Rank Dependent Weighted Average Utility Models for Decision Making under Ignorance or Objective Ambiguity*†

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Abstract

The paper provides an axiomatic characterization of a family of rank dependent weighted average utility criteria applicable to decision making under ignorance or objective ambiguity. Decision making under ignorance compares finite sets of final consequences while decision making under objective ambiguity compares finite sets of probability

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distributions over those final consequences. The criteria characterized are those that assign to every element in a set a *weight* that depends upon the rank of this element if it was available for sure (or non-ambiguously) and that compare sets on the basis of their weighted utility for some utility function. A specific subfamily of these criteria that requires the rank-dependent weights to result from a probability weighting function is also characterized.

1 Introduction

Consider the following decision problem, provided by Ahn (2008), of a cancer patient having to choose between two treatments. The first is a conventional and widely used chemotherapeutic treatment associated to a five-year survival rate of 0.5. The second is a new targeted therapeutic treatment that has only been tried on two samples of patients of comparable sizes. On one sample, 80% of the patients have been observed alive after 5 years but on the other sample, only 20% of the patients were alive after 5 years. This is an example of decision making under *objective ambiguity*. There is *ambiguity* because the probabilities (of survival) that enter in the description of the second treatment are not unique. The ambiguity is however *objective* because the probabilities, while multiple, are known to the decision maker and enter therefore in the description of the decision problem. This contrasts with decision making under subjective ambiguity studied in papers such as Gilboa and Schmeidler (1989), Epstein and Zhang (2001), Ghirartado and Marinacci (2002), Ghirartado, Maccheroni, and Marinacci (2004) or Klibanoff, Marinacci, and Mukerji (2005) in which the compared alternatives are described as Savagean acts without any *a priori* probabilities.[1] Another well-known example of an objectively ambiguous decision is the sequence of two choices made in the Ellsberg (1961) experiment.

[1] An excellent survey of the literature on subjective ambiguity is provided in Wakker (2010).

From a formal point of view, decision making under objective ambiguity amounts to *ranking sets* of possible probability distributions over a set of final consequences. The description of alternatives as sets of objects is also made in the literature on decision making under radical uncertainty or ignorance surveyed, for example, in Barberà, Bossert, and Pattanaik (2004), in which the elements of the sets are interpreted as the final consequences of the decisions rather than as probability distributions over those.

This paper contributes to the literature on decision making under ignorance or objective ambiguity under the additional assumption that the compared alternatives can be described as *finite sets* (of either final consequences or probability distributions). This approach therefore differs from that provided, for instance, in Ahn (2008) and Olszewski (2007) in which the compared alternatives are depicted as uncountable sets of objects. As argued in Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2018), we believe that the description of alternatives as finite sets of consequences (probability distributions) is somewhat natural, and clearly in line with experimental contexts in which we may want to test these models. For sure the Ellsberg experiment or the choice faced by the cancer patient above concern finite sets.

A lot of decision making criteria examined in the literature on ranking finite sets are based on the best and the worst consequences of the decisions or on associated lexicographic extensions. There are two obvious limitations of such "extremist" rankings. The first is that it is natural to believe (in line with various "expected utility" hypotheses) that decision makers are concerned with "averages" rather than "extremes". A second drawback of "extremist" rankings is that they do not allow for much diversity of attitudes toward ignorance across decision makers. In situations where the compared alternatives have only monetary consequences, all decision makers who use an "extremist" rule such as Maximin, Maximax or lexicographic extensions of the same and who prefer more money to less would rank lists of these amounts of money in exactly the same way. This is unsatisfactory since the fact for two decision makers to have the same preference over certain outcomes (or unambiguous decisions) should not imply that they have the same attitude toward ignorance or ambiguity. It is with the aim of obtaining less extreme rankings of finite sets that Gravel, Marchant, and Sen (2012) characterizes with three axioms the Uniform Expected Utility (UEU) family of criteria for comparing finite sets of objects. Any criterion from that family results from assigning to every conceivable element of a set a utility number and from comparing sets on the basis of the expectation of the utility of their elements under the (uniform) assumption that all elements in the sets are equally likely. Gravel, Marchant, and Sen (2018) generalizes the UEU family of rankings of finite sets to the Conditional Expected Utility (CEU) family. Any CEU ranking of finite sets assigns to every conceivable element of set both a utility number and a (strictly positive) likelihood, and compares sets on the basis of their expected utility, with expectations taken with respect to the relative likelihood of those elements conditional upon the fact that they are in the sets. A UEU ranking of sets is nothing but a specific CEU ranking for which the likelihood function considers all conceivable elements as equally likely. CEU rankings can be viewed as the finite analogues of the ranking of atomless sets of objects characterized by Ahn (2008) and, before him, by Bolker (1966) and Jeffrey (1983).

While UEU and CEU criteria provide simple criteria for decision making under ignorance or objective ambiguity, they both satisfy an axiom that may be at odd with actual decision making behavior. This axiom, called *Averaging* in Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2018) (and also Fishburn (1972)) and disjoint set betweenness in Ahn (2008)[2], requires the ranking (weak or strict) of two disjoint sets to be equivalent to the requirement that their union be ranked between the two sets. To see

[2] Weaker variants of this axiom are also satisfied by the ranking examined in Olszewski (2007) and, in another context, in Gul and Pesendorfer (2001).

why such an axiom may not always provide an accurate depiction of actual decision making under objective ambiguity, consider again the choice of a cancer treatment described above. Imagine that, in addition to the second targeted therapeutic experimental treatment, the patient be proposed a third treatment tested this time on three samples of sizes comparable to those of the second treatment, and with respective 5-year survival probabilities of 0.8, 0.5 and 0.2. One can then represent the three treatments by the sets $\{0.5\}$, $\{0.8, 0.2\}$ and $\{0.8, 0.5, 0.2\}$ respectively. It is plausible that a patient facing this (horrible) decision could prefer the traditional $\{0.5\}$ treatment to the new $\{0.8, 0.2\}$ treatment because of a (pessimistic) fear that the sample on which the new treatment has performed poorly provides a better assessment of its true effectiveness than the sample on which it has performed well. The Averaging axiom implies that the patient should then also prefer the third $\{0.8, 0.5, 0.2\}$ treatment (which is nothing else than the union of $\{0.5\}$ and $\{0.8, 0.2\}$) to $\{0.8, 0.2\}$. Should he really? This is not clear. Indeed, one could argue that the results of the experimentation of the third treatment $\{0.8, 0.5, 0.2\}$ are noisier than those of the second in terms of the information that they provide on the treatment's effectiveness. Hence, a pessimistic patient who gives more weights to the samples where the treatment performs poorly to those where it performs well could very well choose the second treatment over the third even though he has chosen the first treatment over the second. In the only instance we know where the averaging axiom has been tested in an experimental context (Vridags and Marchant (2015)), it has been rejected by a vast majority of subjects.

In this paper, we accordingly characterize a family of decision criteria that keeps the smoothness associated to the evaluation of alternatives as per their expected utility, while dispensing with the Averaging axiom in any of the forms considered in the literature. The criteria analyzed can be viewed as variants of the rank-dependent expected utility family originally proposed by Quiggin (1982) (see also Quiggin (1993)) that are suitable to the considered finite set theoretic framework. These criteria were hinted at in Vridags and Marchant (2015) where they were referred to as Uniform Rank-Dependent Utility criteria because they are equivalent to a rank-dependent model applied to a uniform probability distribution. However, in this paper, we use the more explicit name of Rank-Dependent Weighted Average Utility (RDWAU) to designate these criteria. A RDWAU criterion compares two alternatives (finite sets) on the basis of their weighted average utility, for some utility function defined over all elements of the universe and some non-negative weights that depend upon both the number of elements in the sets and the ranking of those elements if they were certain and that sum to one. The RDWAU family of criteria characterized in this paper is quite large. It contains as particular cases "extremist" rankings of sets like the Maximin or Maximax criteria mentioned above as well as those based on a weighted average of the min and the max (for example the alpha-maxmin rule widely discussed in Hartmann (2023) or Olszewski (2007) among many others). It also contains the UEU family of rankings, obtained by assigning the same weight to every element in the sets, and allows the weighting schemes of the elements to vary totally freely with the number of elements in the set (for example assigning a weight of 1 to the worst outcome in the case of sets with two elements, but switching to a weight of 1 to the best outcome in the case of set with three elements). In order to restrict a bit the way with which the weights are allowed to vary with the cardinality of the sets, we also provide characterizations of three subfamilies of RDWAU rankings that may be of interest. One of them consists in RDWAU rankings that are mildly optimistic or pessimistic and whose rank-dependent weights are, accordingly, weakly increasing or decreasing with respect to the rank of the outcomes in the set. Another subfamily of RDWAU rankings characterized in this paper are those for which all rank-dependent weights result from the same cumulative weighting function defined on all rational numbers in the $[0, 1]$ interval. RDWAU rankings generated by such a cumulative weighting function have been extensively discussed in the literature on rank-dependent expected utility models (see for example Luce (1988), Luce (1991), Prelec (1998) and Tverski and Wakker (1995)) in contexts where decision makers are comparing single lotteries instead of sets of outcomes. A (very) particular example of RDWAU rankings generated by a common cumulative weighting function—also characterized in this paper by an additional axiom—are those that satisfy the additional restriction that the ratio between any two adjacent weights is constant.

The organization of the remaining of the paper is as follows. The next section introduces the framework, notation and main definition. Section 3 provides the results and section 4 concludes.

2 Formal Framework

2.1 Notation and definitions

We let X be a universe of *outcomes* that we interpret either as possible *final consequences* of decisions (decision making under ignorance) or as *probability distributions* over a more fundamental set of final consequences (decision making under objective ambiguity). We shall nonetheless use the generic term of "outcomes" to designate these elements of X. Many results stated and proved in this paper will actually also ride on the assumption that X is a connected topological space[3]. Relevant examples of a set X could be monetary (possibly negative) consequences ($X = \mathbb{R}$), non-negative commodity bundles ($X = \mathbb{R}^l_+$ for some integer l) or, in an ambiguity context, the $l-1$ dimensional simplex interpreted as the set of all probability distributions over l consequences. An *alternative* or *prospect* is a finite non-empty subset D of X. We denote by $\mathcal{P}(X)$ the set of all such prospects. Prospects made of a single outcome (singletons) are naturally interpreted as *certain*

[3] A set A is connected for the (relevant) topology if it cannot be written as a finite union of pairwise disjoint non-empty open sets.

or *non-ambiguous*. For all integers m and n such that $m \leq n$, the set $\{m, m+1, \ldots, n\}$ is denoted by $[m, n]$. When $m = 1$, we write simply $[n]$ instead of $[1, n]$. A set of the form $\{m, m+1, \ldots\}$ is denoted by $[m, \cdot]$. The set $[1, \cdot]$ is also denoted by $\mathbb{N}$.

Prospects are compared by an ordering[4] $\succsim$ on $\mathcal{P}(X)$ with the usual interpretation that $D \succsim D'$ if and only if the decision maker weakly prefers prospect D to prospect D'. The asymmetric (strict preference) and symmetric (indifference) factors of $\succsim$ are denoted respectively by $\succ$ and $\sim$. For reasons that will soon become clear, whenever we write a prospect D in $\mathcal{P}(X)$ with n possible outcomes in the form $D = \{d_1, \ldots, d_n\}$, we *label* the outcomes of D in such a way that $\{d_1\} \succsim \ldots \succsim \{d_n\}$. There may of course be several such labellings if there are indifferences between some singleton subsets of D. For every set for which such indifferences happen, we choose once and for all any of the several labellings that could do. For any prospect $D \in \mathcal{P}(X)$ labeled in this way and any $x \in D$, we denote by $r_x^D \in [\#D]$ the rank of x in D defined by $r_x^D = i \Longleftrightarrow x = d_i$ for $D = \{d_1, \ldots, d_{\#D}\}$. For any given number $n \in \mathbb{N}$, we say that a rank $i \in [n]$ is essential for prospects with n possible outcomes if there are two distinct outcomes x and $y \in X$ and a prospect $A \in \mathcal{P}(X)$ such that $\#A = n - 1$, $\{x, y\} \cap A = \varnothing$, $r_x^{A \cup \{x\}} = r_y^{A \cup \{y\}} = i$ and $A \cup \{x\} \succ A \cup \{y\}$. In plain English, rank i is essential for prospects with n possible outcomes if one can think of a prospect with n possible outcome for which the replacement of one alternative having rank i in this prospect by another with the same rank would "make the difference". *A contrario*, a non-essential rank for those prospects would be such that all prospects with n outcomes that differ only in the outcome of that rank would be considered indifferent. Almost all results of this paper will be derived under the condition that, for any integer n, there is at least one essential rank for prospects with n possible outcomes.

This paper is specifically interested in Rank-Dependent Weighted Average

[4] An ordering is a reflexive, complete and transitive binary relation.

Utility (RDWAU) orderings of $\mathcal{P}(X)$ for which there exist a (continuous) function $u : X \to \mathbb{R}$ and, for any $n \in \mathbb{N}$, n *non-negative* real numbers w_i^n satisfying $\sum_{i\in[n]} w_i^n = 1$ such that, for all $A = \{a_1, \ldots, a_{\#A}\}$ and $B = \{b_1, \ldots, b_{\#B}\}$:

$$A \succsim B \iff \sum_{i\in[\#A]} w_i^{\#A} u(a_i) \geq \sum_{i\in[\#B]} w_i^{\#B} u(b_i). \tag{1}$$

Hence, an RDWAU ordering of prospects *can be* thought of as resulting from the comparisons of a weighted average of the utility of the possible outcomes of those prospects for some utility function, and for some weights that depend upon the ranking of the outcomes in the prospects if these outcomes were obtained for sure. There are obviously many RDWAU orderings, as many in fact as the orderings of prospects that can generated from all logically conceivable ways of assigning utility levels to outcomes and non-negative weights to their ranks in the prospects.

To illustrate how a RDWAU ordering compares prospects, reconsider the introductory example of the cancer patient. In this setting, $X = [0, 1]$, interpreted as the various conceivable five-year probabilities of survival ordered in the obvious way if they were known non-ambiguously. The three prospects faced by the patient would then be $\{1/2\}$, $\{4/5, 1/5\}$ and $\{4/5, 1/2, 1/5\}$ and a RDWAU ordering of the prospects *could be* based on the utility function $u(p) = p^2$ for every $p \in [0, 1]$ and on the weights $w_1^1 = 1$, $w_1^2 = 1/4$, $w_2^2 = 3/4$, $w_1^3 = 1/9$, $w_2^3 = 2/9$ and $w_3^3 = 2/3$. In this case, we would have $\{1/2\} \succ \{1/5, 4/5\}$ because

$$u(1/2) = \frac{1}{4} > w_1^2 u(4/5) + w_2^2 u(1/5) = \frac{19}{100}$$

and we would have $\{1/5, 4/5\} \succ \{1/5, 1/2, 4/5\}$ because

$$w_1^2 u(4/5) + w_2^2 u(1/5) = \frac{19}{100} > w_1^3 u(4/5) + w_2^3 u(1/2) + w_3^3 u(1/5) = \frac{23}{150}.$$

As mentioned earlier, this ranking of the three prospects violates the averaging axiom used in Gravel, Marchant, and Sen (2012) and Gravel, Marchant, and Sen (2018) (and also in Ahn (2008) and, in some weakened forms, Olszewski (2007) and Gul and Pesendorfer (2001)) according to which $D \succsim D' \Leftrightarrow D \succsim D \cup D' \succsim D'$ for any two disjoint prospects D and D' (such as $\{1/2\}$ and $\{4/5, 1/5\}$). Hence, this ranking would not be agreed upon by Uniform Expected Utility criteria characterized in Gravel, Marchant, and Sen (2012) or Conditional Expected Utility criteria characterized in Gravel, Marchant, and Sen (2018) (and Ahn (2008) in a setting with atomless prospects). This ranking could not even be produced by the convex combination of the (utility of the) best and the worst outcomes of a prospect characterized by Olszewski (2007) (again in a setting where prospects are sets with uncountable outcomes).

We observe that the class of RDWAU orderings contains the class of UEU orderings, who are nothing else than RDWAU orderings for which the weights w_i^k are equal to $1/k$ for every $i = 1, \ldots, k$. Since some of the rank-dependent weights can be zero, the RDWAU orderings also contain rankings of sets based on their sole worst outcome (by putting all weights except the last one at zero), their sole best outcomes or a convex combination *à la* Olszewski (2007) or Hartmann (2023) of their best and their worst outcomes. It should be observed, however, that there is no inclusion relation between the classes of CEU and RDWAU orderings. The example just given shows that there are RDWAU orderings that are not CEU orderings. The following example provides a CEU ordering that is not a RDWAU ordering.

Example 1 *Let $X = [0, 1]$ (interpreted again as the five-year survival probabilities ordered in the obvious way), and define $\succsim$ on $\mathcal{P}(X)$ by*

$$D \succsim D' \Longleftrightarrow \frac{\sum_{p \in D} \rho(p) u(p)}{\sum_{p \in D} \rho(p)} \geq \frac{\sum_{p' \in D'} \rho(p') u(p')}{\sum_{p' \in D'} \rho(p')},$$

for the functions ρ and u defined (on $[0,1]$) by

$$\rho(p) = 1 + p - p^2 \quad \textit{and } u(p) = p.$$

Observe that this CEU ordering ranks prospect $\{19/20, 1/2, 1/80\}$ above prospect $\{9/10, 1/2, 1/15\}$ because

$$\frac{(1+19/20-361/400)19/20+(1+1/2-1/4)1/2+(1+1/80-1/6400)1/80}{1+19/20-361/400+1+1/2-1/4+1+1/80-1/6400} = 0.49331 >$$

$$0.49286 = \frac{(1+9/10-81/100)9/10+(1+1/2-1/4)1/2+(1+1/15-1/225)1/15}{1+9/10-81/100+1+1/2-1/4+1+1/15-1/225}.$$

This CEU ordering would also rank prospect $\{19/20, 1/5, 1/80\}$ below prospect $\{9/10, 1/5, 1/15\}$ because

$$\frac{(1+19/20-361/400)19/20+(1+1/5-1/25)1/5+(1+1/80-1/6400)1/80}{1+19/20-361/400+1+1/5-1/25+1+1/80-1/6400} = 0.38504 <$$

$$\frac{(1+9/10-81/100)9/10+(1+1/5-1/25)1/5+(1+1/15-1/225)1/15}{1+9/10-81/100+1+1/5-1/25+1+1/15-1/225} = 0.38760.$$

These two rankings however cannot be produced by a RDWAU ordering. Indeed, if they were, the first ranking would imply, for some numbers w_i^3 ($i \in [3]$) and utilities $u(19/20)$, $u(1/2)$ and $u(1/80)$,

$$\begin{aligned} w_1^3u(19/20)+w_2^3u(1/2)+w_3^3u(1/80) &> w_1^3u(9/10)+w_2^3u(1/2)+w_3^3u(1/15) \\ &\Leftrightarrow \\ w_1^3u(19/20)+w_3^3u(1/80) &> w_1^3u(9/10)+w_3^3u(1/15) \end{aligned}$$

while the second ranking would imply, as the reader can verify, the reverse inequality.

Before presenting the axioms that characterize the RDWAU family, we find useful to introduce the following notion of revealed (by the decision maker's ordinal preferences) *preference strength for one outcome over another* as applicable to the various possible ordered pairs of those outcomes. We formulate successively the definitions of weak, strict and equivalent revealed preference strength.

Definition 1 *Let x, y, x' and y' be outcomes in X. The ordering $\succsim$ on $\mathcal{P}(X)$ is said to reveal a weakly larger preference strength for x over y than for x' over y', which we write formally as $(x, y)\ \Delta^{\succsim}\ (x', y')$, if there are two sets A and B not containing x, y, x' and y' and satisfying $\#A = \#B = n - 1$ for some integer n and $r_x^{A\cup\{x\}} = r_y^{B\cup\{y\}} = r_{x'}^{A\cup\{x'\}} = r_y^{B\cup\{y'\}} = i$ for some rank $i \in [n]$ that is essential for prospects with n outcomes such that:*

$$A \cup \{x\} \succsim B \cup \{y\} \text{ and } A \cup \{x'\} \precsim B \cup \{y'\}. \tag{2}$$

If at least one of $\succsim$ and $\precsim$ in (2) is strict, we then say that the preference strength for x over y is revealed strictly *larger than that for x' over y', which we write formally as $(x, y)\ \Delta_{\mathbf{s}}^{\succsim}\ (x', y')$.*

If both $\succsim$ and $\precsim$ in (2) *are replaced by $\sim$, we then say that the two preference strengths are revealed to be equivalent, which we write formally as $(x, y)\ \Delta_{\mathbf{e}}^{\succsim}\ (x', y')$.*

In words, $\succsim$ reveals a preference strength for x over y to be weakly larger than the preference strength for x' over y' if there are two sets A and B with the same number of outcomes to which the respective addition of x and y—under the condition that the rank of x and y in the two enlarged sets is the same and is essential for prospects with the same numbers of outcomes than those enlarged sets—lead to a preference for the enlarged A to the enlarged B while the similar addition of x' and y' to the two sets lead, under the same condition on the ranks, to the opposite preference. Hence, it seems that x "does more" with respect to y than x' does with respect to y', at least as judged by their addition to some sets A and B that do not contain these outcomes. Observe that Definition 1 does not preclude the two sets A and

B to which the outcomes are added to be the same. It does not even rule out the possibility that these two sets be both empty. In this latter case, the "addition" of two outcomes to the same empty set amount simply to comparing those outcomes as if they were available for sure. One can also observe that the requirement of essentiality of the rank used in the definition is important for the appraisal of preference strength. If the common rank of x, y, x' and y' in the enlarged sets would be allowed to be inessential, then no inference whatsoever could be extracted from the pairs of rankings (2) that would both be indifferences.

The interpretation of the quaternary relation $\Delta^{\succsim}$ as evaluating the preference strength for one outcome over another is particularly clear if one assumes from the start that the ordering $\succsim$ is itself an RDWAU ordering. Indeed, for a RDWAU ordering, the fact to have, for some rank $i \in [n]$, $\{a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n\} \succsim \{b_1, \ldots, b_{i-1}, y, b_{i+1}, \ldots, b_n\}$ can be written, thanks to (1), as

$$\sum_{g=1}^{i-1} w_g^n u(a_g) + w_i^n u(x) + \sum_{h=i+1}^{n} w_h^n u(a_h) \geq \sum_{g=1}^{i-1} w_g^n u(b_g) + w_i^n u(y) + \sum_{h=i+1}^{n} w_h^n u(b_h)$$

$$\Leftrightarrow$$

$$u(x) - u(y) \geq \frac{\sum_{g=1}^{i-1} w_g^n \big(u(b_g) - u(a_g)\big) + \sum_{h=i+1}^{n} w_h^n \big(u(b_g) - u(a_g)\big)}{w_i^n} \quad (3)$$

On the other hand,

$$\{a_1, \ldots, a_{i-1}, x', a_{i+1}, \ldots, a_n\} \precsim \{b_1, \ldots, b_{i-1}, y', b_{i+1}, \ldots, b_n\}$$

can be similarly written, for the same RDWAU ordering, as

$$u(x') - u(y') \leq \frac{\sum_{g=1}^{i-1} w_g^n\big(u(b_g) - u(a_g)\big) + \sum_{h=i+1}^{n} w_h^n\big(u(b_g) - u(a_g)\big)}{w_i^n}. \quad (4)$$

Hence, the combination of Inequalities (3) and (4) reveals indeed that $u(x) - u(y) \geq u(x') - u(y')$.

A few additional remarks can be made about the quaternary relation $\Delta^{\succsim}$ of Definition 1. First, this quaternary relation has very little structure. When viewed as a binary relation on $X \times X$, it is not reflexive since one may well have, for some distinct outcomes x and y, that $A \cup \{x\} \succ B \cup \{y\}$ for all sets A and B with the same number of outcomes containing neither x and y such that $r_x^{A\cup\{x\}} = r_y^{B\cup\{y\}}$. It is not complete since, again, nothing rules out the possibility that, for some outcomes x, y, x' and y' both $A \cup \{x\} \succ B \cup \{y\}$ and $A \cup \{x'\} \succ B \cup \{y'\}$ hold for all sets A and B with the same cardinality that do not contain any of these outcomes and that are such that $r_x^{A\cup\{x\}} = r_y^{B\cup\{y\}} = r_{x'}^{A\cup\{x'\}} = r_{y'}^{B\cup\{y'\}}$. For sure, $\Delta^{\succsim}$ is not transitive when viewed as a binary relation on $X \times X$.

However, $\Delta^{\succsim}$ viewed as a binary relation on $X \times X$ does satisfy Property 2 of what Krantz, Luce, Suppes, and Tversky (1971) call an *algebraic difference structure* (Definition 3, chapter 4) as established in the following (obvious) remark.

Remark 1 *For any outcomes x, y, x' and y' in X, the two following statements are equivalent:*

(i) $(x, y)\ \Delta^{\succsim}\ (x', y')$ *and,*

(ii) $(y', x')\ \Delta^{\succsim}\ (y, x)$.

We observe also that $\Delta_{\mathbf{s}}^{\succsim}$ (strictly larger preference strength) of Definition 1 is *not* the asymmetric factor of $\Delta^{\succsim}$, even though it is compatible with it. Somewhat dually, the symmetric factor of $\Delta^{\succsim}$ is compatible with $\Delta_{\mathbf{e}}^{\succsim}$ of Definition 1 (equivalent preference strength) but is not equivalent to it.

2.2 Axioms

We now state and discuss the axioms that characterize the whole family of RDWAU orderings. The first one is an adaptation to the present setting of Peter Wakker's "trade-off consistency" condition (see e.g. Wakker (1989)). The axiom imposes minimal consistency among comparative statements of revealed preference strength performed by the weak $\Delta^{\succsim}$ and the strict $\Delta_{\mathbf{s}}^{\succsim}$ as defined above. Specifically, the axiom requires that if the ordering $\succsim$ reveals a preference strength for x over y that is weakly larger than the preference strength for x' over y', then this ordering should never reveal a preference strength for x over y that is strictly smaller than the preference strength for x' over y'. We state this axiom as follows.

Axiom 1 *Consistency in Comparisons of Preference Strength. For no x, y, x' and y' in X should we observe both $(x,y)\Delta^{\succsim}(x',y')$ and $(x',y')\Delta_{\mathbf{s}}^{\succsim}(x,y)$.*

While (relatively) natural, this consistency condition has strong implications. For one thing, it implies an "independence" axiom that has been widely discussed in the literature on additive numerical representation of orderings. In the current rank-dependent context, the independence axiom requires, in substance, that the ranking of prospects with the same number of outcomes be independent from any outcome that they have in common when the outcome has the same rank in the two prospects. For future reference, we state formally as follows this notion of *comonotonic independence.*

Condition 1 *Comonotonic Independence. For any distinct α and $\beta \in X$, and prospects D and D' such that $\#D = \#D'$, $(D \cup D') \cap \{\alpha, \beta\} = \varnothing$ and $r_{\alpha}^{D\cup\{\alpha\}} = r_{\beta}^{D\cup\{\beta\}} = r_{\alpha}^{D'\cup\{\alpha\}} = r_{\beta}^{D'\cup\{\beta\}}$, we have*

$$D \cup \{\alpha\} \succsim D' \cup \{\alpha\} \iff D \cup \{\beta\} \succsim D' \cup \{\beta\}.$$

One can observe that this condition is a (significant) weakening of the restricted independence condition used by Gravel, Marchant, and Sen (2012) (and also by Nehring and Puppe (1996)), which requires the independence to hold even for a common element that may not have the same rank in the two considered prospects. The fact that Consistency in Comparison of Preference Strength implies Comonotonic Independence is established in the following lemma proved, like all formal results of the paper, in the Appendix

Lemma 1 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ that satisfies Consistency in Comparisons of Preference Strength. Then $\succsim$ satisfies Comonotonic Independence.*

Another noteworthy and immediate implication of the Consistency in Comparisons of Preference Strength axiom is the following condition of weak monotonicity, which says that the attractiveness of a prospect is always increased if a possible outcome of the prospect is replaced by an even more preferable outcome of the same rank when the considered rank is essential. The formal statement of this condition is as follows.

Condition 2 *Weak monotonicity. $D \cup \{x\} \succ D \cup \{y\}$ for every set $D \in \mathcal{P}(X)$, and every two outcomes x and $y \in X \backslash D$ such that $\{x\} \succ \{y\}$ and $r_x^{D\cup\{x\}} = r_y^{D\cup\{y\}} = i$ for some rank $i \in [\#D+1]$ that is essential for prospects with $\#D + 1$ outcomes.*

The following lemma, proved in the Appendix, establishes that this weak monotonicity condition is indeed implied by Consistency in Comparisons of Preference Strength.

Lemma 2 *Let $\succsim$ be an ordering of $\mathcal{P}(X)$ that satisfies the Consistency in Comparisons of Preference Strength axiom. Then $\succsim$ satisfies the Weak Monotonicity condition.*

The second axiom used in the characterization of the family of (continuous) RDWAU orderings is a specific continuity requirement. It requires X to be a connected set with respect to the order topology induced by the restriction of $\succsim$ to singletons.

Axiom 2 *Fixed Cardinality Continuity. For any prospect D, the sets*

$$\{(b_1, \ldots, b_{\#D}) \in X^{\#D} : \{b_1, \ldots, b_{\#D}\} \succsim D\}$$

and

$$\{(b_1, \ldots, b_{\#D}) \in X^{\#D} : \{b_1, \ldots, b_{\#D}\} \precsim D\}$$

are closed in the product topology.

We observe that this continuity axiom is limited to comparisons of prospects with the same number of outcomes. It does not impose any continuity on the comparisons of prospects with a different number of outcomes.

The third—and last—axiom is a (significant) weakening of the Gärdenfors (1976) principle discussed in the literature on ignorance (and notably in Barberà and Pattanaik (1984), Bossert (1989), Fishburn (1984) and Kannai and Peleg (1984)). This axiom, also called *internality* by some—for example Gneezy, List, and Wu (2006)—is formulated as follows.

Axiom 3 *Internality . For every prospect $D = \{d_1, \ldots, d_n\} \in \mathcal{P}(X)$, one has $\{d_1\} \succsim D \succsim \{d_n\}$.*

In words, Internality requires any prospect to be weakly better than its worst outcome received certainly and, symmetrically, to be weakly worse than its best outcome received certainly. It is important to observe that Internality is the only axiom that restricts the ranking of prospects with different numbers of outcomes. The fact that this restriction is limited to the ranking of any uncertain (ambiguous) prospect *vis-à-vis* certain (non-ambiguous) ones is also noteworthy.

In order to prove our main result, we introduce some additional terminology. Let $a_1, \ldots, a_k$ be some finite list of outcomes for some integer $k \geq 3$. We say that $a_1, \ldots, a_k$ form a *standard sequence* if $(a_i, a_{i+1}) \, \Delta_{\mathbf{e}}^{\succsim} \, (a_{i+1}, a_{i+2})$ for all $i \in [k-2]$. In plain English, $a_1, \ldots, a_k$ form a standard sequence if any two pairs of adjacent outcomes in the sequence exhibit the same preference strength for their first outcome over their second. Hence, a standard sequence is made of outcomes who are either increasingly favorable or decreasingly favorable (when received for sure) at a "constant rate".

3 Results

3.1 The general family of RDWAU orderings

The most general theorem proved in this paper is the following.

Theorem 1 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that, for every integer n, there is at least one rank that is essential for sets with n outcomes. Then $\succsim$ satisfies Consistency in Comparisons of Preference Strength, Fixed Cardinality Continuity and Internality iff $\succsim$ is a RDWAU ordering as in (1) for non-negative weights summing to 1. Moreover, if there are prospects containing at least two outcomes that have at least two ranks that are essential, the mapping u is unique up to a positive affine transformation and the weights w_i^n are unique.*

The proof of this theorem, provided in the Appendix, proceeds in several steps. We first prove, under the conditions of the theorem, that the Consistency in Comparisons of Preference Strength and Fixed Cardinality Axioms alone—without therefore Internality—characterize the RDWAU family of rankings of sets containing any specifically given number of outcomes. However, this first result does not say anything about comparisons of sets containing different numbers of outcome. It does not even connect the numerical representation obtained for the ranking of sets with, say, m outcomes with that which enables the ranking of sets with, say, n outcomes. This first step of the proof is summarized in the following proposition.

Proposition 1 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that, for every integer n, there is at least one rank that is essential for sets with n outcomes. Then $\succsim$ satisfies Consistency in Comparisons of Preference Strength and Fixed Cardinality Continuity iff there exist w_i^n as in (1) and a continuous function $u^n : X \to \mathbb{R}$ such that, for all prospects $A, B \in \mathcal{P}(X)$ with the same cardinality n, one has*

$$A \succsim B \iff \sum_{i \in [n]} w_i^n u^n(a_i) \geq \sum_{i \in [n]} w_i^n u^n(b_i). \tag{5}$$

For any $n \geq 2$, if the prospects with n outcomes have at least two ranks that are essential, then the mapping u^n is unique up to a positive affine transformation for every $n > 1$. The weights w_i^n are unique.

The second step of the proof consists in showing that any of the functions u^n that enters in the numerical representation (5) of the ordering $\succsim$ restricted to prospects with n outcomes provides a numerical representation of the ordering $\succsim$ restricted to singletons. The required result for this step is the following Lemma.

Lemma 3 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that, for every integer n, there is at least one rank that is essential for sets with n outcomes. If $\succsim$ satisfies Consistency in Comparisons of Preference Strength and Fixed Cardinality Continuity, then, for any $n \in \mathbb{N}$, the function u^n in (5) also numerically represents the restriction of the ordering $\succsim$ to singletons. Formally, for any $x, y \in X$ and $n \in \mathbb{N}$, $x \succsim y$ iff $u^n(x) \geq u^n(y)$.*

Proposition 1 establishes the validity of the numerical representation (5) for the ranking of prospects with a given number of outcomes. However it does not connect together the functions u^n that enter in the definition of the numerical representation for prospects involving different numbers of outcome. With the help of Lemma 3, the next Lemma establishes that all the functions u^n that enter in the numerical representations of the orderings of prospects containing n outcomes for variable n can actually all be taken to be the same (up to a positive affine transformation). The formal statement of this lemma, proved in the Appendix, is as follows.

Lemma 4 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that, for every integer n, there is at least one rank that is essential for sets with n outcomes. If $\succsim$ satisfies Consistency in Comparisons of Preference Strength and Fixed Cardinality Continuity, then, for any $n \in \mathbb{N}$, and any prospects D and D' with n possible outcomes, one has:*

$$D \succsim D' \Longleftrightarrow \sum_{i\in[n]} w_i^n u(d_i) \geq \sum_{i\in[n]} w_i^n u(d_i')$$

for some continuous function $u : X \rightarrow \mathbb{R}$ uniquely defined up to a positive affine transformation (if there are prospects with n outcomes with at least two essential ranks) or up to an increasing transformation (if only one rank is essential for all prospect size). Moreover, the weights w_i^n are unique.

While Proposition 1 and Lemmas 3 and 4, which roughly establish the validity of the numerical representation RDWAU as per (1) for prospects with the same number of outcomes, makes no use of the Internality axiom, the rest of the proof, which establishes the validity of that same representation for comparing sets with different number of outcomes, will use this axiom extensively. The last intermediate result that is required to prove Theorem 1 is the following lemma that establishes, under all the axioms, the existence of a "certain" (or non-ambiguous) equivalent to any prospect.

Lemma 5 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that, for every integer n, there is at least* *one rank that is essential for sets with n outcomes. If $\succsim$ satisfies Consistency in Comparisons of Preference Strength, Fixed Cardinality Continuity and Internality, then, for any prospect $D \in \mathcal{P}(X)$, there exists an outcome* $\mathrm{CE}(D) \in X$ *such that* $\{\mathrm{CE}(D)\} \sim D$.

The proof of Theorem 1 is then completed by showing that the numerical representation (1) shown so far to represent the ordering $\succsim$ on any two prospects containing the same number of outcomes is also valid for comparing prospects with different numbers of outcomes.

In the appendix, we also establish the logical independence of the three axioms used in the characterization.

3.2 Some subclasses of RDWAU orderings

The family of RDWAU orderings of prospects characterized in Theorem 1 is very large. It is so large that the rank-dependent weights used by RDWAU to calculate average utility are not restricted at all, if we except the fact that they are all non-negative and sum to 1, and are the same for all sets with the same number of outcomes. However, the weights are allowed to vary in a completely arbitrary way when possible outcomes are added—or deleted—from a prospect. For example, one could imagine a RDWAU ordering that puts a weight of 1 on the worst possible outcome of two-outcome prospects but yet reverses perspective when evaluating three-outcomes prospects by putting a weight of 1 on the best outcome in those cases. In this subsection, we explore possibilities of restricting the rank-dependent weights of RDWAU orderings without of course going as far as making them identical as they are in the family of UEU orderings discussed previously.

3.2.1 Optimism and pessimism

A possible way of restricting the weights is through the specification of the decision maker's *optimism* with respect to uncertain or ambiguous prospects. We are using here the term "optimism" in the common sense of "the quality of being full of hope and emphasizing the good parts of a situation, or a belief that something good will happen" (Cambridge Dictionary). There are various ways by which we can introduce this notion and its opposite—pessimism—in the current setting.

A somewhat plausible definition of optimism (pessimism) for a RDWAU decision maker is the requirement that the rank-dependent weights be increasing (decreasing) with the ranking of outcomes if they were certain. Such a definition would be at least compatible with the definition of optimism/pessimism given in the literature on rank-dependent expected utility models in terms of the super (sub) additivity of the Choquet capacity (see e.g. Dillenberger, Postlewaite, and Rozen (2017) or Wakker (1990)), even though nothing in the current radical uncertainty or objective ambiguity context enables the definition of such a capacity as the source of the rank-dependent weights.

There is an easy ordinal test—and definition—of optimism (pessimism) in our finite set ranking context that leads precisely to this monotonicity of the weights as definition of optimism. Consider indeed any outcomes w, x, y and z such that $(w, x) \ \Delta_{\mathbf{e}}^{\succsim} \ (y, z)$ and $\{w\} \succ \{x\} \succsim \{y\} \succ \{z\}$. Hence, when certain, these four outcomes are ranked in decreasing order from w to z (with strict preference between the first two outcomes and the last two) and the preference strength for the best w over the second best x has been revealed the same—as per Definition 1—as the the preference strength for y over z. Consider then a prospect D with at least two possible outcomes among which are x and y (but not w nor z) and such that the simultaneous replacement of x by w and of y by z would not affect any rank of the outcomes. Observe that the replacement of x by a more favorable

w is appealing to the decision maker while the simultaneous replacement of y by z is detrimental to him/her. However, since $(w,x)\ \Delta_{\mathbf{e}}^{\succsim}\ (y,z)$, the preference benefit of replacing x by w is exactly the same as the preference cost of replacing y by z. Since x is ranked weakly above y and the rank of the two options is not affected by their respective replacement by w and z, an optimistic agent—who tends to believe that something good will happen—should favour such a simultaneous replacement, while a pessimistic agent should find this very same simultaneous replacement detrimental overall. Hence we find plausible to define formally optimism (pessimism) as follows.

Definition 2 *An ordering $\succsim$ on $\mathcal{P}(X)$ is said to be weakly optimistic if for every four distinct outcomes w,x,y and $z\in X$ such that $(w,x)\ \Delta_{\mathbf{e}}^{\succsim}\ (y,z)$ and $\{w\}\succ\{x\}\succsim\{y\}\succ\{z\}$ and every prospect $A\in\mathcal{P}(X)$ such that $\{x,y\}\subset A$, $\{w,z\}\cap A=\varnothing$, $r_x^A=r_w^{(A\setminus\{x,y\})\cup\{w,z\}}$ and $r_y^A=r_z^{(A\setminus\{x,y\})\cup\{w,z\}}$ with the ranks r_x^A and r_y^A essential for prospects with $\#A$ outcomes, we have $(A\setminus\{x,y\})\cup\{w,z\}\succsim A$. The ordering is strictly optimistic if the last comparison is strict.*

Weak pessimism and strict pessimism are defined similarly, with the last comparison replaced by $\precsim$ (or $\prec$).

We leave to the reader the task of verifying the following implication of this definition of weak optimism/pessimism for a RDWAU decision maker.

Claim 1 *Let $\succsim$ be a RDWAU ordering of $\mathcal{P}(X)$ that is numerically represented as per (1) for some utility function $u: X\longrightarrow\mathbb{R}$ and some collection of non-negative weights w_i^n ($n\in\mathbb{N}$ and $i\in[n]$) satisfying $\sum_{i\in[n]}w_i^n=1$ for any n. Then $\succsim$ is weakly optimistic (pessimistic) if and only if $w_i^n\geq(\leq)\ w_{i+1}^n$ for every $n\in\mathbb{N}\backslash\{1\}$ and $i\in[n-1]$ and is strictly optimistic (pessimistic) if and only the inequality is strict.*

3.2.2 RDWAU generated by a probability weighting function

The monotonicity of the rank-dependent weights applied to prospects with the same number of outcomes just discussed does not impose any restriction on the behavior of the weights when the number of possible outcomes changes. Yet, it seems plausible that a decision maker would not use radically different weighting schemes when facing prospects with, say, three possible outcomes then when facing prospects with four outcomes.

A possible way to impose consistency on the weights used to compare prospects with varying number of possible outcomes would be to assume that these weights are all generated by the same *probability weighting function.* We specifically consider the following subclass of RDWAU orderings.

Definition 3 *A RDWAU ordering is said to be generated by a probability weighting function if there exists an increasing function* $\psi : [0,1] \to [0,1]$ *satisfying* $\psi(0) = 0$ *and* $\psi(1) = 1$ *such that, for all integer* n*, and all* $i \in [n]$*,* $w_i^n = \psi(\frac{i}{n}) - \psi(\frac{i-1}{n})$ *or, equivalently,* $\psi(\frac{i}{n}) = \sum_{h=1}^{i} w_h^n$*.*

The term probability weighting function comes from the fact that the function ψ takes its value in the $[0,1]$ interval (set of all probabilities), and may thus be thought of as "weighting" those probabilities. Most rank-dependent expected utility criteria examined in the decision theoretic literature—albeit in a different setting than the finite set framework considered here—have also considered weighting schemes that are generated by a given probability weighting function (see e.g. Quiggin (1982), Quiggin (1993), Schmeidler (1989), Yaari (1987)). Heuristically, a decision maker who compares prospects by means of a RDWAU generated by a probability weighting function may be viewed as assigning to all ranks of the outcomes of a prospect an equal probability of occurrence and distorting this probability by some weighting function.

It can be observed that a RDWAU ordering generated by a probability weighting function satisfies the following *strong dominance* axiom, which is, as its name suggests, a significant strengthening of the *dominance* axiom widely discussed by the literature on decision making under ignorance surveyed in Barberà, Bossert, and Pattanaik (2004) (see their section 3.2).

Axiom 4 *Strong Dominance. For every n and $r \in \mathbb{N}$, every prospect A such that $\#A = n$ and every list of n prospects $B^1, ..., B^n$ satisfying $B^j \cap A = \varnothing$ and $\#B^j = r$ for every $j = 1, ..., n$:*
if $\{a_j\} \succ \{b_1^j\}$ for all $j \in [n]$, then $A \succ A \cup B^1 \cup ... \cup B^n$ and,
if $\{b_r^j\} \succ \{a_j\}$ for all $j \in [n]$, then $A \prec A \cup B^1 \cup ... \cup B^n$.

This axiom requires that if any possible outcome of a prospect is better (worse), if certain, than every outcome in a collection of r outcomes distinct from the outcomes of the prospect, then the prospect should be better (worse) than the (large) prospect formed by the merging of the prospect with the union of all those collections of outcomes. The dominance axiom discussed in Barberà, Bossert, and Pattanaik (2004) is a weakening of this axiom that restricts its applicability to the case where $r = 1$ and the unique outcome of the collection is the same for all outcomes of the initial prospect.

As it turns out, adding this axiom to those characterizing the whole family of RDWAU ordering as per Theorem 1 suffices to eliminate from that family all RDWAU orderings that do not result from a probability weighting function. The formal statement of this is as follows.

Proposition 2 *Let X be a set of outcomes and $\succsim$ be an ordering of $\mathcal{P}(X)$ and assume that X is connected for the order topology associated to $\succsim$ when restricted to singletons. Assume also that, for every integer n, there is at least one rank that is essential for sets with n outcomes. Then $\succsim$ satisfies Consistency in Comparisons of Preference Strength, Fixed Cardinality Continuity, Internality and Strong Dominance iff $\succsim$ is a RDWAU ordering generated by a probability weighting function.*

A particularly simple family of RDWAU orderings that result from a probability weighting function are those satisfying the (very stringent) restriction that the weights w_i^n of Expression (1) can be written as

$$\frac{w_{i+1}^n}{w_i^n} = \rho$$

for any $n \in \mathbb{N}$ and $i \in [n]$ for some strictly positive real number ρ. Any such RDWAU ordering can be seen as being generated by the probability weighting function ψ defined by:

$$\psi(x) = \frac{e^{\rho x} - 1}{e^{\rho} - 1}$$

Let us refer to any RDWAU ordering that satisfies this restriction as an *RDWAU ordering with constant ratio*. Optimism for this class would correspond to the requirement that $\rho < 1$. while pessimism would mean $\rho > 1$. Observe finally that if $\rho = 1$ (a limiting case of both optimism and pessimism), then the weights are the same for all outcomes and this brings us back to the UEU family characterized in Gravel, Marchant, and Sen (2012).

A simple observation reveals whether the real number ρ is smaller than, equal to, or larger than 1. Suppose indeed a_1, a_2 and a_3 form a standard sequence; if $\{a_1, a_2, a_3\} \precsim \{a_1, a_3\}$, then $\rho \geq 1$ and if $\{a_1, a_2, a_3\} \succsim \{a_1, a_3\}$, then $\rho \leq 1$.

The following condition is necessary and sufficient for a RDWAU ordering to exhibit a constant ratio.

Condition 3 *Rank-dependent preservation of equivalences among pairs. Let x_1, x_2, x_3, x_4 be outcomes such that $\{x_1\} \succ \{x_2\} \succsim \{x_3\} \succ \{x_4\}$. Let A be a set such that $A \cap \{x_1, x_2, x_3, x_4\} = \varnothing$ and $r_{x_i}^{A \cup \{x_i\}} = r_{x_j}^{A \cup \{x_j\}}$ for all $i, j \in [4]$. Then*

$$\{x_1, x_4\} \sim \{x_2, x_3\} \iff A \cup \{x_1, x_4\} \sim A \cup \{x_2, x_3\}.$$

Proposition 3 *Let $\succsim$ be a RDWAU ordering of $\mathcal{P}(X)$ as in Theorem 1. It satisfies Condition 3 iff it exhibits constant ratio.*

4 Conclusion

This paper has axiomatically characterized the rather large family of criteria for decision making under ignorance or objective ambiguity that result from comparing rank-dependent weighted average utilities of the prospects, for some utility function and some rank-dependent weighting scheme. It has done so by describing prospects as finite sets of outcomes—that could be either final consequences or lotteries over the same.

While the rank-dependent weighted average of utility criteria are evocative of the rank-dependent expected utility criteria *à la* Quiggin (1993) or Yaari (1987) considered in decision making under risk (when prospects are described as probability distributions) or uncertainty (when prospects are described as functions from a set of states of nature to a set of consequences), they are more general than those because they can not meaningfully be described as resulting from a Choquet or otherwise capacity. The rank-dependent weights are, in this paper, completely arbitrary. And of course, the finite set theoretic framework in which we analyze decision making makes our characterization very different from those obtained in the literature.

Moreover, we have provided additional restrictions that one may want to impose on the weights to make them more structured. One of them, taking the form of a rather mild dominance axiom, has been sufficient to single out the class of RDWAU rankings of prospects that have been generated by a given probability weighting function. This class is clearly reminiscent of the rank dependent expected utility criteria considered in the decision theoretic literature.

We finally notice that the two main axioms—if we leave aside Fixed Cardinality Continuity—used in the characterization are quite easily amenable to experimental testing. It is our hope that future work in the area—including possibly our own—will enable progress in these directions.

A Proofs

A.1 Lemma 1

Consider two distinct outcomes α and $\beta \in X$, and two prospects D, D' such that $\#D = \#D'$, $(D \cup D') \cap \{\alpha, \beta\} = \varnothing$ and $r_{\alpha}^{D\cup\{\alpha\}} = r_{\beta}^{D\cup\{\beta\}} = r_{\alpha}^{D'\cup\{\alpha\}} = r_{\beta}^{D'\cup\{\beta\}} = i$ for some rank $i \in [\#D+1] = [\#D'+1]$. We observe that if i is not essential for sets with $\#D+1 = \#D'+1$ outcomes, then $D \cup \{\alpha\} \sim D \cup \{\beta\}$ and $D' \cup \{\alpha\} \sim D' \cup \{\beta\}$ so that $D \cup \{\alpha\} \succsim D' \cup \{\alpha\} \iff D \cup \{\beta\} \succsim D' \cup \{\beta\}$ always hold in that case. Assume therefore that i is essential for sets with $\#D + 1 = \#D' + 1$ outcomes and, by contraposition, assume that $D \cup \{\alpha\} \succsim D' \cup \{\alpha\}$ but not $D \cup \{\beta\} \succsim D' \cup \{\beta\}$. Since $\succsim$ is complete, this amounts to assuming that $D \cup \{\beta\} \prec D' \cup \{\beta\}$. From Definition 1, we obviously have $(\alpha, \alpha)\ \Delta_s^{\succsim}\ (\beta, \beta)$ and $(\beta, \beta)\ \Delta_s^{\succsim}\ (\alpha, \alpha)$, which contradicts Consistency in Comparisons of Preference Strength. □

A.2 Lemma 2

Suppose for contradiction that Condition 2 is violated. Hence, there are two outcomes x and $y \in X$ for which both $\{x\} \succ \{y\}$ and $\{x\} \cup D \precsim \{y\} \cup D$ hold for some set $D \in \mathcal{P}(X)$ such that $\{x, y\} \cap D = \varnothing$ and $r_x^{D\cup\{x\}} = r_y^{D\cup\{y\}}$ with $r_x^{D\cup\{x\}}$ (or $r_y^{D\cup\{y\}}$) being essential for prospects with $\#D + 1$ outcomes. From the statement $\{x\} \succ \{y\}$, we obtain, using the reflexivity of $\succsim$ restricted to singletons and the definition of $\Delta_{\mathbf{s}}^{\succsim}$ that $(x, y)\Delta_{\mathbf{s}}^{\succsim}(x, x)$. From $\{x\} \cup D \precsim \{y\} \cup D$, we conclude using this time the definition of $\Delta^{\succsim}$, that $(x, x)\Delta^{\succsim}(x, y)$, thus contradicting the Weak Consistency in Comparisons of Preference Strength. □

A.3 Proposition 1

The result being true for $n = 1$ by Debreu (1954) theorem (any continuous ordering on a topological space can be numerically represented by a utility function), consider any integer $n \geq 2$. Any prospect D with $n \geq 2$ ordered elements can be represented as an ordered vector in X^n. The set of all such vectors is a subset of X^n, denoted by $\mathcal{O}^n(X)$. Let us consider n disjoint connected subsets $\{Y_1, \ldots, Y_n\}$ of X such that, for all $i \in [n-1]$ and for all $x \in Y_i$, $y \in Y_{i+1}$, we have $\{x\} \succsim \{y\}$. By construction, the Cartesian product $\Pi_{i=1}^n Y_i$ is a subset of $\mathcal{O}^n(X)$. The restriction of $\succsim$ to $\Pi_{i=1}^n Y_i$ satisfies Consistency in Comparisons of Preference Strength and Continuity. We can therefore apply Theorem III.6.6 in Wakker (1989) (p. 70)—after noticing that Consistency in Comparisons of Preference Strength implies the absence of contradictory trade-offs mentioned in this theorem—and conclude in the existence of n continuous mappings $\{u_i^n\}_{i\in[n]}$ such that, for all $A, B \in \Pi_{i\in[n]} Y_i$

$$A \succsim B \iff \sum_{i\in[n]}^{n} u_i^n(a_i) \geq \sum_{i\in[n]}^{n} u_i^n(b_i). \tag{6}$$

Using then Observation III.6.6′ in Wakker (1989) (p. 71), we conclude that the functions u_i^n are unique up to an affine increasing transformation for all essential ranks if there are two or more such ranks (the functions associated to inessential ranks being constant). If however there is only one such essential rank, then the function of this unique rank is unique up to an increasing transformation, and all the other functions are constant. Let $E(n) \subset [n]$ be the set of ranks that are essential for prospects with n possible outcomes and, since at least one rank is essential, let $\hat{n}$ be the integer defined by $0 < \hat{n} = \#E(n) \leq n$. Consider then the set $\mathcal{O}^{\hat{n}}(X)$ of all ordered vectors in $X^{\hat{n}}$ that correspond to the $\hat{n}$ ordered elements with essential rank in sets with n outcomes for some such set D. Just like its cousin $\mathcal{O}^n(X)$, the set $\mathcal{O}^{\hat{n}}(X)$ can be written as the union of infinitely many Cartesian products of the form $\Pi_{i\in[\hat{n}]}Y_i$ with the sets Y_i defined as above for ranks i that are all essential for sets with n outcomes. The set $\mathcal{O}^{\hat{n}}(X)$ satisfies Assumption 2.1 in Chateauneuf and Wakker (1993). In particular, all the components of this subset of a Cartesian product are essential and, thanks to Lemma 2, the ranking of $\mathcal{O}^{\hat{n}}(X)$ induced by $\succsim$ satisfies the Chateauneuf and Wakker (1993) property of strong monotonicity required by their Assumption 2.1. As a result of Theorem 2.2 in Chateauneuf and Wakker (1993), there is a continuous additive representation of $\succsim$ (restricted to sets of cardinality n). That is to say, Expression (6) provides a numerical representation of the ordering $\succsim$ not only on $\Pi_{i\in[n]}Y_i$ but, also, on the whole set $\mathcal{O}^n(X)$, taking for granted that the functions u_i^n of Expression (6) can (and must) be taken to be constant for all ranks i that are not essential for sets with n outcomes. .

Let $\hat{\jmath} \in E(n)$ be an essential rank for sets with n outcomes such that $\hat{\imath} > \hat{\jmath}$ for any other essential rank $\hat{\imath}$ (if any) in $E(n)$. Assume first that there are at least two essential ranks for sets with n outcomes and consider therefore any $\hat{\imath} \in E(n)$ such that $\hat{\imath} > \hat{\jmath}$. Let A and B be any two sets of cardinality $n-1$ and x_1, x_2 and x_3 be outcomes in X not contained in either A or B such that $\{x_1\} \succsim \{x_2\} \succsim \{x_3\}$ and $r_{x_i}^{A\cup\{x_i\}} = r_{x_{i+1}}^{A\cup\{x_{i+1}\}} = r_{x_i}^{B\cup\{x_i\}} = r_{x_{i+1}}^{B\cup\{x_{i+1}\}} = \hat{\imath}$. Choose also such outcomes in such a way that $A \cup \{x_{i+1}\} \sim B \cup \{x_i\}$ for $i \in [2]$. The existence of these three outcomes and two sets A and B having those features is secured by the continuity of the representation in (6) and the connectedness of X. Hence, using (6), we can write $A\cup\{x_{i+1}\} \sim B\cup\{x_i\}$ for $i \in [2]$ as

$$\sum_{g\in[\hat{\imath}-1]} u_g^n(a_g) + u_{\hat{\imath}}^n(x_3) + \sum_{g\in[\hat{\imath}+1,n]} u_g^n(a_{g-1}) = \sum_{g\in[\hat{\imath}-1]} u_g^n(b_g) + u_{\hat{\imath}}^n(x_2) + \sum_{g\in[\hat{\imath}+1,n]} u_g^n(b_{g-1}) \quad (7)$$

and

$$\sum_{g\in[\hat{\imath}-1]} u_g^n(a_g) + u_{\hat{\imath}}^n(x_2) + \sum_{g\in[\hat{\imath}+1,n]} u_g^n(a_{g-1}) = \sum_{g\in[\hat{\imath}-1]} u_g^n(b_g) + u_{\hat{\imath}}^n(x_1) + \sum_{g\in[\hat{\imath}+1,n]} u_g^n(b_{g-1}). \quad (8)$$

Subtracting (8) from (7) yields $u_{\hat{\imath}}^n(x_2) - u_{\hat{\imath}}^n(x_1) = u_{\hat{\imath}}^n(x_3) - u_{\hat{\imath}}^n(x_2)$. Let now C and D be sets of cardinality $n-1$ not containing x_1, x_2 and x_3 such that the rank of x_1, x_2 and x_3 in $\{x_i\} \cup C$ and in $\{x_i\} \cup D$ (for $i \in [3]$) is $\hat{\jmath}$. Again, the existence of these two sets C and D having those features is secured by the continuity of the representation in ((6)) and the connectedness of the set X. We observe that, thanks to Consistency in Comparisons Preference Strength, one must have $C \cup \{x_{i+1}\} \sim D \cup \{x_i\}$ for $i \in [2]$. The same reasoning as above therefore yields $u_{\hat{\jmath}}^n(x_2) - u_{\hat{\jmath}}^n(x_1) = u_{\hat{\jmath}}^n(x_3) - u_{\hat{\jmath}}^n(x_2)$. In other words, the images of x_1, x_2, x_3 under $u_{\hat{\imath}}^n$ are equally spaced and so are they in $u_{\hat{\jmath}}^n$. Let us say that x_1, x_2 and x_3 form a grid in X with a mesh of size 1. Thanks to the uniqueness remark made above with respect to the two essential ranks $\hat{\jmath}$ and $\hat{\imath}$, we necessarily have that $u_{\hat{\imath}}^n = \alpha_{\hat{\imath}} + \beta_{\hat{\imath}} u_{\hat{\jmath}}^n$ for some real numbers $\alpha_{\hat{\imath}}$ and $\beta_{\hat{\imath}}$ with $\beta_{\hat{\imath}} > 0$.

By continuity and connectedness, there is $x_{1\circ 2}$ 'halfway' between x_1 and x_2. More formally, there is $x_{1\circ 2}$ such that $u_{\hat{\imath}}^n(x_{1\circ 2}) - u_{\hat{\imath}}^n(x_1) = u_{\hat{\imath}}^n(x_2) - u_{\hat{\imath}}^n(x_{1\circ 2})$. There is also $x_{2\circ 3}$ such that $u_{\hat{\imath}}^n(x_{2\circ 3}) - u_{\hat{\imath}}^n(x_2) = u_{\hat{\imath}}^n(x_3) - u_{\hat{\imath}}^n(x_{2\circ 3})$. The images of $x_1, x_{1\circ 2}, x_2, x_{2\circ 3}$ and x_3 under $u_{\hat{\imath}}^n$ are thus equally spaced and so are they under $u_{\hat{\jmath}}^n$. The outcomes $x_1, x_{1\circ 2}, x_2, x_{2\circ 3}$ and x_3 thus form a grid in X with mesh of size 1/2. We can again halve the mesh of this grid by adding the outcomes $x_{1\circ(1\circ 2)}, x_{(1\circ 2)\circ 2}, x_{2\circ(2\circ 3)}$ and $x_{(2\circ 3)\circ 3}$ to the grid. And we can make the mesh as fine as we want by repeating this process.

Let us denote the outcomes of the initial grid (mesh size = 1), by $g_1^1 = x_1$, $g_2^1 = x_2$ and $g_3^1 = x_3$. Similarly, we denote the elements of the second grid (mesh size = 1/2) by $g_1^2 = x_1$, $g_2^2 = x_{1\circ 2}$, $g_3^2 = x_2$, $g_4^2 = x_{2\circ 3}$ and

$g_5^2 = x_3$, those of the third grid (mesh size $= 1/2^2, g_1^3 = x_1$, $g_2^3 = x_{1\circ(1\circ 2)}$, $g_3^3 = x_{1\circ 2}, \ldots, g_8^3 = x_{(2\circ 3)\circ 3}$ and $g_9^3 = x_3$ and so on.

If x_1 is not maximal for $\succsim$ restricted to singletons, we can try to extend the grid to the 'left' of $x_1 = g_1^1$, by looking for an element g_0^1 such that $u_j^n(g_1^1) - u_j^n(g_0^1) = u_j^n(g_2^1) - u_j^n(g_1^1)$. If such a g_0^1 does not exist, then there exists a mesh size s such that there is g_0^s satisfying $u_j^n(g_1^s) - u_j^n(g_0^s) = u_j^n(g_2^s) - u_j^n(g_1^s)$.

If g_0^s is not maximal for $\succsim$, we can again extend the grid to the 'left' of g_0^s (this may require using a finer mesh). By repeating this process, we can extend the grid to the 'left' of x_1 and go as close as we wish to the maximal prospects for $\succsim$ (if any).

We can also extend he grid to the 'right' of x_3 and go as close as we wish to the minimal singleton prospect for $\succsim$ (if any). Since the images of all elements of a grid are equally spaced in $\mathbb{R}$ under $u_{\hat{\imath}}^n$ and $u_{\hat{\jmath}}^n$, we necessarily have that $u_{\hat{\imath}}^n(x) = \alpha_{\hat{\imath}} + \beta_{\hat{\imath}} u_{\hat{\jmath}}^n(x)$ for some real numbers $\alpha_{\hat{\imath}}$ and $\beta_{\hat{\imath}}$ with $\beta_{\hat{\imath}} > 0$ and for any element x of a grid of any mesh size.

Consider now an element x that does not belong to any grid. We have just seen above that we can refine or extend the initial grid in order to be as close as we wish to x. Continuity of $u_{\hat{\imath}}^n$ and $u_{\hat{\jmath}}^n$ then imply that $u_{\hat{\imath}}^n(x) = \alpha_{\hat{\imath}} + \beta_{\hat{\imath}} u_{\hat{\jmath}}^n(x)$ holds for any $x \in X$. The reasoning just made is valid for any essential rank $\hat{\imath} \in E(n)$ distinct from $\hat{\jmath}$. Hence, if there is at least one such an essential rank, we can write Equivalence (5) as

$$A \succsim B \iff \sum_{i \in E(n)} \beta_i u_{\hat{\jmath}}^n(a_i) \geq \sum_{i \in E(n)} \beta_i u_{\hat{\jmath}}^n(b_i)$$

after neglecting all irrelevant constant terms (including the constant functions that represent inessential ranks). Defining $u^n = u_{\hat{\jmath}}^n$ and $w_i^n = \beta_i / \sum_{j \in E(n)} \beta_j$ for all essential ranks i, and $w_i^n = 0$ for all inessential ranks, one can equivalently write

$$A \succsim B \iff \sum_{i \in [n]} w_i^n u^n(a_i) \geq \sum_{i \in [n]} w_i^n u^n(b_i).$$

If however $\hat{\jmath}$ is the only essential rank for prospects with n outcomes, then one sets $w^n_{\hat{\jmath}} = 1$ and $w^n_i = 0$ for all $i \neq \hat{\jmath}$ and $u^n = u^n_{\hat{\jmath}}$. $\square$

A.4 Lemma 3

Let x and y be two outcomes in X such that $\{x\} \succsim \{y\}$. Since for any n, there is at least one rank that is essential for prospects with n outcomes, this applies to $n = 1$ so that the ranking of singletons is not trivial. For any $n \geq 2$, consider a prospect D such that $\#D = n-1, \{x, y\} \cap D = \varnothing$ and $r_x^{D\cup\{x\}} = r_y^{D\cup\{y\}} = i$ for some $i \in E(n)$. Again, the existence of such a prospect for any given outcomes x and y such that $\{x\} \succsim \{y\}$ is guaranteed by the connectedness of X and the essentialness condition. By consistency in Comparisons of Preference Strength, one must have $D \cup \{x\} \succsim D \cup \{y\}$. Thanks to Proposition 1, this latter statement can equivalently be written as

$$\sum_{h\in[i-1]} w^n_h u^n(d_h) + w^n_i u^n(x) + \sum_{h\in[i+1,n]} w^n_h u^n(d_{h-1}) \geq \sum_{h\in[i-1]} w^n_h u^n(d_h) + w^n_i u^n(y) + \sum_{h\in[i+1,n]} w^n_h u^n(d_{h-1}).$$

Since $w^n_i > 0$ (because $i \in E(n)$), this is equivalent to $u^n(x) \geq u^n(y)$.

Using a similar reasoning and the completeness of $\succsim$, one would obtain the strict inequality $u^n(y) > u^n(x)$ if one had assumed $\{x\} \prec \{y\}$ instead. Hence $\{x\} \succsim \{y\}$ if and only if $u^n(x) \geq u^n(y)$ for any $n \geq 1$. $\square$

A.5 Lemma 4

Thanks to Proposition 1, we know that, for any $n \in \mathbb{N}$, there exist non-negative weights w^n_i as in (1) and $u^n : X \to \mathbb{R}$ such that (5) holds. We consider two cases.

Case 1. Suppose there is at least one cardinality m such that two ranks are essential (one of them is rank $\hat{\imath}$). Suppose $(a_i)_{i\in[p]}$ is some standard sequence for the sets B and C of cardinality $m-1$ with $\{a_1,\dots,a_p\}\cap(B\cup C)=\varnothing$, $B\prec C$ and such that $r_{a_i}^{B\cup\{a_i\}}=r_{a_i}^{C\cup\{a_i\}}=\hat{\imath}$ for all $i\in[p]$. By Consistency in Comparisons of Preference Strength, we must have $\{a_i\}\cup B\sim\{a_{i+1}\}\cup C$ for all $i\in[p-1]$ and, applying the numerical representation of Proposition 1 for $n=m$,

$$u^m(a_i)-u^m(a_{i-1})=u^m(a_{i+1})-u^m(a_i) \tag{9}$$

for all those i. Consider now any cardinality $m'\neq m$ with at least two essential ranks (if there is such m'). Let $\hat{\jmath}$ denote one of these essential ranks. One can find, thanks to continuity and connectedness of X, some sets D and E of cardinality $m'-1$ such that $\{a_1,\dots,a_p\}\cap(D\cup E)=\varnothing$ and, for all $i\in[p-1]$, $r_{a_i}^{D\cup\{a_i\}}=r_{a_{i+1}}^{E\cup\{a_{i+1}\}}=\hat{\jmath}$. Thanks to Consistency in Comparisons of Preference Strength, one will have $D\cup\{a_i\}\sim E\cup\{a_{i-1}\}$ for all $i\in[2,p-1]$ and, thanks to (5),

$$u^{m'}(a_i)-u^{m'}(a_{i-1})=u^{m'}(a_{i+1})-u^{m'}(a_i). \tag{10}$$

Thanks to the connectedness of X and continuity of $\succsim$ when restricted to singletons, we can choose the standard sequence $(a_i)_{i\in[p]}$ as fine as we want following the procedure described at the end of the proof of Proposition 1. Hence, the comparison of (9) and (10) shows that $u^m=\gamma^{m'}+\lambda^{m'}u^{m'}$ for any $m'\in[2,\cdot]$ with at least two essential ranks. We have seen in Proposition 1 that u^n is unique up to any positive affine transformation in the case of n with at least two essential ranks. Since u^m is a positive affine transformation of $u^{m'}$, we can use u^m instead of $u^{m'}$ in (5).

For any cardinality n with only one essential rank, (5) is a purely ordinal representation because only one weight w_i^n is non zero. We can therefore replace u^n by u^m because, as shown in Lemma 3, u^m is a strictly increasing transformation of u^n.

Since we can use u^m everywhere instead of u^n for every n (including $n = 1$), we can just define $u = u^m$ and this completes the proof for the case that there is a cardinality m with at least two essential ranks.

Case 2. Suppose there is no cardinality such that two ranks are essential. Then, for every n, only one weight w_i^n is non zero and (5) is a purely ordinal representation. We can therefore replace u^n by u^1 because, as shown in Lemma 3, u^1 is a strictly increasing transformation of u^n. Posing $u = u^1$ completes the proof. □

A.6 Lemma 5

The result being trivially true for a prospect D with a single outcome, consider a prospect D with $n \geq 2$ outcomes. Let us write $D = \{d_1, \ldots, d_n\}$. We define $\text{CE}(D)$ to be any element of the set $u^{-1}(\sum_{i\in[n]} w_i^n u(d_i))$ where the continuous function u and the weights w_i^n are those that define the numerical representation constructed in Lemma 4, which is also a numerical representation of the restriction of the ordering $\succsim$ to singletons thanks to Lemma 3. We need to show that $u^{-1}(\sum_{i\in[n]} w_i^n u(d_i)) \neq \varnothing$ and, also, that $\{\text{CE}(D)\} \sim D$.

The proof of the non-emptiness of $u^{-1}(\sum_{i\in[n]} w_i^n u(d_i))$ (and of the fact that $\{\text{CE}(D)\} \sim D$) being immediate if $u(d_i) = u(d_{i+1})$ for all $i \in [n-1]$, we consider the case where $u(d_1) > u(d_n)$ and, by Lemma 3, where $\{d_1\} \succ \{d_n\}$. We observe that since $w_i^n \geq 0$ for all i (with at least one strict inequality thanks to the assumption that at least one rank is essential for prospects with n outcomes) and $u(d_1) > u(d_n)$ by assumption, one must have (since $1 \geq w_i^n$) $u(d_1) \geq \sum_{i\in[n]} w_i^n u(d_i) \geq u(d_n)$ with at least one of the two inequalities strict. The continuity of u on its connected domain X implies u^{-1} is defined for any real number in the non-degenerate interval $[u(d_n), u(d_1)]$ and thus, in particular, for the real number $\sum_{i\in[n]} w_i^n u(d_i)$. Hence CE$(D)$ exists.

Suppose now that CE$(D) \in X$ is such that $\{\text{CE}(D)\} \nsim D$. Let us specifically assume that $\{\text{CE}(D)\} \succ D$. Let $i \in [n-1]$ be defined by $u(d_i) \geq u(\text{CE}(D))$ and $u(d_{i+1}) < u(\text{CE}(D))$. Consider then the sequence of prospects D^t (for $t \in \mathbb{N}$) defined by

$$D^t = \{\text{CE}(D)^{+\varepsilon_n^t}, ..., \text{CE}(D)^{+\varepsilon_{i+1}^t}, \text{CE}(D)^{-\varepsilon_i^t}, ..., \text{CE}(D)^{-\varepsilon_1^t}\}$$

for some list of n outcomes $\text{CE}(D)^{+\varepsilon_h^t}$ and $\text{CE}(D)^{-\varepsilon_h^t}$ such that $u(\text{CE}(D)^{-\varepsilon_h^t}) = u(\text{CE}(D)) - \varepsilon_h^t$, for $h \in [i+1, n]$, and $u(\text{CE}(D)^{+\varepsilon_h^t}) = u(\text{CE}(D)) + \varepsilon_h^t$, for $h \in [i]$ for some sequence of suitably small positive real numbers ε_h^t (for $h \in [n]$) such that $\varepsilon_1^t < \varepsilon_2^t < \ldots < \varepsilon_i^t$, $\varepsilon_n^t < \varepsilon_{n-1}^t < \ldots < \varepsilon_{i+1}^t$, $\lim_{t\to\infty} \varepsilon_j^t = 0$ for all $j \in [n]$ and

$$\sum_{h\in[i]} w_h^n \varepsilon_h^t = \sum_{h\in[i+1,n]} w_h^n \varepsilon_h^t.$$

It is clear here again that the existence of these $\text{CE}(D)^{+\varepsilon_h^t}$ and $\text{CE}(D)^{-\varepsilon_h^t}$ is secured by the continuity of u, the connectedness of X and the fact that $u(d_1) > \sum_{h\in[n]} w_h^n u(d_h) > u(d_n)$. We then have

$$\sum_{h\in[i]} w_h^n u(\text{CE}(D)^{-\varepsilon_h^t}) + \sum_{h\in[n]:h>i} w_h^n u(\text{CE}(D)^{+\varepsilon_h^t})$$
$$= u(\text{CE}(D)) = \sum_{h\in[n]} w_h^n u(d_h).$$

By Proposition 1, this implies $D \sim D^t$ for all t. By transitivity, we therefore have $\{\text{CE}(D)\} \succ D^t$ for all t. We observe also that the Internality axiom implies that

$$\{\text{CE}(D)^{+\varepsilon_n^t}\} \succsim D^t \succsim \{\text{CE}(D)^{-\varepsilon_1^t}\} \tag{11}$$

while the fact—established in Lemma 3—that u numerically represents the ordering $\succsim$ restricted to singletons implies that

$$\{\mathrm{CE}(D)^{+\varepsilon_n^t}\} \succ \{\mathrm{CE}(D)\} \succ \{\mathrm{CE}(D)^{-\varepsilon_1^t}\}. \tag{12}$$

Since $\{\mathrm{CE}(D)\} \succ D^t$ for all t, it follows from (11) and (12) and transitivity that:

$$\{\mathrm{CE}(D)^{+\varepsilon_n^t}\} \succ \{\mathrm{CE}(D)\} \succ D^t \succsim \{\mathrm{CE}(D)^{-\varepsilon_1^t}\}.$$

Yet both sequences of singletons $\{\mathrm{CE}(D)^{+\varepsilon_n^t}\}$ and $\{\mathrm{CE}(D)^{-\varepsilon_1^t}\}$ converge to $\{\mathrm{CE}(D)\}$. Hence, having $\{\mathrm{CE}(D)^{+\varepsilon_n^t}\} \succ \{\mathrm{CE}(D)^{-\varepsilon_1^t}\}$ for all t holding at the limit is incompatible with the continuity of $\succsim$. The argument for the case where $D \succ \{\mathrm{CE}(D)\}$ is similar. □

A.7 Remaining of Theorem 1

We know from Proposition 1 and Lemma 4 that for any $n \in \mathbb{N}$, there is a continuous utility function $u : X \to \mathbb{R}$ and a set of weights w_i^n (for $i \in [n]$) for which (1) holds for any prospects D and D' having both n possible outcomes. For any prospect D, let $U_D = \sum_{i \in [\#D]} w_i^{\#D} u(d_i)$. Consider two prospects $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ for some n and $m \in \mathbb{N}_{++}$ with $n \neq m$ such that $A \succsim B$. We need to show that $U_A \geq U_B$. By contradiction, suppose that $U_B > U_A$. Choose any suitably small strictly positive real number ε. Using Lemma 5, the continuity of u and the connectedness of X, define the sets C_n^j and C_m^j by

$$C_n^j = \{c_1^j, \ldots, c_n^j\}$$

and

$$C_m^j = \{c_1^j, \ldots, c_m^j\}$$

where, for any $i \in [m] \cup [n]$ and $j \in \mathbb{N}, c_i^j \in X$ is such that:

$$c_i^j = u^{-1}\left(U_A + \frac{\varepsilon}{ij}\right).$$

By Internality, we have that $\{c_1^j\} \succsim C_n^j$. By the numerical representation, $C_n^j \succ A$. We observe also that, for sufficiently large j, $U_{C_m^j}$ can be made arbitrarily close to U_A. Take therefore a fixed j, say $\bar{\jmath}$, sufficiently large for the assumption $U_B > U_A$ to imply $U_B > U_{C_m^{\bar{\jmath}}} > U_A$. Since the numerical representation holds for sets of cardinality m, we must therefore have $B \succ C_m^{\bar{\jmath}}$. By Internality, $C_m^{\bar{\jmath}} \succsim \{c_m^{\bar{\jmath}}\}$ and, by the numerical representation applied to singletons, $\{c_m^{\bar{\jmath}}\} \succ \{c_m^j\}$ for all $j > \bar{\jmath}$. Thanks to Lemma 5 and transitivity, we have

$$\{c_1^j\} \succsim C_n^j \succ A \sim \{\text{CE}(A)\} \succsim B \succ C_m^{\bar{\jmath}} \succsim \{c_m^{\bar{\jmath}}\} \succ \{c_m^j\},$$

for all $j > \bar{\jmath}$. But having $\{c_1^j\} \succ \{\text{CE}(A)\} \succ \{c_m^{\bar{\jmath}}\} \succ \{c_m^j\}$ for all $j > \bar{\jmath}$ contradicts Fixed Cardinality Continuity (applied to singletons), since both sequences $\{c_1^j\}_{j\in\mathbb{N}}$ and $\{c_m^j\}_{j\in\mathbb{N}}$ converge to $u^{-1}(U_A) = u^{-1}(\text{CE}(A))$. The converse implication that $U_A \geq U_B$ implies $A \succsim B$ is proved in the same way. $\square$

A.8 Proposition 2

By our main theorem, $\succsim$ has a RDWAU representation with non-negative weights w_i^n. Let q be any rational number in $[0, 1]$ and let us consider two pairs of integers (i, n) and (j, m) such that $i/n = j/m = q$. Let $A = \{a_1, \ldots, a_n\}$ be a prospect such that, for all $h \in [n-1]$, $u(a_h) = u(a_{h+1}) + \delta$ for some positive real number δ. The existence of such a prospect does not raise any difficulty thanks to the connectedness of the set X and the continuity of u. For $h \in [n]$, let $B^h = \{b_1^h, \ldots, b_{m-1}^h\}$ be a prospect with $m - 1$ outcomes such that $u(b_k^h) = u(a_h) - \varepsilon\delta/k$ for all $k \in [m-1]$, with $0 < \varepsilon < 1$. By construction, $A \cap B^1 \cap \ldots \cap B^n = \emptyset$ and $\{a_h\} \succ \{b_k^h\}$ for all $h \in [n]$ and $k \in [m-1]$. Consider then the prospect $B = B^1 \cup \ldots \cup B^n = \{b_1, ..., b_{\#B}\}$ with its elements ordered as usual. By Strong Dominance one has:

$$A \succ A \cup B^1 \cup \ldots \cup B^n. \tag{13}$$

Symmetrically, for $h \in [n]$, one can construct a prospect $C^h = \{c_1^h, \ldots, c_{m-1}^h\}$ such that $u(c_k^h) = u(a_h) + \varepsilon\delta/k$ for all $k \in [m-1]$, with $\epsilon < 1$. By construction, $A \cap C^1 \cap \ldots \cap C^n = \emptyset$ and $\{a_h\} \prec \{c_k^h\}$ for all $h \in [n]$ and $k \in [m-1]$. Applying again strong dominance dominance, we obtain:

$$A \prec A \cup C^1 \cup \ldots \cup C^n. \tag{14}$$

When ε tends to zero, both $u(b_k^h)$ and $u(c_k^h)$ tend to $u(a_h)$. Hence, since (13) and (14) hold for all $0 < \varepsilon < 1$ and since the RDWAU representation is continuous, we have

$$w_h^n = \sum_{k=(h-1)m+1}^{hm} w_k^{nm}, \quad \forall h \in [n].$$

This implies, in particular,

$$\sum_{h=1}^{i} w_h^n = \sum_{h=1}^{im} w_k^{nm}.$$

A similar reasoning, starting from a set A with m elements and sets B^h, C^h with $n-1$ elements, yields

$$\sum_{h=1}^{j} w_h^m = \sum_{h=1}^{jn} w_k^{nm}.$$

Since $im = jn$, we have:

$$\sum_{h=1}^{i} w_h^n = \sum_{h=1}^{j} w_h^m. \tag{15}$$

For any $q \in [0,1]_{\mathbb{Q}}$, let us therefore define $\psi(q) = \sum_{h=1}^{i} w_h^n$, using any i, n such that $i/n = q$. By (15), $\psi(q)$ does not depend on the particular choice of i and n. By construction, ψ is increasing. □

A.9 Proposition 3

Necessity. Let x_1, x_2, x_3, x_4 and A be as in the premise of Condition 3. Let $\#A = n$ and $r_{x_1}^{A\cup\{x_1\}} = k$. Then $\{x_1, x_4\} \sim \{x_2, x_3\}$ imply

$$w_1^2 u(x_1) + w_2^2 u(x_4) = w_1^2 u(x_2) + w_2^2 u(x_3).$$

Obvious simplifications yield

$$\frac{w_2^2}{w_1^2} = \frac{u(x_4) - u(x_3)}{u(x_2) - u(x_1)}.$$

Assuming a constant ratio ordering, we have

$$\frac{w_{k+1}^n}{w_k^n} = \rho = \frac{u(x_4) - u(x_3)}{u(x_2) - u(x_1)}$$

and

$$\begin{aligned}
&\sum_{h\in[k-1]} w_h^n u(a_h) + w_k^n u(x_1) + w_{k+1}^n u(x_4) + \sum_{h\in[k,n]} w_{h+2}^n u(a_h) \\
&\quad = \sum_{h\in[k-1]} w_h^n u(a_h) + w_k^n u(x_2) + w_{k+1}^n u(x_3) + \sum_{h\in[k,n]} w_{h+2}^n u(a_h).
\end{aligned}$$

Hence $A \cup \{x_1, x_4\} \sim A \cup \{x_2, x_3\}$ and necessity is proved.

Sufficiency. Suppose $\succsim$ is a RDWAU ordering as in Theorem 1, with utility function u and weights w_i^n. Consider any four outcomes x_1, x_2, x_3 and x_4 such that $\{x_1\} \succ \{x_2\} \succsim \{x_3\} \succ \{x_4\}$, $\{x_1, x_4\} \sim \{x_2, x_3\}$ and $\{x\} \succ \{x_1\}$, $\{x_4\} \succ \{y\}$ for some x and $y \in X$. The existence of such outcomes is secured by the Essentialness condition (applied to prospects made of two outcomes) and the continuity of the RDWAU ordering (combined with the connectedness of X for the order topology of $\succsim$ restricted to singletons).

Consider any $n \in [2, \cdot]$ and $k \in [n-1]$. By continuity of $\succsim$ and connectedness of X again, there are $n-2$ outcomes $a_1, \ldots, a_{n-2}$ such that $a_1 \succ \ldots \succ a_{k-1} \succ x_1 \succ x_4 \succ a_{k+2} \succ \ldots \succ a_n$. If we define $A = \{a_1, \ldots, a_{n-2}\}$, then Condition 3 implies $A \cup \{x_1, x_4\} \sim A \cup \{x_2, x_3\}$. Using the numerical representation of Theorem 1, we find

$$\frac{w_{k+1}^n}{w_k^n} = \frac{u(x_4) - u(x_3)}{u(x_2) - u(x_1)}.$$

Since this holds for all $n \in [2, \cdot]$ and $k \in [n-1]$, the proof is complete if we define

$$\rho = \frac{u(x_4) - u(x_3)}{u(x_2) - u(x_1)}.$$

□

B Independence of the axioms

In the following three examples, we prove the independence of the three axioms that characterize RDWAU orderings by exhibiting non-RDWAU orderings that satisfy any two of the three axioms but not the remaining one.

Example 2 *Assume that* $X = \mathbb{R}$, *and consider the ordering* $\succsim^{\text{add}}$ *on* $\mathcal{P}(X)$ *defined by* $A \succsim^{\text{add}} B \iff \sum_{i \in \# A} a_i \geq \sum_{i \in \# B} b_i$. *This ordering obviously satisfies Fixed Cardinality Continuity and Consistency in Comparisons of*

Preference Strength. To see that it violates the Internality axiom, one can simply observe that, contrary to what this principle would require, $\{1,2\} \succ^{\text{add}} \{2\}$.

Example 3 *Assume that* $X = \mathbb{R}$, *and consider the ordering* $\succsim^{\max}$ *on* $\mathcal{P}(X)$ *defined by* $A \succsim^{\max} B \iff a_1 \geq b_1$. *This ordering obviously satisfies Fixed Cardinality Continuity and Internality. To see that it violates Consistency in Comparisons of Preference Strength, one first observes that* $\{6,1\} \succ^{\max} \{5,2\}$ *and* $\{3,1\} \prec^{\max} \{4,2\}$, *which implies through Definition 1 that* $(6,5)\ \Delta_{\mathbf{s}}^{\succsim^{\max}}\ (3,4)$. *The violation of Consistency in Comparisons of Preference Strength is then established by noticing that* $\{7,6\} \sim^{\max} \{7,5\}$ *and* $\{7,3\} \sim^{\max} \{7,4\}$ *and, therefore, that* $(3,4)\ \Delta_{\mathbf{e}}^{\succsim^{\max}}\ (6,5)$.

Example 4 *Assume that* $X = \mathbb{R}^2$ *and, for any* $x \in X$ *and* $i \in [2]$, *let* x^i *denote the* i^{th} *component of* x. *Consider then the lexicographic version of the UEU ordering* $\succsim^{\text{lex}}$ *on* $\mathcal{P}(X)$ *defined, for any prospects* A *and* B, *by*

$$A \sim^{\text{lex}} B \iff \sum_{i \in [\#A]} \frac{a_i^1}{\#A} = \sum_{i \in [\#A]} \frac{b_i^1}{\#A} \text{ and } \sum_{i \in [\#A]} \frac{a_i^2}{\#A} = \sum_{i \in [\#A]} \frac{b_i^2}{\#A}$$

and by $A \succ^{\text{lex}} B$ *if either*

$$\sum_{i \in [\#A]} \frac{a_i^1}{\#A} > \sum_{i \in [\#A]} \frac{b_i^1}{\#A}$$

or

$$\sum_{i \in [\#A]} \frac{a_i^1}{\#A} = \sum_{i \in [\#A]} \frac{b_i^1}{\#A} \text{ and } \sum_{i \in [\#A]} \frac{a_i^2}{\#A} > \sum_{i \in [\#A]} \frac{b_i^2}{\#A}$$

Hence, the ordering $\succsim^{\text{lex}}$ *compares prospects on the basis of a lexicographic combination of the symmetric average of each component of the (two-dimensional) outcomes of those prospects. It is easy to see that this ordering violates Fixed Cardinality Continuity. Indeed, for any outcome* $(a^1, a^2) \in X$, *the set of outcomes* $(x^1, x^2) \in X$ *such that* $\{(x^1, x^2)\} \succsim^{\text{lex}} \{(a^1, a^2)\}$ *is not closed in* X.

To see that $\succsim^{\text{lex}}$ satisfies Internality, consider the prospect $D = \{d_1, \ldots, d_n\}$ for some $n \in \mathbb{N}$. Since $\{d_n\} \precsim^{\text{lex}} \{d_i\}$ for all $i \in [n]$, we have either

$$d_i^1 > d_n^1 \tag{16}$$

or

$$d_i^1 = d_n^1 \text{ and } d_i^2 \geq d_n^2 \tag{17}$$

for all $i \in [n]$. Summing over n the inequalities or equalities (16) and (17) yields either

$$\sum_{i \in [n]} d_i^1 > n d_n^1 \Longleftrightarrow \frac{\sum_{i \in [n]} d_i^1}{n} > d_n^1 \quad \text{if (16) holds for some } i \text{ or}$$
$$\frac{\sum_{i \in [n]} d_i^1}{n} = d_n^1 \quad \text{and} \quad \frac{\sum_{i \in [n]} d_i^2}{n} \geq d_n^2 \quad \text{if (17) holds for all } i.$$

Hence, one has $\{d_1, \ldots, d_n\} \succsim^{\text{lex}} \{d_n\}$ as required by the Internality axiom. The conclusion that $\{d_1\} \succsim^{\text{lex}} \{d_1, \ldots, d_n\}$ can be obtained through a similar reasoning.

We now turn to Consistency in Comparisons of Preference Strength. To show that this axiom is satisfied by the ordering $\succsim^{\text{lex}}$ suppose by contradiction that it is not. This means that there exist prospects A, A', B and B' in $\mathcal{P}(X)$ satisfying $\#A = \#A' = n$ and $\#B = \#B' = m$ for some m, $n \in \mathbb{N}$ and distinct outcomes x, y, x', $y' \in X$ such that $\{x, y, x', y'\} \cap (A \cup A' \cup B \cup B') = \varnothing$ for which one has

$$\{a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n\} \succsim^{\text{lex}} \quad \{a'_1, \ldots, a'_{i-1}, y, a'_{i+1}, \ldots, a'_n\}, \tag{18}$$
$$\{a_1, \ldots, a_{i-1}, x', a_{i+1}, \ldots, a_n\} \precsim^{\text{lex}} \quad \{a'_1, \ldots, a'_{i-1}, y', a'_{i+1}, \ldots, a'_n\}, \tag{19}$$
$$\{b_1, \ldots, b_{j-1}, x, b_{j+1}, \ldots, b_m\} \prec^{\text{lex}} \quad \{b'_1, \ldots, b'_{j-1}, y, b'_{j+1}, \ldots, b'_m\}, \tag{20}$$
$$\{b_1, \ldots, b_{j-1}, x', b_{j+1}, \ldots, b_m\} \succsim^{\text{lex}} \quad \{b'_1, \ldots, b'_{j-1}, y', b'_{j+1}, \ldots, b'_m\}, \tag{21}$$

for some $i \in [n]$ and $j \in [m]$ or, possibly, with the comparison (20) weak and the comparison (21) strict. Yet, we focus on (20) strict and (21) weak in the following sketch. From (18) we conclude

$$\sum_{h=1}^{i-1} a_h^1 + x^1 + \sum_{h=i+1}^{n} a_h^1 > \sum_{h=1}^{i-1} a_h'^1 + y^1 + \sum_{h=i+1}^{n} a_h'^1 \tag{22}$$

or

$$\begin{aligned} \sum_{h=1}^{i-1} a_h^1 + x^1 + \sum_{h=i+1}^{n} a_h^1 &= \sum_{h=1h}^{i-1} a_h'^1 + y^1 + \sum_{h=i+1}^{n} a_h'^1 \text{ and} \\ \sum_{h=1}^{i-1} a_h^2 + x^2 + \sum_{h=i+1}^{n} a_h^2 &\geq \sum_{h=1}^{i-1} a_h'^2 + y^2 + \sum_{h=i+1}^{n} a_h'^2. \end{aligned} \tag{23}$$

Similarly, we obtain from (19)

$$\sum_{h=1}^{i-1} a_h^1 + x'^1 + \sum_{h=i+1}^{n} a_h^1 < \sum_{h=1}^{i-1} a_h'^1 + y'^1 + \sum_{h=i+1}^{n} a_h'^1 \tag{24}$$

or

$$\begin{aligned} \sum_{h=1}^{i-1} a_h^1 + x'^1 + \sum_{h=i+1}^{n} a_h^1 &= \sum_{h=1}^{i-1} a_h'^1 + y'^1 + \sum_{h=i+1}^{n} a_h'^1 \text{ and} \\ \sum_{h=1}^{i-1} a_h^2 + x'^2 + \sum_{h=i+1}^{n} a_h^2 &\leq \sum_{h=1}^{i-1} a_h'^2 + y'^2 + \sum_{h=i+1}^{n} a_h'^2. \end{aligned} \tag{25}$$

Four cases need to be considered:

1. *(22) and (24) imply $x^1 - x'^1 > y^1 - y'^1$;*

2. (22) and (25) imply $x^1 - x'^1 > y^1 - y'^1$ *and*

$$\sum_{h=1}^{i-1} a_h^2 + x'^2 + \sum_{h=i+1}^{n} a_h^2 \leq \sum_{h=1}^{i-1} a_h'^2 + y'^2 + \sum_{h=i+1}^{n} a_h'^2;$$

3. (23) and (24) imply $x^1 - x'^1 > y^1 - y'^1$ *and*

$$\sum_{h=1}^{i-1} a_h^2 + x^2 + \sum_{h=i+1}^{n} a_h^2 \geq \sum_{h=1}^{i-1} a_h'^2 + y^2 + \sum_{h=i+1}^{n} a_h'^2;$$

4. (23) and (25) imply $x^1 - x'^1 = y^1 - y'^1$ *and* $x^2 - x'^2 \geq y^2 - y'^2$.

Similarly, we can derive from (20) that

$$\sum_{h=1}^{i-1} b_h^1 + x^1 + \sum_{h=i+1}^{n} b_h^1 < \sum_{h=1}^{i-1} b_h'^1 + y^1 + \sum_{h=i+1}^{n} b_h'^1 \tag{26}$$

or

$$\begin{aligned} \sum_{h=1}^{i-1} b_h^1 + x^1 + \sum_{h=i+1}^{n} b_h^1 &= \sum_{h=1}^{i-1} b_h'^1 + y^1 + \sum_{h=i+1}^{n} b_h'^1 \text{ and} \\ \sum_{h=1}^{i-1} b_h^2 + x^2 + \sum_{h=i+1}^{n} b_h^2 &\leq \sum_{h=1}^{i-1} b_h'^2 + y^2 + \sum_{h=i+1}^{n} b_h^2, \end{aligned} \tag{27}$$

while (21) leads to

$$\sum_{h=1}^{i-1} b_h^1 + x'^1 + \sum_{h=i+1}^{n} b_h^1 > \sum_{h=1}^{i-1} b_h'^1 + y'^1 + \sum_{h=i+1}^{n} b_h'^1 \tag{28}$$

or

$$\sum_{h=1}^{i-1} b_h^1 + x'^1 + \sum_{h=i+1}^{n} b_h^1 \;=\; \sum_{h=1}^{i-1} b_h'^1 + y'^1 + \sum_{h=i+1}^{n} b_h'^1 \text{ and}$$

$$\sum_{h=1}^{i-1} b_h^2 + x'^2 + \sum_{h=i+1}^{n} b_h^2 \;\geq\; \sum_{h=1}^{i-1} b_h'^2 + y'^2 + \sum_{h=i+1}^{n} b_h^2. \qquad (29)$$

The four implications resulting from all the possible combinations of these expressions are

1. *(a) (26) and (28) yield* $x^1 - x'^1 < y^1 - y'^1$*;*

   (b) (26) and (29) yield $x^1 - x'^1 < y^1 - y'^1$ *and*

   $$\sum_{h=1}^{i-1} b_h^2 + x'^2 + \sum_{h=i+1}^{n} b_h^2 \geq \sum_{h=1}^{i-1} b_h'^2 + y'^2 + \sum_{h=i+1}^{n} b_h^2;$$

   (c) (27) and (28) yield $x^1 - x'^1 < y^1 - y'^1$ *and*

   $$\sum_{h=1}^{i-1} b_h^2 + x^2 + \sum_{h=i+1}^{n} b_h^2 \leq \sum_{h=1}^{i-1} b_h'^2 + y^2 + \sum_{h=i+1}^{n} b_h^2;$$

   (d) (27) and (29) yield $x^1 - x'^1 = y^1 - y'^1$ *and* $x^2 - x'^2 > y^2 - y'^2$*.*

Since any combination of one of the cases (1)–(4) with one the cases (a)–(b) leads to an obvious contradiction, this shows that the ordering $\succsim^{\text{lex}}$ *does indeed satisfy Consistency in Comparisons of Preference Strength.*

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