Group Dominant Networks and Convexity

Christophe Bravard[1], Sudipta Sarangi[2], Hector Tzavellas[2]

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Abstract

This paper investigates the role of convexity of players' payoff functions with respect to their own links on pairwise equilibrium networks (Pwen). We establish that convexity guarantees the existence of Pwen. Next, we extend the work of Goyal and Joshi (2006), who have established that Pwen are always group dominant networks. We show that it is possibly difficult to select Pwen from among the group dominant networks. More precisely, the set of Pwen may contain 'holes'. When payoff functions have strategic complementarity, these holes are eliminated, whereas if these functions have strategic substitutability, these holes may appear. We provide conditions that eliminate the possibility of these holes appearing, thus simplifying the characterization of the Pwen set. The required conditions induce that the function describing the incentive for players in the dominant group to deviate is quasi-concave, and that the function describing the incentive for isolated players to deviate is quasi-convex. Finally, these conditions allow us to establish simple Pwen uniqueness conditions.

Preliminary Version, Do not circulate

[1] *Université de Grenoble-Alpes, GAEL, France.*

[2] *Department of Economics, Virginia Tech Blacksburg VA 24061 - 0316, USA.*

Emails: christophe.bravard@univ-grenoble-alpes.fr, ssarangi@vt.edu, hectortz@vt.edu.

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1 Introduction

In many economic situations, agents establish collaborations before taking part in specific interactions. For example, firms can group together to improve their competitiveness before engaging in a particular market. These collaborations can take the form of multilateral or bilateral agreements, with this paper focusing on the latter. In practice, bilateral agreements are of considerable importance in certain sectors. In the automotive industry, for example, companies often conclude bilateral agreements to strengthen their competitiveness while continuing to compete in the same market. The 2005 partnership between Toyota and the PSA Group for the joint production of small city cars is a case in point.[1]

Goyal and Joshi (GJ, 2003) propose a theoretical model that accounts for this type of situation. Firms strategically form bilateral collaborative links before engaging in quantity or price competition. They establish that each firm's profit function depends on the number of its own links and the total number of links in which it is not involved. Additionally, they provide two crucial properties of each firm's marginal profit:

1. The marginal profit of each firm increases with the number of its own links, implying that the profit function is *convex* with respect to its own links.
2. The marginal profit of each firm decreases with the number of links in which it is not involved, indicating that the profit function exhibits *strategic substitute*.

Westbrock (2010) extends this model to differentiated oligopolies competing on quantity and price. Notably, the convexity of the profit function remains preserved in these contexts. Moreover, GJ (2006) generalizes this model to games where the players are not necessarily firms interacting in a market, but still satisfy the convexity property in their payoff function. They also assume that the payoff function satisfies either the condition of strategic substitute or *strategic complement*, i.e., each player's marginal payoff increases with the number of links in which he is not involved. They establish two types of results. First, they demonstrate that if the payoff function is convex and satisfies either strategic complementarity or substitutability, a *pairwise equilibrium network* (Pwen) always exists. A Pwen refers to a Nash equilibrium where there are no pairs of unlinked players who have an incentive to form a link together. Second, when the payoff function is convex, they provide a necessary condition regarding the architecture of a pairwise equilibrium network. Specifically, a pairwise Nash equilibrium network is a k-group dominant network, where k firms are all linked together, while the remaining firms remain unlinked. Specifically, a pairwise Nash equilibrium network is a k-group dominant network, where k firms are all linked together, while the remaining firms

[1] The agreement involved sharing a common platform and technology for three models, enabling them to be produced at the same manufacturing facility.

remain unlinked.[2] Both empty and complete networks are group dominant networks and are candidates for being Pwen. We refer to interior Pwen those that are distinct from empty or complete networks.

In this paper, we have two objectives. First, we establish the existence of Pwen when the payoff functions are convex but do not satisfy the strategic substitute or strategic complement condition (Proposition 2). Second, we provide a precise characterization of the Pwen set, improving on previous work by GJ (2003, 2006) by selecting from the n different group dominant networks that are candidates to be Pwen. We establish two conditions that are both necessary and sufficient for a Pwen (Proposition 1) enabling us to characterize the set of Pwen. The first condition implies that players in the dominant group have no incentive to remove all their links, while the second condition indicates that players who have formed no links have no incentive to form one link. More precisely, we fully characterize the set of Pwen when the payoff function satisfies the strategic complement in addition to the convexity. In that case, we find that the set of interior Pwen is a discrete convex set (Propositions 3 and 4). This means that if the k_1-group dominant networks and the k_2-group dominant networks, with $k_2 > k_1$, are Pwen, then any k-group dominant networks, with $k > k_1$ and $k < k_2$, are also Pwen. In that case, we are able to provide a condition under which there is a unique interior Pwen. Additionally, we explore the case where the payoff function satisfies strategic substitutability and convexity. Through an example (Example 4), we illustrate that the set of Pwen may not be a discrete convex set. This is why we have added specific conditions to ensure that the set of interior Pwen is a discrete convex set when the payoff function satisfies strategic substitutability. We introduce the function Δ, which measures in each k-group dominant network the difference between the payoff obtained by a player in the dominant group and his payoff if he removes all his links. We establish that when Δ is quasi-concave and the payoff function satisfies strategic substitutability, the set of Pwen is a discrete convex set (see Proposition 6). In this context, we are again able to provide a condition that guarantees the uniqueness of an interior Pwen. Finally, to provide general conditions guaranteeing the interior set of Pwen to be a discrete convex set, we define the function Λ. It measures the difference between the payoff obtained by a player with no links and his payoff if he adds one link for each k-group dominant network. We establish that if both Λ is quasi-convex and Δ is quasi-concave, then the set of Pwen is a discrete convex set (see Proposition 7).

The major value of this article is its contribution to the existing studies on network for-

[2]Pwen are aymetric that is in line with the empirical literature. For example, Hagedoorn and Schakenraad (1992) identify such notable companies as AT&T, IBM, Siemens, Philips, Fujitsu, NEC and Olivetti as key contributors to collaborative partnerships in the information and communications technology sectors during the 1980s. Similarly, Powell, Koput, White and Owen-Smith (2005) note that in the global biotechnology and pharmaceutical industries of the 1990s, a select group of 24 leading entities entered into more than 20 strategic alliances each, while the majority of companies established fewer than two such alliances.

mation game where the payoff function is convex (e.g., models of collaboration networks in oligopolistic markets). We emphasize a significant property to this type of game: characterizing the set of equilibria can be challenging, even if we know that it contains only group dominant networks. Indeed, if the Pwen set is not a discrete convex set, i.e., contains '*holes*', as seen in games with payoff functions satisfying strategic substitutability, precisely characterizing the set of Pwen becomes notably difficult. Possibly, characterizing the Pwen set requires testing each of the n group dominant networks. Note that many models of collaborative networks in oligopolistic markets feature reduced profit functions for each firm, exhibiting strategic substitutability. Moreover, the precise characterization of the set of Pwen allows us to establish sufficient conditions that guarantee its uniqueness. Consequently, this paper addresses an essential theoretical question concerning equilibria – in particular the third aspect, which is the uniqueness of Pwen, complementing the existing exploration of its existence and characterization.

This paper completes the literature on network formation games with convex payoff functions. Some studies, such as Goyal and Joshi (2003, 2006), explore the properties of equilibrium networks, while others, such as Westbrock (2010) and Billand *et al.* (2016, 2019), focus on efficient networks. Our paper is directly in line with the first part of the literature, examining the properties of equilibrium networks. Finally, the study by Billand and *et al.* (2023) focuses on the interaction of two networks. In this context, each player's payoff function is characterized by being convex with respect to the number of links within each network, while exhibiting supermodularity or submodularity between these numbers of links. The paper further requires that the payoff function exhibits strategic complementarity in terms of the number of links from other players within each network. The emphasis placed on the interaction between the two networks in their analysis is an interesting difference from previous studies. While Billand *et al.* (2023) provide notable results regarding existence and characterizations, their work does not provide a complete description of the pairwise equilibrium set – which is one of our main goal in this paper. In addition, the proof techniques employed diverge considerably, due to the different frameworks used in their paper and ours.

The paper is organized as follows. In Section 1, we introduce the model setup. In Section 2, we present existence and characterization results based only on the convexity assumption for players' payoff functions. In Section 3, we provide a complete characterization of the Pwen set when the payoff function exhibits strategic complementarity in addition to its convexity. In Section 4, we examine cases where the payoff function satisfies strategic substitutability. In Section 5, we establish conditions that guarantee that the Pwen set is a discrete convex set. In Section 6, we conclude. The appendix contains detailed proofs for all the results.

2 Model Setup

We denote the set $\{a, a+1, \ldots, b-1, b\}$ with $a, b \in \mathbb{N}$ by $[\![a, b]\!]$.

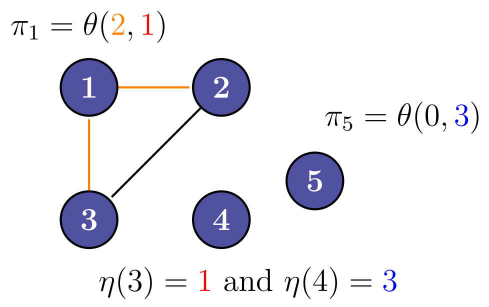
Link formation game. Let $N = [\![1, n]\!]$, with $n \geq 3$, denote a finite set of ex-ante identical players. Every player makes an announcement of intended links. An intended link $s_{i,j} \in \{0, 1\}$, where $s_{i,j} = 1$ means that player i intends to form a link with player j, while $s_{i,j} = 0$ means that player i does not intend to form such a link. A link between two players i and j is formed if and only if $s_{i,j} = s_{j,i} = 1$. A strategy of player i is given by $s_i = (s_{i,j})_{j \in N \setminus \{i\}}$, and a strategy profile is denoted by $s = \{s_1, s_2, \ldots, s_n\}$. We denote by S_i the set of strategies of player i. We denote the link between i and j by ij, when this link exists, we have $G_{i,j} = 1$. In the absence of the link ij, we have $G_{i,j} = 0$. Clearly, a strategy profile s induces a network $G[s]$. For simplicity, we often omit the network's dependence on the underlying strategy profile. A network $G = (G_{i,j})_{i,j \in N}$ is a formal description of the pairwise links that exist between the players. We let $g_i = \sum_{j \in N} G_{i,j}$ denote the number of players with whom player i has a link in the network, or the *degree* of player i in network G. We define G_{-i} as the network obtained from G by removing player i and all his links, and $g_{-i} = \frac{1}{2}\left(\sum_{\ell \in N \setminus \{i\}} g_\ell - g_i\right)$ is the total number of links in this network; g_{-i} is interpreted as the number of links that player i faces in G. We denote by $G + ij$ the network identical to G except that ij does not belong to G and belongs to $G + ij$, and $(g + ij)_i = \sum_{j \in N} (G + ij)_{i,j}$.

Networks. In a *k-group dominant network*, k players are linked together, and $n - k$ players are not involved in any links. We denote by G^k a typical k-dominant network. Let us denote by $D(G^k)$ the set of players who have formed links in G^k and $E(G^k)$ the set of players who have formed no links in G^k. In the empty network, G^1, no pair of players are linked. In the complete network, G^n, all pairs of players are linked. We define the function $\eta(x) = \frac{(x-2)(x-1)}{2}$, for $x \geq 2$. This function allows us to calculate the number of links in G^k_{-i}, that is g^k_{-i}, for every $i \in N$ and $k \geq 2$. More precisely, each player $i \in D(G^k)$ faces $\eta(k)$ links and player $i \in E(G^k)$ faces $\eta(k+1)$ links. It is worth noting that η is not defined at $x = 1$ since $D(G^1) = \emptyset$. Finally, each player in $D(G^k)$ is involved in $k - 1$ links, and each player in $E(G^k)$ is involved in zero links.

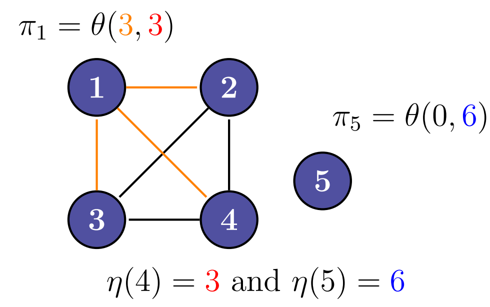
Payoffs. In our analysis, we use a payoff function that closely resembles the GJ payoff functions but with slightly more generality.[3] We assume that

$$\pi_i(s) = \theta(g_i[s], g_{-i}[s]).$$

[3] In GJ (2006), the payoff function is defined as an additive separable function, in which the cost of forming links is assumed to be linear.



(a) 3-group dominant network



(b) 4-group domnant network

Figure 1: Payoffs in some group dominant networks with 5 players

when there is no possible confusion, we write $\theta(g_i[s], g_{-i}[s]) = \theta(g_i, g_{-i})$. We illustrate the payoffs in group dominant networks in Figure 1.

Second order properties of the payoff function. Let $\theta_1(x, y) = \theta(x, y) - \theta(x - 1, y)$. We can further define $\theta_{11}(x, y) = \theta_1(x, y) - \theta_1(x - 1, y)$, $\theta_{12}(x, y) = \theta_1(x, y) - \theta_1(x, y - 1)$. θ is *convex* when $\theta_{11}(x + 1, y) \geq 0$ for all admissible x. It has *strategic complement* when $\theta_{12}(x, y) \geq 0$ for all admissible x, and *strategic substitute* when $\theta_{12}(x, y) \leq 0$ for all admissible x. These properties imply that the marginal profit, θ_1, is monotonic. Specifically, convexity implies that the marginal payoff of each player increases with his own number of links. Strategic complementarities imply that the marginal payoff of each player increases with the total number of links in which he is not involved. Conversely, strategic substitutes imply that the marginal payoff of each player decreases with the total number of links in which he is not involved. From convexity, we obtain the following useful result.

Remark 1 *Suppose θ is convex in its first argument.*

R1. *If $\theta(x, y) - \theta(x_0, y) \geq 0$, then $\theta(x', y) - \theta(x_0, y) \geq 0$, for $x' > x$.*

R2. *If $\theta(x, y) - \theta(x_0, y) \geq 0$, then $\theta(x, y) - \theta(x_0', y) \geq 0$, for $x_0' > x_0$.*

Pairwise equilibrium network. A strategy profile s is said to be a Nash equilibrium if $\pi_i(s_i, s_{-i}) \geq \pi_i(s_i', s_{-i}), \forall s_i' \in S_i, \forall i \in N$. We supplement the idea of Nash equilibrium with the requirement of pairwise stability (which is taken from Jackson and Wolinsky, 1996). A Nash equilibrium network G is said to be pairwise stable if any pair of players have no incentive to form a link that does not exist in G.

Definition 1 *A network g is a pairwise equilibrium network (Pwen) if the following conditions hold:*

1. *There is a Nash equilibrium strategy profile which supports G.*

2. For $g_{i,j} = 0$, $\theta((g+ij)_i, g_{-i}) - \theta(g_i, g_{-i}) \geq 0 \Rightarrow \theta((g+ij)_j, g_{-j}) - \theta(g_j, g_{-j}) < 0$.

The second condition means that if player i has a weak incentive to form the link ij, then player j has no incentive to form this link.

In the following we say that G^k is the unique Pwen if there is no other non-isomorphic network that is a Pwen, i.e., there is only one unlabeled network that is a Pwen.

Discrete convex set. In this paragraph, we refer to the definitions provided by Murota (2008, p.227). The indicator function of a set $M \subseteq \mathcal{M}$ is a function $\delta_M : \mathcal{M} \to \{0, +\infty\}$ defined by

$$\delta_M(x) = \begin{cases} 0 & \text{if } x \in M, \\ +\infty & \text{otherwise.} \end{cases}$$

A function $g : \mathcal{M} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *L-convex* when for $p, q \in N$,

$$g(p) + g(q) \geq g\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + g\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right).$$

M is a *discrete convex set* if and only if δ_M is an L-convex function.[4] Specifically, $M = [\![3, 6]\!]$ with $a, b \in \mathbb{N}$ and $a < b$, is a discrete convex set, and $S' = [\![3, 6]\!] \cup [\![8, 12]\!]$ since $\delta_{M'}(6) + \delta_{M'}(8) = 0 < +\infty = \delta_{M'}\left(\left\lceil \frac{6+8}{2} \right\rceil\right) + \delta_{M'}\left(\left\lceil \frac{6+8}{2} \right\rceil\right) = \delta_{M'}(7) + \delta_{M'}(7)$. More generally, we have the following result.

Remark 2 *Let* $M \subseteq N$, $M \neq \emptyset$. M *is a convex discrete set if and only if there are* $a, b \in N$, *with* $a \leq b$, *such that* $M = [\![a, b]\!]$.

Applications. We now present several economic applications where the payoff function of player i exhibits convexity with respect to his number of links. In the first application, we consider a two-step game where players form links in the initial stage and then engage in a linear oligopoly game in the subsequent stage. This application was initially introduced by Goyal and Joshi (2003, 2006), who presented a model where the links formed by firm i enable it to reduce its marginal cost. They specifically examined a linear Cournot/Bertrand model with homogeneous goods. Westbrock (2010) extended these examples by incorporating additional considerations. First, he accounted for cases where the marginal cost of firm i depends not only on its own links but also on the number of links formed by other firms. Second, he addressed scenarios involving differentiated goods.

Application 1. *Collaborations in Oligopolies (Westbrock, 2010, GJ, 2003, 2006).* We assume that in the second stage of the game is an oligopoly game where each firm $i \in N$ sells a,

[4] Murota calls this set an L-convex set.

possibly differentiated, product to a continuum of homogeneous consumers. Let q_i denote the quantity and p_i the price of good i. A representative consumer maximizes

$$U(I, q_1, \ldots, q_n) = I + \zeta_1 \sum_{i \in N} q_i - \frac{1}{2} \sum_{i \in N} q_i^2 - \frac{\zeta_2}{2} \sum_{i \in N} \sum_{j \in N \setminus \{i\}} q_i q_j,$$

with $\zeta_1 > 0$, $\zeta_2 \in (0, 1]$ subject to the budget constraint $\sum_i p_i q_i + I \leq m$, where m denotes income, and the price of the composite good is normalized to one. From standard utility maximization, we arrive at a system of inverse demand functions $p_i = \zeta_1 - q_i - \zeta_2 \sum_{j \in N \setminus \{i\}} q_j$. The profit, gross of linking costs, is given by $\pi_i = (p_i - c_i) q_i$. The firms can compete either in quantities or in prices. In any case, $\zeta_2 = 1$ depicts a market of perfect substitutable goods. To avoid the homogeneous price competition situation, if firms compete in prices, we assume $\zeta_2 \in (0, 1)$. In either case, the Nash equilibrium quantities can be expressed in the form $\omega q_i = \alpha_1 \zeta_1 - \alpha_2 c_i + \alpha_3 \sum_{j \in N \setminus \{i\}} \mathrm{MC}_j$, where MC_j is the marginal cost of firm j. Using superscript C and B to denote Cournot and Bertrand, respectively, the parameters are given by $\omega^C = 1$, $\alpha_1^C = \frac{1}{2+(n-1)\zeta_2} > 0$, $\alpha_2^C = \frac{2+(n-2)\zeta_2}{(2+(n-1)\zeta_2)(2-\zeta_2)} > 0$, $\alpha_3^C = \frac{\zeta_2}{(2+(n-1)\zeta_2)(2-\zeta_2)} > 0$. Similarly, $\omega^B = \frac{(1-\zeta_2)(1+(n-1)\zeta_2)}{1+(n-2)\zeta_2} > 0$, $\alpha_1^B = \frac{(1-\zeta_2)}{2+(2n-3)\zeta_2} > 0$, $\alpha_2^B = \frac{2+(3n-6)\zeta_2+(n^2-5n+5)\zeta_2^2}{(2+(n-3)\zeta_2)(2+(2n-3)\zeta_2)} > 0$, $\alpha_3^B = \frac{(1+(n-2)\zeta_2)\zeta_2}{(2+(n-3)\zeta_2)(2+(2n-3)\zeta_2)} > 0$. Prices and profits are in equilibrium, $p_i = \omega q_i + c_i$, and $\pi_i = \omega q_i^2$, respectively.

We assume that the marginal cost of firm i is $\mathrm{MC}_i(g_i, g_{-i}) = \gamma_0 - \gamma_1 g_i - \gamma_2 g_{-i}$. Thus, each collaboration link has an impact on the whole industry, as in d'Aspremont and Jacquemin (1988). It follows that the equilibrium quantity in the Cournot competition can be written as:

$$q_i = \frac{\alpha_1 \zeta_1 - \alpha_2 (\gamma_0 - \gamma_1 g_i - \gamma_2 g_{-i}) + \alpha_3 \sum_{j \in N \setminus \{i\}} (\gamma_0 - \gamma_1 g_j - \gamma_2 g_{-j})}{\omega}.$$

We have $\sum_{j \in N \setminus \{i\}} g_j + g_i = 2(g_{-i} + g_i)$, i.e., $\sum_{j \in N \setminus \{i\}} g_j = 2g_{-i} + g_i$. Moreover, $\sum_{j \in N \setminus \{i\}} g_{-j} + \sum_{j \in N \setminus \{i\}} g_j = 2 \sum_{j \in N \setminus \{i\}} (g_{-i} + g_i) = 2(n-1)(g_{-i} + g_i)$, i.e., $\sum_{j \in N \setminus \{i\}} g_{-j} = 2(n-1)(g_{-i} + g_i) - (2g_{-i} + g_i) = 2(n-2) g_{-i} + (2n-3) g_i$. Let $a = \frac{\alpha_1 \zeta_1 - \alpha_2 \gamma_0 + (n-1)\alpha_3 \gamma_0}{\omega}$, $b = \frac{(\alpha_2 \gamma_1 - \alpha_3 \gamma_1 - (2n-3)\alpha_3 \gamma_2)}{\omega}$, and $c = \frac{\alpha_2 \gamma_2 - 2\alpha_3 \gamma_1 - 2(n-2)\alpha_3 \gamma_2}{\omega}$. We have $q_i = a + b g_i + c g_{-i}$. Suppose that the cost of forming each link is F. Then, the profit function of firm i is given by $\theta(g_i, g_{-i}) = (a + b g_i + c g_{-i})^2 - F g_i$. Hence, θ is convex, and it exhibits strategic complementarity if $b \times c \geq 0$, and strategic substitutability if $b \times c \leq 0$. In the price competition scenario, the functional form is similar, but the parameters a, b, and c have different values.

Let us explore the case of the homogeneous Cournot oligopoly (see GJ, p. 64, 2003), i.e., $\zeta_2 = 1$. We have $q_i = a + b g_i + c g_{-i}$, where $\omega^C = 1$, $\alpha_1^C = \frac{1}{n+1}$, $\alpha_2^C = \frac{n}{n+1}$, $\alpha_3^C = \frac{1}{n+1}$, and $a = \frac{\zeta_1 - \gamma_0}{n+1}$, $b = \frac{(n-1)\gamma_1 - (2n-3)\gamma_2}{n+1}$, $c = \frac{n\gamma_2 - 2\gamma_1 - 2(n-2)\gamma_2}{n+1}$. If in addition, $\gamma_2 = 0$, then $q_i = \frac{\zeta_1 - \gamma_0 + (n-1)\gamma_1 g_i - 2\gamma_1 g_{-i}}{n+1}$. ◇

In the previous application, the payoff function of each player has the following functional form

$$\theta(g_i, g_{-i}) = (a + b g_i + c g_{-i})^2 - F g_i. \tag{1}$$

Therefore, we will refer to this functional form in most of our examples throughout the paper.

In the second application, we consider a two-step game where players form links in the initial stage and then participate in a public good game in the subsequent stage. Unlike in the previous application, the payoff function of players is convex in the links formed by them, but it does not exhibit strategic complementarity or strategic substitutability.

Application 2. *Provision of a Public Good.* We consider a specific two-stage game where players first form their links and then choose their levels of effort, denoted as $e_i \geq 0$, for producing a public good. The production of the public good is standard and depends on the efforts of the players, given by $Y(e_i, e_{-i}) = e_i + \sum_{j \neq i} e_j$. Assuming complementarities between player i's effort and the number of links formed by others, we simplify the analysis by letting the cost of effort, denoted by $C(e_i) = \frac{1}{2(g_{-i})^2} e_i^2$, decrease with the number of links in which player i is not involved. The payoff function of player i is:

$$\theta(g_i, g_{-i}) = g_i B_i(e_i, e_{-i}) - F g_i,$$

where $B_i(e_i, e_{-i}) = Y(e_i, e_{-i}) - C(e_i)$. Clearly, in the second step of the game, the best response of player i is $e_i = g_{-i}$. Consequently,

$$\theta(g_i, g_{-i}) = g_i \left(g_{-i} + \sum_{j \neq i} g_{-j} \right) - F g_i.$$

Since $\sum_{j \neq i} g_{-j} = (n-3)g_{-i} + (n-2)g_i$, we have

$$\theta(g_i, g_{-i}) = g_i \left(\frac{(g_{-i})^2}{2} + (n-3)g_{-i} + (n-2)g_i - F \right). \tag{2}$$

We observe that $\theta_{22}(g_i, g_{-i}) = 2(n-2) > 0$, and θ is convex. Moreover, $\theta_{21}(g_i, g_{-i}) = g_{-i} + \frac{n-7}{2}$. Hence, $\theta_{21}(g_i, g_{-i}) > 0$ for $g_{-i} < \frac{n-7}{2}$, and $\theta_{21}(g_i, g_{-i}) \geq 0$ for $g_{-i} \geq \frac{n-7}{2}$, for $n \geq 9$. Therefore, θ is neither complement strategic nor substitute strategic. $\diamond$

3 Pairwise Equilibrium Under Convexity

In this section, we only assume that θ is convex in its first argument. As noted by Goyal and Joshi (2006, Theorem 3.1), if G is a Pwen and $g_i > 0$, then $\theta_1(g_i, g_{-i}) \leq \theta_1(g_i + 1, g_{-i})$. This implies that if player i has formed at least one link, then this player has an incentive to form an additional link. As a result, all players who have formed links are linked together in a Pwen. Consequently, *when θ is convex in its first argument, a Pwen is a k-group dominant network.* A k-group dominant network, $k \in [\![2, n-1]\!]$, which is a Pwen, is called an *interior Pwen.*

We first provide necessary and sufficient conditions for G^k, $k \in [\![2, n-1]\!]$, to be an interior Pwen. The first condition ensures that players in $D(G^k)$ have no incentive to remove links, while the second condition ensures that players in $E(G^k)$ have no incentive to form a link.

Proposition 1 *Suppose that θ is convex.*

1. *Let G^k be a k-group dominant network, $k \in [\![2, n-1]\!]$. Network G^k is a Pwen equilibrium if and only if*

   C1. $\theta(k-1, \eta(k)) \geq \theta(0, \eta(k))$, *and*

   C2. $\theta_1(1, \eta(k+1)) < 0$.

2. *The empty network, G^1, is a Pwen if and only if $\theta_1(1, 0) < 0$.*

3. *The complete network, G^n, is a Pwen if and only if $\theta(n-1, \eta(n)) \geq \theta(0, \eta(n))$*

Condition C2 is obvious since players in $E(G^k)$ have no incentive to form a link in a Pwen. Condition C1 comes from the fact that if players in $D(G^k)$ have no incentive to remove all their links, then they have no incentive to remove some of their links by R2.

In the following example, we examine conditions C1 and C2 in the specific case where $\theta(x, y)$ is given by Equation (1).

Example 1 *Consider $\theta : [\![0, n-1]\!] \times [\![0, (n-1)(n-2)/2]\!]$, $\theta(x, y) = (a + bx + cy)^2 - Fx$, with $a, b, F > 0$, and $c \in \mathbb{R}$. Then, $\theta(x, y) - \theta(0, y) \geq 0$ if and only if $x(2ab + b^2x + 2bcy - F) \geq 0$, i.e., $2a + bx + 2cy - F_b \geq 0$, where $F_b = F/b$. A k-group dominant network satisfies C1 if and only if $2a + b\,(k-1) + c\,(k-2)\,(k-1) - F_b \geq 0$*

$$\lambda_1(k) = ck^2 + (b - 3c)k - F_b + 2a - b + 2c \geq 0. \tag{3}$$

Moreover, $\theta(1, y) - \theta(0, y) < 0$ if and only if $2a + b + 2cy - F_b < 0$. A k-group dominant network satisfies C2 if and only if $2a + b + ck(k-1) - F_b < 0$, that is

$$\lambda_2(k) = ck^2 - ck - F_b + 2a + b < 0. \tag{4}$$

Note that C2 never holds when $c > 0$ and $F_b < 2a + b$, in particular when $c > 0$ and $F = 0$.

C1 and C2 allow us to establish the existence of a Pwen. More precisely, we establish that the existence of a Pwen can be obtained without requiring the strategic substitutes or strategic complementarity property of θ. These assumptions were used by Goyal and Joshi (Proposition 3.1, 2006) in addition to the convexity of θ. The proposition 2 generalizes their existence result, and our proof is more concise, relying on arguments similar to those used to prove Tarski's theorem (1955).[5]

Proposition 2 *Suppose that θ is convex. Then, there always exists a Pwen.*

The proof strategy is to use the set of k-group dominant networks, where players belonging to $D(G^k)$ prefer to have one link rather than zero, i.e., $k \in [\![2, n]\!]$ for which $\theta_1(1, \eta(k)) \geq 0$. We call this set Ξ. There are two possibilities:

[5]We adopt Vives' (1999) theorem presentation.

1. Ξ is non-empty. Then, Ξ admits a maximum as a finite set, denoted $\hat{k}$. Because of R1, players in $D(G^{\hat{k}})$ have no incentive to remove links in $G^{\hat{k}}$. Since $\hat{k}$ is the maximum of Ξ, players in $E(G^{\hat{k}})$ have no incentive to form a link in $G^{\hat{k}}$ since they face the same number of links as players in $D(G^{\hat{k}+1})$. Consequently, $G^{\hat{k}}$ is a Pwen.

2. Ξ is empty. Then, it is clear that the empty network is a Pwen by observing that $\theta_1(1, \eta(2)) = \theta_1(1, 0)$, and no player has an incentive to form a link in the empty network.

Let us now illustrate the significance of Proposition 2. Thanks to this proposition, we state that Application 2 admits a Pwen for all values of n and F. The payoff function of players in this application is convex, but it does not exhibit strategic complementarity or strategic substitutability. Consequently, we cannot apply Theorem 3.1 of Goyal and Joshi (2006) in this case, making Proposition 2 an essential tool for establishing the existence of Pwen.

Finally, due to the necessary and sufficient conditions that determine whether the complete network or the empty network is an equilibrium, we immediately obtain that if $\theta(n-1, \eta(n)) < \theta(0, \eta(n))$ and $\theta_1(1, 0) \geq 0$, then there is an interior Pwen.

Although we have a result on the existence of Pwen and a necessary condition for identifying candidate pairwise equilibrium networks (provided by GJ, 2006), it is not straightforward to select Pwen from these candidates. We illustrate this in the following example, where obtaining the Pwen set requires testing each of the n dominant group networks.

Example 2 *Suppose that* $n = 7$*, and* $\theta(g_i, g_{-i}) = (g_i)^2 + (-1)^{g_{-i}} n^2 g_i$. *We know that the only candidates for being Pwen are* k*-group dominant networks, with* $k \in [\![1, 6]\!]$. *We have* $\theta(1, y) - \theta(0, y) = 37 > 0$, *for* y *even, and* $\theta(1, y) - \theta(0, y) = \theta(1, y) - \theta(0, y) = -35 < 0$ *for* y *odd. Similarly,* $\theta(2, 1) - \theta(0, 1) = -68$, $\theta(3, 3) - \theta(0, 3) = -99$, $\theta(4, 6) - \theta(0, 6) = 160$, $\theta(5, 10) - \theta(0, 10) = 205$, $\theta(6, 15) - \theta(0, 15) = -180$. *Consequently, the set of Pwen is* $\{G^2, G^6\}$. *Figure 2 illustrates the set of Pwen within the set of* k*-group dominant networks.*

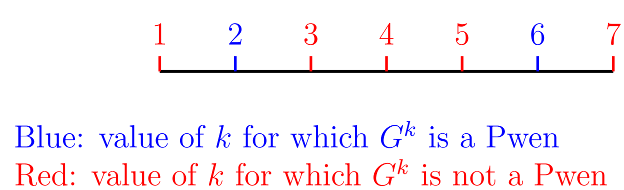


Figure 2: The Set of Pwen in Example 2

In the following, we explore the conditions that facilitate the characterization of the Pwen set. We begin our analysis with the cases most common in the economic literature, namely when the payoff function satisfies either strategic complementarity or strategic substitutability.

4 Pairwise Equilibrium Under Strategic Complement

In order to achieve a more precise characterization of the set of Pwen, we introduce an additional property to θ, in addition to its convexity: the strategic complementarity. We now use C1 and C2 to define the two following sets $\psi_1 = \{k \in [\![2, n]\!] : \theta(k-1, \eta(k)) \geq \theta(0, \eta(k))\}$ and $\psi_2 = \{k \in [\![1, n-1]\!] : \theta_1(1, \eta(k+1)) < 0\}$. Clearly, ψ_1 and ψ_2 determine the set of networks G^k that satisfy C1 and C2 respectively. When ψ_1 and ψ_2 are non-empty, they have maximal and minimal elements because they are finite sets. For $\ell \in \{1, 2\}$, we denote the maximal element of ψ_ℓ by $k_\ell^{\max}$ and its minimal element by $k_\ell^{\min}$. With this in mind, we can present a useful result.

Lemma 1 *Suppose that θ is convex in its first argument and satisfies strategic complementarity.*

1. *If $k \in \psi_1$ and $\kappa > k$, then $\kappa \in \psi_1$.*
2. *If $k' \in \psi_2$ and $\kappa < k'$, then $\kappa \in \psi_2$.*

Let us explain this result. The first part means that for $\kappa > k$, we have $\theta(k-1, \eta(k)) \geq \theta(0, \eta(k))$, then $\theta(\kappa - 1, \eta(\kappa)) \geq \theta(0, \eta(\kappa))$. This result is derived from the convexity R1 since $\theta(\kappa-1, \eta(k)) - \theta(0, \eta(k)) \geq \theta(k-1, \eta(k)) - \theta(0, \eta(k))$, and from the strategic complementarity of θ since θ_1 is increasing in its second argument and $\theta(\kappa - 1, \eta(\kappa)) - \theta(0, \eta(\kappa)) \geq \theta(\kappa - 1, \eta(k)) - \theta(0, \eta(k))$. The second part of the result follows the same reasoning.

We can now present the main result of this section, it focuses on interior Pwen. More precisely, we establish a necessary and sufficient condition for the existence of interior Pwen, and we characterize the set of interior Pwen. This set consists of all k-group dominant networks such that $k \in [\![k_1^{\min}, k_2^{\max}]\!]$ when $k_1^{\min}$ and $k_2^{\max}$ exist. In other words, the set of interior Pwen is a discrete convex set. Recall that by construction of ψ_1 and ψ_2, we have $k_1^{\min} \geq 2$, and $k_2^{\max} \leq n-1$.

Proposition 3 *Suppose that θ is convex in its first argument and satisfies strategic complementarity.*

1. *There exists an interior Pwen if and only $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$ and $k_2^{\max} \geq k_1^{\min}$.*
2. *Suppose that $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$. Network G^k, $k \in [\![2, n-1]\!]$, is a Pwen if and only if $k \in [\![k_1^{\min}, k_2^{\max}]\!]$.*

The first part of the proposition is straightforward since when G^k is a Pwen, it satisfies C1 and C2. Specifically, G^k satisfies C1 if $k \in \psi_1$, and it satisfies C2 if $k \in \psi_2$. Therefore, $k \geq k_1^{\min}$ and $k \leq k_2^{\max}$. The second part of the proposition can be derived from Lemma 1. Indeed, we know that if G^k satisfies C1 and C2, then $G^{\bar{\kappa}}$, with $\bar{\kappa} > k$, satisfies C1, and $G^{\underline{\kappa}}$, with $\underline{\kappa} < k$, satisfies C2.

Proposition 3 provides a critical understanding of the possibility of uniqueness for interior Pwen. We present this in the subsequent corollary.

Corollary 1 *Suppose that θ is convex in its first argument and is strategic complement. Network G^k is the unique interior Pwen if and only if $k = k_1^{\min} = k_2^{\max}$.*

Finally, we establish a result that provides additional properties of the Pwen set when θ is convex with respect to its first argument and satisfies strategic complementarity. Specifically, we show that it is not possible to have a unique Pwen when there exists an interior Pwen.

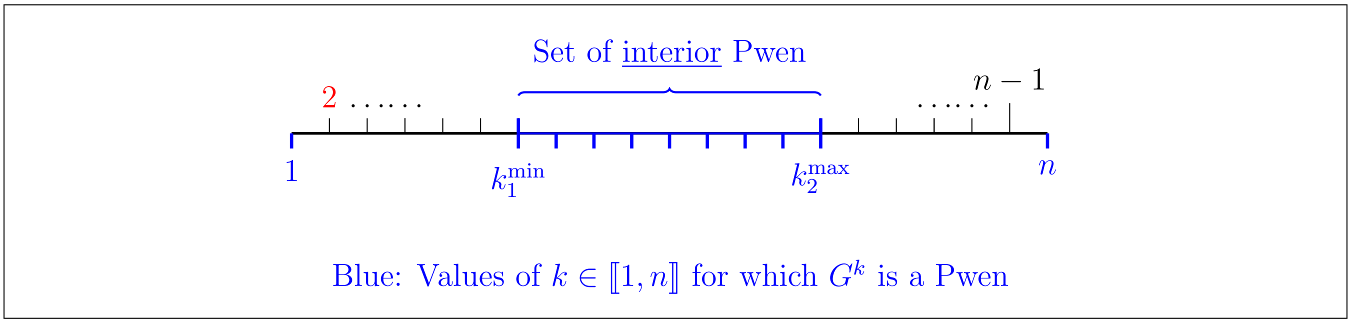


Figure 3: Pwen set for strategic complement θ

Proposition 4 *Suppose that θ is convex in its first argument and is strategic complement.*

1. *If G^k, $k \in [\![3, n-1]\!]$, is a Pwen, then G^1 and G^n are Pwen.*
2. *Network G^2 cannot be a Pwen.*

The first part of the proposition can again be directly deduced from Lemma 1. If G^k is a Pwen, then it satisfies C1 and C2. According to Lemma 1, every network $G^{\bar{\kappa}}$, with $\bar{\kappa} > k$, satisfies C1, and in particular when $\bar{\kappa} = n$. Similarly, every network $G^{\underline{\kappa}}$, with $\underline{\kappa} < k$, satisfies C2 and in particular when $\underline{\kappa} = 1$. The second part follows from the fact that the players have the same payoff function. To establish a contradiction suppose that G^2 is a Pwen. In this case, two players have formed links, implying $\theta(1, 0) - \theta(0, 0) \geq 0$. On the other hand, other players are involved in zero links, hence $\theta(1, 1) - \theta(0, 1) \geq 0$. Clearly, these two inequalities cannot hold simultaneously.

We now illustrate the above results using specific parameter values for the payoff functions of players, represented as $\theta(x, y) = (a + bx + cy)^2 - Fx$, with $a, b, c, F > 0$ following Example 1.

Example 3 *Let $\theta(x, y) = (a + bx + cy)^2 - Fx$, with $2a + F_b = -1145$, $b = 7$, $c = 0.1$. In Figure 4, we use $\lambda_1(k) = -1145 + 7 \times (k - 1) + 0.1 \times (k - 1) \times (k - 2)/2$ and $\lambda_2(k) = -1145 + 7 + 0.1 \times (k - 1) \times k/2$, and Inequalities (3) and (4) to represent the sets ψ_1 and ψ_2 associated with these values. The set of interior Pwen is the set of G^k such that $k \in [\![98, 151]\!]$.*

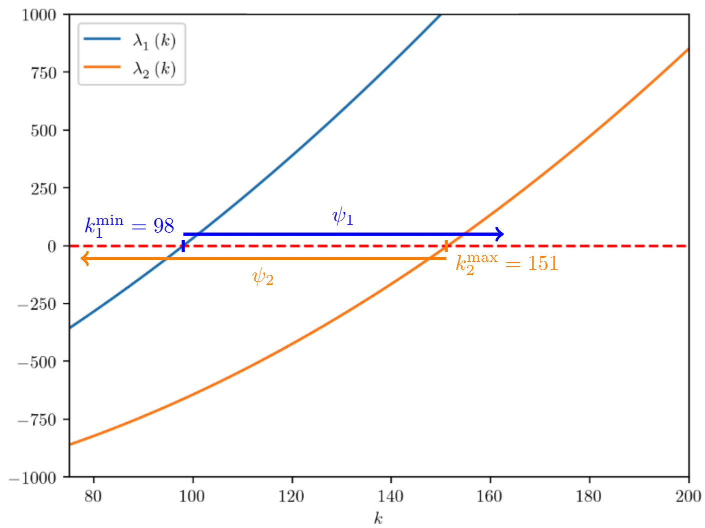


Figure 4: The set of interior Pwen of Example 3

5 Pairwise Equilibrium Under Strategic Substitute

We now assume that θ satisfies strategic substitute. Under this assumption, obtaining results becomes more challenging without introducing additional assumptions. The difficulty comes from the loss of the monotonicity result stated in Lemma 1. First, when network G^k is a Pwen, i.e., satisfies C1 and C2, then $G^{\bar{\kappa}}$, with $\bar{\kappa} > k$ satisfies C2. Indeed, by strategic substitute, we have $\theta_1(1, \eta(x)) \geq \theta_1(1, \eta(x+1))$. We state this result in the following Lemma.

Lemma 2 *Suppose that θ is convex in its first argument and satisfies strategic substitute. If $k \in \psi_2$ and $\kappa > k$, then $k \in \psi_2$.*

However, we cannot state that $G^{\underline{\kappa}}$, with $\underline{\kappa} < k$, satisfies C1 when G^k satisfies C1. In other words, we do not have $\theta(k, \eta(k)) - \theta(0, \eta(k)) \geq 0 \Rightarrow \theta(\underline{\kappa}, \eta(\underline{\kappa})) - \theta(0, \eta(\underline{\kappa})) \geq 0$. Indeed, on the one hand, because of the convexity of θ, we have $\theta(k, \eta(k)) - \theta(0, \eta(k)) \geq \theta(\underline{\kappa}, \eta(k)) - \theta(0, \eta(k))$. In the other hand since θ satisfies strategic substitute, we have $\theta(\underline{\kappa}, \eta(k)) - \theta(0, \eta(k)) \leq \theta(\underline{\kappa}, \eta(\underline{\kappa})) - \theta(0, \eta(\underline{\kappa}))$. To summarize, strategic substitutability and convexity act in opposite directions. In the next result, we use the monotonicity result concerning C2 to provide necessary and sufficient conditions for obtaining the existence of an interior Pwen.

Proposition 5 *Suppose that θ is convex in its first argument and satisfies strategic substitute.*

1. *There exists an interior Pwen if and only $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$ and $[\![k_2^{\min}, n-1]\!] \cap \psi_1 \neq \emptyset$.*
2. *Suppose that $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$. Network G^k, $k \in [\![2, n-1]\!]$, is a Pwen if and only if $k \geq k_2^{\min}$ and $k \in \psi_1$.*

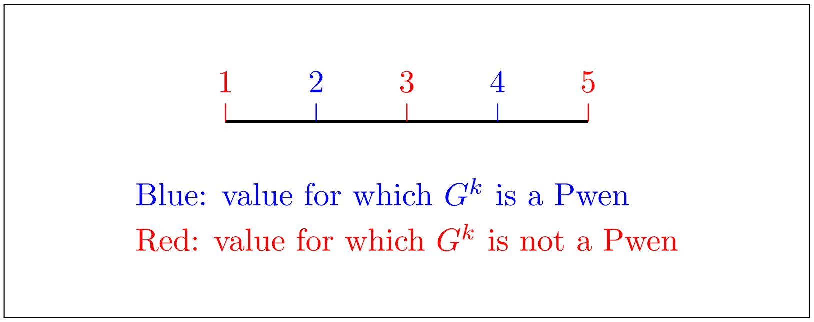
Proposition 5 appears to allow for the possibility of a set of interior Pwen that contains 'holes', i.e., the existence of a non-convex set of interior Pwen. We demonstrate this possibility with an example.

	$g_i = 0$	$g_i = 1$	$g_i = 2$	$g_i = 3$	$g_i = 4$
$g_{-i} = 0$	0	1	2.5	4.6	7.5
$g_{-i} = 1$	-1	-1.1	-1.1	1	3.5
$g_{-i} = 2$	-2	-2.1	-2.1	0	2.4
$g_{-i} = 3$	-1	-1.1	-1.1	0	2.4
$g_{-i} = 4$	-3	-13	-23	-33	-43
$g_{-i} = 5$	-4	-14	-24	-34	-44
$g_{-i} = 6$	-5	-15	-25	-35	-45
$g_{-i} = 7$	-6	-16	-26	-36	-46
$g_{-i} = 8$	-7	-17	-27	-37	-47
$g_{-i} = 9$	-8	-18	-28	-38	-48
$g_{-i} = 10$	-9	-19	-29	-39	-49

(a) Payoffs of Example 4

	$\theta_1(1, g_{-i})$	$\theta_1(2, g_{-i})$	$\theta_1(3, g_{-i})$	$\theta_1(4, g_{-i})$
$g_{-i} = 0$	1.0	1.5	2.1	2.9
$g_{-i} = 1$	-0.1	0.0	2.1	2.5
$g_{-i} = 2$	-0.1	0.0	2.1	2.4
$g_{-i} = 3$	-0.1	0.0	1.1	2.4
$g_{-i} = 4$	-10.0	-10.0	-10.0	-10.0
$g_{-i} = 5$	-10.0	-10.0	-10.0	-10.0
$g_{-i} = 6$	-10.0	-10.0	-10.0	-10.0
$g_{-i} = 7$	-10.0	-10.0	-10.0	-10.0
$g_{-i} = 8$	-10.0	-10.0	-10.0	-10.0
$g_{-i} = 9$	-10.0	-10.0	-10.0	-10.0
$g_{-i} = 10$	-10.0	-10.0	-10.0	-10.0

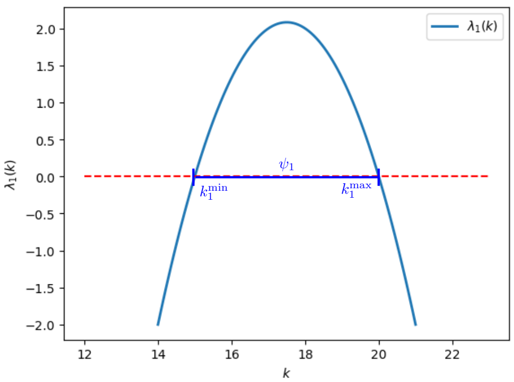
(b) Marginal payoffs of Example 4



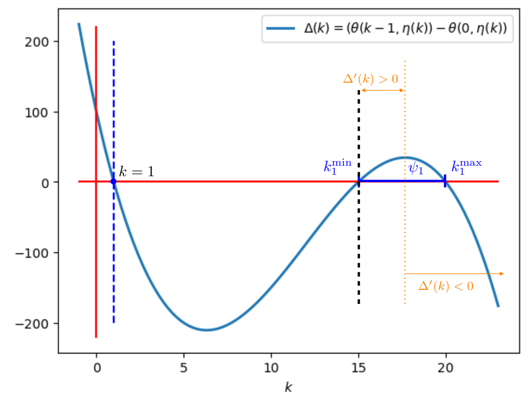
(c) The set of Pwen of Example 4

Figure 5: Payoffs and the set of Pwen of Example 4

Example 4 *Let* $N = [\![1, 5]\!]$*, and the payoffs associated with* θ *are given in Table (a) in Figure 5. For example, the value of* $\theta(1, 0) = 1$*. Table (b) summarizes the marginal payoffs associated to Table (a),* θ_1*. By observing Table (b) in Figure 5, function* θ *satisfies convexity and strategic substitutability since the values exhibit a non-decreasing sequence in each row and a non-increasing sequence in each column. There are five networks candidates for being Pwen:* G^1*,* G^2*,* G^3*,* G^4*,* G^5*. The empty network,* G^1 *is not an equilibrium since* $\theta(1, 0) = 1 > 0 = \theta(0, 0)$*. The 2-group dominant network,* G^2*, is an equilibrium since* $\theta(1, 0) = 1 > 0 = \theta(0, 0)$ *and* $\theta(1, 1) = -1.1 < -1 = \theta(0, 1)$ *The 3-group dominant network,* G^3*, is not an equilibrium since* $\theta(2, 1) = -1.1 < -1 = \theta(0, 1)$*. The 4-group dominant network,* G^4*, is an equilibrium since* $\theta(3, 3) = 0 > -1 = \theta(0, 3)$ *and* $\theta(0, 6) = -5 > -15 = \theta(1, 6)$*. The complete network,* G^5*, is not an equilibrium since* $\theta(4, 6) = -45 < -5 = \theta(0, 6)$*. It follows that the set of Pwen is* $\{G^2, G^4\}$*.*



(a) The set ψ_1



(b) Curve of $\theta(k-1, \eta(k)) - \theta(0, \eta(k))$

Figure 6: Example 5. Functions $\lambda_1(k)$, and $\theta(k-1, \eta(k)) - \theta(0, \eta(k))$

In the previous example, it is important to note that $\psi_1 = [\![2, 5]\!]$, and $\psi_2 = \{2, 4\}$. This means that the 'holes' in the interior Pwen set are a result of the 'holes' in ψ_1, i.e., ψ_1 is not a discrete convex set. Now, we present an example to demonstrate that the set of interior Pwen can actually have no holes when θ satisfies strategic substitutability. This is possible when ψ_1 itself is a discrete convex set. We illustrate this possibility by using the payoff function given in Equation (1) for specific parameters.

Example 5 *Let* $N = [\![1, 25]\!]$*,* $2a - F_b = -265/3$*,* $b = 32/3$*, and* $c = -1/3$*. C1 holds when* $\lambda_1(k) = -(1/3)k^2 + (11 + 2/3)k - 100 \geq 0$*, and C2 holds when* $\lambda_2(k) = -(1/3)k^2 + 1/3k + 32/3 - 265/3 < 0$*. First,* $\lambda_1(k) \geq 0$ *for* $k \in [\![15, 20]\!]$*. We draw* $\lambda_1(k)$ *in Figure 6. Second,* $\lambda_2(k) < 0$ *for* $k \in [\![1, 25]\!]$*. Clearly, the set of Pwen is* $\{G^1, G^{15}, G^{16}, G^{17}, G^{18}, G^{19}, G^{20}\}$*.*

6 Convexity of the Set of Interior Pwen

Proposition 5 offers less information compared to Proposition 3 and its corollary. This limitation arises from the fact that ψ_1 is not a discrete convex set. Hence, it is crucial to identify conditions that enable the convexification of the set ψ_1, in particular when θ satisfies strategic substitutability.

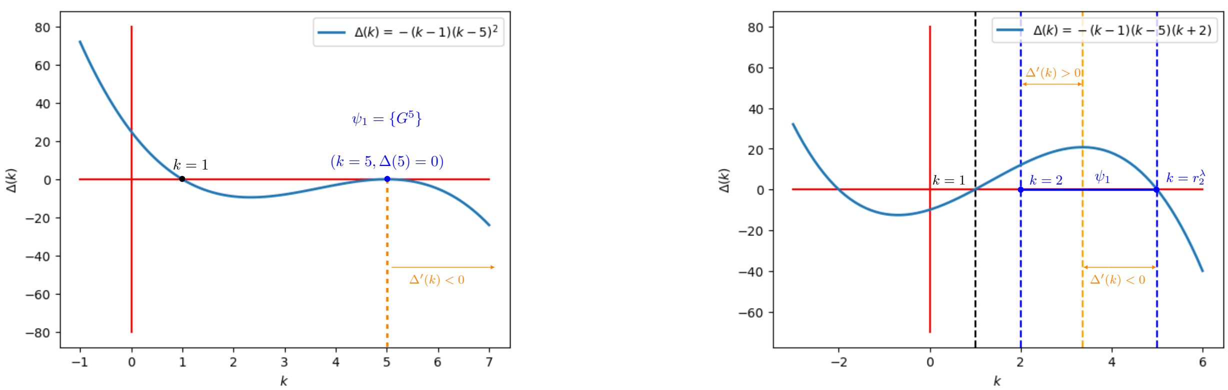
To address this, we introduce an additional condition inspired by observations made on the function $\lambda_1(k)$ in Example 1. There are three possibilities to consider. First, $\lambda_1(k)$ has no real roots, resulting in an empty set ψ_1. Second, $\lambda_1(k)$ has one real root, denoted as r_1^λ, with a multiplicity of 2. In this case, ψ_1 is empty when $r_1^\lambda < 2$, and $\psi_1 = \left\{G^{r_1^\lambda}\right\}$, when $r_1^\lambda \geq 2$ – see Figure 7 (a). Third, $\lambda_1(k)$ has two real roots, r_1^λ and r_2^λ, with $r_1^\lambda < r_2^\lambda$. There are three possibilities to consider in this case. If $r_1^\lambda \geq 2$, it corresponds to the case examined in Example

5. If $r_1^\lambda < 2$ and $r_2^\lambda \geq 2$, networks G^k with $k \in [\![2, r_2^\lambda]\!]$ are Pwen – see Figure 7 (b). If $r_2^\lambda < 2$, ψ_1 is empty.

In Figure 7, we illustrate the function $\Delta(k) = \Delta_\theta(k) = \theta(k-1, \eta(k)) - \theta(0, \eta(k))$ in the two cases where $\psi_1 \neq \emptyset$, which are distinct from the case presented in Example 5. In both cases, ψ_1 is a discrete convex set. In the first case, $\Delta(k)$ increases up to a certain point where it decreases. In the second case, Δ is a monotonic function after $k_1^{\min}$, as shown in Figure 7(b). Furthermore, in Figure 6 (b), it is observed that Δ is strictly increasing when $n \in [\![15, 16]\!]$.

Based on all these observations, we have the intuition that ψ_1 is a discrete convex set when Δ is strictly quasi-concave over the interval $[\![k_1^{\min}, n-1]\!]$. However, we need to adapt the standard definition of quasi-concavity for functions defined over a discrete set. Function Δ is said to be a *discrete quasi-concave function* if for all $[\![\kappa_1, \kappa_2]\!] \subseteq [\![k_1^{\min}, n-1]\!]$, the following holds:

$$\forall k \in [\![\kappa_1 + 1, \kappa_2 - 1]\!], \Delta(k) \geq \min\{\Delta(\kappa_2), \Delta(\kappa_1)\}.$$



(a) $\Delta(k)$ when $r_1^\lambda = r_2^\lambda \geq 2$

(b) $\Delta(k)$ when $r_2^\lambda < 2$ and $r_2^\lambda \geq 2$

Figure 7: Example of functions $\Delta(k)$ where $\psi_1 \neq \emptyset$

Let us establish that the discrete quasi-concavity of Δ guarantees that ψ_1 is a discrete convex set.

Lemma 3 *Suppose that Δ_θ is a discrete quasi-concave function. Then, ψ_1 is a discrete convex set, i.e., if $k_1^{\min} < k_1^{\max} + 1$, then for $k \in [\![k_1^{\min} + 1, k_1^{\max} - 1]\!]$, $k \in \psi_1$.*

The payoff function in Application 2, as described by (2), generates a strictly increasing Δ function over $[\![k^\star, n]\!]$, where $k^\star = \arg\min_{k \in [\![1,n]\!]}\{\Delta\ (k)\} \geq 0$. This establishes that Δ is a discrete quasi-concave function, making ψ_1 associated with Application 2 a discrete convex set.

We can now characterize the set of Pwen when Δ_θ is quasi-concave in addition to the convexity and the strategic substitutability of θ. Indeed, we know from Lemma 2 that $\psi_2 =$

$[\![k_2^{\min}, n-1]\!]$, and from Lemma 3, we know that $\psi_1 = [\![k_1^{\min}, k_1^{\max}]\!]$. We use these observations to characterize the set of Pwen.

Proposition 6 *Suppose θ is convex in its first argument, exhibits strategic substitute, and Δ_θ is a discrete quasi-concave function. Moreover, $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$.*

1. *There exists an interior Pwen if and only if $k_2^{\min} \geq k_1^{\max}$ and $k_1^{\min} < n$. The set of interior Pwen is a discrete convex set.*
2. *G^k is an interior Pwen if and only if $k \in \{\underline{k}, \bar{k}\}$, with $\underline{k} = \max\{k_1^{\min}, k_2^{\min}\}$ and $\bar{k} = \min\{k_1^{\max}, n-1\}$.*

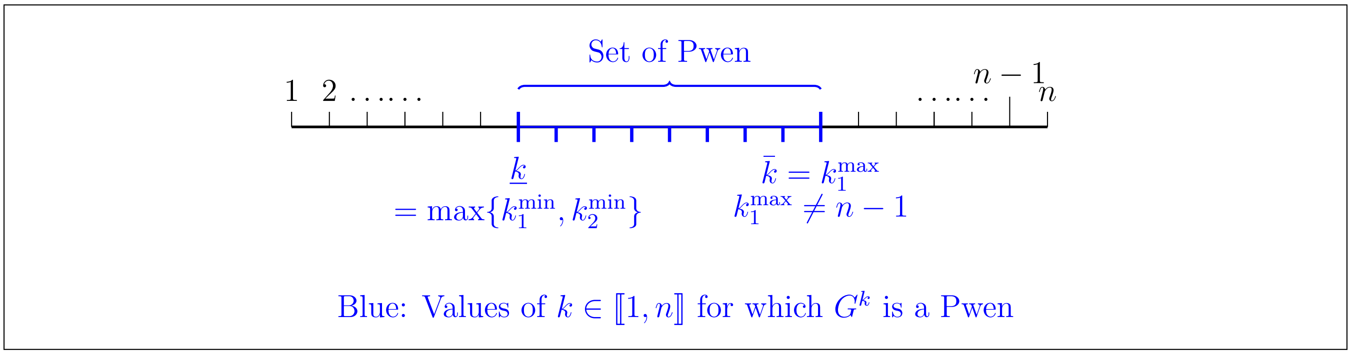


Figure 8: Pwen set for θ with strategic substitutability and Δ_θ quasi-Concave

Let us explain Proposition 6. Suppose that $\psi_1 \cap \psi_2 \neq \emptyset$. Then, $\psi_2 \neq \emptyset$ and when θ exhibits strategic substitute, by Lemma 2, $k_2^{\max} = n-1$. It is worth noting that since $\psi_1 \cap \psi_2 = [\![k_1^{\min}, k_1^{\max}]\!] \cap [\![k_2^{\min}, n-1]\!]$, and $k_1^{\max} \in [\![2, n]\!]$, we have $\psi_1 \cap \psi_2 = [\![\underline{k}, \bar{k}]\!]$, with $\underline{k} = \max\{k_1^{\min}, k_2^{\min}\}$ and $\bar{k} = n-1$ if $k_1^{\max} = n$, and $\bar{k} = k_1^{\max}$ otherwise. We illustrate these observations in Figure 8.

Proposition 6 allows us to derive a straightforward condition for the uniqueness of Pwen. It is important to note that, by construction, $k_1^{\max} \neq 1$ and $k_2^{\min} \neq n$

Corollary 2 *Suppose θ is convex in its first argument, exhibits strategic substitute, and Δ_θ is a discrete quasi-concave function. If $k_2^{\min} = k_1^{\max} = k$, then G^k is the unique Pwen.*

Finally, we observe that when θ satisfies either strategic complementarity or strategic substitutability, the set ψ_2 is a discrete convex set. We now proceed to establish that if $\Lambda(k) = \Lambda_\theta(k) = \theta(1, \eta(k+1)) - \theta(0, \eta(k+1))$ is quasi-convex, then ψ_2 is a discrete convex set. To achieve this, we adapt the concept of quasi-convexity for functions defined over a discrete set. We define Λ as a discrete quasi-convex function if, for all $[\![\kappa_1, \kappa_2]\!] \subseteq [\![k_2^{\min}, n-1]\!]$, the following condition holds:

$$\forall k \in [\![\kappa_1 + 1, \kappa_2 - 1]\!], \Lambda(k) \leq \max\{\Lambda(\kappa_2), \Lambda(\kappa_1)\}.$$

Note that Λ_θ is a discrete quasi-convex function when θ exhibits strategic complementarity. Indeed, $\Lambda_\theta(k) = \theta(1, \eta(k+1)) - \theta(0, \eta(k+1))$. Given that θ exhibits strategic complementarity, Λ_θ is non-decreasing, and as a result, it is a discrete quasi-convex function. Similarly, when θ exhibits strategic substitutability, Λ_θ does not increase, and is therefore a discrete quasi-convex function. In the next result, we establish that if θ is such that Δ_θ is a quasi-concave function and Δ_θ is a quasi-convex function, then the set of interior Pwen is a discrete convex set. Thus, it is easy to characterize the set of Pwen in this case.

Proposition 7 *Suppose that Λ_θ is discrete quasi-convex, then ψ_2 is a discrete convex set. Moreover, if in addition Δ_θ is discrete quasi-concave, then the set of interior Pwen is a discrete convex set. Suppose that $k_1^{\min} < k_1^{\max}$ and $k_2^{\min} < k_2^{\max}$*

1. *Suppose $k_2^{\min} = 1$ and $k_1^{\max} = n$. Networks G^k are interior Pwen if and only if $k \in [\![k_1^{\min}, k_2^{\max}]\!]$. Moreover, networks G^1 and G^n are Pwen.*
2. *Suppose $k_2^{\min} > 1$ and $k_1^{\max} = n$. Networks G^k are interior Pwen if and only if $k \in [\![\underline{k}, k_2^{\max}]\!]$, with $\underline{k} = \max\{k_1^{\min}, k_2^{\min}\}$. Moreover, network G^n is a Pwen.*
3. *Suppose $k_2^{\min} = 1$ and $k_1^{\max} < n$. Networks G^k are interior Pwen if and only if $k \in [\![\underline{k}, \hat{k}]\!]$, with $\underline{k} = \max\{k_1^{\min}, k_2^{\min}\}$ and $\hat{k} = \min\{k_1^{\max}, k_2^{\max}\}$. Moreover, networks G^1 is a Pwen.*
4. *Suppose $k_2^{\min} > 1$ and $k_1^{\max} < n$. Networks G^k are interior Pwen if and only if $k \in [\![k_1^{\min}, k_2^{\max}]\!]$, and $\underline{k} = \max\{k_1^{\min}, k_2^{\min}\}$. Moreover, networks G^1 and G^n are not Pwen.*

Function Λ associated with the payoff function (2) in Application 2 is a discrete quasi-convex function. Specifically, $\Lambda(k) = \frac{(\eta(k+1))^2}{2} + (n-3)\eta(k+1) + (n-2) - F$. Moreover, we observe that $\Lambda(k+1) - \Lambda(k) = (\eta(k+1) - \eta(k))\left(\frac{\eta(k+1)+\eta(k)}{2} + n - 3\right) > 0$, since η is strictly increasing with k. It follows that the set of interior Pwen of Application 2 is a discrete convex set. This is because Δ_θ is a quasi-concave function, and Λ_θ is a quasi-convex function. Proposition 7 implies the following corollary.

Corollary 3 *Suppose that θ induces that Δ_θ is a discrete quasi-concave function and Λ_θ is a discrete quasi-convex function. Let the set of interior Pwen be nonempty. There is a unique interior Pwen if and only if $\max\{k_1^{\min}, k_2^{\min}\} = \min\{k_1^{\max}, k_2^{\max}\}$.*

7 Conclusion

In this paper, we have shown that the convexity of the payoff function of players in the number of their own links is a property that guarantees the existence of Pwen. As emphasized by GJ (2006), this property implies that a Pwen is a group dominant network. We have shown that this result, which seems powerful for characterizing the Pwen, is limited in certain cases, in particular when the set of interior Pwen contains holes. We have established that such holes are absent when the payoff function exhibits strategic complementarity. Then, we have presented an example illustrating the possibility of holes occurring when the payoff function exhibits strategic substitutability. This possibility led us to formulate conditions simplifying the characterization of Pwen sets by eliminating the possibility of holes. As a result, the Pwen set becomes a discrete convex set. These conditions are as follows. First, the payoff function θ induces the function Δ_θ to be a discrete quasi-concave function. Second, θ induces the function Λ_θ to be a discrete quasi-convex function. The fact that the set of interior Pwen is a discrete convex set is an important result, since it makes it easy to obtain uniqueness conditions for Pwen.

We have focused on the convexity of players' payoff functions with respect to their own links. The characterization of the set of Pwen when the payoff function is concave, as an extension, is an interesting investigation. Moreover, the study of the potential uniqueness of the Pwen under concavity adds further interest to our study.

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Appendix

A. Remarks given in model setup

Proof of Remark 1. We prove successively the two parts of this remark.

1. We have $\theta(x,y) - \theta(x_0,y) = \sum_{\ell=0}^{x-x_0-1} \theta_1(x-\ell,y)$. Since θ is convex, $\theta_1(x-\ell,y)$ is decreasing in ℓ. It follows that $\theta_1(x,y) = \max_{\ell\in[\![0,x-x_0-1]\!]} \{\theta_1(x-\ell,y)\}$. Since $\sum_{\ell=0}^{x-x_0-1} \theta_1(x-\ell,y) \geq 0$, $\theta_1(x,y) = \max_{\ell\in[\![0,x-x_0-1]\!]}\{\theta_1(x-\ell,y)\} \geq 0$. Moreover, $\theta(x',y) - \theta(x_0,y) = \sum_{\ell=0}^{x_0-1} \theta_1(x'-\ell,y) = \sum_{\ell=0}^{x'-x-1} \theta_1\ (x'-\ell,y) + \sum_{\ell=0}^{x-x_0-1} \theta_1(x-\ell,y)$. By convexity, for every $\ell \in [\![0,x'-x-1]\!]$, $\theta_1(x'-\ell,y) \geq \theta_1(x,y) \geq 0$. It follows that if $\sum_{\ell=0}^{x-x_0-1} \theta_1(x-\ell,y) \geq 0$, then $\sum_{\ell=0}^{x'-x-1} \theta_1(x'-\ell,y) \geq 0$ and the result follows.

2. We have $\theta(x,y)-\theta(x_0,y) = \sum_{\ell=0}^{x-x_0-1} \theta_1(x-\ell,y) = \sum_{\ell=0}^{x-x_0'-1} \theta_1(x-\ell,y)+\sum_{\ell=x-x_0'}^{x-x_0-1} \theta_1(x-\ell,y)$, and $\theta(x,y)-\theta(x_0',y) = \sum_{\ell=0}^{x-x_0'-1} \theta_1(x-\ell,y)$. Since $\theta_1(x-\ell,y)$ is decreasing in ℓ, we have $\min_{\ell\in[\![0,x-x_0'-1]\!]} \theta_1(x-\ell,y) = \theta_1(x_0'+1,y) \geq \theta_1(x_0',y) = \max_{\ell\in[\![x-x_0',x-x_0-1]\!]} \theta_1(x-\ell,y)$. It follows that if $\theta(x,y)-\theta(x_0,y) \geq 0$, then $\sum_{\ell=0}^{x-x_0'-1} \theta_1(x-\ell,y) \geq 0$ and $\theta(x,y) - \theta(x_0',y) \geq \theta(x,y) - \theta(x_0,y)$. The result follows.

□

Proof of Remark 2. Suppose that $M = [\![a,b]\!]$, with $a,b \in N$, and $a \leq b$. We establish that δ_M is L-convex. For every $p,q \in [\![a,b]\!]$, we have $\left\lfloor \frac{p+q}{2} \right\rfloor, \left\lceil \frac{p+q}{2} \right\rceil \in [\![a,b]\!]$. It follows that

$$\delta_M(p) + \delta_M(q) = 0 \geq 0 = \delta_M\left(\left\lceil \frac{p+q}{2} \right\rceil\right) + \delta_M\left(\left\lfloor \frac{p+q}{2} \right\rfloor\right),$$

and δ_M is L-convex.

Let $M \subseteq N$, $M \neq \emptyset$. We establish that if δ_M is L-convex, i.e.,

$$[p,q \in M, M \subseteq N] \Rightarrow \left\lfloor \frac{p+q}{2} \right\rfloor, \left\lceil \frac{p+q}{2} \right\rceil \in M,$$

then there are $a, b \in N$, with $a \leq b$ such that $M = [\![a, b]\!]$.

Note that M is finite and bounded as a subset of a bounded and finite set. Hence, it admits a minimal element $\sigma^{\min}$ and a maximal element $\sigma^{\max}$ since it is nonempty.

To introduce a contradiction, suppose that there is $\sigma \in [\![\sigma^{\min}, \sigma^{\max}]\!]$ such that $\sigma \notin M$. Let $M^-(\sigma) = \{j \in M : j < \sigma\}$ and $M^+(\sigma) = \{j \in M : j > \sigma\}$, and $\sigma^- = \arg\min_{j \in M^-(\sigma)}\{\sigma - j\}$ and $\sigma^+ = \arg\min_{j \in M^+(\sigma)}\{j - \sigma\}$.

We establish that $\sigma^- = \sigma - 1$ and $\sigma^+ = \sigma + 1$. Suppose it is not the case, then $\sigma^+ - \sigma^- \geq 3$. Let $\sigma_1 = \left\lfloor \frac{\sigma^+ + \sigma^-}{2} \right\rfloor$ and $\sigma_2 = \left\lceil \frac{\sigma^+ + \sigma^-}{2} \right\rceil$, we know that $\sigma_1, \sigma_2 \in M$. We show that $\sigma_1 \in [\![\sigma^- + 1, \sigma^+ - 2]\!]$.

We have
$$\left\lfloor \frac{\sigma^+ + \sigma^-}{2} \right\rfloor \geq \left\lfloor \frac{3 + \sigma^- + \sigma^-}{2} \right\rfloor = \left\lfloor \frac{3}{2} + \sigma^- \right\rfloor = \sigma^- + 1.$$

Similarly,
$$\left\lfloor \frac{\sigma^+ + \sigma^-}{2} \right\rfloor \leq \left\lfloor \frac{\sigma^+ + \sigma^+ - 3}{2} \right\rfloor = \left\lfloor \sigma^+ - \frac{3}{2} \right\rfloor = \sigma^+ - 2.$$

There are two possibilities, either $\sigma < \sigma_1$ and $\sigma^+ \neq \arg\min_{j \in M^+(\sigma)}\{j - \sigma\}$, or $\sigma > \sigma_1$ and $\sigma^- \neq \arg\min_{j \in M^-(\sigma)}\{j - \sigma\}$, a contradiction. We conclude that $\sigma^- = \sigma - 1$ and $\sigma^+ = \sigma + 1$. We have $\sigma \in M$, since $\left\lfloor \frac{\sigma^+ + \sigma^-}{2} \right\rfloor \in M$, and $\left\lfloor \frac{\sigma^+ + \sigma^-}{2} \right\rfloor = \left\lfloor \frac{\sigma + 1 + \sigma - 1}{2} \right\rfloor = \lfloor \sigma \rfloor = \sigma$.

□

B. Convexity of θ

Proof of Proposition 1. We prove successively the three parts of the proposition

1. Consider G^k a k-group dominant network, $k \in [\![2, n-1]\!]$.
   - (a) Suppose that C1 and C2 are satisfied. When C1 holds, by R2, we have $\theta(k-1, \eta(k)) \geq \theta(k - 1 - \kappa, \eta(k))$, $\kappa \in [\![1, k-1]\!]$. It follows that players in $D(G^k)$ have no incentive to remove some of their links in G^k. When C2 holds, players in $E(G^k)$ have no incentive to form a link in G^k. Therefore, when C1 and C2 hold, G^k is a Pwen.
   - (b) Suppose that G^k is a Pwen. Since G^k is a Pwen, players in $D(G^k)$ have no incentive to remove $k-1$ links in g and C1 holds. Moreover, due to the convexity of θ, for all players in $D(G^k)$, we have for $k \leq n-2$, $\theta(k, \eta(k)) - \theta(0, \eta(k)) \geq \theta(k-1, \eta(k)) - \theta(0, \eta(k)) \geq 0$ by R1, i.e., $\theta(k, \eta(k)) - \theta(k-1, \eta(k)) \geq 0$. It follows that players in $D(G^k)$ have an incentive to form an additional link in G^k. Thus, since G^k is a Pwen, players in $E(G^k)$ have no incentive to form a link in G^k. We conclude that C2 holds.
2. Consider the empty network, G^1. Suppose that $\theta_1(1, 0) < 0$. Then, no player has an incentive to form a link in G^1, and G^1 is a Pwen. Conversely, suppose that G^1 is a Pwen. Then, no player has an incentive to form a link in G^1. Consequently, $\theta_1(1, 0) < 0$.

3. Consider the complete network G^n. Suppose that $\theta(n-1, \eta(n)) \geq \theta(0, \eta(n))$. By R2, we have $\theta(n-1, \eta(n)) \geq \theta(n-1-\kappa, \eta(n))$, i.e., no player has an incentive to remove links in G^n, and G^n is a Pwen. Conversely, suppose that G^n is a Pwen. Then, no player has an incentive to remove all his link in G^n. Hence we have $\theta(n-1, \eta(n)) \geq \theta(0, \eta(n))$.

□

Proof of Proposition 2. Let $\Xi = \{k \in [\![2, n]\!] : \theta_1(1, \eta(k)) \geq 0\}$. Suppose that Ξ is empty. Then, $\theta_1(1, \eta(2)) = \theta_1(1, 0) < 0$. By Proposition 1, the empty network is a Pwen. Suppose Ξ is non-empty, i.e., there exists k such that $\theta_1(1, \eta(k)) \geq 0$. Then, Ξ admits a maximal element, say $\hat{k}$, since Ξ is finite. We show that $G^{\hat{k}}$ is a Pwen. We have $\theta(\hat{k}-1, \eta(\hat{k})) - \theta(0, \eta(\hat{k})) \geq 0$ since $\theta_1(1, \eta(\hat{k})) \geq 0$ by $R1$, and $G^{\hat{k}}$ satisfies C1. The proof is over if $\hat{k} = n$ since the complete network is a Pwen by Proposition 1. Suppose that $\hat{k} \neq n$. Since $\hat{k}$ is the maximal element of Ξ, we have $\theta_1(1, \eta(\hat{k}+1)) < 0$, and $G^{\hat{k}}$ satisfies C2. It follows that $G^{\hat{k}}$ is a Pwen by Proposition 1. □

C. Convexity of θ and Strategic Complement

Proof of Lemma 1. We prove the two parts of the result successively.

1. We show that if $\kappa > k$ and $k \in \psi_1$, then $\kappa \in \psi_1$. Because of strategic complementarity of θ, we have $\theta(\kappa-1, \eta(\kappa)) - \theta(0, \eta(\kappa)) = \sum_{\ell=0}^{\kappa-2} \theta_1(\kappa-1-\ell, \eta(\kappa)) \geq \sum_{\ell=0}^{\kappa-2} \theta_1(\kappa-1-\ell, \eta(k)) = \theta(\kappa-1, \eta(k)) - \theta(0, \eta(k))$. By R1, we have $\theta(\kappa-1, \eta(k)) - \theta(0, \eta(k)) \geq \theta(k-1, \eta(k)) - \theta(0, \eta(k))$ since $\theta(k-1, \eta(k)) - \theta(0, \eta(k)) \geq 0$. Moreover, since $k \in \psi_1$, we have $\theta(k-1, \eta(k)) - \theta(0, \eta(k)) \geq 0$. It follows that $\theta(\kappa-1, \eta(k)) - \theta(0, \eta(k)) \geq 0$, and $\kappa \in \psi_1$.
2. We show that if $\kappa < k'$ and $k' \in \psi_2$, then $\kappa \in \psi_2$. Because of strategic complementarity of θ, we have $\theta_1(1, \eta(\kappa)) \leq \theta_1(1, \eta(k'))$. Since $k' \in \psi_2$, we have $\theta_1(1, \eta(k')) < 0$. It follows that $\theta_1(1, \eta(\kappa)) < 0$, and $\kappa \in \psi_2$.

□

Proof of Proposition 3. We show successively the two parts of the proposition.

1. We establish that there is an interior Pwen if and only $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$ and $k_2^{\max} \geq k_1^{\min}$. Recall that $\psi_1, \psi_2 \subseteq [\![2, n-1]\!]$.
   (a) Suppose that $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$ and $k_2^{\max} \geq k_1^{\min}$. Since $\psi_1 \neq \emptyset$ and is finite existence of $k_1^{\min}$ is guaranteed. By Lemma 1.1, for every $k \geq k_1^{\min}$, $k \in \psi_1$, and G^k satisfies C1. Since $\psi_2 \neq \emptyset$ and is finite existence of $k_2^{\max}$ is guaranteed. By Lemma 1.2, for every $k \leq k_2^{\max}$, $k \in \psi_2$, and G^k satisfies C2. If $k_2^{\max} \geq k_1^{\min}$, then there exists $k \in [\![k_1^{\min}, k_2^{\max}]\!]$. Clearly, for $k \in [\![k_1^{\min}, k_2^{\max}]\!]$, we have $k \in \psi_1 \cap \psi_2$, and G^k satisfies C1 and C2. It follows that G^k is a Pwen by Proposition 1.

(b) Suppose that there exists an interior Pwen, G^k. By Proposition 1, G^k satisfies C1 and C2, that is $k \in \psi_1 \cap \psi_2$. It follows that $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$. Moreover, by construction, $k \geq k_1^{\min}$ and $k \leq k_2^{\max}$. It follows that $k_2^{\max} \geq k_1^{\min}$.

2. Suppose that $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$. We show that G^k, $k \in [\![2, n-1]\!]$, is a Pwen if and only if $k \geq k_1^{\min}$ and $k \leq k_2^{\max}$. Suppose that $k \geq k_1^{\min}$ and $k \leq k_2^{\max}$. Then, by Lemma 1, $k \in \psi_1 \cap \psi_2$, and G^k satisfies C1 and C2. Network G^k is a Pwen by Proposition 1. Conversely, suppose that G^k is a Pwen. Then, G^k satisfies C1 and C2. Thus, $k \in \psi_1 \cap \psi_2$. By construction, $k \geq k_1^{\min}$ and $k \leq k_2^{\max}$.

□

D. Convexity of θ and Strategic Substitute

Proof of Lemma 2. Suppose that $k \in \psi_2$, we have $\theta_1(1, \eta(k)) < 0$. By strategic substitute, we have $\theta_1(1, \eta(k)) \geq \theta_1(1, \eta(\kappa))$ for $\kappa > k$. Hence, $\kappa \in \psi_2$. □

Proof of Proposition 5. We prove the two parts of the result successively.

1. Suppose that $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$ and $[\![k_2^{\min}, n-1]\!] \cap \psi_1 \neq \emptyset$. Since ψ_2 is non-empty and finite, $k_2^{\min}$ exists. For every $k' \in [\![k_2^{\min}, n-1]\!]$, we have $k' \in \psi_2$ by Lemma 2, and $G^{k'}$ satisfies C2. Since $[\![k_2^{\min}, n-1]\!] \cap \psi_1 \neq \emptyset$, there is $k \in [\![k_2^{\min}, n-1]\!] \cap \psi_1$. It follows that there is $k \in \psi_1 \cap \psi_2$ and G^k satisfies C1 and C2. Consequently, G^k is a Pwen by Proposition 1. Conversely, If $\psi_1 = \emptyset$ or $\psi_2 = \emptyset$, then there is no interior Pwen since no G^k, $k \in [\![2, n-1]\!]$, satisfies C1 and C2 simultaneously. Similarly, if $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$ and $[\![k_2^{\min}, n-1]\!] \cap \psi_1 = \emptyset$, then no G^k, $k \in [\![2, n-1]\!]$, satisfies C1 and C2.
2. Suppose that $\psi_1 \neq \emptyset$, $\psi_2 \neq \emptyset$ and $[\![k_2^{\min}, n-1]\!] \cap \psi_1 \neq \emptyset$. If $k \geq k_2^{\min}$, then k satisfies C2. Moreover $k \in \psi_1$, consequently G^k is a Pwen by Proposition 1. Conversely, if G^k is a Pwen, then it satisfies C1 and C2. Consequently, $k \in \psi_1 \cap \psi_2$, that is $k \in [\![k_2^{\min}, n-1]\!] \cap \psi_1$.

□

E. Discrete Convexity of the Set of Interior Pwen

Proof of Lemma 3. Suppose that ψ_1 is non-empty. Then, ψ_1 is a convex discrete set if and only if for every $\kappa \in [\![k_1^{\min}+1, k_1^{\max}-1]\!]$, $\kappa \in \psi_1$, i.e., $\Delta(\kappa) \geq 0$, for $\kappa \in [\![k_1^{\min}+1, k_1^{\max}-1]\!]$. Suppose that Δ is discrete quasi-concave over $[\![k_1^{\min}+1, n]\!]$, $\Delta(k_1^{\min}) \geq 0$, and $\Delta(k_1^{\max}) \geq 0$. Since $\kappa \in [\![k_1^{\min}+1, k_1^{\max}-1]\!]$, $\Delta(\kappa) \geq \min\{\Delta(k_1^{\min}), \Delta(k_1^{\max})\} \geq 0$, and $\kappa \in \psi_1$. □

Proof of Proposition 6. We prove successively the two parts of the proposition.

1. Since $\psi_1, \psi_2 \neq \emptyset$, we know that $\psi_1 = [\![k_1^{\min}, k_1^{\max}]\!]$, and $\psi_2 = [\![k_2^{\min}, n-1]\!]$. Clearly, the set of interior Pwen consists in G^k with $k \in \psi_1 \cap \psi_2$. Consequently, the set of interior Pwen is nonempty if and only if $\psi_1 \cap \psi_2 \neq \emptyset$, i.e., $k_2^{\min} \geq k_1^{\max}$ and $k_1^{\min} < n$.

   Suppose $k_2^{\min} \leq k_1^{\max}$ and $k_1^{\min} < n$. Thus, $\psi_1 \cap \psi_2 \neq \emptyset$. By construction, G^k, $k \in [\![2, n-1]\!]$, satisfies C1 and C2 simultaneously if and only if $k \in \psi_1 \cap \psi_2$. We establish that $\psi_1 \cap \psi_2$ is a discrete convex set. Let $\kappa_1, \kappa_2 \in \psi_1 \cap \psi_2$, with $\kappa_1 < \kappa_2 + 1$. We have to show that for $k \in [\![\kappa_1 + 1, \kappa_2 - 1]\!]$, $k \in \psi_1 \cap \psi_2$. Since $\kappa_1, \kappa_2 \in \psi_1$, we have for $k \in [\![\kappa_1 + 1, \kappa_2 - 1]\!]$, $k \in \psi_1$ since ψ_1 is a discrete convex set by Lemma 3. Moreover, since $k > \kappa_1$, by Lemma 2, $k \in \psi_2$. It follows that $k \in \psi_1 \cap \psi_2$, and $\psi_1 \cap \psi_2$ is a discrete convex set.

2. G^k is an interior Pwen if and only if it satisfies both C1 and C2, i.e., $k \in \psi_1 \cap \psi_2$. Note that by Lemma 2, $k_2^{\max} = n-1$, otherwise C2 never holds. It follows that with $\psi_1 \cap \psi_2 = [\![k_1^{\min}, k_1^{\max}]\!] \cap [\![k_2^{\min}, n-1]\!]$. We know that $k_1^{\max} \geq 2$. Hence, we have $\psi_1 \cap \psi_2 = [\![\underline{k}, \bar{k}]\!]$, with $\underline{k} = \max\{k_1^{\min}, k_2^{\min}\}$ and $\bar{k} = \min\{k_1^{\max}, n-1\}$ otherwise.

□

Proof of Proposition 7. Suppose that ψ_2 is non-empty and admits $k_2^{\min}$ and $k_2^{\max}$ such that $k_2^{\min} \neq k_2^{\max}$. It is sufficient to show that if Λ is a discrete convex function, then $k \in [\![k_2^{\min} + 1, k_2^{\max} - 1]\!]$ is in ψ_2. We have $\Lambda(k) \leq \max\{\Lambda(\kappa_2), \Lambda(\kappa_1)\}$ and $\max\{\Lambda(\kappa_2), \Lambda(\kappa_1)\} < 0$ since $k_2^{\min}, k_2^{\max} \in \psi_2$. It follows that $\Lambda(k) = \theta_1(1, \eta(k+1)) < 0$ and $k \in \psi_2$. The fact that the set of Pwen is a discrete convex set is straightforward from Lemma 3. Let us now deal with the four parts. First, we know by Proposition 1 that G^n is a Pwen if and only if $k_1^{\max} = n$ and G^1 is a Pwen if and only if $k_2^{\min} = 1$. Moreover, G^k is an interior Pwen if and only if $k \in \psi_1 \cap \psi_2 = [\![a, b]\!]$. We know that $\psi_1 \subseteq [\![2, n]\!]$ and $\psi_2 \subseteq [\![1, n-1]\!]$ are discrete convex sets. Consequently, when $k_2^{\min} = 1$, we have $\psi_2 = [\![k_2^{\min}, k_2^{\max}]\!] = [\![1, k_2^{\max}]\!]$ and $a = k_1^{\min}$. Similarly, when $k_1^{\max} = n$, we have $\psi_2 = [\![k_2^{\min}, k_2^{\max}]\!] = [\![k_1^{\min}, n]\!]$ and $b = k_2^{\max}$. Finally, when $k_2^{\min} \neq 1$ and $k_1^{\max} \neq n$, $a = \max\{k_1^{\min}, k_2^{\min}\}$ and $b = \min\{k_1^{\max}, k_2^{\max}\}$. The result follows. □

Proof of Corollary 3. Suppose that Δ_θ is quasi-concave and Λ_θ is quasi-convex. Since the set of Pwen is nonempty, by Proposition 7, we know that $\psi_1 = [\![k_1^{\min}, k_1^{\max}]\!]$ and $\psi_2 = [\![k_2^{\min}, k_2^{\max}]\!]$. It follows that $\psi_1 \cap \psi_2 = [\![\max\{k_1^{\min}, k_2^{\min}\}, \min\{k_1^{\max}, k_2^{\max}\}]\!]$ since $\psi_1 \cap \psi_2 \neq \emptyset$. Clearly, $|\psi_1 \cap \psi_2| = 1$ if and only if $\max\{k_1^{\min}, k_2^{\min}\} = \min\{k_1^{\max}, k_2^{\max}\}$.

□