Intervention Under Network Uncertainty

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Abstract

We study a network intervention problem where a fully informed intervener removes a player or a set of links from a network whose participants can only observe their direct neighborhoods. The intervener's objective is to structurally perturb the network to maximize the change in aggregate equilibrium effort levels stemming from an incomplete information linear quadratic game. Under the assumption that intervention is not anticipated by network participants, we show that uncertainty over network architecture restricts the number of agents who become aware of the fact that intervention has transpired. In particular, the set of agents revising equilibrium actions is restricted to those whose links are directly affected. Consequently, popular metrics such as inter-centrality or Katz-Bonacich centrality are no longer sufficient in characterizing key agents and key links. We demonstrate our model when agents have under uniform prior beliefs over network topologies. In this case, key agents as well as key links are completely characterized by degree centrality.

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1 Introduction

Identifying the key players in a network is an important exercise with applications from optimal technology adaption to the reduction of criminal activities. Most of the work dealing with the identification of key player in networks assumes that network participants have full knowledge of the network topology they are embedded in. However complete information about the network structure is too strong an assumption. Participants of real world economic and social networks are rarely if ever aware of the entire network architecture. Nonetheless, identifying optimal intervention strategies under incomplete information is still an unsolved problem. In this paper we aim to fill this gap.

We employ the incomplete information linear quadratic game of Chaudhuri, Sarangi, and Tzavellas (2023) in which we introduce a fully informed planner who structurally intervenes in the network. The planners objective is to suitably select targets to minimize aggregate actions in equilibrium. The game proceeds as follows: Nature moves first and picks a graph from an ex-ante distribution over all possible unweighted and undirected networks on n vertices. Once a network is realized, each individual sees their direct connections but do not their indirect connections. It is at this stage that the external planner intervenes. Under the assumption that intervention is not anticipated by agents, it is only observed locally in the network. That is, only by those individuals whose local neighborhood is altered by it. Affected agents update their beliefs regarding what other agents were also affected by the intervention. Lastly, all agents simultaneously exert actions to maximize their interim linear quadratic payoffs.

We consider two kinds of intervention. In the first kind of intervention the intervenor removes a player. This player removal is observed only locally by those agents who are directly connected to the removed player. These agents in turn form beliefs regarding what other players observed the removal. The player whose removal reduces the aggregate activity the most is termed as the *key player*. In the second kind of intervention the intervenor removes a set of links. On removing a link, only the players involved in the link will observe it being removed. The set of links whose removal reduces aggregate activity the most is termed as the *key links*. We show that the optimal intervention strategy depends not only on the maximum action exerted by a player but also the expected complementarity strengths that arises due to the removal of nodes or links.

Optimal structural intervention in networks under complete information has been a well studied problem. Ballester, Calvo-Armengol, Zenou (2006) introduces the inter-centrality metric to determine the key player in the network. Sun, Zhao and Zhou (2023) introduce the link index, a function of Katz-Bonacich centrality and influence matrix, to determine the optimal structural intervention in terms of link creation. Unlike their work, we find that when agents have incomplete information about network architecture, neither inter-centrality nor Katz-Bonacich centrality can characterize the optimal structural intervention. Instead, when there is uncertainty about the network architecture, the player that induces the maximum expected complementarity strength for all others in the network is the key player. Similarly, the links that give rise to maximum complementarity strengths are the key links.

We explore optimal structural intervention when the underlying probability distribution is uniform over all possible networks. Under such beliefs, the one with the highest degree turns out to be the key player. For uniform priors, expected complementarity strengths arising in the network are same for all the players. The optimal intervention strategy, therefore, does not depend on the complementarity strengths arising from indirect connections, and depend only on the number of connections a player has. Since expected complementarities do not depend on agent identity, it also follows that all links are equally important under key link intervention.

The rest of the paper is organized as follows. Section 2 depicts the incomplete information linear quadratic model a la Chaudhuri, Sarangi, and Tzavellas (2023) and the two types of intervention. Section 3 explores these concepts through an application with an uniform prior.

2 Model

Let $N = \{1, 2, ..., n\}$ denote the set of players. Letting $i \sim j$ denote a link between players i and j, a *network* (or *graph*) $\mathbf{g}$ is the collection of all pairwise links that exist between the players. The links are undirected such that $i \sim j \in \mathbf{g}$ implies $j \sim i \in \mathbf{g}$. The network can be represented by its adjacency matrix which, with some abuse of notation, is also denoted as $\mathbf{g} = [g_{ij}]$, where $g_{ij} = 1$ if a link exists between players i and j, and $g_{ij} = 0$ otherwise. There are no self-loops and thus $g_{ii} = 0$ for all $i \in N$. The fact that links are undirected implies $\mathbf{g} = \mathbf{g}^T$. Let E denote the set of edges present in the network, i.e. $E = \{(i, j) : g_{ij} = 1\}$. Since, links are undirected, $(i, j) \in E$ implies $(j, i) \in E$. We denote by $\mathcal{G}_n$ the set of all unweighted and undirected networks on n vertices whose cardinality is $2^{\frac{n(n-1)}{2}}$. Given the adjacency representation of a network $\mathbf{g} \in \mathcal{G}_n$ we let $\mathbf{g}_i$ denote its i^{th} row. That is, $\mathbf{g}_i = (g_{i1}, g_{i2}, ..., g_{in}) \in \{0, 1\}^n$, where it is understood that $g_{ii} = 0$. In the following section, it will be convenient to represent any network $\mathbf{g}$ by the rows of its adjacency matrix:

$$\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, .., \mathbf{g}_n) \tag{1}$$

2.1 Effort Game

We employ the incomplete information network game of Chaudhuri, Sarangi, and Tzavellas (2023), in which agents embedded in a network have access to information only on their local neighborhood and play a linear quadratic game. At the beginning of the game Nature, a non-strategic player, draws a network from the set of all possible networks on n nodes. The network is chosen from an ex-ante distribution that is common knowledge among all agents. Following Nature's draw, players realize their direct connections (they can see the agents with who they are linked), but do not know the network's architecture beyond that. In other words, they do not observe the links of their neighbors. Using the information on their direct connections, agents proceed to update their beliefs about the network chosen by Nature according to Bayes' rule. Given these updated beliefs, agents simultaneously exert actions to maximize their interim payoffs.

Type Space

Agents' **types** are representative of their corresponding row in the adjacency representation of the network over which the game will be played. That is, each player's **type set** assumes the following form:

$$G_i = \{(g_{i1}, g_{i2}, ..., g_{in})_i \in \{0, 1\}^n : g_{ii} = 0\}$$

where $g_{ij} = 1$ if player i is connected to j and 0 otherwise. The cardinality of each agent's type set is:

$$|G_i| \equiv \gamma = 2^{n-1}$$

and we denote its elements by $\mathbf{g}_i^{t_i} \in G_i$. Whenever the context is clear and we need not enumerate the elements of each type set we suppress the superscript t_i. Given each player's type set, we can write down the ***type space*** of the game:

$$G = \times_{i \in N} G_i$$

Ex-ante Beliefs

We denote by $p \in \Delta(G)$ the probability distribution over the type space, with $\Delta(G)$ denoting the set of all probability distributions over G. In our game, Nature moves first and chooses an element of the type space $\mathbf{g} \in G$. We want to restrict Nature's choice to those elements in G that have valid network representations. Towards this, we define the following set of admissible distributions, and impose the assumption that Nature draws a network from a distribution in this set.

Definition 1. We say that the probability distribution $p \in \Delta(G)$ is admissible if it satisfies:

$$p(\mathbf{g}) = 0 \ \forall \ \mathbf{g} \in G \text{ s.t } \mathbf{g} \neq \mathbf{g}^T$$

and denote the set of all admissible distributions by $\Delta_A(G)$.

***Assumption 1*:** $p \in \Delta_A(G)$ and this is common knowledge.

The imposition of assumption 1 implies that $p(\mathbf{g}) > 0$ if and only if $\mathbf{g} \in \mathcal{G}_n$. Consequently, Nature will choose an unweighted and undirected network with certainty, and the fact that the agents are part of one such network is common knowledge.

Belief Updating

Given assumption 1, agents know that Nature draws a network and proceed to **update their beliefs** regarding its true topology according to Bayes' Rule. These ex-post updated beliefs can be written as:

$$p(\mathbf{g}_j|\mathbf{g}_i) = \frac{p(\mathbf{g}_i, \mathbf{g}_j)}{p(\mathbf{g}_i)} = \frac{\sum_{\mathbf{g}\in G} p(\mathbf{g})\mathbb{I}\{\mathbf{g}_i = \mathbf{g}|_i \wedge \mathbf{g}_j = \mathbf{g}|_j\}}{\sum_{\mathbf{g}\in G} p(\mathbf{g})\mathbb{I}\{\mathbf{g}_i = \mathbf{g}|_i\}} \;\forall i, j \in N, \tag{2}$$

where $\mathbb{I}$ is the indicator function. Specifically, for $\mathbf{g} \in G$, $\mathbb{I}\{\mathbf{g}_i = \mathbf{g}|_i\} = 1$ if $\mathbf{g}_i$ is the projection of $\mathbf{g}$ (i.e., $\mathbf{g}|_i$) on its i^{th} component and 0 otherwise.

State Game and Equilibrium

Given the above, conditional on a state $\mathbf{g} \in G$ being realized, agents play the **state game**:

$$s_{\mathbf{g}} = (N, A, (u_i(\mathbf{a}_i,\ \mathbf{a}_{-i}))_{i\in N})$$

where every agent has the same action set $A \equiv \mathbb{R}_+$. Let $\mathbf{a}_j = (a_j(\mathbf{g}_j^1), ..., a_j(\mathbf{g}_j^\gamma))$, $\mathbf{a}_{-i} = (\mathbf{a}_1, ..., \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, ..., \mathbf{a}_n)$ and $\mathbf{a}_i(\mathbf{g}_i^{-t_i}) = (a_i(\mathbf{g}_i^1), ..., a_i(\mathbf{g}_i^{t_i-1}), a_i(\mathbf{g}_i^{t_i+1}), ..., a_i(\mathbf{g}_i^\gamma))$. Interim utilities assume a linear-quadratic form:

$$u_i(a_i(\mathbf{g}_i^{t_i});\ \mathbf{a}_i(\mathbf{g}_i^{-t_i}),\ \mathbf{a}_{-i}) = a_i(\mathbf{g}_i^{t_i}) - \frac{1}{2}a_i(\mathbf{g}_i^{t_i})^2 + \lambda a_i(\mathbf{g}_i^{t_i}) \sum_{j=1}^{n} g_{ij}^{t_i} \sum_{\mathbf{g}_j\in G_j} p(\mathbf{g}_j|\mathbf{g}_i^{t_i}) a_j(\mathbf{g}_j) \tag{3}$$

Agents simultaneously exert actions to maximize (3). For each agent i, a pure strategy σ_i maps each possible type to an action. That is,

$$\sigma_i = (a_i(\mathbf{g}_i^1), ..., a_i(\mathbf{g}_i^\gamma))$$

This is a simultaneous move game of incomplete information so we invoke Bayes-Nash as the equilibrium notion.

Definition 2. The pure strategy profile $\sigma^* = (\sigma_i^*,\ \sigma_{-i}^*)$ where $\sigma_i = (a_i^*(\mathbf{g}_i^1), ..., a_i^*(\mathbf{g}_i^\gamma))$ is a Bayesian-Nash equilibrium (BNE) if:

$$a_i^*(\mathbf{g}_i^{t_i}) = arg \max_{a_i(\mathbf{g}_i^{t_i})} u_i(a_i(\mathbf{g}_i^{t_i}),\ \mathbf{a}_i^*(\mathbf{g}_i^{-t_i}),\ \sigma_{-i}^*)\ \forall\ i \in N,\ \forall\ \mathbf{g}_i^{t_i} \in G_i$$

Best Responses and BNE

Given the payoff structure, the best response of the i^{th} player whose is of type $\mathbf{g}_i^{t_i}$ is given by:

$$a_i(\mathbf{g}_i^{t_i}) = 1 + \lambda \sum_{j=1}^{n} g_{ij}^{t_i} \sum_{\mathbf{g}_j\in G_j} p(\mathbf{g}_j|\mathbf{g}_i^{t_i}) a_j(\mathbf{g}_j) \tag{4}$$

The system characterizing the best responses for all players can be written in vector notation as follows:

$$\mathbf{a} = \mathbf{1}_{n\gamma} + \lambda\mathbb{B}\mathbf{a} \tag{5}$$

where $\mathbf{1}_{n\gamma}$ is the $n\gamma$-dimesnional column vector of 1's, $\mathbf{a} = [\mathbf{a}_i]_{i=1}^n$, $\mathbf{a}_i = \left[a_i(\mathbf{g}_i^{t_i})\right]_{t_i=1}^\gamma$, $\gamma = 2^{n-1}$ is the total number of types of each player, and $\mathbb{B}$ is a block matrix that assumes the following form:

$$\mathbb{B} = \begin{pmatrix} \mathbf{0} & G_{1\sim 2} & \dots & G_{1\sim n} \\ G_{2\sim 1} & \mathbf{0} & \dots & G_{2\sim n} \\ \dots & \dots & \dots & \dots \\ G_{n\sim 1} & G_{n\sim 2} & \dots & \mathbf{0} \end{pmatrix}_{n\gamma\times n\gamma}$$

with

$$[G_{i\sim j}]_{t_i t_j} = g_{ij}^{t_i} p(\mathbf{g}_j^{t_j}|\mathbf{g}_i^{t_i}) \quad \forall\ t_j, t_i = 1, .., \gamma \quad \text{and} \quad \forall\ \mathbf{g}_j^{t_j} \in G_j, \mathbf{g}_i^{t_i} \in G_i$$

For any ex-ant prior and any realized network, Chaudhuri et al. (2023) characterize the unique pure strategy BNE given by

$$a_i^*(\mathbf{g}_i^{t_i}) = \sum_{s=0}^{\infty} \lambda^s \beta_{i,t_i}^{(s)} \quad \forall\ i \in N,\ \forall\ \mathbf{g}_i^{t_i} \in G_i, \tag{6}$$

where

$$\beta_{i,t_i}^{(s)} = \sum_{j_1, j_2, .., j_s = 1}^{n} \quad \sum_{t_{j_1}, t_{j_2}, .., t_{j_{s-1}} = 1}^{\gamma} g_{ij_1}^{t_i} g_{j_1 j_2}^{t_{j_1}} ... g_{j_{s-1} j_s}^{t_{j_{s-1}}} p(\mathbf{g}_{j_{s-1}}^{t_{j_{s-1}}}|\mathbf{g}_{j_{s-2}}^{t_{j_{s-2}}}) p(\mathbf{g}_{j_{s-2}}^{t_{j_{s-2}}}|\mathbf{g}_{j_{s-3}}^{t_{j_{s-3}}}) ... p(\mathbf{g}_{j_1}^{t_{j_1}}|\mathbf{g}_i^{t_i})$$

2.2 Intervention

We introduce a fully informed planner whose objective is to structurally intervene in the network and remove suitably selected targets to minimize aggregate actions in equilibrium. By fully informed, we mean that the intervener has complete information of the effort game played by agents and can thus compute BNE actions over any realized network.

Assumption 1. Intervention is not anticipated by players.

The key implication of this assumption is that the intervener's actions are only observed locally i.e., by players whose links are directly affected. Stated differently, only those agents whose realized types are altered by the structural intervention become aware that such intervention has transpired.

Intuitively, assumption 1 provides us with a framework for interpreting the nature of the intervention exercise itself. This framework, is along the lines of standard network intervention analysis as in the seminal Ballester et al. (2006) paper. In their set up, agents playing the complete information variant of the linear quadratic game are subjected to agent removal. In particular, an intervener will strategically remove a player in the network to maximize the change in aggregate equilibrium actions. Similar to our set up, the planner's intention to intervene is not anticipated by agents, and hence the intervention exercise reduces to comparing aggregate actions between pre and post removal Nash equilibria. Assumption 1 implies that we may think of intervention in a similar manner. That is, as a counterfactual or a comparative statics exercise on equilibrium actions to changes in the network topology.

The timing of events is as follows: (1) Nature selects a network and agents realize their types, (2) The planner structurally intervenes in the network, and (3) Agents exert actions. We consider two kinds of targeting strategies: (i) node removal (and all its links), and (ii) links removal. While these two types of intervention may appear to be structurally identical in certain cases (when a player is taken out of the network versus when all its links are removed), they are distinct in their degree of local informativeness. Given agents incomplete information on network architecture, player removal drives adjacent agents to internalize the intervention, while link removal does not. This stems from the fact that by removing a node i from the network, adjacent agents will need to form beliefs about which other agents observed $i's$ removal. On the other hand simply removing a link, say $i \sim j$, does not remove the node itself. Consequently agents i and j (whose link was removed) still observe one another participating in the network, and internalize the fact that the only two players who can observe the link being removed are themselves.

Key Player

Suppose a network $\tilde{\mathbf{g}} = (\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2, .., \tilde{\mathbf{g}}_n) \in G$ is realized. The intervenor removes an agent $k \in N$ to minimize the total activity in the network. This removal is observed by all $i \in N_k := \{i : \tilde{g}_{ik} = 1\}$i.e., all players directly connected to k (the player removed). On the other hand, all $i \in N \setminus N_k \cup \{k\}$ realize no change in their local neighborhoods and hence their actions remain unaltered. For any $j \in N$, we can partition the type set G_j into $G_j^1 := \{\mathbf{g}_j \in G_j : g_{jk} = 0\}$, i.e. the types for which j doesn't observe the removal of k, and $G_j^2 := \{\mathbf{g}_j \in G_j : g_{jk} = 1\}$, i.e. types for which j observes the removal of k.

Upon realizing the removal of k, all $i \in N_k := \{i : \tilde{g}_{ik} = 1\}$ update their best-responses according to,

$$\tilde{a}_i(\tilde{\mathbf{g}}_i) = 1 + \lambda \sum_{j \neq k} \tilde{g}_{ij} \sum_{\mathbf{g_j} \in G_j^2} p(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i)\, \tilde{a}_j(\mathbf{g}_j) + \lambda \sum_{j \neq k} \tilde{g}_{ij} \sum_{\mathbf{g_j} \in G_j^1} p(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i)\, a_j(\mathbf{g}_j)$$

On the other hand, all $k \neq i \in N \setminus N_k$ who do not realize the removal of k, have the same best-responses as given by (4),

$$a_i(\tilde{\mathbf{g}}_i) = 1 + \lambda \sum_{j=1}^{n} \tilde{g}_{ij} \sum_{\mathbf{g_j} \in G_j} p(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i)\, a_j(\mathbf{g}_j)$$

Aggregate activity in the network after the removal of k is given by $\Sigma_{-k} = \sum_{i \in N_k} \tilde{a}_i(\tilde{\mathbf{g}}_i) + \sum_{\substack{i \notin N_k \\ i \neq k}} a_i(\tilde{\mathbf{g}}_i)$. The change in aggregate activity is thus

$$\Sigma - \Sigma_{-k} = \sum_{i \in N_k} [a_i(\tilde{\mathbf{g}}_i) - \tilde{a}_i(\mathbf{g}_i)] + a_k(\tilde{\mathbf{g}}_k)$$

The key player is the agent for which this change is maximal. We denote this agent by $k^* \in N$ where

$$k^* \in \arg\max_{k \in N} \Psi(k) := \Sigma - \Sigma_{-k}$$

The change in actions of a player i upon removal of another k where player i in k's neighborhood is given by $\Psi_k(\mathbf{g}_i) = a_i(\mathbf{g}_i) - \tilde{a}_i(\mathbf{g}_i)$. Using the best responses, this change in actions can be written as:

$$\begin{aligned}\Psi_k(\mathbf{g}_i) &= a_i(\mathbf{g}_i) - \tilde{a}_i(\mathbf{g}_i) \\ &= \lambda \sum_{j \neq k} g_{ij} \sum_{\mathbf{g_j} \in G_j^2} p(\mathbf{g}_j \mid \mathbf{g}_i)(a_j(\mathbf{g}_j) - \tilde{a}_j(\mathbf{g}_j)) + \lambda \sum_{\mathbf{g_k} \in G_k} p(\mathbf{g}_k \mid \mathbf{g}_i) a_k(\mathbf{g}_k) \\ &= \lambda \sum_{j \neq k} g_{ij} \sum_{\mathbf{g_j} \in G_j^2} p(\mathbf{g}_j \mid \mathbf{g}_i) \Psi_k(\mathbf{g}_j) + \lambda \Phi_k(\mathbf{g}_i)\end{aligned}$$

where $\Phi_k(\mathbf{g}_i) = \sum_{\mathbf{g_k} \in G_k} p(\mathbf{g}_k \mid \mathbf{g}_i) a_k(\mathbf{g}_k)$ is player i's expectation about the action exerted by player k given he is of type $\mathbf{g}_i$. It follows that

$$\mathbf{\Psi}_k = \lambda \tilde{\mathbb{B}} \mathbf{\Psi}_k + \lambda \mathbf{\Phi}_k \tag{7}$$

where $\mathbf{\Psi}_k = \left[\mathbf{\Psi}_k^i\right]_{i \in N_k}$, $\mathbf{\Psi}_k^i = [\Psi_k(\mathbf{g}_i)]_{\mathbf{g}_i \in G_i^2}$, and $\tilde{\mathbb{B}}$ is a block matrix that assumes the following form:

$$\tilde{\mathbb{B}} = \begin{pmatrix} \mathbf{0} & G^2_{k_1 \sim k_2} & \cdots & G^2_{k_1 \sim k_q} \\ G^2_{k_2 \sim k_1} & \mathbf{0} & \cdots & G^2_{k_2 \sim k_q} \\ \cdots & \cdots & \cdots & \cdots \\ G^2_{k_q \sim k_1} & G^2_{k_q \sim k_2} & \cdots & \mathbf{0} \end{pmatrix}_{q\gamma' \times q\gamma'}$$

where $q = |N_k|$, $N_k := \{k_1, k_2, \ldots, k_q\}$, $\gamma' = \left|G_i^2\right| = 2^{n-2}$,and

$$\left[G^2_{k_i \sim k_j}\right]_{t_i t_j} = g^{t_i}_{k_i k_j} p\left(\mathbf{g}^{t_j}_{k_j} \mid \mathbf{g}^{t_i}_{k_i}\right) \qquad \forall t_j, t_i = 1, \ldots, \gamma' \text{ and } \forall \mathbf{g}^{t_j}_{k_j} \in G^2_{k_j}, \mathbf{g}^{t_i}_{k_i} \in G^2_{k_i}$$

Note that matrix $\tilde{\mathbb{B}}$ is structurally equivalent to $\mathbb{B}$ and can be thought of as a weighted and directed network over the types of k's neighbors that are directly connected to k. That is, those types who would observe the removal of k. The following theorem characterizes the key player.

Theorem 1. *For any admissible probability distribution, and for any realized network $\tilde{\mathbf{g}} \in G$, the key-player k^* is the one for whom*

$$\Psi(k^*) := a_{k^*}(\tilde{\mathbf{g}}_{k^*}) + \lambda \sum_{i \in N_{k^*}} \tilde{\Psi}_{k^*}(\tilde{\mathbf{g}}_i)$$

is the maximum, i.e. $\Psi(k^*) = \max_{k \in N} \Psi(k)$ *, with*

$$\tilde{\Psi}_{k^*}(\tilde{\mathbf{g}}_i) = \sum_{s=0}^{\infty} \lambda^s \psi^{(s)}_{i,t_i} \quad \forall\ i \in N$$

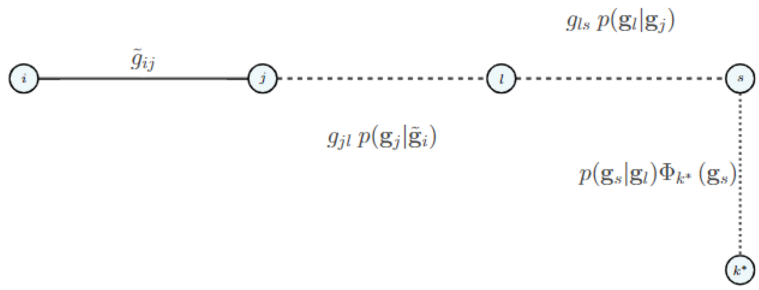

Figure 1: Expected complementarity strength from walks of length 3 when k is removed

where

$$\psi_{i,t_i}^{(s)} = \sum_{j_1,j_2,..,j_s=1}^{n} \sum_{t_{j_1},t_{j_2},..,t_{j_{s-1}}=1}^{\gamma} g_{ij_1}^{t_i} g_{j_1 j_2}^{t_{j_1}} ... g_{j_{s-1}j_s}^{t_{j_{s-1}}} p(\mathbf{g}_{j_{s-1}}^{t_{j_{s-1}}} | \mathbf{g}_{j_{s-2}}^{t_{j_{s-2}}}) p(\mathbf{g}_{j_{s-2}}^{t_{j_{s-2}}} | \mathbf{g}_{j_{s-3}}^{t_{j_{s-3}}}) ... p(\mathbf{g}_{j_1}^{t_{j_1}} | \mathbf{g}_i^{t_i}) \Phi_{k^*}(\mathbf{g}_{j_1})$$

$\Psi(k^*)$ gives the total complementarity strength arising in the network $\tilde{\mathbf{g}}$ due to the presence of k^*. This complementarity comprises of three parts, (i) the direct complementarity arising due to the action exerted by the player k^*, (ii) the direct complementarity arising due to the action exerted by the post-removal isolated players whose only connection was with k^* and (iii) the indirect complementarity strength that arises due to the complementarity that k^*'s neighbors expect to extract from it. While (i) arises due to the local complementarities in the network, the belief structure in (iii) is a consequence of the prevailing incomplete information. Observe that in the expression of $\Psi_{k^*}(\tilde{\mathbf{g}}_i)$, only those players whose types are such that $g_{ik^*} = 1$ adjust their action. Thus, $\tilde{\Psi}_{k^*}(\tilde{\mathbf{g}}_i)$ gives the total complementarity strength that player i expects to extract from player k^* through the walks of all lengths.

As an example, consider the complementarity that player i expects to extract from player k^* through walks of length 3:

$$\sum_{j,l,s \in N_k} \sum_{\mathbf{g}_j \in G_j^2} \sum_{\mathbf{g}_l \in G_l^2} \sum_{\mathbf{g}_s \in G_s^2} \tilde{g}_{ij} g_{jl} g_{ls} p(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i) \, p(\mathbf{g}_l \mid \mathbf{g}_j) \, p(\mathbf{g}_s \mid \mathbf{g}_l) \, \Phi_k(\mathbf{g}_s)$$

To compute the expected complementarity from player k^* through the walks of length 3, player i performs a similar calculation as in Chaudhuri et al. (2023) to reach player s. As only those types are considered here where k^* is a neighbor of s, hence the complelementarity arising from this walk is the product of the complementarity arising through the links of $s \sim l \sim j \sim i$ (given by $\tilde{g}_{ij} g_{jl} p(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i) \, g_{ls} p(\mathbf{g}_l \mid \mathbf{g}_j))$ and the complementarity that s expects to exert from k^*, i.e. $p(\mathbf{g}_s \mid \mathbf{g}_l) \, \Phi_{k^*}(\mathbf{g}_s)$.

Key Links

Instead of removing a player as in the previous case, suppose the intervenor now removes links between two players. This can be thought of as an intervention resulting in an obstruction of the direct connection between any two players. Let $L \subseteq E$ be the links removed by the intervenor and $V(L) := \{i \in N : \exists j \in N \text{ such that } (i,j) \in E\}$ denote the nodes involved in this removal process. Since we consider undirected links, $(i,j) \in L$ implies $(j,i) \in L$. The removal of a link is only observed by the two nodes involved in that link, i.e. if a link $(k,l) \in L$ is removed, then this is observed only by the agents k and l. Thus, for all $i \in V(L)$ upon realizing the removal of their respective links will update their best responses according to

$$\tilde{a}_i(\tilde{\mathbf{g}}_i) = 1 + \lambda \sum_{j \notin L_i} g_{ij} \sum_{\mathbf{g}_j \in G_j} p(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i) a_j(\mathbf{g}_j)$$

where $L_i := \{j \in N : (i,j) \in L\}$ is the identity of i's neighbors the direct connections to whom has been removed and $l_i = |L_i|$ is the number of links removed in which i is involved in. For all $i \in N \setminus V(L)$ the best-response remains unaltered as given by (4),

$$a_i(\tilde{\mathbf{g}}_i) = 1 + \lambda \sum_{j=1}^{n} \tilde{g}_{ij} \sum_{\mathbf{g_j} \in G_j} p(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i) a_j(\mathbf{g}_j)$$

The aggregate activity in the network after the removal of the links L is given by, $\Sigma_{-L} := \sum_{i \in N \setminus V(L)} a_i(\tilde{\mathbf{g}}_i) + \sum_{i \in V(L)} \tilde{a}_i(\tilde{\mathbf{g}}_i)$. The change in aggregate action due to the removal of the links takes the form

$$\Sigma - \Sigma_{-L} = \sum_{i \in V(L)} [a_i(\tilde{\mathbf{g}}_i) - \tilde{a}_i(\tilde{\mathbf{g}}_i)]$$

The impact of the removal of links on an individual player $i \in V(L)$ is given by

$$a_i(\tilde{\mathbf{g}}_i) - \tilde{a}_i(\tilde{\mathbf{g}}_i) = \lambda \sum_{j \in L_i} \mathbb{E}_i[a_j \mid \tilde{\mathbf{g}}_i]$$

where, $\mathbb{E}_i[a_j \mid \tilde{\mathbf{g}}_i] = \sum_{\mathbf{g_j} \in G_j} p(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i) a_j(\mathbf{g}_j)$ is player i's expectation about the action exerted by j, when its type is $\tilde{\mathbf{g}}_i$. Thus, the change in aggregate actions is given by

$$\Sigma - \Sigma_{-L} = \lambda \sum_{i \in V(L)} \sum_{j \in L_i} \mathbb{E}_i[a_j \mid \tilde{\mathbf{g}}_i]$$

If the planner is performing under the constraint that they can remove at most $\bar{l}$ links, then the set of *key links* L^* are given by

$$L^* \in \arg\max_{\substack{L \subseteq E \\ |L| = \bar{l}}} \lambda \sum_{i \in V(L)} \sum_{j \in L_i} \mathbb{E}_i[a_j \mid \tilde{\mathbf{g}}_i]$$

Note that the *key player* and *key links* may not correspond to the same optimal intervention strategy. To demonstrate consider the following example. Let $N = \{1,2,3,4\}$ and consider networks $\mathbf{g}^1, \mathbf{g}^2 \in$

$\mathcal{G}_4$ as shown in Figure 2.

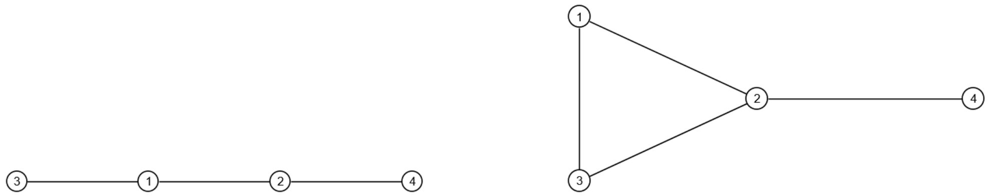

Figure 2: $\mathbf{g}^1$ (left) and $\mathbf{g}^2$ (right)

Suppose that the ex-ante distribution p satisfies:

$$p(\mathbf{g}) = \begin{cases} 0.3 & \text{if } \mathbf{g} = \mathbf{g}^1 \\ 0.7 & \text{if } \mathbf{g} = \mathbf{g}^2 \\ 0 & \text{otherwise} \end{cases}$$

If the graph $\mathbf{g}^1$ is realized, Theorem 1 shows that player 2 is the key player. If the intervenor has a constraint of removing only two links, then the key links are $1 \sim 2$ and $1 \sim 3$. Thus removing the key player would result in removing the links $1 \sim 2$ and $2 \sim 3$, which is not the same as the key links.

In the following section we show how the key players and key links are determined when the underlying probability distribution is uniform.

3 Uniform Priors

Definition 3. The probability distribution $p \in \Delta_A(G)$ is *uniform* if it satisfies:

$$p(\mathbf{g}) = \begin{cases} \frac{1}{2^{\frac{n(n-1)}{2}}} & \text{if } \mathbf{g} \in \mathcal{G}_n \\ 0 & \text{otherwise} \end{cases}$$

Suppose the underlying probability distribution be uniform over all networks and a graph $\tilde{\boldsymbol{g}}$ is realized. Moreover, let d_i denote the realized degree of agent i in the graph. Then from Chaudhuri, Sarangi, Tzavellas (2023) we know that the equilibrium action exerted by an agent i is given by:

$$a_i^* (d_i) = 1 + \frac{\lambda d_i}{1 - \frac{n\lambda}{2}} \tag{8}$$

Key Player

Suppose the intervenor removes a player $k \in N$ whose neighborhood is N_k. The updated best responses of these agents can also characterized by degrees. These are given by

$$\tilde{a}_i(d_i) = 1 + \lambda(d_i - 1)\left[\sum_{d=2}^{n-1} \frac{\binom{n-3}{d-2}}{2^{n-2}}\tilde{a}(d) + \sum_{d=1}^{n-2} \frac{\binom{n-3}{d-1}}{2^{n-2}} a(d)\right]$$

For all other agents who do now observe the intervention, their best responses are unaltered:

$$a_i(d_i) = 1 + \frac{\lambda d_i}{1 - \frac{n\lambda}{2}} \quad \forall k \neq i \in N \setminus N_k$$

Solving for updated equilibrium actions, we get

$$\begin{aligned}\tilde{a}_i(d_i) &= 1 + \lambda(d_i - 1)[f_1 + f_2] \\ a_i(d_i) &= 1 + \lambda d_i f_3\end{aligned}$$

where

$$f_1 = \frac{1}{2}\left(\frac{2-\lambda}{2-n\lambda}\right), \qquad f_2 = \frac{1 + \lambda f_1(n-1)}{2 - \frac{\lambda(n-1)}{2}}, \qquad f_3 = \frac{1}{1 - \frac{n\lambda}{2}}$$

The change in aggregate activity due to the removal of k then takes the form:

$$\Sigma - \Sigma_{-k} = 1 + \lambda\left[w_1 \sum_{i \in N_k} d_i + w_2 d_k\right] \tag{9}$$

where, $w_1 = f_3 - (f_1 + f_2)$ is the difference in complementarity strengths that each player in N_k expects to extract and $w_2 = f_3 + (f_1 + f_2)$ represents the total complementarity strengths that arises due to the presence of player k.

Lemma 1. *For any graph realization with $n \geq 3$ an individual with the highest degree is the key player.*

This lemma shows that under the uniform probability distribution, the expected complementarity that each player expects to extract through their neighbors is same for everyone. As a result, the intervention is most effective when the player with the highest degree is removed.

Key Links

Suppose the intervenor removes a set of links $L \subseteq E$ where they are performing under the constraint that they can remove only $\bar{l}$ links. Then for any player i of type $\tilde{\mathbf{g}}_i$ and degree d_i, their expectation about the action exerted by their neighbor j is given by,

$$\mathbb{E}_i[a_j \mid \tilde{\mathbf{g}}_i] = \frac{1}{1 - \frac{n\lambda}{2}} = f_3 \tag{10}$$

Remark 1. Any set of $\bar{l}$ links is the key links.

With the underlying distribution being uniform, the complementarity strengths that the individuals involved in the link removal expects to extract is same for everyone. As a result the complementarity strength involved in a link is same for all the links, which is illustrated in (10). As a result, any set of $\bar{l}$ links is the key links.

Key player vs Key links

In the previous sections we have seen who is the key player and what are the key links. We have also seen through an example that the key player and the key links might not be the same. Now the question arises, if the intervenor is faced with a strategic decision to make, which type of intervention will be more fruitful, i.e. which type of intervention will result in reducing the aggregate activity by a higher margin. For comparing the two types of intervention, we assume that when all the links of a key player is removed, they become inactive in the network (i.e. they exert no action). This can be thought of as an externality of intervention on the player whose links are removed.

Proposition 1. *For any realized graph $\tilde{\mathbf{g}}$ if k^* is the player with the highest degree, then according to Lemma 1, k^* is the key player. Denote by $\Delta\Sigma_{key}$ as the change in aggregate activity in the network due to the removal of k^*. Alternatively, if the intervenor had instead removed all the links of k^*, i.e. $L = \{(k^*, i) : g_{ik^*} = 1\}$, then denote $\Delta\Sigma_{links}$ as the change in aggregate activity due to the removal of the links of k^*. Then we have $\Delta\Sigma_{links} \geq \Delta\Sigma_{key}$.*

The above proposition discusses about the efficiency of intervention. Instead of removing a player, if the links of the same player is removed then the aggregate activity is reduced by a higher margin. This is because of the fact that, link removal is only observed by the individuals involved in the link and there is no anticipation of any other links being removed. Where as, for the node removal the direct neighbors anticipates about who else might have seen this node removal and resulting in updating their actions accordingly. Containing the information within the players involved in a link thus turns out to be more effective in the absence of speculation and updating actions.

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Appendix

Proof of Theorem 1

From (7) we have that

$$\begin{aligned}\mathbf{\Psi}_k &= \lambda\left(\mathbf{I}-\lambda\tilde{\mathbb{B}}\right)^{-1}\cdot\mathbf{\Phi}_k\\ &= \lambda\left(\mathbf{\Phi}_k+\lambda\tilde{\mathbb{B}}\cdot\mathbf{\Phi}_k+\lambda^2\tilde{\mathbb{B}}^2\cdot\mathbf{\Phi}_k+\dots\right)\\ &= \lambda\tilde{\mathbf{\Psi}}_k\end{aligned}$$

And hence for any $i \in N_k$,

$$\begin{aligned}\Psi_k\left(\tilde{\mathbf{g}}_i\right) &= \lambda\left(\Phi_k\left(\tilde{\mathbf{g}}_i\right)+\lambda\sum_{j\in N_k}\sum_{\mathbf{g}_j\in G_j^2}\tilde{g}_{ij}p\left(\mathbf{g}_j \mid \tilde{\mathbf{g}}_i\right)\Phi_k\left(\mathbf{g}_j\right)+\dots\right)\\ &= \lambda\tilde{\Psi}_k\left(\tilde{\mathbf{g}}_i\right)\end{aligned}$$

Thus, the key-player $k^*$is the one for whom

$$\Psi\left(k^*\right) := a_{k^*}\left(\tilde{\mathbf{g}}_{k^*}\right)+\lambda\sum_{i\in N_{k^*}}\tilde{\Psi}_{k^*}\left(\tilde{\mathbf{g}}_i\right)$$

is the maximum, i.e. $\Psi\left(k^*\right) = \max_{k\in N}\Psi\left(k\right)$.

■

Proof of Lemma 1

Note that $w_1 < 0$ and $w_2 > 0$ for any $n \geq 3$ and $\lambda < \frac{1}{n-1}$. If $d_{k^*} \geq d_k$ and $\sum_{i\in N_{k^*}} d_i \geq \sum_{i\in N_k} d_i$ for all $k \in N$ which proves the result, as $\Sigma - \Sigma_{-k}$ is maximum at $k = k^*$. On the other hand, if $d_{k^*} > d_{k'}$ and $\sum_{i\in N_{k^*}} d_i \leq \sum_{i\in N_{k'}} d_i$ for some $k' \in N$ and $\Sigma - \Sigma_{-k}$ is maximum at $k = k'$ then

$$\begin{aligned}w_1\sum_{i\in N_{k'}} d_i + w_2 d_{k'} &> w_1\sum_{i\in N_{k^*}} d_i + w_2 d_{k^*}\\ w_1\left[\sum_{i\in N_{k'}} d_i - \sum_{i\in N_{k^*}} d_i\right] &> w_2\left[d_{k^*} - d_{k'}\right]\end{aligned}$$

But this is not a possibility as $w_1 < 0$ and $w_2 > 0$. Hence the player with the highest degree is the key player.

■

Proof of Proposition 1

From (9) we see that if k^* is the player with the highest degree

$$\Delta\Sigma_{key} = \Sigma - \Sigma_{-k^*} = 1 + \lambda \left[w_1 \sum_{i \in N_{k^*}} d_i + w_2 d_{k^*} \right]$$

Suppose the links in $L = \{(k^*, i) : g_{ik^*} = 1\}$ are removed. Then for all $i \in V(L) \setminus \{k^*\}$, $L_i = \{k^*\}$ and $L_{k^*} = \{j \in N : \tilde{g}_{jk^*} = 1\}$. And as we assume that the removal of all links of k^* will make them inactive, the change in aggregate activity due to the removal of links in L is given by

$$\begin{aligned}
\Delta\Sigma_{links} = \Sigma - \Sigma_{-L} &= a_{k^*}(\tilde{\mathbf{g}}_{k^*}) + \lambda \sum_{\substack{i \in V(L) \\ i \neq k^*}} \sum_{j \in L_i} \mathbb{E}_i \left[a_j \mid \tilde{\mathbf{g}}_i \right] \\
&= 1 + \lambda f_3 d_{k^*} + \lambda \sum_{\substack{i \in V(L) \\ i \neq k^*}} \sum_{j \in L_i} f_3 \\
&= 1 + 2\lambda f_3 d_{k^*}
\end{aligned}$$

Hence,

$$\Delta\Sigma_{links} - \Delta\Sigma_{key} = 2\lambda f_3 d_{k^*} - \lambda w_1 \sum_{i \in N_{k^*}} d_i - \lambda w_2 d_{k^*}$$

And we know that

$$\sum_{i \in N_{k^*}} d_i \geq d_{k^*} \implies -w_1 \sum_{i \in N_{k^*}} d_i \geq -w_1 d_{k^*}$$

where the second inequality holds because $w_1 < 0$. As a result we have

$$\begin{aligned}
\Delta\Sigma_{links} - \Delta\Sigma_{key} &\geq 2\lambda f_3 d_{k^*} - \lambda w_1 d_{k^*} - \lambda w_2 d_{k^*} \\
&= 2\lambda f_3 d_{k^*} - \lambda \left(w_1 + w_2 \right) d_{k^*} \\
&= 0
\end{aligned}$$

where the last equality is due to the fact that $w_1 + w_2 = 2f_3$. And hence, $\Delta\Sigma_{links} \geq \Delta\Sigma_{key}$.

■