The Effects of Declining Population Growth on Patents and Economic Growth Dynamics*

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Abstract

The paper analyzes how the patent-economic growth relationship changes as population growth dynamics change in the variety expansion model with overlapping generations. The literature on this relationship has not focused on the role of population growth despite data showing that countries' population growth trends have recently shifted from positive to declining and even negative. The declining population growth trend decreases the economy's new demand for output, which reduces the producers' demand for inputs. As patent owners seek profits by providing differentiated inputs to the producers, decreasing input demand reduces patent owners' rewards, restricting entry into research and development (R&D) entrepreneurship and hence economic growth. As a result, we conclude that, with diminishing population growth, the existing patent policy must be tightened in order to increase entry into R&D entrepreneurship and to boost economic growth.

Keywords: Economic Growth, Overlapping Generations Economy, Patent, Physical Capital, Population Growth, Variety Expansion Model

JEL classifications: O31, O34, O40

1 Introduction

Several countries have taken positive steps in recent decades to tighten their patent protection policies.[1] The global agreement on trade-related aspects of intellectual property rights (TRIPS), which came into force in 1995; has influenced countries' adoption of such tighter patent protection policies. For example, Park (2008) obtains that the strength of India's patent protection policies has increased from 1.23 in 1995 to 3.76 in 2005.[2] However, recent studies suggests that the relationship between the economy's strength of patent protection policies and the rate of economic growth is ambiguous.[3] For instance, Falvey et al. (2006) found that the relationship between the economy's patent policy tightness and growth rate depends on its development level. It is positive and significant for high- and low-income countries but not for middle-income countries, as middle-income countries primarily engage in imitation rather than innovation. Further, Iwaisako and Futagami (2013) obtained a non-monotone relationship between the economy's patent policy tightness and growth rate in an endogenous growth model.

We theoretically investigate why countries have continued to tighten patent protection policies despite theoretical and empirical findings indicating that doing so may not be a panacea for economic growth. Does the rate of population growth play a role in this? For instance, in a world with positive population growth and large rates of immigration, like the United States, does optimal patent protection loosen because there is a large pool of potential entrepreneurs? In a world with negative population growth and no immigration, like Japan, is optimal patent protection tightened? Therefore, the study adds to the existing literature by looking into how growth-maximizing patent policies interact with changing population growth dynamics. This is especially important at a time when most countries are experiencing declining population growth trends (see Figure 1), with some, such as Japan, Germany, Italy, and Spain, already experiencing negative population growth.[4] As a result, the implications of changing population growth dynamics, specifically the declining population growth trend, on the patent-growth nexus must be investigated. Furthermore, we allow for a negative population growth rate and investigate its impact on growth-maximizing patent protection policies and, consequently, on the optimal output per capita growth. Although there are growth models that allow for negative population growth rates; see Sasaki and Hoshida (2017), Jones (2022), and Bucci (2023), but they do not examine the effects of patent policies on economic growth given the negative population growth rates.

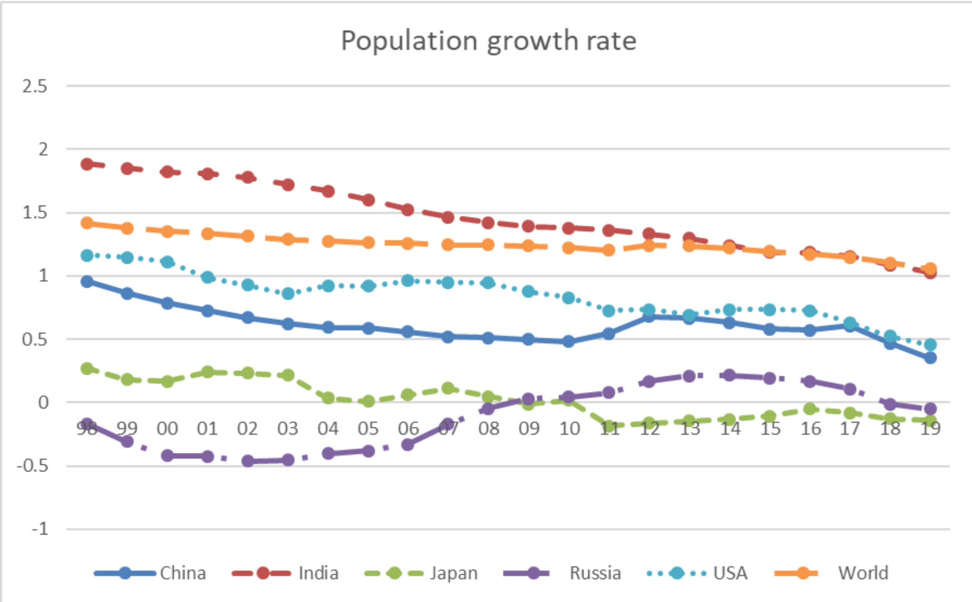

Fig. 1: The population growth trends for various regions since 1998.
Source: The World Bank

We employ the variety expansion model in a finite horizon overlapping generations (OLG) economy with physical capital and lab-equipment type R&D specification as in Rivera-Batiz and Romer (1991). Most studies on patent protection policies and economic growth have focused on economies of infinitely lived households.[5] However, the infinite horizon model, by design, implies positive population growth rate and is not entirely consistent with declining population growth trend or negative population growth rate. In short, a finite horizon model, such as an OLG model, can capture the effects of a declining population growth rate or a negative population growth rate more accurately. Furthermore, because the literature on endogenous growth has emphasized the role of R&D in economic growth and the role of patents in incentivizing these R&Ds, the endogenous growth model is a natural framework for capturing the impact of patent policies on growth. Therefore, following Diwakar et al. (2021), we employ a variety expansion endogenous growth model in a finite horizon OLG economy to analyze the effects of declining population growth on patents and economic growth dynamics.[6] Moreover, the government regulates patent protection through several patent policy instruments, the most well-known of which are patent length and patent breadth in growth theory. The patent breadth limits the ability of the patent owner to charge an unconstrained monopolist's price, whereas the patent length is the duration for which a patent is valid. This study focuses on patent breadth policy for simplicity and tractability. Furthermore, patent breadth has a more immediate impact on the value of patented varieties, as it affects both current and future profits. In contrast, patent length merely affects future profit, see Chu (2022). Our primary objective is to analyze three key aspects: First, we examine the impact on the output per capita growth resulting from changes in patent breadth policy. Second, we explore how growth-maximizing patent breadth policy responds to shifting population growth trends, which have transitioned from positive to declining and, in some cases, negative. Third, we investigate whether the patent breadth policy that maximizes economic growth also maximizes the economy's welfare.

Only a few theoretical analyses in the literature on patent policy and economic growth used a finite-period OLG economy, pioneered by Samuelson (1958) and Diamond (1965).[7] These are Chou and Shy (1993), Sorek (2011), and Diwakar et al. (2021). Chou and Shy (1993) analyze the effects of different patent lengths, particularly one-period and infinite patent length, on the growth in a variety expansion model. Sorek (2011) analyzes the effects of both patent length and patent breadth on growth in a quality-ladder model. He obtained that both Inter-temporal Elasticity of Substitution (IES) and patent length determine the effects of loosening patent breadth on growth. Diwakar et al. (2021) also analyze the effects of both patent length and patent breadth on growth but in a variety expansion model. They obtained that unlike Sorek (2011), IES has no role in their analysis. Furthermore, they discovered that the growth-maximizing patent breadth is incomplete for any positive depreciation rate and tightened with an increase in effective labor supply. We may deduce from this that, under certain special assumptions, growth maximizing patent breadth tightens with an increase in labor supply (Young population).[8] Does this also imply that growth-maximizing patent protection policies tighten with population or labor supply growth rates? However, this is not supported by data because there are countries that, on the one hand, have declining population growth rates (See Figure 1) and, on the other hand, are tightening their patents (See Figure 2).[9] For example, while India's population growth rate fell from 1.89 in 1998 to 1.03 in 2019, the Patent Enforcement Index (PEI) rose from 3.9 in 1998 to 5.0 in 2017.[10]

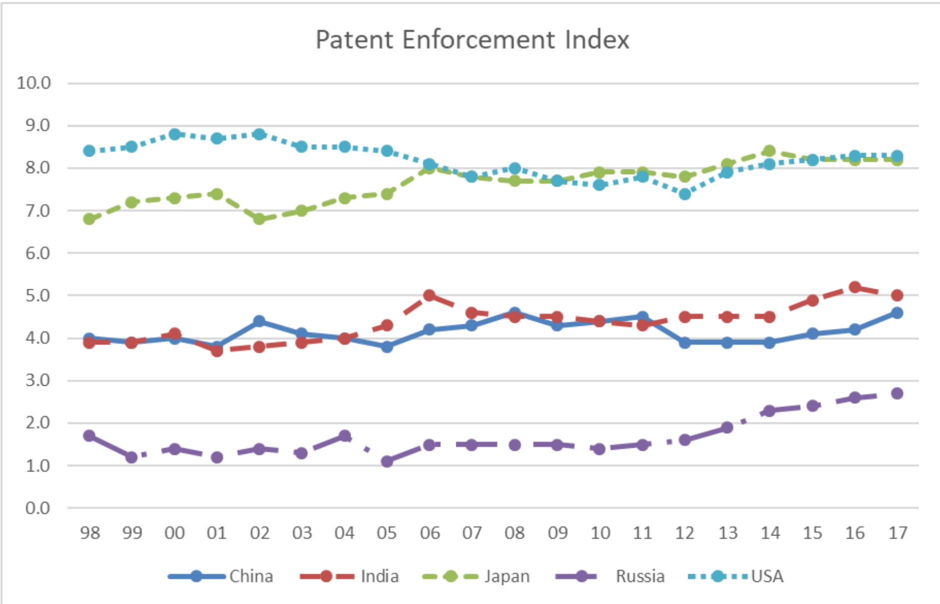

Fig. 2: The Patent Enforcement Index (PEI) trends for various regions since 1998. Source: Papageorgiadis and Sofka (2020).

To the best of our knowledge, this is the first study that investigates how growth-maximizing patent policy is related to declining population growth dynamics. The declining population growth decreases the economy's new demand for final output; therefore, the output producer reduces their demand for new inputs (machines),[11] and subsequently, it decreases the profit flow of patent owners that enter the R&D entrepreneurship to seek profits. So, the existing patent policy must be revised to enhance entry in the R&D entrepreneurship and economic growth. In addition, we will also examine how growth-maximizing patent policy is related to other parameters, such as the depreciation of machines and the patent's per capita R&D cost.

In our model, the production involves decreasing returns to machine's utilization. It implies that, the rising machine-variety per capita increases the output per capita at a decreasing rate. Therefore, an optimal machine-variety per capita must exist that maximizes machines' utilization and output per capita. Further, rising machine-variety per capita beyond the optimal level does not increase the machines' utilization level but crowds out investment from innovation,[12] thereby lowering the output per capita. As expected, our model obtains an optimal machine-variety per capita that maximizes the output per capita growth; it also corresponds with machine-variety per capita at an optimal patent breadth. We find that increasing or decreasing machine-variety per capita by loosening or tightening patent breadth respectively,[13] may have two opposing effects on output per capita and two opposing effects on consumption. It improves output per capita growth by narrowing the gap between actual and optimal machine-variety per capita (positive effect).[14] However, it impedes per capita output growth by widening the same gap (negative effect). The relative strength of the aforementioned mechanisms relating to machines' utilization and crowding out of investments, determines whether the gap widens or narrows. On the consumption side, it boosts Young consumption due to increased labour income (a positive effect), as increased machine variety per capita raises labor's marginal productivity. However, it reduces Old consumption due to a lower interest rate on savings (a negative effect) because more machines must be supplied at a low-interest rate. An inverted-U relationship between machine variety per capita and output per capita growth may exist as a result of these two opposing effects. Similarly, an inverted-U relationship between machine variety per capita and lifetime utility of all generations may exist. Therefore, the economy must have a unique growth- and welfare-maximizing patent breadth for any fixed population growth. Our analysis, however, indicates that the machine-variety per capita that maximizes growth may differ from the machine-variety per capita that maximizes lifetime utility of all generations.

We also show that optimal patent breadth may be complete or incomplete, depending on the new demand for machine-varieties.[15] Suppose the economy does not have new demand for machines due to a population decline. In that case, the optimal patent breadth is complete to encourage R&D entrepreneurship. However, the optimal patent breadth is incomplete if the economy has new demand for machines.

Furthermore, we observe whether the declining population growth impacts the optimal patent breadth policy and whether countries should consider shifting population growth dynamics while formulating their patent policy. We obtain that if the economy has a positive demand for machine-varieties, the declining population growth impacts the optimal patent breadth, and it tightens with declining population growth to promote entry in R&D entrepreneurship and to maximize growth. In contrast, declining population growth does not affect the optimal patent policy when there is no new demand for machines. Thus, in contrast to Diwakar et al. (2021), we obtain that the growth-maximizing patent breadth can be complete for any positive depreciation rate if there is no new demand for machines, and it tightens with a decrease in population growth (labor supply).

This study follows the subsequent structure: Section 2 introduces the model and explores the growth implications of patent breadth in declining population growth, while Section 3 delves into the welfare implications of patent breadth. Finally, Section 4 concludes the study.

2 The Model

We consider a variety expansion model in a two-period overlapping generations framework with lab-equipment type R&D specification. The economy is consisting of three types of agents: the households, producer of the final output, and R&D entrepreneurs. Households are finitely-lived and can live for at most two-periods defined as their young and old ages.

S/he dies at the start of old age with a probability $1-\mu$ and lives through old age with a probability μ.[16] Therefore, at any point in time, the economy is composed of 2 cohorts: the Young and the Old. Each young agent is endowed with one unit of labor that they supply inelastically. Old agents retire and consume by dis-saving. In each period t, L_t young agents are born and grow at the constant rate $n \in$ (-1, ∞),

$$L_{t+1} = (1+n)L_t; \quad n \in (-1, \infty). \tag{1}$$

The economy's population may increase, remain fixed or decrease according to the positive, zero and negative values of n. The producer of final output employs labor and differentiated capital inputs to produce the final output, which is sold at the normalized unitary price. The R&D entrepreneur devotes resources in R&D or inter-generational patent trade to get a patent for newly invented or old existing varieties' blueprints. After getting a patent, the entrepreneur creates a monopoly and sells the input at the monopolist's price.

2.1 The Household Sector

A representative agent consumes only one good, the final good produced by perfectly competitive firms, and derives utility from his or her lifetime consumption: consumption when young and consumption when old. We assume that the utility specification is intertemporal logarithmic. As a result, the lifetime expected utility of a representative agent born at period t is,

$$u_t = lnc_{Y,t} + \mu lnc_{O,t+1}, \tag{2}$$

where $c_{Y,t}$ is consumption at young and $c_{O,t+1}$ is consumption at old. At Young, the representative agent supplies his or her labor inelastically to the production sector and earns wage w_t that s/he allocates between current consumption $c_{Y,t}$ and saving s_t. The uncertainty regarding old age survival makes old age consumption also uncertain. As a result, each agent obtains insurance by utilizing savings to purchase actuarial notes in order to mitigate risk. Following Blanchard (1985), we assume an actuarially fair annuity market, in which the survivor receives $\frac{1+r_{t+1}}{\mu}s_t$ in exchange for the insurance company's saving, s_t. Moreover, similar to Grossman and Helpman (1991), we assume that capital is held as shares of monopolist firms. The representative agent retires (if s/he survives) and consumes using returns. Thus, the inter-temporal budget constraints are,

$$c_{Y,t} = w_t - s_t, \tag{3}$$

$$c_{O,t+1} = \left[\frac{1 + r_{t+1}}{\mu}\right] s_t. \tag{4}$$

We assume $r_t \in [0, \infty)$ $\forall t$; however only a positive interest rate motivates the agent to invest. Now, maximizing Equation (2) with respect to the inter-temporal budget constraints given by Equations (3) and (4) gives optimal saving,

$$s_t = \frac{w_t}{1 + \mu^{-1}}, \tag{5}$$

which maximizes the expected lifetime utility of an agent. This optimal saving increases with the labor income w_t and survival probability μ. The aggregate saving of the economy is equal to the aggregate saving by the Young:

$$S_t = \frac{w_t L_t}{1 + \mu^{-1}}. \tag{6}$$

2.2 The Final Good Sector

The producer of the final output operates in a perfectly competitive environment, employing labor from the households and differentiated capital inputs from the monopolists to produce a single output in the economy.[17] We assume Constant Returns to Scale (CRS) production technology,

$$Y_t = L_t^{1-\alpha} \int_0^{N_t} K_{i,t}^{\alpha}\, di, \tag{7}$$

where, L_t is the labor supply, N_t is the available varieties, $K_{i,t}$ is the utilization level of i^{th} machine-variety at period t and, $\alpha \in (0, 1)$ determines the share of labor, $1 - \alpha$, in final output.

Let w_t, and $p_{i,t}$ represent the wage rate and the rental price for the labor and the $i^{th}(i \in (0, N_t])$ machine-variety at period t, respectively. The CRS and perfect competition assumptions then imply that final output producer earns a normal profit and assigns a wage and a rental price to their respective marginal productivities.

$$w_t = (1-\alpha)\frac{Y_t}{L_t}, \tag{8}$$

$$p_{i,t} = \alpha L_t^{1-\alpha} K_{i,t}^{\alpha-1} \quad \forall i \in (0, N_t]. \tag{9}$$

The Equation (9) represents the inverse demand for the input $i \in (0, N_t]$ at the rental price $p_{i,t}$, indicating that producer of the final output demand more inputs at a lower price. The final demand for the i^{th} machine-variety at the rental price $p_{i,t}$ can be written as,

$$K_{i,t} = \left(\frac{\alpha}{p_{i,t}}\right)^{\frac{1}{1-\alpha}} L_t. \tag{10}$$

2.3 The Monopolistic Sector

The economy has a continuum of input varieties $i \in (0, N_t]$ at any given time t, each of which is produced by its respective patent owner after creating a monopoly. At each period t, the patent owner of the i^{th} variety borrows raw capital from the annuity market at the net interest rate r_{t+1} and transforms each unit of raw capital into one specialized machine at no additional cost. At period t+1, machines are then rented to the final-good producer at the rental price $p_{i,t+1}$.[18]

Let the specialized machines depreciate at a constant rate $\delta \in [0, 1]$ per period. As a result, the average cost of raw capital is $\delta + r_{t+1}$, and given the demand for i^{th} machine-variety, the profit flow of i^{th} patent owner or monopolist at time t+1 can be written as,

$$\pi_{i,t+1} = [p_{i,t+1} - (\delta + r_{t+1})]\, K_{i,t+1}. \tag{11}$$

A monopolist maximizes profit by setting the optimal price. At period t+1, the i^{th} patent owner or monopolist maximizes profit flow by setting the price of the i^{th} machine equal to,

$$p_{i,t+1} = \frac{\delta + r_{t+1}}{\alpha} \equiv p_{t+1} \;\; \forall \;\; i \in (0, N_{t+1}]. \tag{12}$$

2.3.1 Patent Breadth

Assume that the government limits the patent owner's ability to charge an unconstrained monopolist's price by introducing a patent breadth.[19] We follow Goh and Olivier (2002) to model the patent breadth,[20] using parameter λ, it modifies the monopolist's price to

$$p_{t+1,\lambda} = \frac{\lambda(\delta + r_{t+1})}{\alpha}; \qquad \lambda \in (\alpha, 1]. \tag{13}$$

When $\lambda = \alpha$, the price $p_{t+1,\lambda}$ that the monopolist is allowed to charge is equal to the marginal cost of (input) production $\delta + r_{t+1}$, and the monopolist completely loses his or her market power. However, when λ becomes one, s/he is allowed to charge an unconstrained monopolist price p_{t+1}.[21] Tightening (rising) the patent breadth λ increases the monopolist's market power by enabling the patent owner to charge a higher monopolist's price for each machine-variety.

The final good producer's per capita demand for each machine-variety or machine-variety per capita at the monopolist's price $p_{t,\lambda}$ in period t is,

$$\frac{K_{i,t}}{L_t} = \left[\frac{\alpha^2}{\lambda(\delta + r_t)}\right]^{\frac{1}{1-\alpha}} \equiv k_{t,\lambda}; \;\; \forall \;\; i \in (0, N_t]. \tag{14}$$

Equation (14) gives the number of machine-variety assigned to each labor. After plugging the monopolist's price $p_{t+1,\lambda}$ in Equation (11) and using Equation (14), we get the profit flow of each monopolist in period t+1,

$$\pi_{t+1,\lambda} = \alpha \left(1 - \frac{\alpha}{\lambda}\right) k^{\alpha}_{t+1,\lambda} L_{t+1}. \tag{15}$$

Then, plugging the aggregate demand $K_{i,t} \equiv k_{t,\lambda} L_t$ for each input $i \in (0, N_t]$ in Equation (7), we get the per capita output that is produced in each period t,

$$\frac{Y_{t,\lambda}}{L_t} = k_{t,\lambda}^{\alpha} N_t \equiv y_{t,\lambda}. \tag{16}$$

According to Equation (16), the rising machine-variety per capita increases the output per capita at a decreasing rate. Therefore, an optimal machine-variety per capita must exist that maximizes machines' utilization and output per capita. Further, rising machine-variety per capita beyond the optimal level does not increase the machines' utilization level but crowds out investment from innovation, thereby lowering the output per capita. Thus, the loosening (tightening) patent breadth can increase the output of an economy if the actual machine-variety per capita is lower (higher) than the optimal machine-variety per capita. Because the loosening (tightening) patent breadth decreases (increases) the price for machines, thereby increasing (decreasing) the actual machine-variety per capita, and the gap between actual and optimal machine-variety per capita become small. Moreover, we get the aggregate saving of the economy using Equations (8) and (16) in Equation (6),

$$S_{t,\lambda} = \frac{(1-\alpha)Y_t}{1+\mu^{-1}} = \frac{(1-\alpha)k_{t,\lambda}^{\alpha} N_t L_t}{1+\mu^{-1}}. \tag{17}$$

2.3.2 The R&D Sector

We consider lab-equipment type R&D specification as proposed by Rivera-Batiz and Romer (1991). The R&D entrepreneur devotes η_t units of output in R&D or inter-generational patent trade to get a patent for newly invented or old existing varieties' blueprints. We assume that the R&D cost η_t for getting a patent to machine-variety is given by ηL_t.[22] Additionally, we assume free entry conditions in the R&D sector. An entrepreneur who values the patent above R&D cost and is willing to bear those costs can enter the R&D sector. At equilibrium (or at the zero profit condition), the value of a patent must equal the R&D cost. Therefore, the patent owner of a machine-variety obtains a profit $\pi_{t+1,\lambda}$ and a capital gain or loss $(\eta_{t+1} - \eta_t)$ by investing η_t units of funds in patents.[23] Furthermore, investing η_t units of funds in the risk-free asset gives net return $r_{t+1}\eta_t$.

The no-arbitrage condition, which equates the net rate of return on a risk-free asset to the net rate of return on investment in a patent, can therefore be expressed as follows:

$$r_{t+1}\eta_t = \pi_{t+1,\lambda} + (\eta_{t+1} - \eta_t). \tag{18}$$

By entering the monopolist's profit and R&D cost in the no-arbitrage condition, we can obtain the implicit expression for interest rate.

$$1 + r_{t+1} = \left\{ \frac{\alpha}{\eta}\left(1 - \frac{\alpha}{\lambda}\right) k_{t+1,\lambda}^{\alpha} + 1 \right\} (1+n). \tag{19}$$

***Lemma* 1.** *A positive and unique stationary interest rate exists for sufficiently small patent's per capita R&D cost, η.*[24]

Proof. Let $f(r_t) = 1 + r_t - \left\{ \frac{\alpha}{\eta}\left(1 - \frac{\alpha}{\lambda}\right) k_{t,\lambda}^{\alpha} + 1 \right\}(1+n)$.

Since $f(r_t)$ is continuous and $f(0) = 1 - \left\{ \frac{\alpha}{\eta}\left(1 - \frac{\alpha}{\lambda}\right)\left(\frac{\alpha^2}{\lambda\delta}\right)^{\frac{\alpha}{1-\alpha}} + 1 \right\}(1+n) < 0$ as $\eta \to 0$, and $f(\infty) > 0$. Hence by Intermediate Value Property, $\exists\ r_\lambda^* \in (0, \infty)$ such that $f(r_\lambda^*) = 0$.

We are left with the proof of uniqueness. Since $f'(r_t) = 1 + \frac{\alpha^2(1+n)k_\lambda^\alpha}{\eta(1-\alpha)(\delta+r_t)}\left(1 - \frac{\alpha}{\lambda}\right) > 0$, which implies $f(r_t)$ is increasing and one-one. Moreover, r_t depends on the parameters only; therefore, the unique positive interest rate is stationary. That is, $r_t = r_\lambda^*(> 0), \forall t$.

***Assumption* 1.** *The patent's per capita R&D cost η is sufficiently small.*[25]

Since only a positive interest rate motivates the agent to invest, and the Assumption 1 is necessary to get a positive and unique stationary interest rate. Thus, hereafter we follow the Assumption 1 and therefore the stationary interest rate.

The monopolist's price $p_{t,\lambda}$ and the per capita demand for machine $k_{t,\lambda}$, have both become stationary at the stationary interest rate r_λ^*, and are denoted by p_λ and k_λ respectively. The output per capita growth rate $g_{y,\lambda} = \frac{Y_{t+1,\lambda}}{Y_{t,\lambda}} \frac{L_t}{L_{t+1}} - 1$ can be obtained by using Equation (16). We get the output per capita growth rate $g_{y,\lambda} = \frac{Y_{t+1,\lambda}}{Y_{t,\lambda}} \frac{L_t}{L_{t+1}} - 1$ to be exactly equal to the variety growth rate $g_{N,\lambda} = \frac{N_{t+1}}{N_t} - 1$. We represent this by g_λ.

$$g_{y,\lambda} = g_{N,\lambda} = g_\lambda \tag{20}$$

Thus, the optimal patent breadth λ^o that maximizes variety growth $g_{N,\lambda}$ also maximizes the output per capita growth $g_{y,\lambda}$.

***Lemma* 2.** *The stationary interest rate r^*_λ decreases with the declining population growth n.*

Proof. The implicit expression for the stationary interest rate can be written as,

$$1 + r^*_\lambda = \left\{ \frac{\alpha}{\eta} \left(1 - \frac{\alpha}{\lambda}\right) k^\alpha_\lambda + 1 \right\} (1+n), \tag{21}$$

where $k_\lambda = \left[\frac{\alpha^2}{\lambda(\delta + r^*_\lambda)}\right]^{\frac{1}{1-\alpha}}$. With loosening patent breadth or decreasing depreciation rate, machine-variety per capita k_λ increases.[26] Differentiating the stationary interest rate given in Equation (21) with respect to n, we get

$$\frac{\partial r^*_\lambda}{\partial n} = \frac{(1-\alpha)(1+r^*_\lambda)(\delta + r^*_\lambda)}{(1+n)[(\delta + r^*_\lambda) - \alpha(\delta + n)]}$$

It is clear, from Equation (21), that $r^*_\lambda > n$ which implies $\frac{\partial r^*_\lambda}{\partial n} > 0$. The stationary interest rate r^*_λ decreases with the declining population growth n. This is due to decreased demand for raw capital, eventually leading to a fall in interest rate. A decline in the population growth rate n decreases the market size, leading to lower demand for machine-varieties. The demand for raw capital falls as monopolists use raw capital to create these machines, thereby decreasing interest rates. Moreover, the stationary interest rate increases as the patent breadth rises and decreases as patent's per capita R&D cost or the depreciation rate rises. That is, $\frac{\partial r^*_\lambda}{\partial \lambda} > 0$, $\frac{\partial r^*_\lambda}{\partial \eta} < 0$ and, $\frac{\partial r^*_\lambda}{\partial \delta} < 0$.[27] Tightening patent breadth λ increases the monopolist's market power and encourages the R&D entrepreneur to devote η_t units of funds in R&D or inter-generational patent trade for getting a patent for newly invented or old existing varieties' blueprints. It increases the demand for raw capital, thereby increasing interest rates. In contrast, the stationary interest rate r^*_λ decreases as patent's per capita R&D cost η and depreciation rate δ rise. This is because a rise in patent's per capita R&D cost η and depreciation rate δ increase the cost of the new blueprint invention η_t and the price of machine, respectively, decreasing the demand for raw capital and lowering the interest rate.

2.4 Capital Market Clearing Conditions

At any time t, the aggregate investment $I_{t,\lambda}$ can be obtained by aggregating investment in buying old patents, in acquiring new patents on inventions, and in the formation of differentiated machines. Therefore, the aggregate investment at the time t is given by,

$$I_{t,\lambda} = \int_0^{N_{t+1}} [\eta_t + K_{i,t+1}]\, di = [\eta + (1+n)k_\lambda]\, N_{t+1}L_t. \tag{22}$$

The market clearing condition is an equilibrium point at which the economy's aggregate saving $S_{t,\lambda}$, is translated into the economy's aggregate investment $I_{t,\lambda}$. Now, if we set $S_{t,\lambda}$ equal $I_{t,\lambda}$ at the stationary interest rate, we get the variety growth rate, which is also equal to the the output per capita growth.[28]

$$g_\lambda = \frac{N_{t+1}}{N_t} - 1 = \frac{(1-\alpha)\, k_\lambda^\alpha}{(1+\mu^{-1})[\eta + (1+n)\, k_\lambda]} - 1. \tag{23}$$

***Lemma* 3.** *The output per capita growth is positive under the Assumption 1.*[29]

Proof. See Appendix A3 for the proof of Lemma 3.

A sufficiently low patent's per capita R&D cost makes R&D participation affordable to the entrepreneurs by lowering the R&D cost of getting a patent. This spurs variety growth in the economy and leads to positive output per capita growth.

***Proposition* 1.** *An inverted-U relationship may exist between machine-variety per capita and output per capita growth.*

Proof. Differentiating the output per capita growth g_λ in Equation (23) with respect to the machine-variety per capita k_λ, we get

$$\frac{\partial g_\lambda}{\partial k_\lambda} = \frac{(1-\alpha)\, k_\lambda^{\alpha-1}}{(1+\mu^{-1})\,[\eta + (1+n)k_\lambda]^2}[\alpha\eta - (1+n)(1-\alpha)k_\lambda]. \tag{24}$$

The output per capita growth g_λ increases (decreases) with rise of the machine-variety per capita k_λ, if $k_\lambda < (>) \frac{\alpha\eta}{(1+n)(1-\alpha)}$. However, $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$ is the critical point at which the output per capita growth may have a maxima.

Differentiating Equation (24) with respect to k_λ, at $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$ we get

$$\left[\frac{\partial^2 g_\lambda}{\partial k_\lambda^2}\right]_{k_\lambda = k_{\lambda^o}} = \frac{-(1+n)(1-\alpha)^2 k_{\lambda^o}^{\alpha-1}}{(1+\mu^{-1})\left[\eta + \frac{\alpha\eta}{1-\alpha}\right]^2} < 0.$$

Therefore, the output per capita growth will have maximum at $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$, which is the optimal machine-variety per capita that maximizes machines' utilization level and output. Thus, increasing machine-variety per capita generates an inverted-U relationship with the output per capita growth. Because it initially closes the gap between actual and optimal machine-variety per capita up to a critical level. However, as it continues to increase beyond the critical level, it gradually widens the gap, leading to this distinctive relationship.

Loosening patent breadth λ, a policy variable in our model, raises machine-variety per capita (see Appendix A1), which increases (decreases) the output per capita growth if the actual machine-variety per capita k_λ is lower (higher) than optimal machine-variety per capita k_{λ^o}. Furthermore, the output per capita growth is always maximum at the optimal machine-variety per capita $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$ because the machines' utilization is maximal at the optimal machine-variety per capita. Moreover, at the optimal machine-variety per capita, the proportion of aggregate investment devoted to patents at any period equals the labor share.[30]

Suppose the term "optimal patent breadth" refers to the patent breadth that maximizes the output per capita growth. Since the output per capita growth is maximized at the optimal machine-variety per capita $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$. Therefore, the patent breadth λ^o is optimal if it makes the machine-variety per capita at k_{λ^o}, alternatively if the actual machine-variety per capita becomes the optimal machine-variety per capita.

At the actual patent breadth λ, the machine-variety per capita is $k_\lambda = \left[\frac{\alpha^2}{\lambda(\delta+r_\lambda^*)}\right]^{\frac{1}{1-\alpha}}$. Using Equation (21), it can be written as $k_\lambda = \frac{\alpha\eta(r_\lambda^*-n)}{(1+n)(\lambda-\alpha)(\delta+r_\lambda^*)}$. However, at the optimal patent breadth λ^o, the machine-variety per capita is $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$. Thus, the gap between the machine-variety per capita at the optimal and actual patent breadth can be written as

$$k_{\lambda^o} - k_\lambda = \frac{\alpha\eta}{(1+n)(1-\alpha)(\lambda-\alpha)(\delta+r_\lambda^*)}[\delta(\lambda-\alpha) + n(1-\alpha) - r_\lambda^*(1-\lambda)].$$

For the case of zero patent breadth $\lambda = \alpha$, the gap between the machine-variety per capita at the optimal and zero patent breadth becomes

$$k_{\lambda^o} - k_\alpha < 0.$$

Since the optimal machine-variety per capita k_{λ^o} is much lesser than the actual machine-variety per capita k_α; thus the zero patent breadth can not be optimal. In this case, the tightening patent breadth that reduces the actual machine-variety per capita towards the optimal and reduces the crowding out of investment can enhance the output per capita growth. Furthermore, at the zero patent breadth, the monopolist's price equals the marginal cost of machines' production. Therefore, no one will devote resources to getting a patent, and growth will be zero.

For the case of complete patent breadth $\lambda = 1$, the gap between the machine-variety per capita at the optimal and complete patent breadth becomes

$$k_{\lambda^o} - k_1 = \frac{\alpha\eta(\delta + n)}{(1+n)(1-\alpha)(\delta + r_1^*)},$$

and its sign depends on the factor $(\delta + n)$. The parameters δ and n also show the rate at which the new machines replace the old ones and the new machines are assigned to increased labor over the period, respectively. Thus, $\delta + n$ shows the rate at which new machines are demanded over the period. If the economy has new demand for machines over the period due to the depreciation of machines and/or the population growth, $\delta + n > 0$, then the optimal machine-variety per capita k_{λ^o} is much higher than the actual machine-variety per capita k_1; thus, the complete patent breadth can not be optimal. In this case, the optimal patent breadth λ^o will be incomplete, and the loosening patent breadth that raises the actual machine-variety per capita towards the optimal can enhance the output per capita growth. If the economy does not have new demand for machines due to population decline, even if the depreciation of machines is positive, $\delta + n = (<)\ 0$, then the optimal machine-variety per capita k_{λ^o} is equal to (lower than) the actual machine-variety per capita k_1; thus, the complete patent breadth will be the optimal patent breadth.[31]

Hence, if the economy has new demand for machines over the period then the incomplete patent breadth is an optimal patent policy. An inverted-U relationship exists between machine-variety per capita and output per capita growth. However, the above relationship will be monotonic and the complete patent breadth is an optimal patent policy if the economy does not have any new demand for machines.

***Corollary* 1.** *An inverted-U relationship may exist between patent breadth and output per capita growth.*[32]

***Proof*.** The machine-variety per capita increases with the loosening patent breadth, and from Proposition 1, the machine-variety per capita and output per capita growth may

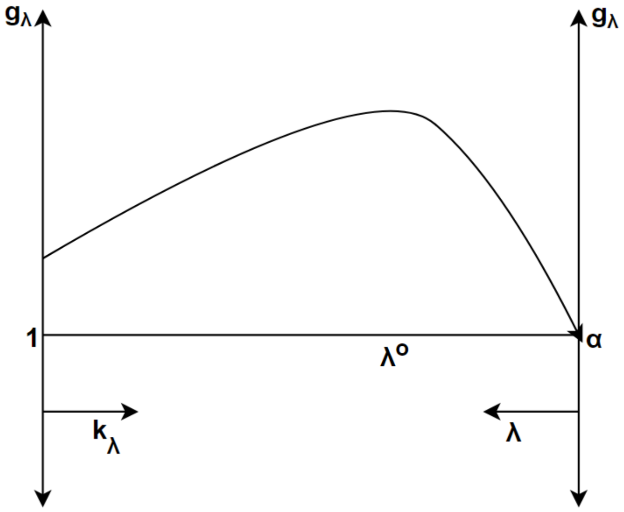

Fig. 3: The relationship between machine-variety per capita and output per capita growth if the economy has new demand for machines, $\delta + n > 0$.
Note: The vertical axis represents the output per capita growth, while the horizontal axis represents machine-variety per capita and the strength of patent breadth. Moving left to right on the horizontal axis increases machine-variety per capita $k_\lambda \in [k_1, k_\alpha)$ and decreases the strength of patent breadth $\lambda \in (\alpha, 1]$. The maximal output per capita growth exists at the incomplete patent breadth λ^o.

have an inverted-U relationship. As a result, Corollary 1 follows from Proposition 1.

***Lemma* 4.** *The stationary interest rate expression is explicit at the optimal patent breadth.*

Proof. At the optimal patent breadth λ^o, the actual machine-variety per capita becomes the optimal machine-variety per capita. That means,

$$\left[\frac{\alpha^2}{\lambda^o(\delta + r^*_{\lambda^o})}\right]^{\frac{1}{1-\alpha}} = \frac{\alpha\eta}{(1+n)(1-\alpha)} = k_{\lambda^o}.$$

This implies,

$$k^\alpha_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}\left[\frac{\alpha^2}{\lambda^o(\delta + r^*_{\lambda^o})}\right]^{-1}.$$

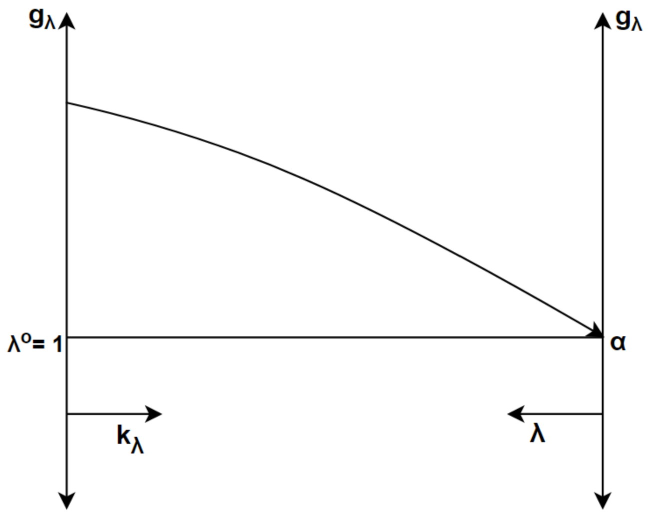

Fig. 4: The relationship between machine-variety per capita and output per capita growth if the economy has no any new demand for machines, $\delta + n \leq 0$.
Note: The vertical axis represents the output per capita growth, while the horizontal axis represents machine-variety per capita and the strength of patent breadth. Moving left to right on the horizontal axis increases machine-variety per capita $k_\lambda \in [k_1, k_\alpha)$ and decreases the strength of patent breadth $\lambda \in (\alpha, 1]$. The maximal output per capita growth exists at the complete patent breadth $\lambda^o = 1$.

Now, plugging $k^\alpha_{\lambda^o}$ in Equation (21) at the optimal patent breadth λ^o, we get an explicit expression for the interest rate:

$$r^*_{\lambda^o} = \frac{(\lambda^o - \alpha)\delta + (1-\alpha)n}{1 - \lambda^o}. \tag{25}$$

***Proposition* 2.** *The optimal patent breadth, λ^o, tightens with declining population growth, n, if the economy has new demand for machines. Moreover, the declining population growth does not have any impact on the optimal patent policy if the economy does not have any new demand for machines.*

Proof. At the optimal patent breadth λ^o, the actual machine-variety per capita becomes the optimal machine-variety per capita.

$$\left[\frac{\alpha^2}{\lambda^o(\delta + r^*_{\lambda^o})}\right]^{\frac{1}{1-\alpha}} = \frac{\alpha\eta}{(1+n)(1-\alpha)}, \tag{26}$$

where $r^*_{\lambda^o}$ represents the stationary interest rate at λ^o. Plugging the explicit expression for stationary interest rate from Equation (25) into Equation (26), we get the expression for optimal patent breadth,

$$\lambda^o = \frac{\alpha^2 \left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha} (1+n)^{1-\alpha}}{(1-\alpha)(\delta+n) + \alpha^2 \left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha} (1+n)^{1-\alpha}}. \tag{27}$$

It is obvious that the optimal patent breadth is incomplete, $\lambda^o \in (\alpha, 1)$, if the economy has new demand for machines over the period due to the depreciation of machines and/or the population growth, $\delta + n > 0$. However, the optimal patent breadth is complete, $\lambda^o = 1$, if the economy does not have new demand for machines, $\delta + n \leq 0$.

Now differentiating the optimal patent breadth expression given in Equation (27) with respect to the population growth rate n, we get

$$\frac{\partial \lambda^o}{\partial n} = \frac{-\alpha^2(1-\alpha) \left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha} (1+n)^{-\alpha} \left[(1-\delta) + \alpha(\delta+n)\right]}{\left[(1-\alpha)(\delta+n) + \alpha^2 \left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha} (1+n)^{1-\alpha}\right]^2}. \tag{28}$$

Clearly $\frac{\partial \lambda^o}{\partial n} < 0$ if $\delta + n > 0$, implying that the optimal patent breadth λ^o that maximizes the output per capita growth becomes stringent with declining population growth n. A declining population growth decreases the new demand for machines thereby less entry into the R&D entrepreneurship. Thus, as the population growth declines, the optimal patent breadth should be tightened to generate more entry into the R&D entrepreneurship and to maximum growth. Furthermore, if $\delta + n \leq 0$ then a declining population growth could not change the status of $\delta + n \leq 0$. Thus, under $\delta + n \leq 0$, the optimal patent breadth remains complete even if the population growth trajectory of the economy is declining.[33]

Since the optimal patent breadth, λ^o tightens with declining population growth, n, if the economy ha new demand for machines, $\delta + n > 0$. Hence, a tightening patent breadth policy up to new optimal patent breadth level can enhance the output per capita growth for this case. However, keeping patent breadth complete is the optimal policy for the case where the economy does not have any new demand for machines.

***Proposition* 3.** *The optimal patent breadth, λ^o, tightens with declining patent's per capita R&D cost, η, if the economy has new demand for machines. Moreover, the declining patent's per capita R&D cost does not have any impact on the optimal patent breadth policy if the economy does not have new demand for machines.*

***Proof*.** Differentiating the optimal patent breadth λ^o given in Equation (27) with respect to the patent's per capita R&D cost η, we get

$$\frac{\partial \lambda^o}{\partial \eta} = \frac{-\alpha^3(1-\alpha)(\delta+n)\left(\frac{1-\alpha}{\alpha\eta}\right)^{2-\alpha}(1+n)^{1-\alpha}}{\left[(1-\alpha)(\delta+n)+\alpha^2\left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha}(1+n)^{1-\alpha}\right]^2}.$$

Clearly $\frac{\partial \lambda^o}{\partial \eta} < 0$ if $\delta + n > 0$, implying that the optimal patent breadth λ^o tightens with declining patent's per capita R&D cost η if $\delta + n > 0$. Since new demand for machines exists, entry into R&D entrepreneurship is profitable even at incomplete patent breadth. However, a declining patent's per capita R&D cost decreases optimal machine-variety per capita. Therefore, a tightening patent breadth that reduces machine-variety per capita to optimal machine-variety per capita can enhance the output per capita growth. Furthermore, if $\delta + n \leq 0$ then a declining patent's per capita R&D cost could not change the status of $\delta + n \leq 0$. Thus, if $\delta + n \leq 0$ then the optimal patent breadth remains complete even if the patent's per capita R&D cost is declining.

The optimal patent breadth λ^o that maximizes the output per capita growth, decreases as depreciation of machines $\delta \in [0, 1]$ rises. That is, $\frac{\partial \lambda^o}{\partial \delta} < 0$.[34]

***Proposition* 4.** *The maximal output per capita growth g_{λ^o}, at the optimal patent breadth, increases with declining population growth n.*

Proof. The output per capita growth is maximal at the optimal patent breadth and can be written as

$$g_{\lambda^o} = \left(\frac{1-\alpha}{1+\mu^{-1}}\right)\left(\frac{1-\alpha}{\eta}\right)^{1-\alpha}\left(\frac{\alpha}{1+n}\right)^{\alpha} - 1. \tag{29}$$

Clearly, $\frac{\partial g_{\lambda^o}}{\partial n} < 0$, implying that the maximal output per capita growth, at the optimal patent breadth, increases with declining population growth. A declining population growth n makes optimal patent breadth stringent. So, the government tightens the patent breadth policy to the optimal patent breadth. This increases new entry in R&D entrepreneurship and maximizes the output per capita growth. Moreover, $\frac{\partial g_{\lambda^o}}{\partial \eta} < 0$ and $\frac{\partial g_{\lambda^o}}{\partial \mu} > 0$ because the optimal patent breadth becomes stringent as the patent's per capita R&D cost declines and the aggregate saving increases as the survival probability increases, respectively.

According to Equation (29), the maximal output per capita growth can be steady at a fixed level of population growth, even if the population is declining if $n \in (-1, 0)$. The negative n ultimately empties the people of the economy. Thus, at a negative population growth, the economy can become steady at a positive output per capita growth but with no population ultimately.

3 Patent and Welfare

This section conducts a welfare analysis to ascertain whether enhancing the lifetime utility of all generations is feasible beyond the optimal patent breadth level. At the optimal patent breadth, actual machine-variety per capita becomes optimal and gives the maximal output per capita growth. We follow Diwakar et al. (2021), the lifetime utility of generation t is,[35]

$$U_t = ln(c_{Y,t}L_t) + \mu ln(c_{O,t+1}\mu L_t).$$

Plugging the consumption flow of a representative agent born at period t, the lifetime utility of generation t becomes,

$$U_t = (1+\mu)ln\left[\left(\frac{1-\alpha}{1+\mu}\right)k_\lambda^\alpha N_t L_t\right] + \mu ln\left[(1+r_\lambda^*)\mu\right]. \tag{30}$$

Equation (30) implies that $U_t = U_{t-1} + (1+\mu)ln[(1+n)(1+g_\lambda)]$, and thus

$$U_t = U_0 + t(1+\mu)ln[(1+n)(1+g_\lambda)], \tag{31}$$

where $U_0 = (1+\mu)ln\left[\left(\frac{1-\alpha}{1+\mu}\right)k_\lambda^\alpha N_0 L_0\right] + \mu ln\left[(1+r_\lambda^*)\mu\right]$ represents the lifetime utility of initial generation.[36] According to Equation (31), adjusting patent breadth, either by loosening or tightening it up to the critical patent breadth level that maximizes the sum of the lifetime utility of initial generation and the output per capita growth, can enhance the lifetime utility of all generations. This improvement in lifetime utility is greater for generations that will be further into the future.

***Proposition* 5.** *An inverted-U relationship may exist between the machine-variety per capita and the lifetime utility of generation t.*

***Proof*.** Differentiating the lifetime utility of generation t given in Equation (31) with respect to machine-variety per capita, we get

$$\frac{\partial U_t}{\partial k_\lambda} = \frac{(1+\mu)\left[(1+t)-\xi\left(\frac{1-\lambda}{\lambda}\right)\right]}{k_\lambda[\eta+(1+n)k_\lambda]}\left[\alpha\eta - (1+n)\left(\frac{t}{(1+t)-\xi\left(\frac{1-\lambda}{\lambda}\right)} - \alpha\right)k_\lambda\right],$$

where $\xi = \frac{\mu(\delta+r_\lambda^*)[(\delta+r_\lambda^*)-(\delta+n)]}{(1+\mu)(1+r_\lambda^*)\left[(\delta+r_\lambda^*)-\frac{\alpha}{\lambda}(\delta+n)\right]} < 1$. Clearly, the lifetime utility of generation t maximizes at $k_{\lambda^w} = \frac{\alpha\eta}{(1+n)\left(\frac{t}{(1+t)-\xi\left(\frac{1-\lambda}{\lambda}\right)}-\alpha\right)}$.[37] Because, it increases (decreases) with the rise of machine-variety per capita if $k_\lambda < (>) k_{\lambda^w}$. Thus, we get an inverted-U relationship between machine-variety per capita and lifetime utility of generations because an increment of machine-variety per capita has two opposing effects on consumption of initial generation and two opposing effects on output per capita.[38] Positive, it increases the Young consumption of initial generation and the output per capita growth due to increased wage and decreased gap between optimal and actual machine-variety per capita. Negative, it decreases the Old consumption of initial generation and the output per capita growth due to decreased interest rate and increased gap between optimal and actual machine-variety per capita. The positive effect dominates over negative effect up to a threshold level, k_{λ^w}, where these two effects cancel out, after that negative effects dominates over positive effect.

***Corollary* 2.** *An inverted-U relationship may exist between the patent breadth and the lifetime utility of generation t.*

Proof. Since machine-variety per capita increases with the loosening patent breadth (see Appendix A1). Thus, Corollary 2 follows from Proposition 5.

***Corollary* 3.** *The welfare-maximizing machine-variety per capita, k_{λ^w}, may differ from the optimal (growth-maximizing) machine-variety per capita, k_{λ^o}.*

Proof. Clearly, $k_{\lambda^w} \gtreqqless k_{\lambda^o}$ if $\xi \lesseqqgtr \frac{\lambda}{1-\lambda}$.

***Corollary* 4.** *Loosening (tightening) patent breadth further up to welfare-maximizing patent breadth level from optimal patent breadth level may benefits all generations if $\xi < (>) \frac{\lambda}{1-\lambda}$.*

Proof. It follows from Proposition 1 and Corollary 3.[39]

4 Conclusion

Using a lab-equipment type variety expansion model with physical capital, this study investigates the impact of declining population growth on the optimal patent breadth policy in an OLG economy. We show that increasing machine-variety per capita due to loosening patent breadth yields two opposing effects on consumption and two opposing effects on output per capita. Thus, increasing machine-variety per capita may generate an inverted-U relationship with the output per capita growth and the lifetime utility of all generations. We obtain unique growth- and welfare-maximizing patent breadth policy. However, the patent breadth policy, which maximizes growth, may differ from the patent that maximizes lifetime utility of all generations.

Additionally, we show that the optimal patent breadth depends on the economy's new demand for machines, which can come from population growth and/or the depreciation of machines. The optimal patent breadth is complete if population decline results in no new demand for machines. However, it is incomplete in the presence of new demand for machines.

Furthermore, we investigate whether the declining population growth impacts optimal patent breadth policy. We obtain that if the economy has a positive demand for machine-varieties, the declining population growth impacts optimal patent breadth, and it tightens with declining population growth to promote entry in R&D entrepreneurship and to maximize growth. In contrast, declining population growth does not affect the optimal patent policy when there is no any new demand for machines.

Notes

[1] Tightening patent rewards the R&D entrepreneurship more.

[2] Park (2008) provides an update to the index of patent protection policies of Ginarte and Park (1997). The index is the unweighted sum of five categories of patent protection (extent of coverage, membership in international treaties, duration of patent protection, enforcement mechanism, and restriction on patent protection) that have been assigned a score ranging from 0 to 1. Thus, the index of patent strength is on a 0-5 scale, where a higher value indicates more robust protection of inventions.

[3] See Gould and Gruben (1996), Thompson and Rushing (1999), Falvey et al. (2006), Qian (2007), Lerner (2009), and for theoretical studies; see O'donoghue and Zweimuller (2004), Furukawa (2007), Horii and Iwaisako (2007), Chu et al. (2012a), Chu et al. (2012b), Iwaisako and Futagami (2013), and Nakabo and Tabata (2018).

[4] Jones (2022) using United Nations 2019 data showed that the natural population growth rates (births minus deaths rate, ignoring immigration) in Japan, Germany, Italy, and Spain are already negative.

[5] See Iwaisako and Futagami (2003), Kwain and Lai (2003), O'donoghue and Zweimuller (2004), Furukawa (2007), Horii and Iwaisako (2007), Chu et al. (2012a), Chu et al. (2012b), Cysne and Turchick (2012), Iwaisako and Futagami (2013), Zeng et al. (2014).

[6] To the best of our knowledge Chou and Shy (1993), Sorek (2011), and Diwakar et al. (2021) are the only studies that analyze the growth implications of patent protection policies in a discrete-time OLG economy with finitely living households, whereas Nakabo and Tabata (2018) analyzes it in a continuous-time OLG economy of perpetual youth households.

[7] See the latest study on patent policy and economic growth in OLG framework by Diwakar et al. (2021).

[8] Note that if $A = 1$ in Proposition 4 of Diwakar et al. (2021), effective labor supply becomes labor supply (Young population) in the model. In two period OLG model, the overall population growth is the same as the Young population growth.

[9] Moreover, Jones (2022) showed that fertility rates of high-income countries as a whole, as well as India, China and the US, have been below the replacement rates.

[10] Papageorgiadis and Sofka (2020) provide a composite index of the patent enforcement system (on a 0-10 scale, with a higher value indicating a strong patent enforcement system). PEI is the equally weighted sum of the scores of the three transaction costs (servicing, property rights protection, and monitoring costs).

[11] In our model, the final output producer employs labor from households and machines from the patent owners to produce output.

[12]Suppose there is an increasing demand for machine-variety per capita beyond the optimal level. In that case, more investment goes to the machine formation to meet the supply-demand condition, which crowds out the investment that can be used in new inventions.

[13] The loosening (tightening) patent breadth lowers (raises) the price of machine-variety, thereby increasing (decreasing) its demand.

[14] The narrowing (widening) gap implies that the actual machine-variety per capita approaches (diverges) optimal.

[15] The differentiated input producer is allowed to charge a price that is less than (equal to) the monopoly price under incomplete (complete) patent breadth.

[16] The households setup is similar to Tabata (2015) and Morimoto et al. (2018).

[17] We follow Iwaisako and Futagami (2013) and Diwakar et al. (2021) and assume differentiated inputs are investment input (physical capital or machine).

[18] Investment input takes one period to form and is then available for rent or use.

[19] In this section, we assume that the government only uses patent breadth as a patent protection tool and takes patent length as fixed and infinite.

[20] This modelling approach is widely used in patent policy and growth literature; for example, see Zeng et al. (2014), Chu et al. (2016) and Diwakar et al. (2021). The subscript λ indicates variables after implementing the patent breadth.

[21] The patent breadths $\lambda = \alpha$ and $\lambda = 1$ are known as zero and complete patent breadth, and the patent breadth λ between α and 1 is known as incomplete patent breadth.

[22] We follow Barro and Sala-i-Martin (2004), Laincz and Peretto (2006), Sorek and Diwakar (2017), and Nakabo and Tabata (2018) to define the cost of a new variety blueprint, which eliminates the scale effect.

[23] We have assumed that the length of a patent is fixed and infinite. Therefore, the old (patent owner of existing machine-varieties) sell their patent to the young at a price equal to the new variety R&D cost. That is, the market value of the old patent is equal to the R&D cost of the new one.

[24] We require an additional assumption, unlike Diwakar et al. (2021), to obtain the unique stationary interest rate. Because, in our study, population can increase or decrease as $n \in (-1, \infty)$.

[25] The similar assumption has been taken by Diwakar et al. (2021) to obtain a positive output growth.

[26] See Appendix A1 for details.

[27] See Appendix A2 for details.

[28] For the sake of simplifying notation, Diwakar et al. (2021) has assumed $\psi = k_{\lambda}^{1-\alpha}$, and the term has no economic meaning. By setting $n = 0$, Equation (23) of this paper matches with Equation (9) of Diwakar et al. (2021) .

[29] Diwakar et al. (2021) obtain the similar result but in aggregate terms. However, our result is in per capita terms.

[30] See Appendix A4 for details.

[31] For the case of $\delta + n < 0$, $\lambda^o = 1$ because monopolists maximize their profit at $\lambda^o = 1$ and will not choose any λ^o beyond 1.

[32] Nakabo and Tabata (2018) obtains an inverted-U relationship between patent breadth and economic growth in a continuous OLG economy. However, we find the same results in discrete OLG economy.

[33] Unlike Diwakar et al. (2021), we obtain that the optimal patent breadth can be complete for a positive depreciation rate $\delta > 0$ if $\delta + n \leq 0$.

[34] See Appendix A5 for details.

[35] The generation t represents all households/agents who born at period t.

[36] A high stationary interest rate decreases the machine-variety per capita. See Appendix A6 for details.

[37] Since $\frac{t}{(1+t)-\xi\left(\frac{1-\lambda}{\lambda}\right)} - \alpha > \frac{t}{1+t} - \alpha$, where $\frac{t}{t+1} \to 1$ and $\alpha = 0.3$ suggested by empirical literature. Thus, the expression $\frac{t}{(1+t)-\xi\left(\frac{1-\lambda}{\lambda}\right)} - \alpha$ is positive.

[38] The two opposing effects on consumption due to increased machine-variety per capita is similar to Diwakar et al. (2021). Moreover, we obtain an additional opposing effects on output per capita.

[39] A part of result, in line with Corollary 4, is obtained by Diwakar et al. (2021): loosening patent breadth further beyond optimal level benefits all generations. However, in our analysis we also get a role for tightening patent breadth.

References

[1] Barro, R. J. and X. Sala-i Martin (2004) Economic Growth, 2nd ed. Cambridge, MA: MIT Press.

[2] Blanchard, O. J. (1985) Debt, deficits, and finite horizons. Journal of Political Economy 93, 223–247.

[3] Bucci, A. (2023) Can a negative population growth rate sustain a positive economic growth rate in the long run? Mathematical Social Sciences 122, 17–28.

[4] Chou, C.-f. and O. Shy (1993) The crowding-out effects of long duration of patents. RAND Journal of Economics 24, 304–312.

[5] Chu, A. C. (2022) Patent policy and economic growth: A survey. Manchester School 90, 237–254.

[6] Chu, A. C., G. Cozzi, and S. Galli (2012a) Does intellectual monopoly stimulate or stifle innovation? European Economic Review 56, 727–746.

[7] Chu, A. C., Y. Furukawa, and L. Ji (2016) Patents, R&D subsidies, and endogenous market structure in a schumpeterian economy. Southern Economic Journal 82, 809–825.

[8] Chu, A. C., S. Pan, and M. Sun (2012b) When does elastic labor supply cause an inverted-U effect of patents on innovation? Economics Letters 117, 211–213.

[9] Cysne, R. P. and D. Turchick (2012) Intellectual property rights protection and endogenous economic growth revisited. Journal of Economic Dynamics and Control 36, 851–861.

[10] Diamond, P. A. (1965) National debt in a neoclassical growth model. American Economic Review 55, 1126–1150.

[11] Diwakar, B., G. Sorek, and M. Stern (2021) Patents and growth in OLG economy with physical capital. Macroeconomic Dynamics 25, 489–508.

[12] Falvey, R., N. Foster, and D. Greenaway (2006) Intellectual property rights and economic growth. Review of Development Economics 10, 700–719.

[13] Furukawa, Y. (2007) The protection of intellectual property rights and endogenous growth: is stronger always better? Journal of Economic Dynamics and Control 31, 3644–3670.

[14] Ginarte, J. C. and W. G. Park (1997) Determinants of patent rights: a cross-national study. Research Policy 26, 283–301.

[15] Goh, A.-T. and J. Olivier (2002) Optimal patent protection in a two-sector economy. International Economic Review 43, 1191–1214.

[16] Gould, D. M. and W. C. Gruben (1996) The role of intellectual property rights in economic growth. Journal of Development Economics 48, 323–350.

[17] Grossman, G. M. and E. Helpman (1991) Innovation and growth in the global economy. Cambridge, MA: MIT Press.

[18] Horii, R. and T. Iwaisako (2007) Economic growth with imperfect protection of intellectual property rights. Journal of Economics 90, 45–85.

[19] Iwaisako, T. and K. Futagami (2003) Patent policy in an endogenous growth model. Journal of Economics 78, 239–258.

[20] Iwaisako, T. and K. Futagami (2013) Patent protection, capital accumulation, and economic growth. Economic Theory 52, 631–668.

[21] Jones, C. I. (2022) The end of economic growth? unintended consequences of a declining population. American Economic Review 112, 3489–3527.

[22] Kwan, Y. K. and E. L.-C. Lai (2003) Intellectual property rights protection and endogenous economic growth. Journal of Economic Dynamics and Control 27, 853–873.

[23] Laincz, C. A. and P. F. Peretto (2006) Scale effects in endogenous growth theory: an error of aggregation not specification. Journal of Economic Growth 11, 263–288.

[24] Lerner, J. (2009) The empirical impact of intellectual property rights on innovation: puzzles and clues. American Economic Review 99, 343–348.

[25] Morimoto, T., Y. Nakabo, and K. Tabata (2018) Population aging, labor market frictions, and payg pension. Kwansei Gakuin University Discussion Paper, 172.

[26] Nakabo, Y. and K. Tabata (2018) An inverted-U effect of patents on economic growth in an overlapping generations model. Kwansei Gakuin University Discussion Paper, 191.

[27] O'donoghue, T. and J. Zweimuller (2004) Patents in a model of endogenous growth. Journal of Economic Growth 9, 81–123.

[28] Papageorgiadis, N. and W. Sofka (2020) Patent enforcement across 51 countries–patent enforcement index 1998-2017. Journal of World Business 55, 101092.

[29] Park, W. G. (2008) International patent protection: 1960–2005. Research Policy 37, 761–766.

[30] Qian, Y. (2007) Do national patent laws stimulate domestic innovation in a global patenting environment? a cross-country analysis of pharmaceutical patent protection, 1978–2002. Review of Economics and Statistics 89, 436–453.

[31] Rivera-Batiz, L. A. and P. M. Romer (1991) Economic integration and endogenous growth. Quarterly Journal of Economics 106, 531–555.

[32] Samuelson, P. A. (1958) An exact consumption-loan model of interest with or without the social contrivance of money. Journal of Political Economy 66, 467–482.

[33] Sasaki, H. and K. Hoshida (2017) The effects of negative population growth: an analysis using a semiendogenous R&D growth model. Macroeconomic Dynamics 21, 1545–1560.

[34] Sorek, G. (2011) Patents and quality growth in OLG economy. Journal of Macroeconomics 33, 690–699.

[35] Sorek, G. and B. Diwakar (2017) Weak scale effects in overlapping generations economy. Economics Bulletin 37, 962–969.

[36] Tabata, K. (2015) Population aging and growth: the effect of pay-as-you-go pension reform. FinanzArchiv/Public Finance Analysis 71, 385–406.

[37] Thompson, M. A. and F. W. Rushing (1999) An empirical analysis of the impact of patent protection on economic growth: an extension. Journal of Economic Development 24, 67–76.

[38] Zeng, J., J. Zhang, and M. K.-Y. Fung (2014) Patent length and price regulation in an R&D growth model with monopolistic competition. Macroeconomic Dynamics 18, 1–22.

Appendix

A1. Effects of patent breadth and depreciation of machines on the machine-variety per capita

The machine-variety per capita, at the stationary interest rate, is given by

$$k_\lambda = \left[\frac{\alpha^2}{\lambda(\delta + r_\lambda^*)}\right]^{\frac{1}{1-\alpha}}. \tag{A.1}$$

Now differentiating Equation (A.1) with respect to the patent breadth λ and depreciation of machines δ, respectively. We get

$$\frac{\partial k_\lambda}{\partial \lambda} = \frac{-k_\lambda^{2-\alpha}\left[(\delta + r_\lambda^*) + \lambda\frac{\partial r_\lambda^*}{\partial \lambda}\right]}{\alpha^2(1-\alpha)} < 0$$

$$\frac{\partial k_\lambda}{\partial \delta} = \frac{-k_\lambda\left[1 + \frac{\partial r_\lambda^*}{\partial \delta}\right]}{(1-\alpha)(\delta + r_\lambda^*)} = \frac{-k_\lambda}{[(\delta + r_\lambda^*) - \alpha(\delta + n)]} < 0$$

As a result, the machine-variety per capita increases as patent breadth is loosened and the depreciation of machines is declined.

A2. Effects of patent breadth, patent's per capita R&D cost and depreciation of machines on the stationary interest rate

Differentiating the stationary interest rate given by Equation (21) with respect to λ, η and δ, we get

$$\frac{\partial r_\lambda^*}{\partial \lambda} = \frac{\alpha(1-\lambda)(r_\lambda^* - n)(\delta + r_\lambda^*)}{\lambda(\lambda-\alpha)[(\delta + r_\lambda^*) - \alpha(\delta + n)]}$$

$$\frac{\partial r_\lambda^*}{\partial \eta} = \frac{-(1-\alpha)(r_\lambda^* - n)(\delta + r_\lambda^*)}{\eta[(\delta + r_\lambda^*) - \alpha(\delta + n)]}$$

$$\frac{\partial r_\lambda^*}{\partial \delta} = \frac{-\alpha(r_\lambda^* - n)}{[(\delta + r_\lambda^*) - \alpha(\delta + n)]}.$$

It is clear, from Equation (21), that $r^*_\lambda > n$. Therefore, $\frac{\partial r^*_\lambda}{\partial \lambda} > 0$, $\frac{\partial r^*_\lambda}{\partial \eta} < 0$ and, $\frac{\partial r^*_\lambda}{\partial \delta} < 0$. These relationship are consistent with Diwakar et al. (2021).

A3. Proof of Lemma 3

A sufficiently small patent's per capita R&D cost, $\eta \to 0$, implies that $r^*_\lambda \to \infty$ for any patent breadth $\lambda \in (\alpha, 1]$. Thus,

$$\lim_{\eta \to 0}(g_\lambda) = \frac{(1-\alpha)}{(1+\mu^{-1})} \lim_{\eta \to 0} \frac{{k_\lambda}^\alpha}{[\eta + (1+n)k_\lambda]} - 1 = \frac{(1-\alpha)}{(1+\mu^{-1})} \lim_{r^*_\lambda \to \infty} \frac{{k_\lambda}^{\alpha-1}}{1+n} - 1 > 0,$$

which implies that the output per capita growth is positive for sufficiently small patent's per capita R&D cost.

A4. Alternative intuition for the optimal machine-variety per capita

The optimal machine-variety per capita k_{λ^o} is given by $\frac{\alpha\eta}{(1+n)(1-\alpha)}$. That is, $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$ which implies,

$$\frac{(1+n)k_{\lambda^o}}{\eta + (1+n)k_{\lambda^o}} = \alpha \implies 1-\alpha = \frac{\eta}{\eta + (1+n)k_{\lambda^o}}$$

$$\implies 1-\alpha = \frac{\eta N_{t+1} L_t}{[\eta + (1+n)k_{\lambda^o}]N_{t+1}L_t}, \tag{A.2}$$

where $1-\alpha$ is the labor share in final output, $\eta N_{t+1} L_t$ is the investment in patents (old and new), and $[\eta + (1+n)k_{\lambda^o}]N_{t+1}L_t$ is the aggregate investment. Therefore, Equation (A.2) suggests that the labor share at any period equals the proportion of aggregate investment devoted to patents at the optimal machine-variety per capita.

A5. Effect of depreciation of machines on optimal patent breadth

Differentiating the optimal patent breadth λ^o in Equation (27) with respect to the depreciation of machines δ, we get

$$\frac{\partial \lambda^o}{\partial \delta} = \frac{-\alpha^2(1-\alpha)\left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha}(1+n)^{1-\alpha}}{\left[(1-\alpha)(\delta+n)+\alpha^2\left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha}(1+n)^{1-\alpha}\right]^2} < 0.$$

Thus, the optimal patent breadth λ^o increases as the depreciation of machines δ decreases. It is consistent with the result of Diwakar et al. (2021).

A6. The machine-variety per capita and stationary interest rate

From Chain-Rule, we have $\frac{\partial r^*_\lambda}{\partial k_\lambda} = \frac{\partial r^*_\lambda}{\partial \lambda}\frac{\partial \lambda}{\partial k_\lambda}$. Thus,

$$\frac{\partial r^*_\lambda}{\partial k_\lambda} = \frac{-\alpha(1-\lambda)(r^*_\lambda - n)(\delta + r^*_\lambda)}{k_\lambda[\lambda(\delta + r^*_\lambda) - \alpha(\delta + n)]} < 0,$$

which implies that the machine-variety per capita and stationary interest rate have negative correlation.