Symmetric auctions with resale

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December 16, 2023

Abstract

In this paper, we consider resale possibilities in symmetric auctions where the object is allocated efficiently. The potential gains from trade arise from a delay in resale which reduces the bidders' values. Specifically, the winner depletes the object before reselling which impacts the loser's value during resale. We characterize an equilibrium of the first- and second-price auction under different information states and capture the impact of resale and information on bids and seller's expected revenues.

JEL classification: D44, D82

Keywords: resale, time delay, symmetry, efficiency, private value, information

1 Introduction

Private-value auctions with resale has its origin based on the doctrine that the first-price auction is inefficient. Precisely, if an object is allocated inefficiently, i.e., the lowest valuation bidder wins the auction, then there can be potential gains from reselling the object to one of the losers. However, when an object is resold after a delay, potential gains from trade can be realized despite the object being allocated efficiently. The potential gains from trade may arise as the winner of the auction obtains value by consuming the object before reselling it which changes his and his opponents' value at the time of resale. In this paper, we incorporate resale possibilities when an auction format is efficient, i.e., the highest value bidder wins the auction.

Resale may be delayed because of the following reasons. First, many governments regulate resale markets for certain objects. At times, certain regulations are relaxed which opens up the possibility of resale. At other times, bidders do find alternate methods to engage in resale – say, for example, changing ownership, mergers, etc. All these frictions delay the

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process of resale, which changes the values of bidders. Second, a bidder may simply choose to use the object before reselling it, as is very common in the real world – say for example, a used car. This may be because better products are available, purpose is solved, etc. A crucial feature in a delayed resale is that certain objects tend to deplete over time – say for example, spectrum licenses, emission rights, cars, etc. Therefore, at the time of resale, the buyer's value declines.

Two motivating examples are as follows:

1. Company A buys spectrum rights in an auction for five years. Upon winning, A is unable to resell the rights on an immediate basis as government regulates the telecommunications industry. In an year's time, A merges with another Company B to transfer the rights. During the period of one year, A earns profits from using the rights while B's value declines as it will enjoy the benefits for four years.
2. A developed country (say, A) buys a technology in an auction and uses it to run its operations. After some years, A innovates and invents a new technology that is more efficient than the existing one. Thus, A transfers its technology to a developing country (say, B). As the technology gets old, B loses value.

Consider a sealed-bid auction followed by a resale trade for one unit of an indivisible object. Two risk neutral bidders with private information about their valuations have interest in the object. Their valuations are drawn from a symmetric probability distribution defined on a real line with full support. The game is designed as follows. At date 1, a sealed-bid auction is conducted by the seller. After date 1, the winner of the auction consumes the object for a fixed amount of time. The game proceeds to date 2 whereat a resale trade between the two bidders may happen. The game ends after date 2 and there is no further resale of the object.

During the interim time, i.e., between dates 1 and 2, the object depletes as the winner is consuming the object. This depletion reduces the value of the loser. At date 2, bidders accept a resale trade based on their revised values which we refer as *resale values*. A winner accepts a resale trade if the price is at least as large as his resale value while a loser accepts a resale trade if the price is at most his resale value.

Information concerning the revelation of bids after date 1 plays a very vital role. If all the bids are revealed after date 1, then the bidders play a game under complete information at date 2 as bidders learn the values of their opponent. For instance, if the winner has all the bargaining power while trading the object at date 2, then he extracts all the surplus of the loser. On the other hand, if no bids are revealed after date 1, then the bidders trade based on the information regarding the ordinal rank of values. In other words, while trading, the bidders' precision about their opponent's value has improved but it is still a game under incomplete information.

In this paper, we consider both complete and incomplete information cases concerning the revelation of bids after date 1. The aim of the present paper is to (a) characterize the equilibria of the first- and second-price auction under complete and incomplete information, (b) study the bidders' bid behavior, and (c) compare the seller's expected revenues under different situations.

Section 3 deals with complete information case where all the bids are revealed after date 1. Under complete information, we consider a continuum of linear trade rules which are linear combinations of the winning and losing values. These rules have two extremes. At one extreme, the winner extracts all the loser's surplus which is referred as the *monopoly rule*. At the other extreme, the loser extracts all the winner's surplus which is referred as the *monopsony rule*. All the other rules lie between the monopoly and monopsony rules.

Theorems 2 and 4 characterize all the equilibria of the first- and second-price auction respectively. Interestingly, the first-price auction is characterized by a formula for a general probability distribution which ensures existence of a unique equilibrium. Proposition 2 shows that resale under the monopoly rule induces bidders to raise their bids while resale under the monopsony rule induces them to lower their bids. Consequently, the seller prefers resale under monopoly rule to no resale and no resale to resale under monopsony rule.

The second-price auction is characterized by a formula which turns out to be linear and *prior-free*, i.e., the formula is independent of probability distributions. This establishes a unique equilibrium in the family of strictly increasing and continuous bid functions. We derive sufficient conditions that include the monopoly rule under which the bidders outbid their values. As a result, the seller's expected revenues raise from resale possibilities. Similarly, we derive sufficient conditions that include the monopsony rule under which the bidders shade their values. As a result, the seller's expected revenues decline from resale possibilities.

Proposition 8 shows that bidders bid higher in the second-price auction than they do in the first-price auction. It is well-known that the property of revenue equivalence between the first- and second-price auction holds whenever resale possibilities are absent. Moreover, with two asymmetric risk neutral bidders, the first-price auction dominates the second-price auction in terms of expected revenues. In contrast to these results, Theorem 5 establishes a striking property of revenue equivalence between the first- and second-price auction. We also show that the seller's optimum trade rule is the monopoly rule.

Section 4 deals with the incomplete information case where no bids are revealed after date 1. By considering that the winner of the object proposes a take-it-or-leave-it offer to the loser, we characterize all the equilibria of the first-price auction. By considering that the loser of the object proposes a take-it-or-leave-it offer to the winner, we characterize all the equilibria of the second-price auction.

Theorem 8 compares the bid functions of the first-price auction between complete and incomplete information. We show that bidders bid higher under complete information than they do under incomplete information. As a result, the seller prefers to reveal information.

1.1 The literature

Auctions with resale have been studied in Gupta and Lebrun [3]; Haile [6, 7]; Hafalir and Krishna [4, 5]; Virág [13]; Garratt and Tröger [2]; Lebrun [9]; Cheng [1]; Zheng [15]; and Khurana [8] among others.

Haile [7] considers symmetric auctions where the value of the object is not known to a bidder while submitting bids. Rather, bidders receive noisy signals about their values at the time of auction. During resale, they get additional information about their values which leads to expected potential gains from trade. Garratt and Tröger [2] also consider symmetric auctions but in their model one of the bidders is a speculator who has no value for the object. His sole purpose is to earn by reselling the object.

Gupta and Lebrun [3] consider asymmetric bidders with complete information in the resale date. They derive a formula for the bid functions of the first-price auction.

Hafalir and Krishna [4, 5]; Virág [13, 14]; Lebrun [9]; Cheng [1]; and Khurana [8] consider asymmetric auctions with incomplete information during the resale date. With two risk neutral bidders, Hafalir and Krishna [4] show that bid symmetrization holds, i.e., the two bidders win with equal probability, and the first-price auction is revenue-dominant to the second-price auction. Virág [14] shows that bid symmetrization fails with two bidders if there are reserve prices. Virág [13] extends the analysis to more than two bidders and shows that bid symmetrization fails. Khurana [8] considers one risk neutral and one risk averse bidder and shows that bid symmetrization may or may not hold.

The paper is organized as follows. In Section 2, we setup the model. In Section 3, we characterize the first- and second-price auction under complete information and derive other properties. In Section 4, we characterize under incomplete information. In Section 5, we conclude. The proofs are relegated to the appendix.

2 Economic model

Consider one unit of an indivisible object that has to be allocated *via* a first- or second-price auction. The set of two risk neutral bidders is denoted by $N = \{1, 2\}$. The values are drawn from a symmetric probability distribution $F : T \to \Re_+$, where $T = [0, \bar{a}] \subset \Re_+$ is the value space for both the bidders. We denote the random variables for bidders 1 and 2 by $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. The probability distribution is twice continuously differentiable and the density function, denoted by f, is bounded away from zero. The seller is risk neutral and reserve prices are 0.

The structure of the game is as follows. Bidders play a two-date game, whereat date 1 – *the bid date*, the seller allocates the object *via* a first- or second-price auction. After date 1, there is a fixed time delay in the game that is exogenous. The game then proceeds to date 2 – *the resale date*, where the two bidders engage in a resale trade. After date 2, the game ends and utilities are realized.

In this paper, we consider two cases:

1. Complete information: In this case, the seller reveals all the bids after date 1.
2. Incomplete information: In this case, the seller does not reveal any bid after date 1.

The complete information case has been discussed in Section 3 while the incomplete information case has been discussed in Section 4. In the complete information case, the game turns into a game of complete information after date 1.

During the interim time, i.e., between dates 1 and 2, the winner of date 1 consumes the object and obtains value from it while the loser loses value as the object depletes. The winner obtains and the loser loses value linearly with their own values. The parameter for winner is denoted by α_R and the parameter for loser is denoted by α_B, where $\alpha_R, \alpha_B \in (0, 1)$. For example, a bidder with value t obtains a value of $\alpha_R t$ in the interim time if he wins, and loses a value of $\alpha_B t$ if he loses. We refer α_R as the *rate of consumption* and α_B as the *rate of depletion*.

A higher consumption rate implies that either the bidder consumes the object very quickly or resale happens very late. One can easily see either of the two situations. A similar interpretation holds for a higher rate of depletion.

The utilities under different circumstances in the first-price auction are as follows.

1. If a bidder with value t wins by bidding b and resells at p during the resale date, then his utility is $p + \alpha_R t - b$, where $\alpha_R t$ is the utility obtained by consuming the object in the interim.
2. If a bidder with value t wins by bidding b and does not resell during

the resale date, then his utility is $t - b$.

3. If a bidder with value t loses and buys at p during the resale date, then his utility is $(1 - \alpha_B)t - p$, where $\alpha_B t$ is the utility lost in the interim.
4. If a bidder with value t loses and does not buy at the resale date, then his utility is 0.

On similar lines, we can define the utilities under second-price auction. An assumption that we follow throughout the paper is as follows.

Assumption 1. $\alpha_R > \alpha_B$

The above assumption says that the rate of depletion does not exceed the rate of consumption.

3 Complete information

In this section, we consider the complete information case where all the bids are revealed after date 1. The resale trade rule is exogenous and it is a linear combination of the winning and losing valuations. Formally, let the resale trade rule be

$$p(w, l) = \lambda_1 w + \lambda_2 l \tag{1}$$

where w is the value of the winner, l is the value of the loser and λ_1 and λ_2 are positive parameters. If a bidder with value w wins while a bidder with value l loses, then $p(w, l)$ is the payment that goes from the loser to the winner.

For notational convenience, let

$$k_1 = \frac{1 - \alpha_R}{\lambda_2}, k_2 = \frac{\lambda_1}{1 - \alpha_B}, k_3 = \max\left\{\frac{1 - \alpha_R - \lambda_1}{\lambda_2}, \frac{\lambda_1}{1 - \alpha_B - \lambda_2}\right\}$$

For tractability, we assume the following.

Assumption 2. *Either of the following must be true:*

1. $\lambda_1 = 0$ *and* $1 - \alpha_R < \lambda_2 \leq 1 - \alpha_B$.
2. $\lambda_2 = 0$ *and* $1 - \alpha_R \leq \lambda_1 < 1 - \alpha_B$.
3. $\lambda_1, \lambda_2 > 0$, $1 - \alpha_R > \lambda_1$, $1 - \alpha_B > \lambda_2$ *and* $k_3 < 1$.

At one extreme where $\lambda_1 = 0$ and $\lambda_2 = 1 - \alpha_B$, the winner extracts all the surplus from the loser. Thus, we refer to this rule as a *monopoly rule*. At the other extreme where $\lambda_2 = 0$ and $\lambda_1 = 1 - \alpha_R$, the loser extracts all the surplus from the winner. Thus, we refer to this rule as a *monopsony rule*.

In Subsection 3.1, we characterize all the equilibria of the first-price auction. In Subsection 3.2, we characterize all the equilibria of the second-price auction.

3.1 First-price auction

Consider the first-price auction. Denote the symmetric bid function, that belongs to the family of strictly increasing, continuous and onto functions, by β^1, i.e., $\beta^1 : T \to [0, \bar{b}^1]$ where $\bar{b}$ is the maximum bid. Let the symmetric inverse bid function be π^1, i.e., $\pi^1 : [0, \bar{b}^1] \to T$. Note that $\beta^1(0) = \pi^1(0) = 0$ and $\beta^1(\bar{a}) = \bar{b}^1$ and $\pi^1(\bar{b}^1) = \bar{a}$.

In the following result, we derive utility functions of bidders. Denote the expected utility function of a bidder in the first-price auction by $U^1 : T \times [0, \bar{b}^1] \to \Re$.

Theorem 1. *Consider a first-price auction under complete information. The expected utility functions of a bidder with value t and bid b under different situations are as follows.*

If $\lambda_1 = 0$ and $1 - \alpha_R < \lambda_2 \leq 1 - \alpha_B$, then

$$\begin{aligned} U^1(t, b) = F(k_1 t)(t - b) + \int_{k_1 t}^{\pi^1(b)} (\alpha_R t + \lambda_2 \omega - b) f(\omega) \mathrm{d}\omega \\ + (1 - \alpha_B - \lambda_2) t [F(t/k_1) - F \circ \pi^1(b)] \end{aligned} \tag{2}$$

If $\lambda_2 = 0$ and $1 - \alpha_R \leq \lambda_1 < 1 - \alpha_B$, then

$$\begin{aligned} U^1(t, b) = (1 - \alpha_R - \lambda_1) t F(k_2 t) + F \circ \pi^1(b)[(\alpha_R + \lambda_1) t - b] \\ + \int_{\pi^1(b)}^{t/k_2} [(1 - \alpha_B) t - \lambda_1 \omega] f(\omega) \mathrm{d}\omega \end{aligned} \tag{3}$$

If $\lambda_1, \lambda_2 > 0$, $1 - \alpha_R > \lambda_1$, $1 - \alpha_B > \lambda_2$ and $k_3 < 1$, then

$$\begin{aligned} U^1(t, b) = F(k_3 t)(t - b) + \int_{k_3 t}^{\pi^1(b)} [(\alpha_R + \lambda_1) t + \lambda_2 \omega - b] f(\omega) \mathrm{d}\omega \\ + \int_{\pi^1(b)}^{t/k} [(1 - \alpha_B - \lambda_2) t - \lambda_1 \omega] f(\omega) \mathrm{d}\omega \end{aligned} \tag{4}$$

For notational convenience, let $p^1(\pi^1(b), \pi^1(b)) = p^1(b)$, $\alpha = (\alpha_R, \alpha_B)$, $\lambda = (\lambda_1, \lambda_2)$ and $E(\alpha, \lambda) = 2\lambda_1 + 2\lambda_2 + \alpha_R + \alpha_B - 1$. The following result characterizes all the perfect Bayesian equilibria of the first-price auction under complete information.

Theorem 2. *Let Assumptions 1 and 2 be satisfied. A pair (π^1, p^1) is a symmetric perfect Bayesian equilibrium in monotone strategies if and only if it solves:*

$$\begin{aligned} \frac{F \circ \pi^1(b)}{\mathrm{D}F \circ \pi^1(b)} &= (\alpha_R + \alpha_B - 1)\pi^1(b) + 2p^1(b) - b \\ p^1(b) &= (\lambda_1 + \lambda_2)\pi^1(b) \end{aligned} \tag{5}$$

The following result provides a formula of the bid function for general probability distributions and ensures the existence of a unique equilibrium.

Proposition 1. *Let the primitives of Theorem 2 be true. Then, the bid function is characterized as*

$$\beta^1(t) = \frac{E(\alpha, \lambda)}{F(t)} \int_0^t \omega f(\omega) \mathrm{d}\omega \tag{6}$$

Remark 1. *From Assumption 2,* $1 + \alpha_B - \alpha_R \leq E(\alpha, \lambda) \leq 1 + \alpha_R - \alpha_B$*. If the trade rule is monopoly, then* $E(\alpha, \lambda) = 1 + \alpha_R - \alpha_B$*. If the trade rule is monopsony, then* $E(\alpha, \lambda) = 1 + \alpha_B - \alpha_R$*. Clearly, bidders bid higher under the monopoly rule than the monopsony rule.*

Remark 2. *Given the monopoly rule, if the rate of consumption is high or the rate of depletion is low, bidders raise their bid. Given the monopsony rule, if the rate of consumption is low or the rate of depletion is high, bidders raise their bid.*

The following result compares β^1 with the standard symmetric independent private valuation model that is given in Riley and Samuelson [12] (henceforth, R-S). In other words, it captures the impact of resale with delays under complete information on the bid behavior. Let β^* be the bid function in the R-S model.

Proposition 2. *Let Assumptions 1 and 2 hold.*

1. *If* $E(\alpha, \lambda) > 1$*, then* $\beta^1(t) > \beta^*(t)$ *for every* $t \in (0, \bar{a}]$.
2. *If* $E(\alpha, \lambda) < 1$*, then* $\beta^1(t) < \beta^*(t)$ *for every* $t \in (0, \bar{a}]$.

The above result says that as long as $E(\alpha, \lambda) > 1$, bidders bid more aggressively in the presence of resale than they do when there are no resale possibilities. On the other hand, as long as $E(\alpha, \lambda) < 1$, bidders bid less aggressively in the presence of resale than they do when there are no resale possibilities.

Remark 3. *Under the monopoly rule, bidders bid higher than the case when resale is absent. Under the monopsony rule, bidders bid lower than the case when resale is absent. If either the monopoly or the monopsony rule is being implemented with* $\alpha_R \downarrow \alpha_B$*, then bids converge to the R-S model.*

To understand the intuition of the above result, we divide the impact of resale after a delay on bids into three effects: consumption effect, depletion effect, and bargaining effect. The consumption effect captures the impact on bids due to the possibility of consuming the object before reselling it. The depletion effect captures the impact on bids due to depletion of the object. The bargaining effect captures the impact on bids due to the bargaining power that a bidder has during resale.

The consumption and depletion effects raise the bid as a bidder has an incentive to reduce his risk of losing. Thus, the consumption and depletion effects are positive. Under the monopoly rule, the bargaining effect induces a bidder to raise his bid as the winner extracts all the loser's surplus. Thus, the bargaining effect is positive. Under the monopsony rule, the bargaining effect induces a bidder to lower his bid as the loser extracts all the winner's surplus. Thus, the bargaining effect is negative.

Under the monopoly rule, the total effect induces a bidder to bid higher. Under the monopsony rule, the negative bargaining effect dominates the positive consumption and depletion effects which reduces the bid.

The following result compares the seller's *ex-ante* expected revenues between a delayed resale under complete information and absence of resale.

Corollary 1. *Let the primitives of Proposition 2 hold. If $E(\alpha, \lambda) > 1$, the seller generates more expected revenues when resale happens after a delay under complete information than when there are no resale possibilities. If $E(\alpha, \lambda) < 1$, the seller generates less expected revenues when resale happens after a delay under complete information than when there are no resale possibilities.*

In Propositions 3 and 4, we compare the bid function of the present model with the standard asymmetric auctions with resale model that is studied in Hafalir and Krishna [4] (henceforth, H-K) among others. In H-K, the two risk neutral bidders have asymmetric probability distributions and resale happens without a delay. The bidders are distinguished as weak (w) and strong (s) where the strong bidder is more likely to draw a high value than the weak bidder. Furthermore, during the resale date, the losing bid is not revealed and the winner makes a take-it-or-leave-it offer to the loser.

Proposition 3. *Let (π^1, p^1) be a symmetric perfect Bayesian equilibrium in monotone strategies when resale happens after a delay and symmetric bidders have a probability distribution F_s. Let (γ_s, γ_w, r) be a perfect Bayesian equilibrium when resale happens without a delay and the probability distribution pair is (F_s, F_w). Let $F_s(0) > 0$ and $E(\alpha, \lambda) > 1$. Then,*

$$\pi^1(b) < \gamma_s(b)$$

for every $b \in (0, \gamma_s^{-1}(\bar{a})]$.

Before we interpret the above result, it is important to note that, in H-K model, the weak bidder bids more aggressively than the strong bidder and the weak bidder acts as a reseller during the resale date while the strong bidder acts as a buyer.

The above result says that as long as $E(\alpha, \lambda) > 1$, the strong bidder who plays against another strong bidder in an environment where resale happens after a delay with complete information bids more aggressively than while playing against a weak bidder in an alternate environment where resale happens without a delay under incomplete information.

Given the monopoly rule, the intuition of the impact of a strong opponent and a delay in resale on bids of a strong bidder is divided into five effects: consumption effect, depletion effect, bargaining effect, information effect and prior effect. The information effect captures the impact of information during resale. The prior effect captures the impact on bids due to a different opponent.

The consumption and depletion effects are positive as they incentivize the strong bidder to raise his bid. The bargaining effect is zero as all the bargaining power is with winner in both the cases. The information effect induces the strong bidder to raise his bid as he can extract all the loser's surplus. Thus, the information effect is positive. As the strong bidder plays with another strong bidder, he bids higher, i.e., the prior effect is also positive. Therefore, the total effect raises the bid of a strong bidder.

Proposition 4. *Let (π^1, p^1) be a symmetric perfect Bayesian equilibrium in monotone strategies when resale happens after a delay and symmetric bidders have a probability distribution F_w. Let (γ_s, γ_w, r) be a perfect Bayesian equilibrium when resale happens without a delay and the probability distribution pair is (F_s, F_w). Let $F_w(0) > 0$ and $E(\alpha, \lambda) < 1$. Then,*

$$\pi^1(b) > \gamma_w(b)$$

for every $b \in (0, \beta^1(\bar{a})]$.

The above result says that as long as $E(\alpha, \lambda) < 1$, the weak bidder who plays against another weak bidder in an environment where resale happens after a delay with complete information bids less aggressively than while playing against a strong bidder in an alternate environment where resale happens without a delay under incomplete information.

Given the monopsony rule, the intuition is as follows. The consumption and depletion effects are positive. The bargaining and information effects induce the weak bidder to reduce his bid, as the loser has a higher bargaining power. Therefore, the bargaining and information effects are negative. Lastly, the prior effect is negative, as the weak bidder plays with another weak bidder which induces him to reduce his bid. In this case, the negative bargaining, information and prior effects dominate the positive consumption and depletion effects which reduces the bid.

In the next two propositions, we compare the bid function of the present model with the standard asymmetric auctions without resale model that has been studied in Maskin and Riley [10] (henceforth, M-R) among others. In the M-R model, the two risk neutral bidders are distinguished as weak and strong.

Proposition 5. *Let (π^1, p^1) be a symmetric perfect Bayesian equilibrium in monotone strategies when resale happens after a delay and symmetric bidders have a probability distribution F_s. Let (ψ_s, ψ_w) be a Bayesian equilibrium when there is no resale possibility and the probability distribution pair is (F_s, F_w). Let $F_s(0) > 0$ and $E(\alpha, \lambda) > 1$. Then,*

$$\pi^1(b) < \psi_s(b)$$

for every $b \in (0, \psi_s^{-1}(\bar{a})]$.

The above result says that as long as $E(\alpha, \lambda) > 1$, the strong bidder who plays against another strong bidder in an environment where resale happens after a delay with complete information bids more aggressively than while playing against a weak bidder in an alternate environment where there are no resale possibilities.

Proposition 6. *Let (π^1, p^1) be a symmetric perfect Bayesian equilibrium in monotone strategies when resale happens after a delay and symmetric bidders have a probability distribution F_w. Let (ψ_s, ψ_w) be a Bayesian equilibrium when there is no resale possibility and the probability distribution pair is (F_s, F_w). Let $F_w(0) > 0$ and $E(\alpha, \lambda) < 1$. Then,*

$$\pi^1(b) > \psi_w(b)$$

for every $b \in (0, \beta^1(\bar{a})]$.

The above result says that as long as $E(\alpha, \lambda) < 1$, the weak bidder who plays against another weak bidder in an environment where resale happens after a delay with complete information bids less aggressively than while playing against a strong bidder in an alternate environment where there are no resale possibilities.

3.2 Second-price auction

In this subsection, we characterize all the equilibria of the second-price auction. Let β^2 be the symmetric bid function in the family of strictly increasing, continuous, and onto functions, i.e., $\beta^2 : T \to [0, \bar{b}^2]$ where $\bar{b}^2$ is the maximum bid. Let π^2 be the symmetric inverse bid function, i.e., $\pi^2 : [0, \bar{b}^2] \to T$.

In the following result, we derive utility functions of bidders. Denote the utility function of a bidder in the second-price auction by $U^2 : T \times [0, \bar{b}^2] \to \Re$.

Theorem 3. *Consider a second-price auction under complete information. The expected utility functions of a bidder with value t and bid b under different situations are as follows.*

If $\lambda_1 = 0$ and $1-\alpha_R < \lambda_2 \leq 1-\alpha_B$, then

$$\begin{aligned} U^2(t,b) = \int_0^{k_1 t} [t-\beta^2(\omega)] f(\omega) \mathrm{d}\omega + \int_{k_1 t}^{\pi^2(b)} [\alpha_R t + \lambda_2 \omega \\ - \beta^2(\omega)] f(\omega)\mathrm{d}\omega + (1-\alpha_B-\lambda_2) t [F(t/k_1) - F\circ \pi^2(b)] \end{aligned} \tag{7}$$

If $\lambda_2 = 0$ and $1-\alpha_R \leq \lambda_1 < 1-\alpha_B$, then

$$\begin{aligned} U^2(t,b) = \int_0^{k_2 t} [t-\beta^2(\omega)] f(\omega) \mathrm{d}\omega + \int_{k_2 t}^{\pi^2(b)} [(\alpha_R + \lambda_1) t \\ - \beta^2(\omega)] f(\omega)\mathrm{d}\omega + \int_{\pi^2(b)}^{t/k_2} [(1-\alpha_B)t - \lambda_1 \omega] f(\omega)\mathrm{d}\omega \end{aligned} \tag{8}$$

If $\lambda_1, \lambda_2 > 0$, $1-\alpha_R > \lambda_1$, $1-\alpha_B > \lambda_2$ and $k_3 < 1$, then

$$\begin{aligned} U^2(t,b) = \int_0^{k_3 t} [t-\beta^2(\omega)] f(\omega) \mathrm{d}\omega + \int_{k_3 t}^{\pi^2(b)} [(\alpha_R + \lambda_1) t + \lambda_2 \omega \\ - \beta^2(\omega)] f(\omega)\mathrm{d}\omega + \int_{\pi^2(b)}^{t/k_3} [(1-\alpha_B-\lambda_2)t - \lambda_1 \omega] f(\omega)\mathrm{d}\omega \end{aligned} \tag{9}$$

The proof of above theorem is based on similar line of that of Theorem 1.

For notational convenience, let $p^2(b) \equiv p^2(\pi^2(b), \pi^2(b))$. In the following result, we characterize the equilibria.

Theorem 4. *A pair (π^2, p^2) is a perfect Bayesian equilibrium in the second-price auction if and only if it solves the following:*

$$\pi^2(b) = \frac{b}{E(\alpha,\lambda)}, \quad p^2(b) = (\lambda_1+\lambda_2)\pi^2(b) \tag{10}$$

The above result gives us a formula for computing bids which is *prior-free*, i.e., the formula is independent of the probability distribution. Furthermore, the above result ensure the existence of a unique equilibrium.

Remark 4. *If the trade rule is monopoly, the equilibrium is characterized as*

$$\beta^2(t) = (1+\alpha_R-\alpha_B)t$$

As $\alpha_R > \alpha_B$, bidders bid more than their values, i.e., overbidding occur. If the trade rule is monopsony, the equilibrium is characterized as

$$\beta^2(t) = (1+\alpha_B-\alpha_R)t$$

As $\alpha_R > \alpha_B$, bidders bid less than their values, i.e., bid shading happens.

If $\alpha_R \downarrow \alpha_B$, the equilibrium converges to bid-your-own-value.

In the following result, we compare the bid functions of the second-price auction under complete information with the standard models.

Proposition 7. *Let Assumptions 1 and 2 hold.*

1. *If $E(\alpha, \lambda) > 1$, the bidders bid more aggressively during a delayed resale than under no resale.*
2. *If $E(\alpha, \lambda) < 1$, the bidders bid less aggressively during a delayed resale than under no resale.*

As it is well-known that bidders bid their value when resale is absent, the above result conveys that as long as $E(\alpha, \lambda) > 1$, bidders outbid their values and as long as $E(\alpha, \lambda) < 1$, bidders shade their values.

Corollary 2. *Let the primitives of Proposition 7 hold. If $E(\alpha, \lambda) > 1$, the seller generates more expected revenue under a delayed resale with complete information than under no resale. If $E(\alpha, \lambda) < 1$, the seller generates less expected revenue under a delayed resale than under no resale.*

In the next result, we compare bid functions between the first- and second-price auction.

Proposition 8. *Let Assumptions 1 and 2 hold. Let (π^1, p^1) be a perfect Bayesian equilibrium of the first-price auction. Let (π^2, p^2) be a perfect Bayesian equilibrium of the second-price auction. Then,*

$$\pi^1(b) > \pi^2(b)$$

for every $b \in (0, \bar{b}^1]$.

The above result says that bidders bid more aggressively in the second-price auction than they do in the first-price auction.

It is well-known from Riley and Samuelson [12] and Myerson [11] that the seller's *ex-ante* expected revenues are equivalent in the first- and second-price auction as long as resale is absent and bidders are symmetric. In the case of asymmetric bidders and absence of resale, Maskin and Riley [10] show that a general revenue ranking principle does not exist for the two auction formats. Whenever the two bidders are asymmetric and resale occurs without a delay, Hafalir and Krishna [4] show that the first-price auction dominates the second-price auction in terms of expected revenues.

In contrast to the aforementioned results in the literature, the following result establishes a striking property that the expected revenues are equivalent in the two auction formats whenever bidders are symmetric and resale occurs after a fixed time delay.

Theorem 5. *Let Assumptions 1 and 2 be true. The seller's* ex-ante *expected revenues are equivalent in the first- and second-price auction and are given by:*

$$R = 2E(\alpha, \lambda) \int_0^{\bar{a}} t f(t)[1 - F(t)] \mathrm{d}t \tag{11}$$

The above result is called the *revenue equivalence theorem.* An immediate corollary is as follows.

Corollary 3. *The seller's optimum trade rule is one where the winner extracts all the loser's surplus.*

4 Incomplete information

In this section, we consider the incomplete information case where no bids are revealed. Section 4.1 deals with the first-price auction while Section 4.2 deals with the second-price auction.

We require an additional assumption.

Assumption 3. *The following must hold for parameters of the model:*

1. $f/(1-F)$ *is non-decreasing everywhere on the value space.*
2. $\alpha_R + \alpha_B > 1$.

4.1 First-price auction

Consider the first-price auction. At date 2, the winner makes a take-it-or-leave-it offer to the loser.

We restrict to the family of symmetric perfect Bayesian equilibria where the bid functions are measurable, strictly increasing, continuous, and onto. Let the bid functions be denoted by μ^1. It may be shown that $\mu^1(0) = 0$ and $\mu^1(\bar{a}) = \hat{b}$ for some $\hat{b}^1 > 0$. Let the inverse bid functions be denoted by σ^1. Therefore, $\mu^1 : T \to [0, \hat{b}]$ and $\sigma^1 : [0, \hat{b}^1] \to T$.

The following claim establishes the direction of resale.

Lemma 1. *Whosoever wins offer the object at the resale date.*

We solve the game by backward induction. Consider the resale date and bidder 1 with value t. Since he chooses an optimum resale price, it must be true that he wins at date 1. Suppose he wins with a bid b. Then, it must be the case that $b > \mu^1(\mathcal{T}_2)$, which is equivalent to $\mathcal{T}_2 < \sigma^1(b)$.

Since bidder 1 wins, he offers the object to bidder 2 at price, say q^1. Bidder 2 accepts if his resale utility at date 2 exceeds the resale price, i.e., $(1 - \alpha_B)\mathcal{T}_2 > q^1$ which is equivalent to $\mathcal{T}_2 > zq^1$, where $z = 1/(1 - \alpha_B)$.

If $\mathcal{T}_2 < zq^1$, bidder 2 rejects the offer. Therefore, the expected utility function of bidder 1 is

$$\begin{aligned} U^1(t,b,q^1) = &\Pr[\mathcal{T}_2 > zq^1 | \mathcal{T}_2 < \sigma^1(b)](q^1 + \alpha_R t - b) \\ &+ \Pr[\mathcal{T}_2 < zq^1 | \mathcal{T}_2 < \sigma^1(b)](t-b) \end{aligned}$$

Since $zq^1 < \sigma^1(b)$, the expected utility function can be rewritten as

$$U^1(t,b,q^1) = \frac{F\circ\sigma^1(b) - F(zq^1)}{F\circ\sigma^1(b)}(q^1 + \alpha_R t - b) + \frac{F(zq^1)}{F\circ\sigma^1(b)}(t-b)$$

The optimization problem is $\max_{q^1} U^1(t,b,q^1)$. The first-order condition is

$$(1-\alpha_R)t = q^1 - \frac{F\circ\sigma^1(b) - F(zq^1)}{zf(zq^1)} \tag{12}$$

Let $q^1(t,\sigma^1(b))$ be the resale price that solves (12). From Lemmas B.1 and B.2, it follows that

1. (12) is also sufficient.
2. There exists a unique q^1 that solves (12).
3. The resale price $q^1(t,\sigma^1(b))$ is strictly increasing in value t and bid b.

Consider the bid date and bidder 1 with value t and bid b. The expected utility function of bidder 1 is

$$\begin{aligned} U^1(t,b) = &[F\circ\sigma^1(b) - F(zq^1(t,\sigma^1(b)))](q^1 + \alpha_R t - b) \\ &+ F(zq^1(t,\sigma^1(b)))[t - q^1(t,\sigma^2(b))] \\ &+ \int_{\sigma^1(b)}^{\bar{a}} \max\{(1-\alpha_B)t - q^1(t,\omega), 0\} f(\omega) \mathrm{d}\omega \end{aligned}$$

Using Envelope theorem and Leibniz integral rule, the first-order differential equation is

$$\frac{F\circ\sigma^1(b)}{\mathrm{D}F\circ\sigma^1(b)} = 2q^1(b) + (\alpha_R + \alpha_B - 1)\sigma^1(b) - b \tag{13}$$

where $q^1(\sigma^1(b),\sigma^1(b)) \equiv q^1(b)$. In the following theorem, we characterize all the perfect Bayesian equilibria.

Theorem 6. *Let Assumptions 1 and 3 be satisfied. A profile (σ^1, q^1) is a perfect Bayesian equilibrium of the first-price auction under incomplete information if and only if it solves the following Dirichlet problem:*

$$\begin{aligned} \mathrm{D}\sigma^1(b) &= \frac{F\circ\sigma^1(b)}{f\circ\sigma^1(b)} \frac{1}{2q^1(b) + (\alpha_R+\alpha_B-1)\sigma^1(b) - b} \\ (1-\alpha_R)\sigma^1(b) &= q^1(b) - \frac{F\circ\sigma^1(b) - F(zq^1(b))}{zf(zq^1(b))} \\ \sigma^1(0) &= 0, \quad \sigma^1(\hat{b}) = \bar{a} \quad \textit{for some} \quad \hat{b}^1 > 0 \end{aligned} \tag{14}$$

4.2 Second-price auction

Consider the second-price auction. If, at date 2, the winner makes a take-it-or-leave-it offer to the loser, then this simply becomes a game of complete information whose equilibria has been characterized in Remark 4. Therefore, we consider that, at date 2, the loser makes a take-it-or-leave-it offer to the winner. This trade rule is called *monopsony rule*.

Let σ^2 be a symmetric inverse bid function, which belongs to the family of strictly increasing, continuous, and onto functions.

In the following lemma, we establish the direction of resale.

Lemma 2. *Whosoever loses offers the object for resale under monopsony rule.*

Let us solve for an equilibrium by using the process of backward induction. Without loss of generality, consider bidder 1 with a value of t. Suppose he bids b and makes a resell offer at a price of q^2.

Since bidder 1 tries to resell, it must be true that he has lost the auction at date 1. This is possible only if $b < \mu^2(\mathcal{T}_2)$ which is equivalent to $\mathcal{T}_2 > \sigma^2(b)$. His offer gets accepted only if the resale price is more than the resale value of bidder 2, i.e., $q^2 > (1-\alpha_R)\mathcal{T}_2$, or equivalently $\mathcal{T}_2 < yq^2$, where $y = 1/(1-\alpha_R)$. On the other hand, his offer gets rejected if $\mathcal{T}_2 > yq^2$. Therefore, the expected utility function of bidder 1 is

$$\begin{aligned} U^2(t,b,q^2) &= \Pr(\mathcal{T}_2 < yq^2 | \mathcal{T}_2 > \sigma^2(b))[(1-\alpha_B)t - q^2] \\ &= \frac{F(yq^2) - F \circ \sigma^2(b)}{1 - F \circ \sigma^2(b)}[(1-\alpha_B)t - q^2] \end{aligned}$$

The first-order condition gives

$$(1-\alpha_B)t = q^2 - \frac{F \circ \sigma^2(b) - F(yq^2)}{yf(yq^2)} \tag{15}$$

Let $q^2(t, \sigma^2(b))$ be the resale price that solves (12). From Lemmas B.1 and B.2, it follows that

1. (15) is also sufficient.
2. There exists a unique q^2 that solves (15).
3. The resale price $q^2(t, \sigma^2(b))$ is strictly increasing in value t and bid b.

Consider date 1 where a second-price auction happens. Consider bidder 1 with value t. Suppose he bids b while bidder 2 implements σ^2. He wins only if $\mathcal{T}_2 < \sigma^2(b)$. Whenever he wins, he receives a resale offer of $q^2(t, \mathcal{T}_2)$ from bidder 2. He accepts only if $q^2(t, \mathcal{T}_2) + \alpha_R > t$, otherwise he rejects. Therefore, with probability that $\mathcal{T}_2 < \sigma^2(b)$, he incurs a utility of $\max\{q^2(t, \mathcal{T}_2) + \alpha_R t, t\} - \mu^2(\mathcal{T}_2)$ where $\mu^2(\mathcal{T}_2)$ are his payments.

Bidder 1 loses only if $\mathcal{T}_2 > \sigma^2(b)$. Whenever he loses, he proposes a resale offer to bidder 2. Bidder 2 accepts only if $q^2(t, \sigma^2(b)) > (1-\alpha_R)\mathcal{T}_2$, otherwise he rejects. Therefore, with probability that $\sigma^2(b) < \mathcal{T}_2 < yq^2(t, \sigma^2(b))$, bidder 1 gets a utility of $(1-\alpha_B)t - q^2(t, \sigma^2(b))$. Thus, the expected utility function of bidder 1 is

$$\begin{aligned} U^2(t,b) &= \Pr(\mathcal{T}_2 < \sigma^2(b))[\max\{q^2(t,\mathcal{T}_2) + \alpha_R t, t\} - \mu^2(\mathcal{T}_2)] \\ &\quad + \Pr(\sigma^2(b) < \mathcal{T}_2 < yq^2(t,\sigma^2(b)))[(1-\alpha_B)t - q^2(t,\sigma^2(b))] \\ &= \int_0^{\sigma^2(b)} [\max\{q^2(t,\omega) + \alpha_R t, t\} - \mu^2(\omega)] f(\omega)\mathrm{d}\omega \\ &\quad + [F(yq^2(t,\sigma^2(b))) - F\circ\sigma^2(b)][(1-\alpha_B)t - q^2(t,\sigma^2(b))] \end{aligned}$$

Using Envelope theorem and Leibniz integral rule, the first-order derivative is

$$\begin{aligned} \mathrm{D}_b U^2(t,b) = \mathrm{D}F\circ\sigma^2(b)[&\max\{q^2(t,\sigma^2(b)) + \alpha_R t, t\} - b \\ &- (1-\alpha_B)t + q^2(t,\sigma^2(b))] \end{aligned} \tag{16}$$

For notational convenience, let $q^2(\sigma^2(b), \sigma^2(b)) \equiv q^2(b)$. In equilibrium, $t = \sigma^2(b)$, $q^2(b) > (1-\alpha_R)\sigma^2(b)$ and $\mathrm{D}_b U^1(\sigma^2(b), b) = 0$. This gives

$$\sigma^2(b) = \frac{b - 2q^2(b)}{\alpha_R + \alpha_B - 1} \tag{17}$$

In the following theorem, we characterize the equilibria of the second-price auction.

Theorem 7. *Let Assumptions 1 and 3 be satisfied. A profile (σ^2, q^2) is a perfect Bayesian equilibrium in monotone strategies if and only if the following holds:*

$$\begin{aligned} \sigma^2(b) &= \frac{b - 2q^2(b)}{\alpha_R + \alpha_B - 1} \\ (1-\alpha_B)\sigma^2(b) &= q^2(b) - \frac{F\circ\sigma^2(b) - F(yq^2(b))}{yf(yq^2(b))} \end{aligned} \tag{18}$$

In the following result, we compare the bid functions of the first-price auctions between complete and incomplete information.

Theorem 8. *Let (σ^1, p^1) be a symmetric perfect Bayesian equilibrium in monotone strategies when resale happens after a delay with complete information. Let (σ^1, q^1) be a symmetric perfect Bayesian equilibrium when resale happens after a delay with incomplete information. Let Assumptions 1, 2 and 3 be satisfied. Let $F(0) > 0$, $\lambda_1 = 0$ and $\lambda_2 = 1 - \alpha_B$. Then,*

$$\pi^1(b) < \sigma^1(b)$$

for every $b \in (0, \hat{b}^1]$.

The above result says that, in the first-price auction, the bidders bid more aggressively under complete information than they do under incomplete information as long as the monopoly rule is implemented. An immediate corollary is as follows.

Corollary 4. *Let the primitives of Theorem 8 be satisfied. Then, the seller prefers to reveal information in the first-price auction.*

The above result says that, in the first-price auction, the seller's *ex-ante* expected revenues are higher under complete information than under incomplete information.

Example 1. *Let $T = [0, 1]$ and let $F(t) = t$. Then, the bid functions under the first- and second price auction are*

$$\sigma^1(b) = 2b, \quad \sigma^2(b) = b \tag{19}$$

5 Conclusion

In this paper, we have considered resale possibilities in symmetric private-value auctions that allocate the object efficiently. The model's salient feature is to include fixed time delays in resale. During the interim time, i.e., between the bid and resale dates, the winner of the auction consumes the object and generates value from it while the loser's value diminishes as the object depletes. This delay in resale leads to expected potential gains from trade. We have characterized the equilibria of the first- and second-price auction under complete information and incomplete information. Under complete information, the seller reveals all the bids while under incomplete information, the seller reveals no bid.

Our main result is that under complete information, the property of revenue equivalence holds for the two auction formats. Other results concerning complete information compare the bid functions under different situations. We also show that, in the first-price auction, the seller's expected revenues dominate under complete information as compared to incomplete information.

A Appendix: Proofs

Proof of Theorem 1. We show 1. As the trade rule is exogenous, consider date 1 and bidder 1 with value t. He wins with bid b only if $\mathcal{T}_2 < \pi^1(b)$. Trade succeeds only if $(1 - \alpha_R)t \leq \lambda_2 \mathcal{T}_2$ and $(1 - \alpha_B)\mathcal{T}_2 \geq \lambda_2 \mathcal{T}_2$. The first condition says that the *resale value* of the winner is less than the resale price and the second condition says that the resale value of the loser must be more than the resale price. The two conditions together imply $\mathcal{T}_2 \geq k_1 t$. Otherwise, trade does not succeed. Therefore, with probability that $\mathcal{T}_2 \leq k_1 t$, trade does not happen and bidder 1 keeps the object which gives him a utility of $t - b$, and with probability that $k_1 t < \mathcal{T}_2 < \pi^1(b)$, trade happens which gives him a utility of $\alpha_R t + \lambda_2 \mathcal{T}_2 - b$.

On the other hand, bidder 1 loses only if $\mathcal{T}_2 > \pi^1(b)$. In this case, trade succeeds only if $(1 - \alpha_B)t \geq \lambda_2 t$ and $(1 - \alpha_R)\mathcal{T}_2 \leq \lambda_2 t$. These imply $\mathcal{T}_2 \leq t/k_1$. Therefore, with probability that $\pi^1(b) < \mathcal{T}_2 < t/k_1$, trade happens thereby giving bidder 1 a utility of $(1 - \alpha_B - \lambda_2)t$.

Thus, the expected utility function of bidder 1 is

$$\begin{aligned} U^1(t, b) = &\Pr(\mathcal{T}_2 < k_1 t)(t - b) + \Pr(k_1 t < \mathcal{T}_2 < \pi^1(b))[\alpha_R t + \lambda_2 \mathcal{T}_2 - b] \\ &+ \Pr(\pi^1(b) < \mathcal{T}_2 < t/k_2)(1 - \alpha_B - \lambda_2)t \end{aligned}$$

which can be rewritten as (2).

We show 2. As the trade rule is exogenous, consider date 1 and bidder 1 with value t. He wins with bid b only if $\mathcal{T}_2 < \pi^1(b)$. Trade succeeds only if $(1-\alpha_R)t \leq \lambda_1 t$ and $(1-\alpha_B)\mathcal{T}_2 \geq \lambda_1 t$. These imply $\mathcal{T}_2 \geq k_2 t$. Otherwise, trade does not succeed. Therefore, with probability that $\mathcal{T}_2 \leq k_2 t$, trade does not happen and bidder 1 keeps the object which gives him a utility of $t-b$, and with probability that $k_2 t < \mathcal{T}_2 < \pi^1(b)$, trade happens which gives him a utility of $(\alpha_R + \lambda_1)t - b$.

On the other hand, bidder 1 loses only if $\mathcal{T}_2 > \pi^1(b)$. In this case, trade succeeds only if $(1 - \alpha_B)t \geq \lambda_1 \mathcal{T}_2$ and $(1 - \alpha_R)\mathcal{T}_2 \leq \lambda_1 \mathcal{T}_2$. These imply $\mathcal{T}_2 \leq t/k_2$. Therefore, with probability that $\pi^1(b) < \mathcal{T}_2 < t/k_2$, trade happens thereby giving bidder 1 a utility of $(1 - \alpha_B)t - \lambda_1 \mathcal{T}_2$.

Thus, the expected utility function of bidder 1 is

$$\begin{aligned} U^1(t, b) = &\Pr(\mathcal{T}_2 < k_2 t)(t - b) + \Pr(k_2 t < \mathcal{T}_2 < \pi^1(b))[(\alpha_R + \lambda_1)t - b] \\ &+ \Pr(\pi^1(b) < \mathcal{T}_2 < t/k_2)[(1 - \alpha_B)t - \lambda_1 \mathcal{T}_2] \end{aligned}$$

which can be rewritten as (3).

We show 3. As the trade rule is exogenous, consider date 1 and bidder 1 with value t. He wins with bid b only if $\mathcal{T}_2 < \pi^1(b)$. Trade succeeds only if $(1 - \alpha_R)t \leq p^1(t, \mathcal{T}_2) = \lambda_1 t + \lambda_2 \mathcal{T}_2$ and $(1 - \alpha_B)\mathcal{T}_2 \geq p^1(t, \mathcal{T}_2)$. These imply $\mathcal{T}_2 \geq k_3 t$. Otherwise, trade does not succeed. Therefore, with probability that $\mathcal{T}_2 < k_3 t$, trade does not happen and bidder 1 keeps the object which incurs him a utility of $t - b$, and with probability that $k_3 t < \mathcal{T}_2 < \pi^1(b)$, trade happens and bidder 1 gets a utility of $\alpha_R t + p^1 - b$.

On the other hand, bidder 1 loses only if $\mathcal{T}_2 > \pi^1(b)$. In this case, trade succeeds only if $(1 - \alpha_B)t \geq p^1(\mathcal{T}_2, t) = \lambda_1 \mathcal{T}_2 + \lambda_2 t$ and $(1 -$

$\alpha_R)\mathcal{T}_2 \le p^1(\mathcal{T}_2, t)$. These imply $\mathcal{T}_2 > t/k_3$. Therefore, with probability that $\pi^1(b) < \mathcal{T}_2 < t/k_3$, trade happens thereby giving bidder 1 a utility of $(1-\alpha_B)t - p^1$.

Thus, the expected utility function of bidder 1 is

$$
\begin{aligned}
U^1(t,b) = \Pr(\mathcal{T}_2 < k_3 t)(t-b) + \Pr(k_3 t < \mathcal{T}_2 < \pi^1(b))[(\alpha_R + \lambda_1)t \\
+ \lambda_2 \mathcal{T}_2 - b] + \Pr(\pi^1(b) < \mathcal{T}_2 < t/k_3)[(1-\alpha_B-\lambda_2)t - \lambda_1 \mathcal{T}_2]
\end{aligned}
$$

which can be rewritten as (4). ■

Proof of Theorem 2. We first show for the case when part 3 of Assumption 2 holds. In this case, $p^1(w,l) = \lambda_1 w + \lambda_2 l$. Suppose (π^1, p^1) is a symmetric perfect Bayesian equilibrium. We can write (4) as

$$
\begin{aligned}
U^1(t,b) = F(k_3 t)(t-b) + \int_{k_3 t}^{\pi^1(b)} [\alpha_R t + p^1(t,\omega) - b] f(\omega) \mathrm{d}\omega \\
+ \int_{\pi^1(b)}^{t/k} [(1-\alpha_B)t - p^1(\omega, t)] f(\omega) \mathrm{d}\omega
\end{aligned}
\tag{20}
$$

Applying Leibniz integral rule, the first-order derivative of (20) is

$$
\begin{aligned}
\mathrm{D}_b U^1(t,b) &= -F(k_3 t) + [\alpha_R t + p^1(t, \pi^1(b)) - b]\mathrm{D}F \circ \pi^1(b) - F \circ \pi^1(b) \\
&\quad + F(k_3 t) - [(1-\alpha_B)t - p^1(\pi^1(b), b)]\mathrm{D}F \circ \pi^1(b) \\
&= [(\alpha_R + \alpha_B - 1)t + p^1(t, \pi^1(b)) - b + p^1(\pi^1(b), t)] \\
&\quad \mathrm{D}F \circ \pi^1(b) - F \circ \pi^1(b)
\end{aligned}
$$

In equilibrium, $t = \pi^1(b)$ and $\mathrm{D}_b U^1(\pi^1(b), b) = 0$. This gives

$$
\frac{F \circ \pi^1(b)}{\mathrm{D}F \circ \pi^1(b)} = (\alpha_R + \alpha_B - 1)\pi^1(b) + 2p^1(b) - b \tag{21}
$$

Conversely, suppose (π^1, p^1) solves (5). We show that (π^1, p^1) is an equilibrium. Suppose bidder 1 with value t and bid b overbids to c where $\pi^1(c) > t$. Then, the derivative of $U^1(t,b)$ implies

$$
\begin{aligned}
\mathrm{D}_c U^1(t,c) &= [(\alpha_R + \alpha_B - 1)t + p^1(t, \pi^1(c)) - c + p^1(\pi^1(c), t)] \\
&\quad \mathrm{D}F \circ \pi^1(c) - F \circ \pi^1(c) \\
&= [(\lambda_1 + \lambda_2 + \alpha_R + \alpha_B - 1)t + (\lambda_1 + \lambda_2)\pi^1(c) - c] \\
&\quad \mathrm{D}F \circ \pi^1(c) - F \circ \pi^1(c) \\
&< [E(\alpha, \lambda)\pi^1(c) - c]\mathrm{D}F \circ \pi^1(c) \\
&\quad - F \circ \pi^1(c) \\
&= 0
\end{aligned}
$$

Therefore, overbids are not profitable. On similar lines, it can be shown that underbids are also not profitable.

We show for the case when part 1 of Assumption 2 holds. Suppose (π^1, p^1) is a symmetric perfect Bayesian equilibrium. Applying Leibniz integral rule, the first-order derivative of (2) is

$$\begin{aligned} \mathrm{D}U^1(t,b) = \mathrm{D}F \circ \pi^1(b)[\alpha_R t + \lambda_2 \pi^1(b) - b] - F \circ \pi^1(b) \\ - \mathrm{D}F \circ \pi^1(b)(1 - \alpha_B - \lambda_2)t \end{aligned}$$

Using $t = \pi^1(b)$ and $\mathrm{D}_b U^1(\pi^1(b), b) = 0$, we arrive at (21) with $\lambda_1 = 0$. On similar lines of part 3, we can show the converse.

We show for the case when part 2 of Assumption 2 holds. Suppose (π^1, p^1) is a symmetric perfect Bayesian equilibrium. Applying Leibniz integral rule, the first-order derivative of (3) is

$$\begin{aligned} \mathrm{D}U^1(t,b) = \mathrm{D}F \circ \pi^1(b)[(\alpha_R + \lambda_1)t - b] - F \circ \pi^1(b) \\ - \mathrm{D}F \circ \pi^1(b)[(1 - \alpha_B)t - \lambda_1 \pi^1(b)] \end{aligned}$$

Using $t = \pi^1(b)$ and $\mathrm{D}_b U^1(\pi^1(b), b) = 0$, we arrive at (21). On similar lines of part 3, we can show the converse. ■

Proof of Proposition 1. As $p^1(b) = (\lambda_1 + \lambda_2)\pi^1(b)$, from (5), we have

$$\mathrm{D}\pi^1(b) = \frac{F \circ \pi^1(b)}{f \circ \pi^1(b)} \frac{1}{E(\alpha, \lambda)\pi^1(b) - b}$$

As $b = \beta^1 \circ \pi^1(b)$ implies $1 = \mathrm{D}\beta^1 \circ \pi^1(b)\mathrm{D}\pi^1(b)$, we have

$$\frac{1}{\mathrm{D}\beta^1 \circ \pi^1(b)} = \frac{F \circ \pi^1(b)}{f \circ \pi^1(b)} \frac{1}{E(\alpha, \lambda)\pi^1(b) - b}$$

Using $t = \pi^1(b)$, we have

$$\frac{1}{\mathrm{D}\beta^1(t)} = \frac{F(t)}{f(t)} \frac{1}{E(\alpha, \lambda)t - \beta^1(t)}$$

This implies

$$E(\alpha, \lambda)tf(t) = \mathrm{D}[F(t)\beta^1(t)]$$

Using the fundamental theorem of calculus, we have

$$\beta^1(t) = \frac{E(\alpha, \lambda)}{F(t)} \int_0^t \omega f(\omega) \mathrm{d}\omega$$

■

Proof of Proposition 2. From Riley and Samuelson [12], the symmetric bid function is

$$\beta^*(t) = \frac{1}{F(t)} \int_0^t \omega f(\omega) \mathrm{d}\omega$$

Comparing this with (6), we have

$$\beta^1(t) > \beta^*(t)$$

for every $t \in (0, \bar{a}]$. ■

Proof of Proposition 3. To contradict, let $\pi^1 \geq \gamma_s$ around a neighborhood of 0. Then, from H-K, we have

$$\begin{aligned}
\frac{F_s \circ \gamma_s(b)}{\mathrm{D}F_s \circ \gamma_s(b)} &= r(b) - b < \gamma_s(b) - b \leq \pi^1(b) - b \\
&< E(\alpha, \lambda)\pi^1(b) - b \\
&= \frac{F_s \circ \pi^1(b)}{\mathrm{D}F_s \circ \pi^1(b)}
\end{aligned}$$

This implies

$$\mathrm{D}\left[\frac{F_s \circ \gamma_s(b)}{F_s \circ \pi^1(b)}\right] > 0$$

As $F_s(0) > 0$ and $\pi^1(0) = \gamma_s(0) = 0$, we have $\gamma_s > \pi^1$ around a neighborhood of 0 which is a contradiction.

Now, suppose that there exists $b^* > 0$ so that $\gamma_s(b^*) = \pi^1(b^*)$ and $\gamma_s(b) > \pi^1(b)$ for every $b \in (0, b^*]$. Then, we have

$$\begin{aligned}
\mathrm{D}\gamma_s(b^*) &= \frac{F_s \circ \gamma_s(b^*)}{f_s \circ \gamma_s(b^*)} \frac{1}{r(b^*) - b^*} \\
&> \frac{F_s \circ \gamma_s(b^*)}{f_s \circ \gamma_s(b^*)} \frac{1}{\gamma_s(b^*) - b^*} \\
&= \frac{F_s \circ \pi^1(b^*)}{f_s \circ \pi^1(b^*)} \frac{1}{\pi^1(b^*) - b^*} \\
&> \frac{F_s \circ \pi^1(b^*)}{f_s \circ \pi^1(b^*)} \frac{1}{E(\alpha, \lambda)\pi^1(b^*) - b^*} \\
&= \mathrm{D}\pi^1(b^*)
\end{aligned}$$

Thus, there exist $\delta > 0$ so that $\gamma_s(b^* - \delta) < \pi^1(b^* - \delta)$, which is a contradiction. ■

Proof of Proposition 4. To contradict, let $\pi^1 \leq \gamma_w$ around a neighborhood of 0. Then, from H-K, we have

$$\begin{aligned}\frac{F_w \circ \gamma_w(b)}{\mathrm{D}F_w \circ \gamma_w(b)} &= r(b) - b > \gamma_w(b) - b \geq \pi^1(b) - b \\ &> E(\alpha,\lambda)\pi^1(b) - b \\ &= \frac{F_w \circ \pi^1(b)}{\mathrm{D}F_w \circ \pi^1(b)}\end{aligned}$$

This implies

$$\mathrm{D}\left[\frac{F_w \circ \pi^1(b)}{F_w \circ \gamma_w(b)}\right] > 0$$

As $F_w(0) > 0$ and $\pi^1(0) = \gamma_w(0) = 0$, we have $\gamma_w < \pi^1$ around a neighborhood of 0 which is a contradiction.

Now, suppose that there exists $b^* > 0$ so that $\gamma_w(b^*) = \pi^1(b^*)$ and $\gamma_w(b) < \pi^1(b)$ for every $b \in (0, b^*]$. Then, we have

$$\begin{aligned}\mathrm{D}\gamma_w(b^*) &= \frac{F_w \circ \gamma_w(b^*)}{f_w \circ \gamma_w(b^*)} \frac{1}{r(b^*) - b^*} \\ &< \frac{F_w \circ \gamma_w(b^*)}{f_w \circ \gamma_w(b^*)} \frac{1}{\gamma_w(b^*) - b^*} \\ &= \frac{F_w \circ \pi^1(b^*)}{f_w \circ \pi^1(b^*)} \frac{1}{\pi^1(b^*) - b^*} \\ &< \frac{F_w \circ \pi^1(b^*)}{f_w \circ \pi^1(b^*)} \frac{1}{E(\alpha,\lambda)\pi^1(b^*) - b^*} \\ &= \mathrm{D}\pi^1(b^*)\end{aligned}$$

Thus, there exist $\delta > 0$ so that $\gamma_w(b^* - \delta) > \pi^1(b^* - \delta)$, which is a contradiction. ■

Proof of Proposition 5. To contradict, let $\pi^1 \geq \psi_s$ around a neighborhood of 0. Then, from M-R, we have

$$\begin{aligned}\frac{F_s \circ \psi_s(b)}{\mathrm{D}F_s \circ \psi_s(b)} &= \psi_w(b) - b < \psi_s(b) - b \leq \pi^1(b) - b \\ &< E(\alpha,\lambda)\pi^1(b) - b \\ &= \frac{F_s \circ \pi^1(b)}{\mathrm{D}F_s \circ \pi^1(b)}\end{aligned}$$

This implies

$$\mathrm{D}\left[\frac{F_s \circ \psi_s(b)}{F_s \circ \pi^1(b)}\right] > 0$$

As $F_s(0) > 0$ and $\pi^1(0) = \psi_s(0) = 0$, we have $\psi_s > \pi^1$ around a neighborhood of 0 which is a contradiction.

Now, suppose that there exists $b^* > 0$ so that $\psi_s(b^*) = \pi^1(b^*)$ and $\psi_s(b) > \pi^1(b)$ for every $b \in (0, b^*]$. Then, we have

$$\begin{aligned}
\mathrm{D}\psi_s(b^*) &= \frac{F_s \circ \psi_s(b^*)}{f_s \circ \psi_s(b^*)} \frac{1}{\psi_w(b^*) - b^*} \\
&> \frac{F_s \circ \psi_s(b^*)}{f_s \circ \psi_s(b^*)} \frac{1}{\psi_s(b^*) - b^*} \\
&= \frac{F_s \circ \pi^1(b^*)}{f_s \circ \pi^1(b^*)} \frac{1}{\pi^1(b^*) - b^*} \\
&> \frac{F_s \circ \pi^1(b^*)}{f_s \circ \pi^1(b^*)} \frac{1}{E(\alpha, \lambda)\pi^1(b^*) - b^*} \\
&= \mathrm{D}\pi^1(b^*)
\end{aligned}$$

Thus, there exist $\delta > 0$ so that $\psi_s(b^* - \delta) < \pi^1(b^* - \delta)$, which is a contradiction. ■

Proof of Proposition 6. To contradict, let $\pi^1 \leq \psi_w$ around a neighborhood of 0. Then, from M-R, we have

$$\begin{aligned}
\frac{F_w \circ \psi_w(b)}{\mathrm{D}F_w \circ \psi_w(b)} &= \psi_s(b) - b > \psi_w(b) - b \geq \pi^1(b) - b \\
&> E(\alpha, \lambda)\pi^1(b) - b \\
&= \frac{F_w \circ \pi^1(b)}{\mathrm{D}F_w \circ \pi^1(b)}
\end{aligned}$$

This implies

$$\mathrm{D}\left[\frac{F_w \circ \pi^1(b)}{F_w \circ \psi_w(b)}\right] > 0$$

As $F_w(0) > 0$ and $\pi^1(0) = \psi_w(0) = 0$, we have $\psi_w < \pi^1$ around a neighborhood of 0 which is a contradiction.

Now, suppose that there exists $b^* > 0$ so that $\psi_w(b^*) = \pi^1(b^*)$ and $\psi_w(b) < \pi^1(b)$ for every $b \in (0, b^*]$. Then, we have

$$\begin{aligned}
\mathrm{D}\psi_w(b^*) &= \frac{F_w \circ \psi_w(b^*)}{f_w \circ \psi_w(b^*)} \frac{1}{\psi_s(b^*) - b^*} \\
&< \frac{F_w \circ \psi_w(b^*)}{f_w \circ \psi_w(b^*)} \frac{1}{\psi_w(b^*) - b^*} \\
&= \frac{F_w \circ \pi^1(b^*)}{f_w \circ \pi^1(b^*)} \frac{1}{\pi^1(b^*) - b^*} \\
&< \frac{F_w \circ \pi^1(b^*)}{f_w \circ \pi^1(b^*)} \frac{1}{E(\alpha, \lambda)\pi^1(b^*) - b^*} \\
&= \mathrm{D}\pi^1(b^*)
\end{aligned}$$

Thus, there exist $\delta > 0$ so that $\psi_w(b^* - \delta) > \pi^1(b^* - \delta)$, which is a contradiction. ■

Proof of Theorem 4. We first show for the case when part 3 of Assumption 1 holds. In this case, $p^2(w,l) = \lambda_1 w + \lambda_2 l$. Suppose (π^2, p^2) is an equilibrium. We can rewrite (9) as

$$\begin{aligned} U^2(t,b) = \int_0^{k_3 t} [t - \beta^2(\omega)] f(\omega) \mathrm{d}\omega + \int_{k_3 t}^{\pi^2(b)} [\alpha_R t + p^2(t,\omega) \\ - \beta^2(\omega)] f(\omega)\mathrm{d}\omega + \int_{\pi^2(b)}^{t/k_3} [(1-\alpha_B)t - p^2(\omega,t)] f(\omega)\mathrm{d}\omega \end{aligned} \tag{22}$$

Applying Leibniz integral rule while differentiating (22), we have

$$\mathrm{D}_b U^1(t,b) = \mathrm{D}F \circ \pi^2(b)[(\alpha_R + \alpha_B - 1)t + p^2(t, \pi^2(b)) - b + p^2(\pi^2(b), t)] \tag{23}$$

In equilibrium, $\pi^2(b) = t$ and $\mathrm{D}_b(\pi^2(b), b) = 0$. As $\mathrm{D}\pi^2(b), f \circ \pi^2(b) > 0$, we have

$$\pi^2(b) = \frac{b - 2p^2(b)}{\alpha_R + \alpha_B - 1}$$

As $p^2(b) = (\lambda_1 + \lambda_2)\pi^2(b)$, we have

$$\pi^2(b) = \frac{b}{E(\alpha, \lambda)} \tag{24}$$

We show the converse. Suppose (π^2, q) solve (10). Consider bidder 1 with value t and bid b. Suppose he underbids to c such that $t > \pi^2(c)$. Then, $p^2(t, \pi^2(c)) > p^2(c)$ and $p^2(\pi^2(c), t) > p^2(c)$ and from (23), we have

$$\begin{aligned} \mathrm{D}_c U^1(t,c) &= \mathrm{D}F \circ \pi^2(c)[(\alpha_R + \alpha_B - 1)t + p^2(t, \pi^2(c)) - c + p^2(\pi^2(c), t)] \\ &> \mathrm{D}F \circ \pi^2(c)[(\alpha_R + \alpha_B - 1)\pi^2(c) + 2p^2(c) - c] \\ &= 0 \end{aligned}$$

Therefore, underbids are not profitable. Similarly, it can be shown that overbids are also not profitable.

We show for the case when part 1 of Assumption 2 holds. Suppose (π^1, p^1) is a symmetric perfect Bayesian equilibrium. Applying Leibniz integral rule, the first-order derivative of (7) is

$$\mathrm{D}_b U^1(t,b) = \mathrm{D}F \circ \pi^2(b)[(\alpha_R + \alpha_B + \lambda_2 - 1)t + \lambda_2 \pi^2(b) - b]$$

Using $t = \pi^1(b)$ and $\mathrm{D}_b U^1(\pi^1(b), b) = 0$, we arrive at (21) with $\lambda_1 = 0$. On similar lines of part 3, we can show the converse.

We show for the case when part 2 of Assumption 2 holds. Suppose (π^1, p^1) is a symmetric perfect Bayesian equilibrium. Applying Leibniz integral rule, the first-order derivative of (24) is

$$\mathrm{D}_b U^1(t, b) = \mathrm{D}F \circ \pi^2(b)[(\alpha_R + \alpha_B + \lambda_1 - 1)t + \lambda_1 \pi^2(b) - b]$$

Using $t = \pi^1(b)$ and $\mathrm{D}_b U^1(\pi^1(b), b) = 0$, we arrive at (24). On similar lines of part 3, we can show the converse. ■

Proof of Proposition 8. Pick an arbitrary $t > 0$. As $\int_0^t \omega f(\omega)\mathrm{d}\omega < \int_0^t t f(\omega)\mathrm{d}\omega = tF(t)$, we have

$$\frac{1}{F(t)} \int_0^t \omega f(\omega)\mathrm{d}\omega < t$$

which is equivalent to

$$\frac{E(\alpha, \lambda)}{F(t)} \int_0^t \omega f(\omega)\mathrm{d}\omega < E(\alpha, \lambda)t$$

Thus, $\beta^1(t) < \beta^2(t)$. ■

Proof of Theorem 5. Consider the first-price auction and bidder 1 with value t. The interim payments generated from him are

$$\begin{aligned} P^1(t) &= \beta^1(t)F(t) \\ &= E(\alpha, \lambda) \int_0^t \omega f(\omega)\mathrm{d}\omega \end{aligned}$$

The *ex-ante* expected revenues generated from bidder 1 are

$$\begin{aligned} \mathbb{E}[P^1] &= \int_0^{\bar{a}} P_1^1(t) f(t)\mathrm{d}t \\ &= E(\alpha, \lambda) \int_0^{\bar{a}} \int_0^t \omega f(\omega) f(t)\mathrm{d}\omega\mathrm{d}t \end{aligned}$$

where $\mathbb{E}$ is the expectation operator. Using Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}[P^1] &= E(\alpha, \lambda) \int_0^{\bar{a}} \int_t^{\bar{a}} t f(t) f(\omega)\mathrm{d}\omega\mathrm{d}t \\ &= E(\alpha, \lambda) \int_0^{\bar{a}} t f(t)[1 - F(t)]\mathrm{d}t \end{aligned}$$

Therefore, the *ex-ante* expected revenues are

$$R^1 = 2E(\alpha, \lambda) \int_0^{\bar{a}} tf(t)[1 - F(t)]\mathrm{d}t$$

Now, consider the second-price auction and bidder 1 with value t. The interim payments generated from him are

$$\begin{aligned} P^2(t) &= \int_0^t \beta^2(\omega) f(\omega)\mathrm{d}\omega \\ &= E(\alpha, \lambda) \int_0^t \omega f(\omega)\mathrm{d}\omega \\ &= P^1(t) \end{aligned}$$

Therefore, the *ex-ante* expected revenues are

$$R^2 = 2E(\alpha, \lambda) \int_0^{\bar{a}} tf(t)[1 - F(t)]\mathrm{d}t$$

■

Proof of Lemma 1. Since bidders are symmetric, without loss of generality, consider bidder 1 with value t_1. Suppose he wins with a bid of b. Since bid functions are symmetric, it must be the case that $t_1 > t_2$, which is equivalent to $(1 - \alpha_R)t_1 > (1 - \alpha_R)t_2$. From part 2 of Assumption 3, we have $(1-\alpha_B)t_2 > (1-\alpha_R)t_2$. Therefore, with positive probability, we have $(1-\alpha_R)t_1 < (1-\alpha_B)t_2$. Note that $(1-\alpha_R)t_1$ is the resale value of bidder 1 (reseller) at the resale date while $(1 - \alpha_B)t_2$ is the resale value of bidder 2 (buyer) at the resale date. Thus, there are expected potential profits if bidder 1 offers the object to bidder 2 at the resale date. ■

Proof of Theorem 6. We show sufficiency. Suppose a pair (σ^1, q^1) solves (14). We argue that (σ^1, q^1) is an equilibrium. Consider bidder 1 with value t. Suppose he overbids to c, where $\sigma^1(c) > t$. Note that $\max\{(1 - \alpha_B)t - q^1(t, \sigma^1(c)), 0\} \geq (1 - \alpha_B)t - q^1(t, \sigma^1(c))$ and $q^1(\sigma^1(c), \sigma^1(c)) > q^1(t, \sigma^1(c))$. Then,

$$\begin{aligned} \mathrm{D}_c U^1(t, c) &= \mathrm{D}F \circ \sigma^1(c)[q^1(t, \sigma^1(c)) + \alpha_R t - c \\ &\quad - \max\{(1 - \alpha_B)t - q^1(t, \sigma^1(c)), 0\}] - F \circ \sigma^1(c) \\ &\leq \mathrm{D}F \circ \sigma^1(c)[2q^1(t, \sigma^1(c)) + (\alpha_R + \alpha_B - 1)t - c] - F \circ \sigma^1(c) \\ &< \mathrm{D}F \circ \sigma^1(c)[2q^1(\sigma^1(c), \sigma^1(c)) + (\alpha_R + \alpha_B - 1)\sigma^1(c) - c] \\ &\quad - F \circ \sigma^1(c) \\ &= 0 \end{aligned}$$

Thus, overbids are not profitable for bidder 1.

Suppose he underbids to c, where $\sigma^1(c) < t$. As $q^1(\sigma^1(c), \sigma^1(c)) < q^1(t, \sigma^1(c))$, we have

$$\begin{aligned}
\mathrm{D}_c U^1(t, c) = {} & \mathrm{D}F \circ \sigma^1(c)[q^1(t, \sigma^1(c)) + \alpha_R t - c \\
& - \max\{(1-\alpha_B)t - q^1(t, \sigma^1(c)), 0\}] - F \circ \sigma^1(c) \\
> {} & \mathrm{D}F \circ \sigma^1(c)[q^1(c) + \alpha_R t - c \\
& - \max\{(1-\alpha_B)t - q^1(c), 0\}] - F \circ \sigma^1(c)
\end{aligned}$$

As $q^1(c) < (1-\alpha_B)\sigma^1(c)$ and $(1-\alpha_B)\sigma^1(c) < (1-\alpha_B)t$, we have $q^1(c) < (1-\alpha_B)t$. This implies

$$\begin{aligned}
\mathrm{D}_c U^1(t, c) > {} & \mathrm{D}F \circ \sigma^1(c)[q^1(c) + \alpha_R t - c \\
& - \max\{(1-\alpha_B)t - q^1(c), 0\}] - F \circ \sigma^1(c) \\
= {} & \mathrm{D}F \circ \sigma^1(c)[2q^1(\sigma^1(c), \sigma^1(c)) + (\alpha_R + \alpha_B - 1)\sigma^1(c) - c] \\
& - F \circ \sigma^1(c) \\
= {} & 0
\end{aligned}$$

Thus, underbids are not profitable for bidder 1. ■

Proof of Lemma 2. Without loss of generality, consider bidder 1 with value t_1. Suppose he loses with a bid of b. Then, $t_1 < t_2$ which is equivalent to $(1-\alpha_B)t_1 < (1-\alpha_B)t_2$. From part 2 of Assumption 3, $(1-\alpha_B)t_2 > (1-\alpha_R)t_2$. This implies, with a positive probability, we have $(1-\alpha_B)t_1 > (1-\alpha_R)t_2$, where $(1-\alpha_B)t_1$ is the resale value of bidder 1 at date 2 while $(1-\alpha_R)t_2$ is the resale value of bidder 2 at date 2. Therefore, there are expected potential gains from trade. ■

Proof of Theorem 7. Consider a pair (σ^2, q^2) that solves (18). We show (σ^2, q^2) is an equilibrium. Consider bidder 1 with value t. Suppose he overbids to c, where $\sigma^2(c) > t$. Then, $q^2(\sigma^2(c), \sigma^2(c)) > q^2(t, \sigma^2(c))$. As $q^2(\sigma^2(c), \sigma^2(c)) > (1-\alpha_R)\sigma^2(c) > (1-\alpha_R)t$, we have $q^2(\sigma^2(c), \sigma^2(c)) + \alpha_R t > t$. Thus,

$$\begin{aligned}
\mathrm{D}_c U^2(t, c) = {} & \mathrm{D}F \circ \sigma^2(c)[\max\{q^2(t, \sigma^2(c)) + \alpha_R t, t\} - c - (1-\alpha_B)t \\
& + q^2(t, \sigma^2(c))] \\
< {} & \mathrm{D}F \circ \sigma^2(c)[\max\{q^2(c) + \alpha_R t, t\} - c - (1-\alpha_B)t + q^2(c)] \\
= {} & \mathrm{D}F \circ \sigma^2(c)[q^2(c) + \alpha_R t - c - (1-\alpha_B)t + q^2(c)] \\
< {} & \mathrm{D}F \circ \sigma^2(c)[2q^2(\sigma^2(c), \sigma^2(c)) + (\alpha_R + \alpha_B - 1)\sigma^2(c) - c] \\
= {} & 0
\end{aligned}$$

Therefore, overbids are not profitable for bidder 1.

Now, suppose bidder 1 underbids to c, where $\sigma^2(c) < t$. Then $q^2(\sigma^2(c), \sigma^2(c)) < q^2(t, \sigma^2(c))$. As

$$\max\{q^2(t, \sigma^2(c)) + \alpha_R t, t\} \geq q^2(t, \sigma^2(c)) + \alpha_R t,$$

we have

$$\begin{aligned}
\mathrm{D}_c U^2(t,c) &= \mathrm{D}F \circ \sigma^2(c)[\max\{q^2(t,\sigma^2(c)) + \alpha_R t, t\} - c - (1-\alpha_B)t \\
&\quad + q^2(t,\sigma^2(c))] \\
&\geq \mathrm{D}F \circ \sigma^2(c)[2q^2(t,\sigma^2(c)) + (\alpha_R + \alpha_B - 1)t - c] \\
&> \mathrm{D}F \circ \sigma^2(c)[2q^2(c) + (\alpha_R + \alpha_B - 1)\sigma^2(c) - c] \\
&= 0
\end{aligned}$$

Thus, underbids are not profitable for bidder 1. Hence, (σ^2, q^2) is an equilibrium. ■

Proof of Theorem 8. To contradict, let $\pi^1 \geq \sigma^1$ around a neighborhood of 0. Then, we have

$$\begin{aligned}
\frac{F \circ \sigma^1(b)}{\mathrm{D}F \circ \sigma^1(b)} &= 2q^1(b) + (\alpha_R + \alpha_B - 1)\sigma^1(b) - b \\
&< 2(1-\alpha_B)\sigma^1(b) + (\alpha_R + \alpha_B - 1)\sigma^1(b) - b \\
&= (1 + \alpha_R - \alpha_B)\sigma^1(b) - b \\
&\leq (1 + \alpha_R - \alpha_B)\pi^1(b) - b \\
&= \frac{F \circ \pi^1(b)}{\mathrm{D}F \circ \pi^1(b)}
\end{aligned}$$

This implies

$$\mathrm{D}\left[\frac{F \circ \sigma^1(b)}{F \circ \pi^1(b)}\right] > 0$$

As $F(0) > 0$ and $\pi^1(0) = \sigma^1(0) = 0$, we have $\sigma^1 > \pi^1$ around a neighborhood of 0 which is a contradiction.

Now, suppose that there exists $b^* > 0$ so that $\sigma^1(b^*) = \pi^1(b^*)$ and $\sigma^1(b) > \pi^1(b)$ for every $b \in (0, b^*]$. Then, we have

$$\begin{aligned}
\mathrm{D}\sigma^1(b^*) &= \frac{F \circ \sigma^1(b^*)}{f \circ \sigma^1(b^*)} \frac{1}{2q^1(b^*) + (\alpha_R + \alpha_B - 1)\sigma^1(b^*) - b^*} \\
&> \frac{F \circ \sigma^1(b^*)}{f \circ \sigma^1(b^*)} \frac{1}{2(1-\alpha_B)\sigma^1(b^*) + (\alpha_R + \alpha_B - 1)\sigma^1(b^*) - b^*} \\
&= \frac{F \circ \sigma^1(b^*)}{f \circ \sigma^1(b^*)} \frac{1}{(1 + \alpha_R - \alpha_B)\sigma^1(b^*) - b^*} \\
&= \frac{F \circ \pi^1(b^*)}{f \circ \pi^1(b^*)} \frac{1}{(1 + \alpha_R - \alpha_B)\pi^1(b^*) - b^*} \\
&= \mathrm{D}\pi^1(b^*)
\end{aligned}$$

Thus, there exist $\delta > 0$ so that $\sigma^1(b^* - \delta) < \pi^1(b^* - \delta)$, which is a contradiction. ■

B Appendix: Technical lemmas

Lemma B.1. *Let Assumption 3 be true. The expression*

$$\frac{f(xq)}{F(a) - F(xq)}$$

is non-decreasing in q for every $x \in \Re_+$ and $a \in (0, \bar{a})$.

Proof. Pick $q_1, q_2 \in \Re_+$ so that $q_1 > q_2$. We show

$$f(xq_1)[F(a) - F(xq_2)] - f(xq_2)[F(a) - F(xq_1)] \geq 0 \tag{25}$$

From part 1 of Assumption 3, we have $f(xq_1)[1 - F(xq_2)] \geq f(xq_2)[1 - F(xq_1)]$ as $x > 0$.

If $f(xq_1) > f(xq_2)$, the the result follows as $F(xq_1) > F(xq_2)$. If $f(xq_1) \leq f(xq_2)$, the derivative of left hand side of (25) with respect to $F(a)$ is $f(xq_1) - f(xq_2) \leq 0$. Thus, the result holds. ■

Lemma B.2. *Let Assumption 3 be true. Then, (12) and (15) are sufficient.*

Proof. We show that (15) is sufficient. On similar lines, one can show that (12) is sufficient. The first-order derivative of (15) gives

$$\begin{aligned}
\mathrm{D}_{q^2} U^1(t, b, q^2) &= \frac{1}{1 - F \circ \sigma^2(b)} \Big\{ yf(yq^2)[(1 - \alpha_B)t - q^2] + \\
&\quad [F \circ \sigma^2(b) - F(yq^2)] \Big\} \\
&= \frac{yf(yq^2)}{1 - F \circ \sigma^2(b)} \Big\{ [(1 - \alpha_B)t - q^2] + \frac{F \circ \sigma^2(b) - F(yq^2)}{yf(yq^2)} \Big\}
\end{aligned}$$

The second-order derivative is

$$\begin{aligned}
\mathrm{D}^2_{q^2} U^1(t, b, q^2) &= \frac{yf(yq^2)}{1 - F \circ \sigma^2(b)} \Big\{ -1 + \mathrm{D}_{q^2} \frac{F \circ \sigma^2(b) - F(yq^2)}{yf(yq^2)} \Big\} \\
&+ \frac{y^2 \mathrm{D} f(yq^2)}{1 - F \circ \sigma^2(b)} \Big\{ [(1 - \alpha_B)t - q^2] + \frac{F \circ \sigma^2(b) - F(yq^2)}{yf(yq^2)} \Big\} \\
&= \frac{yf(yq^2)}{1 - F \circ \sigma^2(b)} \Big\{ -1 + \mathrm{D}_{q^2} \frac{F \circ \sigma^2(b) - F(yq^2)}{yf(yq^2)} \Big\} \\
&< 0
\end{aligned}$$

where the last inequality follows from Lemma B.1. ■

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