the individual-proposing DA mechanism.

The layout of the paper is as follows: the next subsection provides a brief literature review. Section 2 formally introduces the framework of the matching problem at each stage. Section 3 introduces a multi-period matching problem and Section 3.2 considers a class of multi-period matching problems, referred to as gradual matching problems. Section 5 introduces the stability notion for gradual matching problems. Section 6 provides an application of the model.

1.1 Related Literature

The closest paper to ours is that of Haeringer and Iehlé (2021). We extend their framework to a more generalized setting where institutions' choice rules that have slot-specific priorities (SSP) choice rules. They offer a comprehensive review of dynamic matching models and emphasize how their gradual matching problem differs from other dynamic models. While our results are broader in scope, they align closely with the findings presented in their work.

There is a vast literature dedicated to the study of dynamic matching problems. Bo and Hakimov (2022) introduce a family of *iterative deferred acceptance* mechanism, where the students are asked to sequentially make choices or submit partial rankings from set of colleges. These are used to produce a tentative allocation at each step. If a student is unacceptable to their previous choice at some period, she is asked to make another choice among colleges that would tentatively accept her. We refer the reader to Haeringer and Iehlé (2021) for a comprehensive literature review on dynamic matching models.

The SSP framework of Kominers and Sönmez (2016) provides a tool for market designers to handle diversity and affirmative action constraints in two-sided matching models. Aygün and Bó (2021) design SSP choice rules for the Brazilian college admission problem. More recently, Pathak et al (2023) use the SSP framework to design protocols allocating medical resources when there are multiple ethical considerations. Avataneo and Turhan (2021) extend the framework to a more general one by defining SSP choice rules that allow transfers as in many real-world applications.

Our paper also contributes to the recently active literature on affirmative action in India from a market design perspective. Aygün and Turhan (2017) and Aygün and Turhan (2020) focus on IIT admissions and transferring vacant OBC positions to open-category. Similarly, Aygün and Turhan (2022) introduce a new transfer scheme with superior theoretical and practical properties. Aygün and Turhan (2023) offers another choice rule to implement affirmative action constraints and transfer vacant seats. This paper discusses the joint implementation of vertical and horizontal reservations in engineering college admissions in India via position-specific priorities choice rules in a setting where applicants can update their rank order lists (ROLs) for additional periods. Another related paper is Sönmez and Yenmez (2022), in which the authors study the allocation of government jobs in India and relate matching theory to Indian law. Unlike their work, our paper considers engineering college admissions in India. None of the above papers study the sequential implementation of individual-proposing DA in a setting where individuals can update their ROLs. Other papers studying affirmative action implementations include Echenique and Yenmez (2015), Kamada and Kojima (2015), Correa et al. (2021) among others.

This paper is also related to the characterization results for the DA mechanism of Kojima and Manea (2010) and Morrill (2013). Their main axiom is built on individually rational monotonic transformations (i.r.m.t) of a preference relation. A preference profile R' is an i.r.m.t of a preference profile R at an allocation, if for every individual, any object that is acceptable and preferred to this allocation under R' is preferred to the allocation under R. An outcome satisfies IR monotonicity if every individual weakly prefers the new allocation with respect to R' over the earlier allocation whenever R' is i.r.m.t of R. They show that the DA mechanism satisfies *IR monotonicity* when the choice rules are *substitutable* and *acceptant*. We utilize this condition in the sequential framework by referring to i.r.m.t of ROLs of active individuals as *proposal-adhering*. We establish that when the mechanism in each period is individual-proposing DA, coupled with SSP choice rules of institutions, then the outcome will be monotone.

Our gradual stability notion is a generalization of gradual stability introduced in H&I. When institutions' choice rules are responsive, our gradual stability notion reduces to that of H&I. A related stability concept was introduced in Pereyra (2013) in the context of seniority-based allocation rules. Feigenbaum et al. (2020) study the two-stage dynamic matching problem where a main round of admission is followed by a reassignment stage to fill vacancies.

2 Model

There is a finite set of institutions $S = \{s_1, ..., s_m\}$ and a finite set of individuals $I = \{i_1, ..., i_n\}$. Each individual $i \in I$ has an asymmetric, complete and transitive relation (ROL) P_i over $S \cup \{\emptyset\}$, where $\emptyset$ denotes remaining unmatched. We write $sP_i\emptyset$ to mean that institution s is *acceptable* for individual i. Similarly, $\emptyset P_i s$ means institution s is *unacceptable* for individual i. We denote the profile of individual ROLs by $P = (P_i)_{i\in\mathcal{I}}$. We let $\mathcal{P}$ denote the set of all strict ROLs over $S \cup \emptyset$. We denote by R_i the weak relation associated with P_i and by $R = (R_i)_{i\in\mathcal{I}}$ the profile of weak relations.

Institution s has $\overline{q}_s$ positions, and its selection criterion is summarized by a choice rule C_s, which selects a subset from any given set of individuals. That is, $C_s(I) \subseteq I$. A choice rule C satisfies *substitutability* if for every set $A, B \subset I$, $a \in A \subsetneq B$, $a \in C_s(B) \implies a \in C_s(A)$. Also, C_s is *acceptant* if for all $A \subseteq I$, $|C_s(A)| = min(|A|, \overline{q}_s)$. Finally, C_s satisfies *Irrelevance of Rejected Alternatives* (IRA) if for every set $A, B \subset I$ such that $C(B) \subseteq A \subseteq B$, $C(A) = C(B)$.

We let $\Xi = (I, S, (P_i)_{i\in I}, (C_s, \overline{q}_s)_{s\in S})$ denote a **stage problem**. A **stage matching** for a stage problem Ξ is a mapping $\mu : I \cup S \to 2^I \cup S$ such that, for each $i \in I$ and $s \in S$, (i) $\mu(i) \in S \cup \{\emptyset\}$, (ii) $\mu(s) \subseteq I$, and (iii) $\mu(i) = s$ if and only if $i \in \mu(s)$. A stage matching is **feasible** if $|\mu(s)| \leq \overline{q}_s$ for all $s \in S$.

DEFINITION. A feasible stage matching μ is **stage stable** if for all $i \in I$ and $s \in S$,

1. *Individual rationality for individuals:* $\mu(i)R_i\emptyset$,
2. *Individual rationality for institutions:* $C_s(\mu(s)) = \mu(s)$, and
3. *Unbockedness:* $sP_i\mu(i)$ implies $i \notin C_s(\mu(s) \cup \{i\})$.

The first condition, *individual rationality* for individuals, guarantees that no individual is assigned to an institution they find unacceptable. The second condition, *individual rationality* for institutions, ensures that institutions' selection procedures are respected. This condition guarantees the implementation of affirmative action constraints when they are encoded into institutions' choice rules.[8] The last condition is the standard *no blocking pair* condition.

A **stage matching mechanism** φ maps every stage problem Ξ to a feasible stage matching μ. The mechanism φ is stable if $\varphi(\Xi)$ is stable for every stage problem.

2.1 Institutions' Choice Rules

We model institutions' selection criterion to accommodate affirmative action considerations via choice rules that have *slot (position)-specific priorities* (SSP) structure (Kominers and Sönmez (2016)). Institution s has a set of $\overline{q}_s$ positions denoted by $\mathcal{B}_s \equiv \{p_s^1, ..., p_s^{\overline{q}_s}\}$. Each position $p_s^j \in \mathcal{B}_s$ has a linear priority order $\succ_s^j$ over elements of $I \cup \{\emptyset_s\}$, where $\emptyset_s$ represents remaining unassigned and can be assigned to at most one individual. We denote the profile of positions' priority order by $\succ_s = (\succ_s^j)_{j=1}^{j=\overline{q}_s}$. The positions in $\mathcal{B}_s$ are ordered according to a linear order of precedence, denoted by $\rhd_s$. For any $p, p' \in \mathcal{B}_s$, we say $p \rhd_s p'$ to mean that institution s fills position p before filling p', whenever possible. For convenience, if $i < j$ for any p_s^i, p_s^j, it indicates $p_s^i \rhd_s p_s^j$.

Given the priority order profile $\succ_s$ and the precedence order $\rhd_s$, the choice structure of institution s from a given set of individuals $A \subseteq I$, denoted by $C_s(A, \succ_s, \rhd_s)$, is given as follows:

- First, position p_s^1 is assigned to the individual who is $\succ_s^1$–maximal among the individuals in A. Call this individual i_1.
- Then, position p_s^2 is assigned to the individual who is $\succ_s^2$–maximal among the individuals in $A \setminus \{i_1\}$. Call this individual i_2.
- This process continues with each position p_s^k is being assigned to the individual who is $\succ_s^k$–maximal among the remaining individuals in $A \setminus \{i_1, ..., i_{k-1}\}$.

If no individual is assigned to a position $p_s^l \in \mathcal{B}_s$, then p_s^l is assigned $\emptyset_s$.

[8] See Alva and Doğan (2021) for in depth discussion of this point.

LEMMA **1.** *If C_s has an SSP structure, then it satisfies substitutability and IRA.*

In matching with contracts setup, Kominers and Sönmez (2016) establish that choice rules that have SSP structure satisfy *bilateral substitutes* condition proposed by Hatfield and Milgrom (2005). Additionally, Aygün and Sönmez (2013) demonstrated that these rules also satisfy *Independence of Rejected Contracts* property. Thus, choice rules considered here satisfy substitutability and IRA condition.

We say that a choice rule has an SSP structure if it is "generated" by a pair $(\succ_s, \triangleright_s)$. A special case when $\succ_s^i = \succ_s^j$ for all positions $p_s^i, p_s^j \in \mathcal{B}_s$ is referred to as *responsive* choice rules in the literature (see Chambers and Yenmez (2018)). The SSP structure accommodates a variety of other constraints, including reservation policy for admissions in engineering colleges in India (see Section 6).

Unlike H&I that deal with responsive choice rules, we broaden the class of choice rules of institutions to SSP structures. Thus, a stage problem can now be alternatively represented as

$$\Xi = (I, S, (P_i)_{i\in I}, (\succ_s, \triangleright_s, \overline{q}_s)_{s\in S})$$

where $(\succ_s, \triangleright_s, \overline{q}_s)$ encapsulates the choice structure of an institution s. With this generalized structure, a feasible matching incorporates information regarding the assignments of individuals to specific positions within an institution, thereby encoding both the matching of individuals to institutions and the positions to which those individuals are assigned.

Example 1. *Consider a stage matching problem $\Xi = (I, S, (P_i)_{i\in I}, (\succ_s, \triangleright_s, \overline{q}_s)_{s\in S})$. Let $I = \{a, b, c, d, e, f\}$ and $S = \{s_1, s_2\}$ with capacities $\overline{q}_1 = 3$ and $\overline{q}_2 = 2$. Institutions with $\mathcal{B}_{s_1} = \{p_1^1, p_1^2, p_1^3\}$ and $\mathcal{B}_{s_2} = \{p_2^1, p_2^2\}$, have choice rule generated by $\succ_1 = (\succ_1^1, \succ_1^2, \succ_1^3)$ and $\succ_2 = (\succ_2^1, \succ_2^2)$ respectively, and precedence order $p_1^1 \triangleright_{s_1} p_1^2 \triangleright_{s_1} p_1^2$ and $p_2^1 \triangleright_{s_2} p_2^2$ respectively.*

$\succ_1^1$: $d - a - b - e - \emptyset_{s_1}$	$\succ_2^1$: $b - e - c - f - \emptyset_{s_2}$
$\succ_1^2$: $b - c - f - \emptyset_{s_1}$	$\succ_2^1$: $e - a - d - b - c - \emptyset_{s_2}$
$\succ_1^3$: $d - c - f - a - \emptyset_{s_1}$	

Consider a feasible matching μ such that $\mu(s_1) = \{a, c, f\}$ and $\mu(s_2) = \{b, d\}$. Given the

SSP structure, it must be that p_1^1 is associated with individual a, p_1^2 with individual c, and p_1^3 with individual f. Likewise, p_2^1 and p_2^2 must be associated with individuals b and d respectively. This is a "derived" individual-position matching from an individual-institution matching.

Thus, the SSP structure enables us to define an associated matching for a feasible stage matching μ. For a given $(\succ_s, \triangleright_s)$, the associated matching maps each individual to an institution-position pair. We denote this derived matching[9] as $\hat{\mu} : I \cup S \to 2^I \cup (S \times \mathcal{B})$ where $\mathcal{B} = \bigcup_{s \in S}\{\mathcal{B}_s\}$ is the collection of all the positions in the set of institutions S.

DEFINITION. Slot-specific matching (SSM) for a stage matching μ is a mapping $\hat{\mu} : I \cup S \to 2^I \cup (S \times \mathcal{B})$ such that for all $s \in S$ and $i \in I$,

1. $i \in \hat{\mu}(s)$ implies $i \in \mu(s)$.
2. $\hat{\mu}(i) = (s, p)$ for some $p \in \mathcal{B}_s$ implies $\mu(i) = s$.

We illustrate with the example below that not every feasible matching μ can be associated with an SSM.

Example 2. *Consider a feasible matching μ for the stage problem described in Example 1, where $\mu(s_1) = \{a, e, f\}$ and $\mu(s_2) = \{b, d\}$. It can be noted that institution s_2 via SSP structure, allocates b to position p_2^1 and d to p_2^2. On the other hand, there is no associated SSM for institution s_1. This is because s_1 allocates a to p_1^1 and f to p_1^2 from the set $\{a, e, f\}$. However, there is no acceptable alternative for p_1^3 from the remaining set $\{e\}$. Thus, e cannot be assigned to s_1 with the given priority profile and precedence order. Nevertheless, it is evident that each individual in $\{a, e, f\}$ is acceptable by some position in s_1.*

Proposition 1. *Let μ be a stage matching for a stage problem $\Xi = (I, S, (P_i)_{i \in I}, (\succ_s, \triangleright_s, \overline{q}_s)_{s \in S})$. Then μ has an associated SSM $\hat{\mu}$ if and only if μ is individually rational for institutions.*

For a matching μ such that $C_s(\mu(s)) = \mu_s$, each individual $i \in \mu(s)$ can be associated with a position in s. This is because each position finds the maximal individual of the surviving set

[9]The matching function $\hat{\mu}$ is different from the matching outcome $\tilde{\mu}$ of the associated one-to-one matching problem defined in Kominers and Sönmez (2016).

acceptable. It can therefore be concluded that every stage stable matching μ has an SSM $\hat{\mu}$. The 'if' of Proposition 1 is proved in Example 2. Also, the uniqueness is guaranteed through the SSP procedure. In subsequent sections, we restrict our attention to feasible matchings that have an associated SSM [10].

3 Multi-period Matching and Updating Choice Rules

We study a finite period matching problem that consists of a sequence of stage problems $(\Xi^t)_{1\leq t\leq T}$ where $\Xi^t = (I^t, S^t, (P_i^t)_{i\in I}, (\succ_s^t, \rhd_s^t, \overline{q}_s^t)_{s\in S})$ is the problem at stage t. The choice rule of an institution s has an SSP structure with the set of positions $\mathcal{B}_s^t$, profile of linear order $\succ_s^t$ and the precedence order $\rhd_s^t$.

DEFINITION. A sequence of T stage problems Ξ^1, Ξ^2,..., Ξ^T is **nested**[11] if for all $t = 1, ..., T-1$ and $s \in S$,

1. $I^{t+1} \subseteq I^t$
2. $S^t = S^{t+1} = S$
3. $\mathcal{B}_s^{t+1} \subseteq \mathcal{B}_s^t$

Following the terminology of H&I, an individual $i \in I^t$ means that i is **active** at stage t. Individual $i \in I^t \setminus I^{t+1}$ $(t < T)$ means that i **finalize** her assignment at stage t before the final stage. We denote by $t_i := \arg\max_{1\leq t\leq T}\{i \in I^t\}$, the stage at which individual i finalizes her assignment. At this period, the individual i is last active. Once the individual finalizes her assignment, she can not be active in further stages.

A sequence $((\Xi^t), \mu^t)_{1\leq t\leq T}$ is feasible if for each stage $t \in \{1, \ldots, T\}$, μ^t is a feasible stage matching for the problem Ξ^t.

[10] This assumption is inherently satisfied when dealing with stable matchings.
[11] The concept of nestedness in our setup is weaker than the one defined in H&I.

3.1 Updating Institutions' Choice Rules

The institutions update their choice rule, linking the stage problems with the matching "proposed" in the previous period[12]. Given a feasible matching μ^t for the stage problem Ξ^t and $(\succ_s^t, \triangleright_s^t)$ that generates C_s, the choice rule in next period is obtained by removing "positions that are assigned to individuals" who finalize their assignments[13]. The idea is that when an individual finalizes her assignment, she leaves with the position she is assigned to. No other positions are added or removed in this process. The relative precedence order between two positions and the priority order over the set of active individuals remains unchanged. Such a choice update rule is referred to as *consistent*, or alternatively, the choice rule is considered to be *updated consistently*.

DEFINITION. A choice update rule is **consistent** if for all $s \in S$ and $t = 1, \ldots, T-1$, given μ^t,

1. $\triangleright_s^{t+1} = \triangleright_s^t$.
2. For all $p \in \mathcal{B}_s^{t+1}$ and $i, j \in I^{t+1}$, $i(\succ_s^p)^t j$ implies $i(\succ_s^p)^{t+1} j$.
3. if $i \in I^t \setminus I^{t+1}$ and $\hat{\mu}^t(i) = (s, p)$ for some $p \in \mathcal{B}_s^t$, then $p \notin \mathcal{B}_s^{t+1}$.

Let us illustrate this choice update rule with an example.

Example 3. *Consider a nested two-period matching problem* (Ξ^1, Ξ^2) *where* $I^1 = \{a, b, c, d, e, f\}$ *and* $S^1 = \{s_1, s_2\}$. *The capacities of* s_1 *and* s_2 *at* $t = 1$ *is* $\overline{q}_1^1 = 3$ *and* $\overline{q}_2^1 = 2$. *The institutions have choice rules generated by* $((\succ_s)^1, \triangleright_s^1)$ *as described in Example 1. Suppose the stage matching at* $t = 1$ *be* μ^1 *where* $\mu^1(s_1) = \{a, b, c\}$ *and* $\mu^1(s_2) = \{d, e\}$. *The associated SSM is* $\hat{\mu}$ *depicted in the priority profile below.*

$(\succ_1^1)^1 : d - \mathbf{a} - b - e - \emptyset_{s_1}$

$(\succ_1^2)^1 : \mathbf{b} - c - f - \emptyset_{s_1}$

$(\succ_1^3)^1 : d - \mathbf{c} - f - a - \emptyset_{s_1}$

$(\succ_2^1)^1 : b - \mathbf{e} - c - f - \emptyset_{s_2}$

$(\succ_2^1)^1 : e - a - \mathbf{d} - b - c - \emptyset_{s_2}$

[12] We will be using "stage" and "period" interchangeably.

[13] Based on the discussion in Section 2, μ^t is individually rational for institutions to guarantee that each individual matched to an institution is assigned a position at the institution. When we are referring to a feasible matching generated by the SSP structure, this condition is assumed in the background.

Now after the first period, b finalizes her assignment and others participate in the next stage. That is, $I^2 = \{a, c, d, e, f\}$.

If the choice rule of institutions is updated consistently, then the only the position occupied by b, that is p_1^2, is removed from $\mathcal{B}_{s_1}$, and $\mathcal{B}_{s_2}$ is the same. Thus, $\overline{q}_1^2 = \overline{q}_2^2 = 2$. Over the remaining positions, the precedence order is unchanged and the priority orders are updated as follows:

$$(\succ_1^1)^2 : d - a - e - \emptyset_{s_1} \qquad (\succ_2^1)^2 : e - c - f - \emptyset_{s_2}$$

$$(\succ_1^3)^2 : d - c - f - a - \emptyset_{s_1} \qquad (\succ_2^1)^2 : e - a - d - c - \emptyset_{s_2}$$

It can be observed that if the choice rule of the institution is responsive, that is, the choice rule is generated by a preference order P_s, and top $\overline{q}_s$ individuals are chosen, updating the choice rule consistently reduces to the following: $P_s^t = P_s^1|_{I^t}$ where $= P_s^1|_{I^t}$ is the restriction of P_s^t to the set of active individuals I^t. As all the positions are "homogenous", the updating rule is not dependent on the SSM associated with the feasible matching at previous period. It is noteworthy that in case of non-homogenous positions, uniqueness of SSM is critical in updating the choice rule.

The second observation is that this rule implicitly puts a restriction on the relationship between the capacity of the institutions at every stage. The capacity of an institution is the number of unassigned seats plus the number of active individuals from the previous period. That is, for all $t \leq T - 1$,

$$\overline{q}_s^{t+1} = (\overline{q}_s^t - |\mu^t(s)|) + |\{i \in I^{t+1} : \mu^t(i) = s\}|$$

This relationship between the capacities across stages is assumed explicitly in H&I, which is an implication of our choice update rule. It should be noted in the subsequent sections that if the choice update rule is inconsistent but still adheres to the capacity constraints as described above, our main result will not be applicable.

3.2 Gradual Matching Mechanisms

The multi-period matching problem that we are interested in has the following properties (i) the sequence of stage problems $(\Xi^t)_{1 \leq t \leq T}$ is nested, and (ii) choice rule of institutions is updated consistently. In other words, individuals participate in a college admissions problem by submitting their ROL. Each individual decides to either finalize their match, allocated to them via stage mechanism, or to participate in the next period. The college admissions problem then comprises of active individuals with updated ROLs and choice rule of institutions that is updated consistently. All the individuals finalize their match at some period, after which exit the mechanism. Section 6 examines an application of this matching mechanism.

Thus, the nestedness of matching problems and the consistency of institutions enable us to reduce this sequential problem to a simpler framework. Instead of individuals deciding to continue or finalize the match at every period, this centralized framework only requires the first stage problem Ξ^1 of the sequence $(\Xi^t)_{1\leq t\leq T}$ and a list of ROLs $(P_i^1, P_i^2, \ldots, P_i^{t_i})$ for every $i \in I^1$, denoted by $\mathbf{P}_i = (P_i^t)_{t\leq t_i}$. We refer to this class of multi-period matching problems as **gradual matching problems** and is represented as $\mathbf{\Xi} = (I^1, S, (\mathbf{P}_i)_{i\in I}, (\succ_s^1, \rhd_s^1, \overline{q}_s^1)_{s\in S})$.

An outcome of a gradual matching problem consists of a sequence $(\Xi^t, \mu^t)_{t\leq T}$ that associates a feasible matching to every stage problem at every $t = 1, \ldots, T$. This outcome implicitly defines a matching $\nu : I \cup S \to 2^I \cup S$ such that $\nu(i) = \mu^{t_i}(i)$ for all $i \in I^1$. We refer to this sequence as a **gradual outcome**.

We are interested in imposing restrictions on the ROLs of individuals across different periods (till they are active) to ensure desirable properties of feasible matchings. H&I propose the concept of a *refitting rule* that defines the set of admissible ROLs based on both the ROL submitted by the individual and the match proposed to her in the preceding period. Formally, a refitting rule $\Gamma : \mathcal{P} \times (S \cup \{\emptyset\}) \to \mathcal{P}$ is a selection correspondence that ensures consistency in the updating process as follows: for every $(P, s) \in \mathcal{P} \times (S \cup \{\emptyset\})$,

- If $s \in A_P$, then $A_{P'} \neq \emptyset$ for some $P' \in \Gamma(P, s)$;
- For each $P' \in \Gamma(P, s)$, we have $A_{P'} \subseteq A_P$.

These mild conditions require that an institution —unacceptable at previous stages —can not be expressed as acceptable in updated ROL. There is a wider class of refitting rules that satisfy these mild conditions, including identity mapping[14], truncation mapping[15] and others[16]. It is noteworthy that we take decision of the individuals to finalize their match as given. It is reflected in the length of the sequence of ROLs submitted. Thus, the *strategic behaviour* of the individuals is outside the scope of this model.

For a given refitting rule Γ and a stage mechanism φ, a **gradual matching mechanism**, denoted by $\mathcal{M}_\Gamma^\varphi$, maps every gradual matching problem to a gradual outcome $\mathcal{M}_\Gamma^\varphi(\Xi) \equiv (\Xi^t, \mu^t)_{t\leq T}$ such that (i) $\mu^t = \varphi(\Xi^t)$ for all $t = 1, \ldots, T$, and (ii) $P_i^t = \Gamma(P_i^{t-1}, \mu^{t-1}(i))$ for all $i \in I^1$ and $t = 2, \ldots, T$. We restrict our attention to stable stage mechanisms, that play a key role in the matching literature. When C_s is generated by some $(\succ_s, \rhd_s)$, one such mechanism is *individual-proposing deferred acceptance*, that follows an iterative procedure described below.

Step 1: Every individual applies to her highest ranked acceptable institution under P_i (if any). Let $\hat{I}_s^1$ be the set of agents applying to institution s. The institution tentatively accepts the set $I_s^1 = C_s\left(\hat{I}_s^1\right)$ and rejects the applicants in $\hat{I}_s^1 \backslash I_s^1$.

Step t: $(t \geq 2)$. Every individual not rejected at step $t-1$ applies to her next ranked acceptable institution. Let $\hat{I}_s^t$ be the new set of individuals applying to institution s. The institution tentatively accepts the set $I_s^t = C_s\left(I_s^{t-1} \cup \hat{I}_s^t\right)$ and rejects the applicants in $(I_s^{t-1} \cup \hat{I}_s^t) \backslash I_s^t$.

The deferred acceptance algorithm terminates when every individual—not tentatively accepted by some institution —is rejected by some institution acceptable to her.

4 Monotonic Outcomes

A desirable property of a gradual matching outcome is that, for each period when an individual is active, they are assigned to an institution that is ranked weakly higher than the institution to which they were previously assigned. In effect, the current assignment is acceptable for individuals with respect to the "outside option"—assignment proposed in the previous period. We demonstrate through Example 4 that not every gradual outcome has this property.

[14] An identity mapping is a singleton-valued selection correspondence such that for all $(P, s) \in \mathcal{P} \times (S \cup \{\emptyset\})$, $\Gamma(P, s) = \{P\}$.

[15] A truncation mapping is a singleton-valued selection correspondence discussed in Manjunath and Turhan (2016) such that $\Gamma(P, s) = \{P'\}$ where $sPs' \implies \emptyset P's'$ and $s'Ps''Ps \implies s'P's''P's$

[16] H&I provides a detailed discussion on refitting rules.

Example 4. *Consider the two-stage gradual matching problem in Example 3. The choice rule and capacity of institutions in first period is as described before. The ROLs P^1 of individuals at $t = 1$ is as follows:*

P_a^1	P_b^1	P_c^1	P_d^1	P_e^1	P_f^1
$\underline{s_1}$	$\underline{s_1}$	$\underline{s_1}$	$\underline{s_2}$	$\underline{s_2}$	s_2
s_2	$\emptyset$	s_2	s_1	$\emptyset$	s_1
$\emptyset$		$\emptyset$	$\emptyset$		$\underline{\emptyset}$

Let the stage mechanism φ be individual-proposing DA. The matching $\mu^1 = \varphi(\Xi^1)$ is such that $\mu^1(s_1) = \{a, b, c\}$, $\mu^1(s_2) = \{d, e\}$, and f is unassigned, as underlined in the profile above. The associated SSM is $\hat{\mu}$ depicted in the priority profile $(\succ)^1$ of Example 3.

Now, let b finalize its allocation at $t = 1$ and other individuals participate in the next period. The updated choice rule and capacities of institutions at $t = 2$ is depicted in Example 3 and the ROLs of active individuals is updated to $P_i^2 = \Gamma(P_i^1, \mu^1(i))$ for all $i \in \{a, c, d, e, f\}$ as follows:

P_a^2	P_c^2	P_d^2	P_e^2	P_f^2
s_1	s_2	$\underline{s_1}$	$\underline{s_2}$	s_1
$\underline{s_2}$	$\underline{s_1}$	$\emptyset$	$\emptyset$	s_2
$\emptyset$	$\emptyset$			$\underline{\emptyset}$

The stage mechanism φ results in allocation μ^2 such that $\mu^2(s_1) = \{c, d\}$, $\mu^2(s_2) = \{a, e\}$, and f is unassigned, as underlined in the profile above. It can be noted that outcome for a is strictly worse in second period, that is, $s_1 = \mu^1(a) P_3^2 \mu^2(a) = s_2$. However for d, the matching is strictly better in the second period with respect to P_d^2.

We refer to such gradual outcomes as *monotone* outcomes.

DEFINITION. A gradual outcome $(\Xi^t, \mu^t)_{t \leq T}$ is **monotone**[17] if, for each $2 \leq t \leq T$ and $i \in I^t$,

$$\mu^t(i) R_i^t \mu^{t-1}(i).$$

[17] H&I refer to this property as *gradual safety*. Since our model is motivated by Baswana et al. (2019), we adopt their terminology.

A mechanism $\mathcal{M}_\Gamma^\varphi$ is monotone if for every gradual matching problem Ξ, the gradual outcome $(\Xi^t, \mu^t)_{t\leq T}$ is monotone.

One approach to ensure that every individual is matched to a weakly higher institution than the one proposed in the previous period is that they "adhere" to the proposal made in the preceding period. What is meant by this is that individuals update their ROLs in such a way that no institution, ranked lower than the proposal in the prior period —if acceptable in the current period —is now ranked higher. This property of updating structure, we refer to as *proposal-adhering* within the framework of refitting rules. Formally,

DEFINITION. A refitting rule Γ is **proposal-adhering**[18] if for all $(P, v) \in \mathcal{P} \times (S \cup \{\emptyset\})$ and $s \in S \setminus \{v\}$, if $P' \in \Gamma(P, v)$ then,

$$sP'v \text{ and } sP'\emptyset \implies sPv$$

Refitting rules like identity mapping, truncation mapping are proposal-adhering. The idea for individuals to be "individually rational" with respect to the match in the previous round.

Example 5. *In Example 4, Γ is a singleton-valued correspondence. Note that it is not a proposal-adhering correspondence for individual d because for $(P, v) = (P_d^1, s_2)$ and $s = s_1$, it is true that $s_1 P_d^2 s_2$ and $s_1 P_d^2 \emptyset$, yet $s_2 P_d^1 s_1$. Likewise, for $(P, v) = (P_c^1, s_1)$ and $s = s_2$, the condition is not satisfied. However, updated ROL of individual f satisfies the condition of proposal-adhering refitting rules.*

We now state our main result, that generalizes Theorem 1 of H&I.

Theorem 1. *Let $\mathcal{M}_\Gamma^\varphi$ be the gradual mechanism where φ is individually-optimal stage stable mechanism (IOSSM) and Γ is the refitting rule. Then the following statements are equivalent:*

1. *$\mathcal{M}_\Gamma^\varphi$ is monotone.*

2. *Γ is proposal-adhering.*

[18] While H&I refer to this property as *regularity*, we assert that the terminology *proposal-adhering* better aligns with the interpretation of the rule.

This equivalence result holds significant policy implications, particularly in the context of designing admission procedures, where students are offered several options after every round to participate in subsequent rounds.

It is important to note that the existence of an IOSSM is not guaranteed when employing general choice rules. However, the existence of such mechanisms, specifically in the case of choice rules with an SSP structure, does follow from the results presented in Hatfield and Milgrom (2005) and Aygün and Sönmez (2013). Refer to Appendix 2.A.1 for the proof.

It is worth noting that refitting rules have also been employed in the static framework to model properties "globally"[19]. Kojima and Manea (2010) introduce a notion of *individually rational monotonic transformation (i.r.m.t)* of a preference profile. A preference profile R is an *i.r.m.t* of R at $s \in S \cup \{\emptyset\}$ if for every individual i, any institution that is ranked above both s and $\emptyset$ under R_i' is ranked above s under R_i. In other words, R *i.r.m.t* R at $s \in S \cup \{\emptyset\}$ if $R_i' = \Gamma(R_i, s)$ for all $i \in I$ and Γ is proposal-adhering. A stage mechanism φ satisfies *IR monotonicity* if every individual weakly prefers the allocation $\varphi(R')$ to the allocation $\varphi(R)$ under R' whenever R' i.r.m.t R at $\varphi(R)$. Note that monotonicity of outcomes in a sequential setting differs from that in a static framework as the underlying stage problems at every period is different. The first characterization result of the individual-proposing DA algorithm by Kojima and Manea (2010) relies on IR monotonicity as a necessary axiom. Theorem 1 in this section can be seen as an extension of this result in a sequential framework.

Proposition 2. *Let φ be the individual-proposing DA mechanism and C_s be an acceptant, substitutable choice rule. Then, for all Ξ and proposal-adhering Γ,*

$$P_i' \in \Gamma(P_i, \mu(i)) \quad \implies \quad \mu'(i) P_i' \mu(i) \quad \forall\, i \in I$$

[19] A property is modelled "globally" by considering and linking changes in allocations due to changes in preference profiles (for instance, strategy-proofness, maskin monotonicity etc). However, a property is modelled "locally" via properties of allocations (for instance, efficiency, fairness etc).

5 Gradual Stability

In this section, we introduce a stability notion for gradual matching problems. Our definition of gradual stability generalizes the notion of *gradual stability* introduced in H&I. When institutions' choice rules are responsive, our definition reduces to theirs.

DEFINITION. A gradual outcome $(\Xi^t, \mu^t)_{t \le T}$ is **gradually stable** if for all $i \in I^1,\ t' \le t \le t_i$,

1. *Individual Rationality:* $\mu^t(i) R_i^t \emptyset$ and $\mu^t(i) R_i^t \mu^{t'}(i)$,
2. *Non-wastefulness:* if $|\mu^{t'}(s)| < \overline{q}_s^{t'}$ for some $s \in S$, then $s P_i^t \mu^t(i)$ implies $\emptyset_s \succ_s^p i$ where $p \in \mathcal{B}_s^{t'}$ is unassigned at t',
3. *No justified envy:* for all $j \in I^{t'} \setminus I^{t'+1}$, if $\hat{\mu}^{t'}(j) = (s, p)$ for some $s \in S$, $p \in \mathcal{B}_s^{t'}$, then $s P_i^t \mu^t(i)$ implies $j \succ_s^p i$.

The stability of the stage matching mechanism considers only the final assignments of the individuals and institutions. However, in the gradual matching mechanism, individuals finalize their matchings at different stages. As a result, the definition above extends the notion of stage stability to also include the claims by individuals across stages as long as they are active.

The first condition requires that each individual's assignment must be *individually rational* for each individual at every stage. Here, individual rationality is defined by comparing an assignment with the "outside option". In the case of a stage problem, this outside option corresponds to being unmatched or having an empty matching. In the sequential problem, proposals from previous periods also serve as an outside option.

The second condition is a sequential version of *non-wastefulness*. If an individual i prefers institution s to her stage t assignment, this condition ensures that all positions in s that are deemed acceptable by individual i, must be assigned to someone else in the current period and all of the previous periods.

Finally, the last condition is a *no justified envy* condition adapted to our sequential environment with an SSP structure. Consider two individuals i and j such that j finalizes her assignment before i. Then, justified envy by i against j is checked for all periods $t_j \le t \le t_i$ using the priority order of the position that is assigned to j when she finalizes her assignment.

A gradual mechanism is gradually stable if the outcome for each gradual problem is stable. The theorem we propose in this section establishes a relationship between gradual stability, stage stability, and the monotonicity of gradual outcomes.

Theorem 2. *Let Γ be a proposal-adhering refitting rule and φ be a stage mechanism. Then the following statements are equivalent:*

1. *$\mathcal{M}_{\Gamma}^{\varphi}$ is gradually stable.*

2. *$\mathcal{M}_{\Gamma}^{\varphi}$ is monotone and φ is stage stable.*

Theorem 1 and 2 together indicate that when the stage mechanism is stage stable and Γ is a proposal-adhering refitting rule, the monotonicity of the gradual outcome implies the stability of the mechanism within the sequential context. The proof of this theorem can be found in Appendix 2.A.2.

6 Admissions to Engineering Colleges in India

The admission process in engineering colleges in India is subject to a comprehensive affirmative action program, which has been implemented via a reservation system. The reservation scheme at an institution s partitions the set of positions $\mathcal{B}_s$ in various categories $\mathcal{R}$ and the set of individuals in categories $\mathcal{C}$. It consists of the following key components:

- $\mathcal{R} = \{SC, ST, OBC, EWS\}$ denote the set of reserved categories. The students that belong to no reserved category are in General Category (GC).

- $\mathcal{C} = \{o, SC, ST, OBC, EWS\}$ denote the set of all position categories (o is the open category).

- The vector $q_s = \left(q_s^o, q_s^{SC}, q_s^{ST}, q_s^{OBC}, q_s^{EWS}\right)$ describes the *initial distribution* of positions over reserved categories where $q_s^o = \overline{q}_s - q_s^{SC} - q_s^{ST} - q_s^{OBC} - q_s^{EWS}$. The profile of vectors for the initial distribution of positions over categories at institutions is denoted by $\mathbf{q} = (q_s)_{s \in \mathcal{S}}$.

- The function $t : I \to \mathcal{R} \cup \{GC\}$ denotes the category membership of individuals. For every individual $i \in I$, $t(i)$, or t_i, denotes the category individual i belongs to. We denote a profile of reserved category membership by $T = (t_i)_{i \in \mathcal{I}}$, and let $\mathcal{T}$ be the set of all possible reserved category membership profiles.

Merit scores induce strict meritorious ranking of individuals at each institution s, denoted by $\succ_s$, which is a linear order over $\mathcal{I} \cup \{\emptyset\}$. $i \succ_s j$ means that applicant i has a higher priority (higher merit score) than applicant j at institution s. We write $i \succ_s \emptyset$ to say that applicant i is acceptable for institution s. Similarly, we write $\emptyset \succ_s i$ to say that applicant i is unacceptable for institution s. The profile of institutions' priorities is denoted $\succ = (\succ_{s_1}, ..., \succ_{s_m})$.

For each institution $s \in \mathcal{S}$, the merit ordering for individuals of type $r \in \mathcal{R}$, denoted by $\succ_s^r$, is obtained from $\succ_s$ in a straightforward manner as follows:

- for $i, j \in \mathcal{I}$ such that $t_i = r$, $t_j \neq r$, $i \succ_s \emptyset$, and $j \succ_s \emptyset$, we have $i \succ_s^r \emptyset \succ_s^r j$, where $\emptyset \succ_t^r j$ means individual j is unacceptable for category r at institution s.

- for any other $i, j \in \mathcal{I}$, $i \succ_s^r j$ if and only if $i \succ_s j$.

The *over-and-above* implementation requires filling open-category positions before the reserved categories. Formally, given an initial distribution of positions q_s, a set of applicants $A \subseteq I$, and a category membership profile $T \in \mathcal{T}$ for the members of A, the set of chosen applicants $C_s^{\text{Res}}(A, q_s)$, is computed as follows:

Step 1: Unreserved positions are considered first. Individuals are chosen one at a time following $\succ_s$ up to the capacity q_s^o. Let us call the set of chosen applicants $C_s^o(A, q_s^o)$.

Step 2: Among the remaining applicants $A' = A \backslash C_s^o(A, q_s^o)$, for each reserve category $r \in \mathcal{R}$, applicants are chosen one at a time following $\succ_s^r$ up to the capacity q_s^r. Let us call the set of chosen applicants for reserve category r as $C_s^r(A', q_s^r)$.

Then, $C_s^{Res}(A, q_s)$ is defined as the union of the set of applicants chosen in Steps 1 and Step 2.

That is,

$$C_s^{Res}(A, q_s) = C_s^o(A, q_s^o) \cup \bigcup_{t \in \mathcal{R}} C_s^t(A', q_s^t)$$

This leads to our first lemma of this section, which we state formally below.

LEMMA **2.** *The selection rule of engineering colleges in India can be modeled via choice rule, C_s^{Res} that admits SSP structure.*

6.1 The Multi-round Deferred Acceptance Mechanism

Admissions to engineering colleges in India implement a multi-round matching procedure. Each individual submits an ROL over all programs, including IITs and non-IITs. Each program provides the number of available positions and a merit list of eligible individuals. After collecting this information, the individual-proposing DA mechanism is run in each period for IITs and non-IITs separately[20]. Some individuals may finalize their assignments, while others may want to participate in future rounds. The inputs of individual-proposing DA are modified in each round. The options available to individuals at the end of each round are *freeze*, *float*, *slide*, *reject*, and *withdraw* (see Baswana et al. (2019)).

LEMMA **3.** *The permitted refitting rule is proposal-adhering.*

This lemma is validated by the description of the "options" available to the individuals. The ROLs of candidates in this application can be understood over all the possible (university, program) pairs. However, the institutions are the universities offering seversal programs. The notion of refitting rules applies to this setup too.

Reject. If a candidate rejects an assigned program, then the candidate is completely removed from the assignment process by setting his/her rank-ordered list to an empty set.

$$\Gamma(P, s) = P', \text{ where } \emptyset P' s' \text{ for all } s' \in S.$$

Freeze. Candidates who choose this option accept the program assigned to them in that round. Their ROLs are modified so the assigned program and all other programs ranked below it are kept

[20]Akin to the algorithm proposed by Manjunath and Turhan (2016).

while the rest are removed. Formally,

$$\Gamma(P,s) = P' \text{ where } \emptyset P's' \text{ for all } s' \in S \text{ such that } s'Ps, \text{ and } P, P' \text{ agree for the rest.}$$

Float. ROLs of the individuals who choose the float option remain unchanged. These candidates are willing to participate in future rounds in the hope of getting assigned to a better program.

$$\Gamma(P,s) = P.$$

Slide. Individuals who choose the slide option opt to participate in the future rounds but be considered only for the programs in the same university as the assigned program. In this case, programs in all other universities above the assigned program are removed. The assigned program and other programs below it remain unchanged. Let $U(s)$ be the set of programs that are in the same university as program s.

$$\Gamma(P,s) = P' \text{ where } \emptyset P's' \text{ for all } s' \notin U(s) \text{ such that } s'Ps, \text{ and } P, P' \text{ agree for the rest.}$$

Withdraw. Individuals who choose this option are removed from the problem. The ranking of these applicants are set to an empty set.

$$\Gamma(P,s) = P', \text{ where } \emptyset P's' \text{ for all } s' \in S.$$

Note that the 'reject' and 'withdraw' options have the same effect on the ROLs. The difference is about the timing. A candidate who previously accepts an offered program may withdraw in later stages.

Lemmas 2 and 3 suggest that we can employ Theorems 1 and 2 to state our final result.

Proposition 3. *Multi-round deferred acceptance mechanism implemented in engineering colleges in India is monotone and gradually stable.*

7 Conclusion

This chapter studies a special class of multi-period matching mechanisms. By generalizing the framework of H&I, we can explain a wider range of applications, including the college admissions process in engineering colleges in India. The French college admission system (studied in H&I) is another application that can be explained by our model.

One possible approach to understanding our results is to consider an associated one-to-one matching market that corresponds to the original many-to-one matching market. This technique is utilized in Kominers and Sönmez (2016), where positions, rather than institutions, compete for individuals. In the one-to-one market with unit capacity, the priority structure of institutions is responsive. Therefore, the results of H&I can be employed to understand the one-to-one market. However, these results do not straightforwardly extend to the original many-to-one market. In our study, we take an alternate approach to comprehend our results. Our first result to a great extent relies on the proof of the first characterization result of DA by Kojima and Manea (2010). This enables us to draw a comparison of their static problem with our sequential model.

Appendix 2.A

2.A.1 Proof of Theorem 1

Proof. Let $\Xi = (I^1, S, (\mathbf{P}_i)_{i\in I}, (\succ_s^1, \rhd_s^1, \overline{q}_s^1)_{s\in S})$ be a gradual matching problem with the outcome $\mathcal{M}_\Gamma^{IOSSM}(\Xi) \equiv (\Xi^t, \mu^t)_{1\leq t\leq T}$. We first prove that IOSSM exists when the choice rule has an SSP structure.

LEMMA **4** (Hatfield and Milgrom (2005), Aygün and Sönmez (2013))**.** *When C_s satisfies substitutability and IRA, then IOSSM exists, and it is unique. Moreover, it is the outcome of the deferred acceptance algorithm.*

Lemma 1 and 4 prove the existence of IOSSM.

To prove that (1.) $\implies$ (2.), let $\mathcal{M}_\Gamma^{IOSSM}$ be monotone and Γ be not proposal-adhering. That is, there exists a Ξ such that for some $t \leq T-1$, and $i \in I^t$, $P_i^{t+1} \notin \Gamma(P_i^t, \mu^t(i))$. The counter-example provided in Proposition 1 of H&I suffices as responsive choice rules are a special case of choice rules with SSP structure where each institution is decomposed to identical multiple copies with unit demand.

LEMMA **5** (Proposition 1, H&I)**.** *Let $\mathcal{M}_\Gamma^\varphi$ be a gradual matching mechanism such that $\succ_s^p = \succ_s^{p'}$ for all $s \in S$ and $p, p' \in \mathcal{B}_s$. If $\mathcal{M}_\Gamma^\varphi$ is monotone, then Γ is proposal-adhering.*

Following the methodology used in Kojima and Manea (2010), we now show (2.) $\implies$ (1.). Let Γ be a proposal-adhering refitting rule. Consider Ξ^t for some $t \geq 2$. Let $\mu^{t-1} = \varphi(\Xi^{t-1})$ and $\mu^t = \varphi(\Xi^t)$ be stage matchings at period $t-1$ and t respectively. We need to prove that for all $i \in I^t$, $\mu^t(i) R_i^t \mu^{t-1}(i)$. We prove this with the steps below.

Step 1: Define x_0 as the allocation μ^{t-1} restricted to I^t. That is, $x_0(s) = \mu^{t-1}(s) \cap I^t$.

Define for all $i \in I^t$,

$$x_1(i) = \begin{cases} x_0(i), & \text{if } x_0(i) P_i^t \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

If x_1 is a stable matching at $(P^t_{I^t}, (C^t_s)_{s\in S})$, then using the fact that φ generates individual optimal stable matching, $\mu^t(i) R^t_i x_1(i)$ for all $i \in I^t$ holds and we are done.

Let us define a sequence $(x_k)_{k\geq 1}$ as follows:

DEFINITION (Step-wise unblocking process). Define for all $k \geq 1$ and $i \in I^t$,

$$x_{k+1}(i) = \begin{cases} s_k, & \text{if } i \in C^t_{s_k}\left(x_k(i) \cup \left\{j \in I^t \mid s_k P^t_j x_k(j)\right\}\right), \\ x_k(i), & \text{otherwise.} \end{cases}$$

where s_k is an arbitrary institution that is part of a blocking pair if x_k can be blocked at $(P^t_{I^t}, (C^t_s)_{s\in S})$. If x_k cannot be blocked, then $x_{k+1} = x_k$.

We now prove the following lemma.

LEMMA **6.** *The sequence $(x_k)_{k\geq 0}$ satisfies for every $k \geq 1$:*

(I) x_k *is a feasible stage matching.*

(II) $x_k(i) R^t_i x_{k-1}(i)$ *for all* $i \in I^t$.

(III) $x_k(s) \subseteq C^t_s\left(x_k(s) \cup \left\{j \in I^t \mid s P^t_j x_k(j)\right\}\right)$ *for all* $s \in S$.

As $x_{k+1}(i) R^t_i x_k(i)$ for all $i \in I^t$, the step-wise unblocking process generates a sequence $(x_k)_{k\geq 0}$ that converges to a matching x_K in finite number of steps K. Each iteration results in a different allocation if the initial matching within the iteration is not stable. Hence, x_K is stable at $(P^t_{I^t}, (C^t_s)_{s\in S})$.

Because $x_K(i) R^t_i x_1(i) R^t_i \emptyset$ for all $i \in I^t$, the matching x_K is individually rational for agents. Also, as the outcome $\mathcal{M}^{SOSM}_F(\Xi)$ at time period t is the individual optimal among all the stable outcomes at t, we get for all $i \in I^t$,

$$\mu^t(i) R^t_i x_K(i) R^t_i \mu^{t-1}(i)$$

It remains to prove Lemma 6.

Proof. We prove the lemma by induction with the base case $k = 1$. (I) and (II) hold for $k = 1$ by definition of x_1. Consider any $s \in S$. We now prove that $x_1(s) \subseteq C_s^t\big(x_1(s) \cup \big\{j \in I^t \mid sP_j^t x_1(j)\big\}\big)$. By definition of $x_1(i)$, we have

$$x_1(s) \subseteq x_0(s) \subseteq \mu^{t-1}(s)$$

As Γ is proposal-adhering,

$$\{j \in I^t \mid sP_j^t x_1(j)\} \subseteq \{j \in I^t \mid sP_j^{t-1} x_0(j)\}$$

Together we get

$$x_1(s) \cup \{j \in I^t \mid sP_j^t x_1(j)\} \subseteq x_0(s) \cup \{j \in I^t \mid sP_j^{t-1} x_0(j)\} \tag{1}$$

As the period $t-1$ outcome is stable at $(P_{I^{t-1}}^{t-1}, (C_s^{t-1})_{s\in S})$, we have

1. $C_s^{t-1}(\mu^{t-1}(s)) = \mu^{t-1}(s)$ for all $s \in S$, and
2. $i \notin C_s^{t-1}(\mu^{t-1}(s) \cup \{i\})$, for all $i \in \{j \in I^t \mid sP_j^{t-1} x_0(j)\}$.

By consistency of C_s^{t-1}, we have

$$C_s^{t-1}(\mu^{t-1}(s) \cup \{j \in I^t \mid sP_j^{t-1} x_0(j)\}) = \mu^{t-1}(s).$$

In order to state the set-inclusion property at stage t, we state our next lemma.

LEMMA **7.** *If C_s^t has an SSP structure and the choice update rule is consistent, then for all $A, B \subset I^t$,*

$$A \subseteq C_s^t(B) \implies A \cap I^{t+1} \subseteq C_s^{t+1}(B \cap I^{t+1})$$

Proof. Let C_s^t be a choice rule that is generated by $(\succ_s^t, \rhd_s^t)$ and C_s^{t+1} is updated as per Definition 3.1. Let $i \in A \cap I^{t+1}$ such that $\hat{\mu}^t(i) = (s, p_s^j)$ for some $p_s^j \in \mathcal{B}_s^t$. As i is active at stage $t+1$, $p_s^j \in \mathcal{B}_s^{t+1}$. Since i is $\succ_s^j$ maximal at stage t of the surviving set of individuals at Step j, it must be that i is $\succ_s^j$ maximal at stage $t+1$ as no individual is added to the set. Thus, $\hat{\mu}^{t+1}(i) = (s, p_s^k)$ for some $k \leq j$. This completes the proof. $\square$

Using Lemma 7, we have

$$x_0(s) \subseteq C_s^t(x_0(s) \cup \{j \in I^t \mid sP_j^{t-1}x_0(j)\})$$

By substitutability of choice fuction C_s^t derived by updating structure of choice function,

$$x_1(s) \subseteq C_s^t\left(x_1(s) \cup \left\{j \in I^t \mid sP_j^t x_1(j)\right\}\right).$$

This concludes our proof for the base case.

Assuming the conclusions of step $k \geq 1$ hold, we now prove it for $k+1$ (the only case to prove is when $x_k \neq x_{k+1}$).

Let us prove (I) first. Consider $s \neq s_k$. Observe that $x_{k+1}(s) \subseteq x_k(s)$ by construction. As x_k is an allocation, by the inductive hypothesis we get $|x_{k+1}(s)| \leq |x_k(s)| \leq q_s^t$.

For institution s_k, $x_k(s_k) \subseteq C_{s_k}^t\left(x_k(s_k) \cup \left\{j \in I^t \mid s_k P_j^t x_k(j)\right\}\right)$ holds by inductive hypothesis at k. Then using definition of $x_{k+1}(i)$,

$$x_{k+1}(s_k) = C_{s_k}^t\left(x_k(s_k) \cup \left\{j \in I^t \mid s_k P_j^t x_k(j)\right\}\right)$$

Feasibility of choice rule $C_{s_k}^t$ thus guarantees that $|x_{k+1}(s_k)| \leq q_{s_k}^t$.

We now prove (II). Observe that

$$x_{k+1}(s_k)\backslash x_k(s_k) \subseteq \left\{j \in I^t \mid s_k P_j^t x_k(j)\right\} \tag{2}$$

Thus, for $j \in x_{k+1}(s_k)\backslash x_k(s_k)$, we get

$$s_k = x_{k+1}(j) P_j^t x_k(j) \tag{3}$$

Each agent outside of $x_{k+1}(s_k)\backslash x_k(s_k)$ is assigned the same institution under x_{k+1} and x_k. Therefore, $x_{k+1}(i) R_i^t x_k(i)$ for all $i \in I^t$.

We now show (III) for all $s \neq s_k$. By construction, we have $x_{k+1}(s) \subseteq x_k(s)$. By Equation 3, we have $\left\{j \in I^t \mid sP_j^t x_{k+1}(j)\right\} \subseteq \left\{j \in I^t \mid sP_j^t x_k(j)\right\}$. Therefore,

$$x_{k+1}(s) \cup \{j \in I^t \mid sP_j^t x_{k+1}(j)\} \subseteq x_k(s) \cup \{j \in I^t \mid sP_j^t x_k(j)\} \tag{4}$$

Now, substitutability of C_s^t, inductive hypothesis for k (condition (II)), and Equation 4 implies

$$x_{k+1}(s) \subseteq C^t_s \left(x_{k+1}(s) \cup \left\{ j \in I^t \mid sP^t_j x_{k+1}(j) \right\} \right)$$

Let us now consider institution s_k. By Equation 2, agents in $x_{k+1}(s_k)\backslash x_k(s_k)$ prefer s_k over their allocation in x_k. Individuals who are not chosen from this set in this iteration are those who still prefer s_k over their allocation in x_{k+1}. This is because $x_{k+1}(i)R^t_i x_k(i)$ for all $i \in I^t$. This implies

$$x_{k+1}(s_k)\backslash x_k(s_k) = \left\{ j \in I^t \mid s_k P^t_j x_k(j) \right\} \setminus \left\{ j \in I^t \mid s_k P^t_j x_{k+1}(j) \right\}.$$

Or equivalently,

$$x_{k+1}(s_k) \cup \left\{ j \in I^t \mid s_k P^t_j x_k(j) \right\} = x_k(s_k) \cup \left\{ j \in I^t \mid s_k P^t_j x_{k+1}(j) \right\} \tag{5}$$

Using substitutability of C^t_s, (III) for k and Equation 5, we obtain

$$x_{k+1}(s_k) \subseteq C^t_{s_k} \left(x_{k+1}(s_k) \cup \left\{ j \in I^t \mid s_k P^t_j x_{k+1}(j) \right\} \right).$$

This concludes our proof of Lemma 6. □

□

2.A.2 Proof of Theorem 2

We employ the technique utilized by H&I in the proof of Theorem 2. The notions of stage stability and gradual stability introduced in this chapter are generalized versions of spot stability and gradual stability, respectively, as defined by H&I. Thus, it remains to show the equivalence between Definition 2 and Definition 5 for an arbitrary stage t and $t = t'$ for the stage mechanism φ. We refer to the conditions of Definition 2 as C1, C2, and C3, and the conditions of Definition 5 as C1', C2', and C3', respectively.

We first show that Definition 5 implies Definition 2. Then, C1 directly follows from C1'. If possible, assume that C2 is not true. That is, there exists an $s \in S$ such that $\mu^t(s) \subsetneq C_s(\mu^t(s))$. As $|\mu^t(s)| \leq \overline{q}_s$, this implies $|C_s(\mu^t(s))| < \overline{q}_s$. Suppose $i \notin C_s(\mu^t(s))$. Since C_s is an SSP choice rule, at each position $p_k \in \left\{p_1, \ldots, p_{\overline{q}_s}\right\}$, either (i) $\emptyset_s \succ^{p_k}_s i$ or (ii) there exists some other individual j such that $\hat{\mu}(j) = (s, p_k)$ and $j \succ^{p_k}_s i$. Both cases contradicts with our supposition that

$(\Xi^t, \mu^t)_t$ is gradually stable. Thus C2 is true.

If possible, assume that $C3$ is not true. That is, there exists an institution-individual pair (s, i) such that $sP_i^t\mu^t(i)$ and $i \in C_s^t(\mu^t(s) \cup \{i\})$. Let i be assigned to the position $p_s^k \in \mathcal{B}_s$. If p_s^k is unassigned at μ, then C2' is violated and if $\hat{\mu}(j) = (s, p_s^k)$, then C3' is violated. Thus, φ is stage stable.

We now prove that Definition 2 implies Definition 5. first, C1' follows from C1. If possible, C2' is violated. That is, there exists an institution s and individual i such that $sP_i^t\mu^t(i)$. Also, for some unassigned position $p \in \mathcal{B}_s$, $i \succ_s^p \emptyset_s$. This contradicts C3 as this implies $i \in C_s^t(\mu^t(s) \cup \{i\})$. We now prove C3' by contradiction. Consider $i, j \in I^t$ such that $\hat{\mu}^t(j) = (s, p)$, $\mu^t(i) \neq s$ and $i \succ_s^p j$ for some $p \in \mathcal{B}_s^t$. This implies $i \in C_s^t(\mu^t(s) \cup \{i\})$. If $sP_i^t\mu^t(i)$, C3 is violated. Thus, Definition 5 is true for $t = t'$.

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