Queueing Problem with Heterogeneous Opportunity Costs

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Abstract

We study queueing problems where agents have heterogeneous, per-period waiting costs that are private-information and may vary over queue positions. Our goal is to implement a Rawlsian allocation rule that minimises the maximum individual waiting cost across all agents. Under complete-information, we introduce the *Just Algorithm*, a simple method that always selects a Rawlsian queue. However, in settings with incomplete-information where agents possess *multidimensional private types*, we demonstrate that no Dominant Strategy Incentive-Compatible (DSIC) mechanism can implement the Rawlsian queueing rule over an unrestricted domain of agent types. This result underscores the challenges of designing fair mechanisms in multidimensional environments with quasi-linear preferences. To address this impossibility, we explore the necessary domain restrictions that allow for the existence of Deterministic DSIC mechanisms. We do this by using the *Weak-Monotonicity* condition from Bikhchandani et al (2006), which is both necessary and sufficient for the existence of deterministic DSIC mechanisms in our setting. Further, we restrict the domain to *one-dimensional private-information*, where agents' per-period waiting costs evolve according to publicly known, agent-specific functions based on their private first-period waiting cost. Within this framework, we construct a DSIC mechanism that implements the *Just Algorithm*, thereby ensuring the Rawlsian queue objective is achieved.Our findings contribute to the literature on mechanism design in queueing problems by providing insights into the necessary and sufficient conditions for achieving fairness under strategic behaviour with heterogeneous waiting costs. This work highlights the complexities involved in mechanism design with multidimensional types and offers a viable solution within a significant and non-trivial restricted multidimensional domain with one-dimensional private-information.

Keywords: Queueing, Dominant Strategy Implementation, Rawlsian

JEL Classification: D63 , D72 , D81

1 Introduction

Queueing theory, a fundamental area within operations research, examines the intricate dynamics of service systems where jobs are sequentially processed by servers. In the mechanism design approach to queueing problems, jobs are modelled as strategic agents possessing private-information about their characteristics, particularly their waiting costs. Because agents incur disutility while waiting, they may misreport their information to gain an advantage, which poses challenges for designing fair and efficient allocation mechanisms.

Models of queueing have been scrutinised from various game-theoretic perspectives. In particular, a growing literature (see Subsection 1.1) on queueing problems with one-dimensional agents' types offers insight into mechanisms that are optimal, fair or both. For example, Mitra (2001) shows that First-Best is achievable with one-dimensional private type and a variety of cost functions. De and Mitra (2017) provides a justification of Rawlsian allocation in sequencing problems with each agent having a constant private per-period opportunity cost. They introduce an algorithm that proposes an order consistent with Rawlsian fairness.

In this study, we aim to extend these insights to more complex scenarios where agents have heterogeneous opportunity costs or per-period waiting costs that vary over time. Specifically, we consider a class of queueing problems involving a finite set of agents characterised by agent-specific waiting cost vectors, representing their *multidimensional types*. The waiting cost for each agent evolves over discrete periods or queue positions.

The total waiting cost incurred by an agent is the sum of their per-period waiting costs until their job is processed. The agents utility is quasi-linear in total waiting time and monetary transfers. Our primary objective is to introduce an algorithm that ensures the allocation is Rawlsian, minimising the maximum individual waiting cost among all agents.

Under complete-information, we develop the *Just Algorithm*, a simple yet effective method that consistently identifies a Rawlsian queue.

However, under incomplete-information, the problem essentially becomes one of *multi-dimensional private-information*. The strategy space of multidimensional type agents is more sophisticated than the one-dimensional agents case, and hence achieving the objective is a difficult task. We demonstrate the impossibility of any Dominant Strategy Incentive-Compatible (DSIC) mechanism implementing our algorithm when agents' types are unrestricted. In fact, truth-telling is not even a Nash equilibrium, hence, no ex-post Incentive-Compatible (EPIC) mechanism exists. This result underscores the difficulty of achieving fairness in multidimensional settings, even within quasi-linear environments like ours.

To address this challenge, we restrict the domain to one-dimensional private-information, where agents' per-period waiting costs evolve according to publicly known, agent-specific functions based on their initial private cost. This approach allows agents' opportunity costs to remain heterogeneous while simplifying the strategic complexity of the problem. Within this restricted domain, we propose a DSIC mechanism that successfully implements the Just Algorithm, thereby ensuring the realisation of the Rawlsian queue.

The findings presented here lay the groundwork for a comprehensive exploration of fair mechanisms in queueing problems with multidimensional private-information. Our work contributes to the literature by highlighting the limitations of implementing fairness in complex settings and providing a viable solution within a restricted but significant domain. The rest of the paper is organised as follows. In subsection 1.1, we review the existing literature to place our work in context, highlighting how our contributions extend the current understanding of queueing problems and mechanism design. Section 2 explains the framework of queueing problems with heterogeneous waiting costs along with some necessary definitions. In Section 3, we develop the Just Algorithm. Subsection 3.1 contains our impossibility result for the unrestricted domain. Section 5 introduces the necessary and sufficient domain restriction characterised by the *Weak-Monotonicity condition* presented in Bikhchandani et al (2006). We propose a transfer rule that implements a Rawlsian queueing rule in Dominant Strategies. Appendix A contains Example 4 demonstrating the difference between Rawlsian and efficient queue and Example 5 demonstrating the impossibility of DSIC mechanism with two agents and unrestricted types. Section 6 concludes.

1.1 Related Literature

In this subsection, we survey the existing literature on mechanism design in queueing problems, focusing on both strategic and fairness considerations. The mechanism design literature for optimal resource allocation rules (mechanisms) is rich. Myerson (1981) studies optimal mechanisms for single-item auctions and one-dimensional continuous type spaces of agents. In Hartline and Karlin (2007), the authors introduce optimal mechanism design with one-dimensional continuous types under Dominant Strategy Incentive Compatibility. The literature covers queueing problems involving strategic as well as fairness considerations.

Works such as Chun (2006b), Moulin (2007), Mishra and Rangarajan (2007), Maniquet (2003), Chun (2006a), Chun (2011) study fairness aspects. These works address concepts like equitable sequencing, consistency in allocation, and the design of rules that satisfy various fairness criteria.

From a strategic standpoint, researchers have investigated mechanisms that encourage truthful reporting and efficient outcomes. Mitra (2001) examines efficient and budget-balanced mechanisms in queueing models, demonstrating that first-best outcomes are attainable under certain conditions when agents have private-information about their waiting costs. Similarly, Dolan (1978) and Suijs (1996) contribute to understanding incentive-compatible mechanisms in queueing systems, focusing on how to align individual incentives with social efficiency. Mitra (2002) explores the implementation of efficient allocation rules when agents have private waiting costs, emphasising the challenges of designing mechanisms that are both efficient and strategy-proof.

However, much of the existing literature tends to focus on agents with one-dimensional types, where each agent's private-information is represented by a single parameter typically their constant per-period waiting cost. This simplification facilitates the design of mechanisms but does not capture the complexity inherent in scenarios where agents have multidimensional private-information. One departure from this is the work by Mitra (2001), who address efficient and budget-balanced mechanism design in a multidimensional queueing model. In their study, agents' waiting costs depend on their position in the queue, introducing a multidimensional aspect to their private-information. However, even in Mitra (2001), unrestricted domain does not admit First-Best mechanisms and two conditions : *Independence Property*, and *Combinatorial Property* characterise the domain admitting First-Best mechanisms.

Duives et al (2015) examines the problem in a setting where the optimal mechanism minimises the total expected transfers to all jobs while being Bayesian-Nash incentive-compatible.Recent progress in deriving optimal mechanisms for multidimensional settings often assumes that the type space is discrete. For example, Armstrong (2000) investigates multi-object auction models where valuations are additive and drawn from a binary distribution (i.e., high or low), highlighting the challenges inherent in multidimensional, discrete type spaces. Similarly, Malakhov and Vohra (2009), Pai and Vohra (2014), Cai et al (2012), and Hoeksma and Uetz (2013) make advances in optimal mechanism design under the assumption of discrete types, acknowledging the increased complexity compared to one-dimensional cases. Mishra and Roy (2013) consider deterministic Dominant Strategy implementation in multidimensional dichotomous domains with private values and quasi-linear utility, providing insights into mechanism design when agents have limited types.

The complexity of optimal mechanism design with multidimensional types is well-established, and the challenges are compounded when agents' private-information is continuous, making strategic reporting a significant challenge. In such environments, designing mechanisms incentive-compatible and satisfy additional desiderata becomes significantly more difficult. It is not uncommon to find cut-off(s) based mechanisms in settings with multidimensional types. Armstrong (2000) discusses how the seller can use personalised pricing schemes (akin to cut-off(s)) to maximise revenue. The mechanisms involve setting different prices or cut-off points for different bidders based on their multidimensional types. Armstrong and Rochet (1999) provides a comprehensive guide to multidimensional screening models, where a principal designs mechanisms to screen agents with private-information along multiple dimensions. The authors discuss how cut-off strategies can be employed when agents have heterogeneous types and how these cut-off(s) can vary among agents. Thanassoulis (2004) paper examines bargaining and mechanism design when agents have private-information about substitutable goods. The mechanisms involve setting individualised thresholds for agreement, which can be interpreted as agent-specific cut-off(s). Manelli and Vincent (2007) study revenue-maximising mechanisms in a multi-good monopoly setting. They show that optimal mechanisms may require offering menus of options (contracts) where different agents self-select based on their types, leading to differing cut-off(s). While Mussa and Rosen (1978) is a classic paper on quality differentiation, it introduces the concept of screening consumers through non-linear pricing, which effectively sets different cut-off(s) for consumers based on their willingness to pay. Other valuable works shedding light on personalised threshold mechanisms which are essentially cut-off(s) based mechanisms include Wilson and Institute (1993), Jehiel et al (1999) etc. These studies demonstrate that personalised mechanisms are a common feature in such settings. In complex mechanism design problems involving multidimensional types, it is common for agents to have different numbers of cut-off points due to heterogeneity in their private-information and the design of optimal contracts. In Armstrong (1996), the optimal pricing scheme involves offering a menu of bundles with different prices, effectively creating different cut-off(s) for different consumers. The number of cut-off points (i.e., the number of bundles or pricing tiers) can vary depending on the heterogeneity of consumer types. In Rochet and Choné (1998), optimal mechanism partitions the type space into different regions (akin to cut-off points). Due to the multidimensionality and heterogeneity of agents' types, the number and structure of these regions can differ among agents, implying that agents may face different numbers of cut-off(s). In our paper as well, although the private-information is restricted to the first-period waiting cost, the evolution of costs remains heterogeneous across agents and hence, the cut-off(s) of agent types to obtain queue positions is not the same. In fact, there may be agents who can obtain only a subset of the queue positions. Given other agents type, the functions determining the evolution of costs for an agent may exclude him from getting some of the queue positions, no matter what his type turns out to be. In the queueing and sequencing problems literature, this variation in cut-off(s) and the variation in number of cut-off(s) for different agents is a novel feature. It follows from the heterogeneity of agents' waiting costs.

Our work contributes to this line of research by exploring fair mechanisms for queueing problems where agents have heterogeneous and position dependent waiting costs which is a setting where agents' types are multidimensional and continuous. Unlike previous studies that prioritise efficiency or budget balance, we aim to implement a Rawlsian allocation rule that minimises the maximum individual waiting cost among all agents. This focus on Rawlsian fairness distinguishes our work from that of Mitra (2001), who primarily seek to identify cost structures that enable first-best implementability in terms of aggregate cost minimisation. We present an example to distinguish the two kinds of queuing rules in Appendix A as Example 4.

Consequentially, implementing fairness notions like the Rawlsian criterion in multidimensional settings is difficult and less explored. Bikhchandani et al (2006) show that a necessary condition for the existence of deterministic DSIC mechanisms is that the social choice rule satisfies *weak monotonicity* (W-Mon) on its domain. Furthermore, on convex domains, Saks and Yu (2005) establish that W-Mon is also sufficient for the existence of DSIC mechanisms implementing the rule. In the context of queueing problems with unrestricted multidimensional types, which form a convex set as noted in Mitra (2001), the Rawlsian allocation rule does not satisfy the W-Mon condition. This lack of compliance leads to the impossibility of designing DSIC mechanisms that implement the Rawlsian queueing rule in such settings.

To overcome this impossibility, we introduce a domain restriction to one-dimensional private-information, allowing agents' per-period waiting costs to evolve according to publicly known, agent-specific functions based on their initial private cost. This restriction maintains the heterogeneity and dynamic nature of agents' waiting costs while simplifying the mechanism design problem. By doing so, we are able to design a DSIC mechanism that implements the Rawlsian queue, contributing to the broader understanding of mechanism design in complex, multidimensional environments.

Our study not only highlights the limitations of implementing fairness in multidimensional settings but also provides a viable solution within a significant and non-trivial restricted domain. This work opens avenues for further research into necessary and sufficient conditions for the existence of DSIC mechanisms in such contexts, potentially aligning with the weak monotonicity conditions identified by Bikhchandani et al (2006) and others.

2 The Framework

Consider a finite set of agents $N = \{1, 2, \ldots, n\}$ who need to get their jobs processed using a single server. The server can serve only one agent at a time, and a job, once started, can not be stopped unless finished. Each agent's job takes one unit of time to get processed. Hence, the server needs to design a queue which is an assignment of agents to queue positions [1].

Each agent incurs disutility while waiting for their job to be processed. The cost incurred by every agent in every period is variable, and is the private-information of the agents. A representative agent-i has per-period waiting cost θ_{i1} in the first period, θ_{i2} in the second period, and so on. $\theta_{ik} \in \mathbb{R}_{++}$ [2] indicates the k^{th} period unit waiting cost of agent-i. The vector $\theta_i = (\theta_{i1}, \theta_{i2}, \ldots, \theta_{in}) \in \mathbb{R}^n_{++}$ is the waiting cost vector of agent-i. If agent-i is served in the k^{th} period (or position), his disutility is given by the sum of waiting cost incurred in each period until job completion i.e. $\sum_{j=1}^{k} \theta_{ik}$. The $n \times n$ positive matrix $\theta = [\theta_{ik}]_{1 \leq i,k \leq n}$ is called the waiting cost profile. Let $\Sigma(N)$ denote the set of all $n!$ possible orderings (queues) over N. We denote by $\sigma(\theta) \in \Sigma(N)$ a particular queue, and write $\sigma_i(\theta) = k$ to mean that agent-i has position k in the queue $\sigma(\theta)$. A *queueing rule* is a function $\sigma : \mathbb{R}^{n\times n}_{++} \to \Sigma(N)$ that specifies, for each profile θ, a unique order $\sigma(\theta) = (\sigma_1(\theta), \ldots, \sigma_n(\theta)) \in \Sigma(N)$ [3]. A *transfer rule* is a function $\tau : \mathbb{R}^{n\times n}_{++} \to \mathbb{R}^n$ that specifies for each profile $\theta \in \mathbb{R}^{n\times n}_{++}$ a transfer vector $\tau(\theta) = (\tau_1(\theta), \ldots, \tau_n(\theta)) \in \mathbb{R}^n$, where $\tau_i(\theta) \in \mathbb{R}$ is the monetary transfer made to the

[1] Through out the paper we only consider assignments which are feasible and maximal. Every agent is assigned to a position. One and only one agent is assigned to each position. We will refer to these simply as queues.

[2] $\mathbb{R}_{++}$ denotes the positive orthant of real line $\mathbb{R}$.

[3] Since the queueing rule is a function and not a correspondence, tie-breaking may be required at some profiles.

agent. The term $\tau_i(\theta)$ is negative if the agent pays and positive if he receives monetary compensation. A *mechanism* $\mu = (\sigma, \tau)$ constitutes a queueing rule σ and a transfer rule τ. The bundle of any agent-i under the mechanism μ at reported profile θ is written as $\mu_i(\theta) = (\sigma_i(\theta), \tau_i(\theta))$. The agents have quasi-linear utility functions of the form $u_i(\mu_i(\theta)) = -\sum_{k=1}^{\sigma_i(\theta)} \theta_{ik} + \tau_i(\theta)$. For any mechanism $\mu = (\sigma, \tau)$, if the reported profile is $(\hat{\theta}_i, \theta_{-i})$ [4] when the true waiting cost vector of agent-i is θ_i, then the utility of agent-i is $u_i(\mu_i(\hat{\theta}_i, \theta_{-i}); \theta_i) = -\sum_{k=1}^{\sigma_i(\hat{\theta}_i, \theta_{-i})} \theta_{ik} + \tau_i(\hat{\theta}_i, \theta_{-i})$.

$\mathcal{Q}^D$ denotes the class of Queueing Problem with heterogeneous waiting costs, $\mathcal{Q}^D(N)$ denotes an instance of such problem with a given set of agents (hence profile). If $\forall j, k \in N, \theta_{ik} = \theta_{ij}$ then agent-i has a constant per-period waiting cost. If all agents have constant per-period waiting cost, we have the class of queueing problems $\mathcal{Q} \subset \mathcal{Q}^D$ with constant per-period waiting cost.

The heterogeneous waiting cost setting implies that each agent-$i \in N$ reports a vector $\theta_i = (\theta_{i1}, \theta_{i2}, \ldots, \theta_{in}) \in \mathbb{R}^n_{++}$. Hence, the agents are multidimensional and $\mathcal{Q}^D$ are problems in *multidimensional mechanism design*. The profile $\theta \in \mathbb{R}^{n\times n}_{++}$ can be visualised as an $n \times n$ matrix where agents are labelled along the rows and periods along the columns. Thus agent-i's report is row-i in the matrix.

$$[\theta] = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix} = \begin{bmatrix} \theta_{11} & \theta_{12} & \ldots & \theta_{1(n-1)} & \theta_{1n} \\ \theta_{21} & \theta_{22} & \ldots & \theta_{2(n-1)} & \theta_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta_{n1} & \theta_{n2} & \ldots & \theta_{n(n-1)} & \theta_{nn} \end{bmatrix} \tag{1}$$

The agents have a quasi-linear utility function. For any profile θ and any queueing rule σ and transfer rule τ, if any agent-i is served in position $\sigma_i(\theta)$ and obtains a transfer $\tau_i(\theta)$, his utility is given by $u_i(\mu_i(\theta)) = -\sum_{k=1}^{\sigma_i(\theta)} \theta_{ik} + \tau_i(\theta)$.

[4]Here, $\theta_{-i} \in \mathbb{R}^{n\times(n-1)}_{++}$ is the set of waiting cost vector announcements by the other $(n-1)$ agents in $N \setminus \{i\}$.

We focus our attention on the queueing rule $\sigma^R \in \Sigma(N)$, which minimises the maximum waiting cost incurred by any agent out of all possible orders. We call such a queueing rule *Rawlsian* in keeping with Rawls' *Maxi-Min*[5] Principle.

Definition 1 Rawlsian queueing rule σ^R: A queueing rule σ^R is called a Rawlsian queueing rule if, for every profile $\theta \in \mathbb{R}^{n\times n}++$, we have $\sigma^R(\theta) \in \arg\min \sigma(\theta) \in \Sigma(N) \max_{i\in N} \sum_{k=1}^{\sigma_i(\theta)} \theta_{ik}$.

For an example of queueing problem with heterogeneous waiting costs, and identification of Rawlsian queue, see Appendix A, Example 4.

We now turn our attention to defining mechanisms $\mu = (\sigma, \tau)$, that *implement* the queueing rule σ. As we are interested in truth-telling mechanisms, by the revelation principle we restrict attention to direct mechanisms. Implementation of a rule σ in Dominant Strategies via a mechanism (σ, τ) requires that the transfer rule τ be such that for any agent, truthful reporting (weakly) dominates false reporting *irrespective* of what others report, where as ex-post implementation requires that truthful reporting for any agent weakly dominates false reporting *conditional* upon all other agents reporting their waiting cost vectors truthfully. The ex-post implementability requires that the transfer rule τ be such that truth-telling is a Nash equilibrium for any agent and every true type profile θ. A Mechanism $\mu = (\sigma, \tau)$ is called a Dominant Strategy Incentive-Compatible (DSIC) Mechanism if it implements the queueing rule σ in Dominant Strategies and an ex-post Incentive-Compatible (EPIC) mechanism if it ex-post implements the rule σ. Every DSIC mechanism is an EPIC mechanism hence if DSIC mechanisms exist, they guarantee existence of EPIC mechanism but the converse may not be true.

[5]The Maxi-Min Principle seeks to maximise the minimum utility obtained by any agent. In the case of disutility, it seeks to minimise the maximum disutility obtained by any agent.

Definition 2 Dominant Strategies Implementation: A Mechanism $\mu = (\sigma, \tau)$ is Dominant Strategy Incentive-Compatible (DSIC) implementable if $\forall i \in N$, $\forall \theta_i, \hat{\theta}_i \in \mathbb{R}^n_{++}$, and $\forall \theta_{-i} \in \mathbb{R}^{n\times(n-1)}_{++}$:

$$u_i(\mu_i(\theta_i, \theta_{-i}); \theta_i) \geq u_i(\mu_i(\hat{\theta}_i, \theta_{-i}); \theta_i)$$

Definition 3 ex-post Implementation: A Mechanism $\mu = (\sigma, \tau)$ is ex-post implementable if $\forall i \in N$, $\forall \theta \in \mathbb{R}^{n\times n}_{++}$, and $\forall \hat{\theta}_i \in \mathbb{R}^n_{++}$:

$$u_i(\mu_i(\theta); \theta_i) \geq u_i(\mu_i(\hat{\theta}_i, \theta_{-i}); \theta_i)$$

3 Unrestricted Domain

In order to implement the Rawlsian queueing rule, we need an algorithm to identify a Rawlsian queue at all profiles. In algorithm 1, we propose a method which always selects a unique queue $\sigma^{JA}(\theta)$ given any profile θ. This is followed by example 1 to demonstrate the working of the algorithm in a 4-agent case. It is easy to verify that the algorithm would select the queue **kij** when applied to example 4. To illustrate how

Algorithm 1 Just Algorithm

Tie-breaking rule

The tie-breaking order is given by $\succ_{TB} := 1 \succ_{TB} 2 \succ_{TB} \ldots \succ_{TB} n$. For all $i, k \in N$, and any $m \in \{1, \ldots, n\}$, if $\{i, k\} \subseteq \arg\min_{j \in N^1(\theta)} \sum_{l=1}^{n} \theta_{jl}$, then $\sigma_i^{JA}(\theta) = m$ whenever $k \succ_{TB} i$.

First step

Let $N^1(\theta) = N$ be the set of agents and $\theta^1 = \theta$ be the reported profile for step-1. Let $i = \arg\min_{j \in N^1(\theta)} \sum_{l=1}^{n} \theta_{jl}$. Assign $\sigma_i^{JA}(\theta) = n$. Let $N^2(\theta) = N^1(\theta) \setminus \{i\}$. Update θ^1 to θ^2 by deleting the last column of θ^1 and the row corresponding to such agent-i.

$\mathbf{k}^{th}$ step ($2 \leq k \leq n-1$)

$N^k(\theta) = N \setminus \bigcup_i \{i\} : \sigma_i^{JA}(\theta) \in \{n+2-k, n\}$. Let $i = \arg\min_{j \in N^k(\theta)} \sum_{l=1}^{n-k+1} \theta_{jl}$. Assign $\sigma_i(\theta) = n - k + 1$. Update θ^k to θ^{k+1} by deleting the last column of θ^k, and the row corresponding to such agent-i.

$\mathbf{n}^{th}$ step

$N^n(\theta) = N \setminus \bigcup_i \{i\} : \sigma_i^{JA}(\theta) \in \{2, n\}$. $||N^n(\theta)|| = 1$. For $i \in N^n(\theta)$, assign $\sigma_i^{JA}(\theta) = 1$.

the Just Algorithm operates in practice, consider Example 1 with four agents.

Example 1 Working of Just Algorithm: Consider a four-agent case. $N^1(\theta) = N = \{i, j, k, l\}$. Let the reported profile be θ. We use the tie-breaking rule $i \succ_{TB} j \succ_{TB} k \succ_{TB} l$.

$$\theta = \theta^1 = \begin{bmatrix} \theta_i \\ \theta_j \\ \theta_k \\ \theta_l \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 5 & 8 & 12 \\ 1 & 3 & 6 & 10 \\ 1 & 3 & 6 & 7 \\ 1 & 3 & 6 & 10 \end{bmatrix} = \bar{\theta}^1 \tag{2}$$

We have transformed the matrix θ into $\bar{\theta}^1$ as follows: $\forall p \in N, \forall q \in \{1, 2, 3, 4\}$, $\bar{\theta}^1_{pq} = \sum_{m=1}^{q} \theta_{pm}$. The cost incurred by agent-$p \in N$ when served in period $q \in \{1, 2, 3, 4\}$ can be read off directly as $\bar{\theta}^1_{pq}$. The algorithm works as follows.

In the first step, we calculate the cumulative waiting costs for each agent if they were to be served last. Agent-k has the lowest total cost of 7, so agent-k is assigned to the last position. Thus, $\sigma_k^{JA}(\theta) = 4$. $N^2(\theta) = N^1(\theta) \setminus \{k\} = \{i, j, l\}$. We update θ^1 to θ^2 by removing the agent-k row and last column of θ^1.

$$\theta^2 = \begin{bmatrix} \theta_i \\ \theta_j \\ \theta_l \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 5 & 8 \\ 1 & 3 & 6 \\ 1 & 3 & 6 \end{bmatrix} = \bar{\theta}^2 \tag{3}$$

In the second step, the algorithm calculates the cost incurred by each of the remaining agents if they were to be served in the third period. The minimum cost which will be incurred by any agent getting served in the last period is 6 if either agent-j or agent-l is served in period 3. The tie-breaking rule, $i \succ j \succ k \succ l$, favours agent-j, so he continues to be in the problem for an earlier period assignment, and agent-l losing the tie is awarded the third position, $\sigma_l^{JA}(\theta) = 3$. We update θ^2 to θ^3 by removing the agent-l row and last column of θ^2.

$$\theta^3 = \begin{bmatrix} \theta_i \\ \theta_j \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix} = \bar{\theta}^3 \tag{4}$$

In the third step, agent-j is assigned to period two since $1 + 2 < 3 + 2$. Thus, $\sigma_j^{JA}(\theta) = 2$. There is one remaining agent, and the agent is served in the first period, $\sigma_i^{JA}(\theta) = 1$.

The maximum cost is incurred by agent-k in the queue $\sigma^{JA}(\theta)$ and is equal to 7. Of the 24 possible queues, it is easily verified that there are six queues which serve agent-3 in period four, and the maximum cost in the other 18 queues will be either 12 or 10 depending upon which of the other agents, i, j or l, is served last. All six queues serving agent-k in period 4 are Rawlsian queues, and the Just Algorithm for perceptive agents selected a queue which is Rawlsian. This example demonstrates that the Just Algorithm systematically assigns positions to minimise the maximum individual waiting cost, resulting in a Rawlsian queue.

Example 1 demonstrates the step-by-step working of the *Just Algorithm* including a tie-breaking situation for queue position 3 between agents j and l.

Proposition 1 The Just Algorithm always selects a Rawlsian queue.

Proof Consider the set of agents N, with any reported profile $\theta \in \mathbb{R}^{n\times n}_{++}$. Let $\sigma^{JA}(\theta)$ be the queue selected by the Just Algorithm. Let $p = \arg\max_{i\in N} \sum_{k=1}^{\sigma_i^{JA}(\theta)} \theta_{ik}$. Let agent-$p$, incurring the maximum cost in σ^{JA}, be served in position-q, i.e. $\sigma_p^{JA}(\theta) = q$.

For brevity of notation, we write $c_i(\sigma(\theta))$ to denote the cost incurred by agent-i in the queue $\sigma(\theta)$. Suppose that $\sigma^{JA}(\theta)$ is not a Rawlsian queue. Let $\sigma(\theta) \neq \sigma^{JA}(\theta)$ be one of the Rawlsian queues such that the maximum of individual cost borne by agents in $\sigma(\theta)$ is less than $c_p(\sigma^{JA}(\theta))$. Suppose $c_r(\sigma(\theta)) < c_p(\sigma^{JA}(\theta))$, where $r = \arg\max_{i\in N} c_i(\sigma(\theta)) = \arg\max_{i\in N} \sum_{k=1}^{\sigma_i(\theta)} \theta_{ik}$.

We have the following cases:

Case 1 Given $\sigma(\theta) \neq \sigma^{JA}(\theta)$, let $\sigma_p(\theta) \geq q$. Then, by definition $c_r(\sigma(\theta)) = \max_{i\in N} c_i(\sigma(\theta))$, and hence $c_r(\sigma(\theta)) \geq c_p(\sigma(\theta))$. But, $c_p(\sigma(\theta)) = \sum_{k=1}^{\sigma_p(\theta)} \theta_{pk} \geq \sum_{k=1}^{q} \theta_{pk} = \sum_{k=1}^{\sigma_p^{JA}(\theta)} \theta_{pk}$. This contradicts the claim that $c_r(\sigma(\theta)) < c_p(\sigma^{JA}(\theta))$, thus completing the proof.

Case 2 Let $\sigma_p(\theta) < q$. Then at least one of the predecessors of agent-p in the queue $\sigma^{JA}(\theta)$ is served in a position $s \geq q$. Let agent-$m(\neq p)$ be such an agent, i.e. $\sigma_m(\theta) = s \geq q$. Then, $c_r(\sigma(\theta)) = \max_{i\in N} c_i(\sigma(\theta)) \geq c_m(\sigma(\theta)) = \sum_{k=1}^{\sigma_m(\theta)} \theta_{mk} = \sum_{k=1}^{s} \theta_{mk} \geq \sum_{k=1}^{q} \theta_{mk} \geq \sum_{k=1}^{q} \theta_{pk}$. The last inequality follows from the algorithm. This contradicts the claim that

$c_r(\sigma(\theta)) < c_p(\sigma^{JA}(\theta))$. This completes the proof.

□

Remark 1 In Mitra (2001), the efficient sequencing rule is studied and First-Best mechanisms satisfying efficiency, Strategy-proofness, and Budget-Balance are investigated. It is shown that a necessary restriction on the domain of agent types to achieve First-Best is the *Independence Property*. The *Independence Property* demands the following: Given a queuing problem with three or more agents in set N, the relative position of any two agents $\{j, l\} \in N$, does not depend upon the presence or absence of another agent-$i \in N$ where $i \neq j, i \neq l$. In the reduced problem obtained by deleting agent-i, j remains a predecessor (or follower) of l, if he was the predecessor (respectively follower) of l when agent-i was present under the same queueing rule. It is easy to verify that the externality of agents in our setting is much severe, and it is possible that change in one agent's report changes the relative position of any other pair of agents. For queueing problems with heterogeneous costs and Rawlsian queueing rule, the *Independence Property* is violated.

3.1 Impossibility Results

Are there DSIC mechanisms that implement the queueing rule σ^{JA}? With the unrestricted type spaces, no such DSIC mechanism exists.

Theorem 1 Consider any problem $\mathcal{Q}^D(N)$, where N is the set of agents with reported profile $\theta \in \mathbb{R}^{n\times n}_{++}$. There is no DSIC mechanism $\mu = (\sigma^{JA}, \tau)$.

Proof We prove this by construction of a generic counter-example.
Consider the set of agents $N = \{1, \ldots, n\}$. Arbitrarily choose any agent-i from N. Construct an admissible waiting cost vector $\theta_i \in \mathbb{R}^n_{++}$ such that $\theta_{i(k+1)} - \theta_{ik} > 0$ for some $k \in \{1, \ldots, n-1\}$. Because of unrestricted domain, such construction is allowed. Let $\epsilon = \frac{\theta_{i(k+1)} - \theta_{ik}}{5} > 0$. We can write $\theta_i = (\theta_{i1}, \ldots, \theta_{ik}, \theta_{ik} + 5\epsilon, \theta_{i(k+2)}, \ldots, \theta_{in})$. Construct $\theta_m = (\theta_{i1}, \ldots, \theta_{i(k-1)}, \theta_{ik} + 4\epsilon, \theta_{ik} + 2\epsilon, \theta_{i(k+2)}, \theta_{in})$. Construct $\hat{\theta}_i = (\theta_{i1}, \ldots, \theta_{i(k-1)}, \theta_{ik} +$

$3.5\epsilon, \theta_{ik} + 3.5\epsilon, \theta_{i(k+2)}, \theta_{in})$). The vectors θ_m, $\hat{\theta}_i$, and θ_i differ only in the k^{th} and $(k+1)^{th}$ coordinate. Let the report $\theta_{-i-m} \in \mathbb{R}^{n\times(n-2)}_{++}$ of agents other than agent-i and agent-m be such that 6, 5 and 7 hold. Consider the profiles profile $\theta = (\theta_i, \theta_m, \theta_{-i-m})$ and another profile $\hat{\theta} = (\hat{\theta}_i, \theta_m, \theta_{-i-m})$.

$$i = \arg \min_{j \in N^{n-k}(\theta)} \sum_{l=1}^{k+1} \theta_{jl} \implies \sigma_i^{JA}(\theta) = k+1 \tag{5}$$

$$m = \arg \min_{j \in N^{n-k+1}(\theta)} \sum_{l=1}^{k} \theta_{jl} \implies \sigma_m^{JA}(\theta) = k \tag{6}$$

$$m = \arg \min_{j \in N^{n-k}(\theta)\setminus\{i\}} \sum_{l=1}^{k+1} \theta_{jl} \tag{7}$$

Equation 5 means that, under Just Algorithm, when queue position $(k+1)$ is to be assigned to one of the agents in the set $N^{n-k}(\theta)$, agent-i has the least cost of getting served in period $(k+1)$ amongst the agents in N^{n-k}. Equation 6 means that at the stage when queue position k is to be assigned to one of the agents in the set $N^{n-k+1}(\theta) = N^{n-k}(\theta) \setminus \{i\}$, agent-m has the least cost of getting served in period k amongst the agents in $N^{n-k+1}(\theta)$. Equation 7 states that if agent-i had not been present in the set $N^{n-k}(\theta)$, agent-m would have been the minimum cost agent to get served in period $(k+1)$.

Note $\theta_{ml} = \hat{\theta}_{il}$, for any $l \in \{1, 2, \ldots, k-1, k+2, \ldots, n\}$. Given equation 7 is true, 8 holds because $\sigma_i^{JA}(\hat{\theta}) > k+1$ cannot be true and $\sum_{l=1}^{k+1} \hat{\theta}_{il} = \epsilon + \sum_{l=1}^{k+1} \theta_{ml} > \sum_{l=1}^{k+1} \theta_{ml}$. Also, given that 6 holds and $\sum_{l=1}^{k} \hat{\theta}_{il} = \sum_{l=1}^{k} \theta_{ml} - 0.5\epsilon$, 9 holds.

$$m = \arg \min_{j \in N^{n-k}(\hat{\theta})} \sum_{l=1}^{k+1} \theta_{jl} \implies \sigma_m^{JA}(\hat{\theta}) = k+1 \tag{8}$$

$$i = \arg \min_{j \in N^{n-k+1}(\hat{\theta})} \sum_{l=1}^{k} \theta_{jl} \implies \sigma_i^{JA}(\hat{\theta}) = k \tag{9}$$

Implementation in Dominant Strategies requires 10 $\geq$ 11, and 12 $\geq$ 13. Both conditions together demand: $\theta_{ik} + 3.5\epsilon = \hat{\theta}_{i(k+1)} \geq \tau_i(\theta) - \tau_i(\hat{\theta}) \geq \theta_{i(k+1)} = \theta_{ik} + 5\epsilon$.

$$u_i(\mu_i(\theta); \theta_i) = -\sum_{l=1}^{k+1} \theta_{il} + \tau_i(\theta) \tag{10}$$

$$u_i(\mu_i(\hat{\theta}); \theta_i) = -\sum_{l=1}^{k} \theta_{il} + \tau_i(\hat{\theta}) \tag{11}$$

$$u_i(\mu_i(\hat{\theta});\hat{\theta}_i) = -\sum_{l=1}^{k}\hat{\theta}_{il} + \tau_i(\hat{\theta}) \tag{12}$$

$$u_i(\mu_i(\theta);\hat{\theta}_i) = -\sum_{l=1}^{k+1}\hat{\theta}_{il} + \tau_i(\theta) \tag{13}$$

For any $\epsilon > 0$, it is *impossible* to find any functions $\tau_i(\theta), \tau_i(\hat{(\theta)})$ satisfying the implementation conditions. Hence, for the constructed profiles, allowed by unrestricted domain, no DSIC mechanism can exist. This completes the proof.

□

In Appendix A, Example 5 shows a two-agent case where DSIC mechanisms are impossible.

Remark 2 In Theorem 1, let the reported profile θ_{-i} be the true waiting cost vectors of agents other than agent-i. Then the same proof implies that for such a profile, no EPIC mechanism exists for the unrestricted domain of types.

4 Domain Restrictions : Necessary

While we have achieved a negative result for the existence of DSIC or even EPIC mechanisms implementing Rawlsian queueing, it is well known that Rawlsian queueing can be implemented by DSIC mechanisms when the types of agents are restricted to have only constant per period waiting costs (see, for example, De and Mitra (2017)). Exactly what domain restrictions are necessary for the existence of DSIC mechanisms?

Social choice rules that allow the existence of deterministic mechanisms must satisfy a necessary condition outlined in Bikhchandani et al (2006) as the *Weak-Monotonicity (W-Mon)* condition. While Bikhchandani et al (2006) establish the necessity of *W-Mon*, Saks and Yu (2005) establish the sufficiency of *W-Mon* over convex domains. Hence, for queueing problems with unrestricted multidimensional types (which are convex as noted in Mitra (2001)), *W-Mon* is a necessary and sufficient condition for the existence of deterministic DSIC mechanisms. The *W-Mon* requirement is the following: If changing one agent's type (while keeping the types of other agents fixed) changes the outcome under the social choice function, then the resulting difference in utilities of the new and original outcomes evaluated at the new type of this agent must be no less than this difference evaluated at the original type of this agent. We present below the definition of *W-Mon* borrowed from Bikhchandani et al (2006), in line with our notation. Then, we apply this definition to the utility structure of agents within the mechanism (σ^{JA}, τ) and obtain the necessary and sufficient condition for the domain of type of agents for which the rule σ^{JA} satisfies *W-Mon* because we know it does not satisfy *W-Mon* over unrestricted domain.

Definition 4 Weak-Monotonicity (W-Mon): A social choice function $\sigma(\cdot)$ is weakly monotone (W-Mon) if, for every $i \in N$, $\theta_i, \theta_i' \in \Theta_i$, and every $\theta_{-i} \in \prod_{j \in N \setminus \{i\}} \Theta_j$,

$$U_i(\sigma(\theta_i', \theta_{-i}); \theta_i') - U_i(\sigma(\theta_i, \theta_{-i}); \theta_i') \geq U_i(\sigma(\theta_i', \theta_{-i}); \theta_i) - U_i(\sigma(\theta_i, \theta_{-i}); \theta_i) \tag{14}$$

Bikhchandani et al (2006) prove (Theorem 2 in their paper) that a social choice function on a completely ordered, bounded domain is truthful if and only if it is weakly monotone. The bounded restriction implies that θ_{ij} is finite $\forall i \in N$, and $\forall j \in \{1, \ldots, |N|\}$. The complete ordering restriction is already satisfied for our framework. All agents prefer a queue position earlier than later.[6] We let the rule be σ^{JA}, and restrict $\theta_i, \theta_i' \in \Theta_i \subset (0, \infty)^n$, then condition 14 requires, for every $i \in N$, $\theta_i, \theta_i' \in \Theta_i$, and every $\theta_{-i} \in \prod_{j \in N \setminus \{i\}} \Theta_j$,

$$-\sum_{k=1}^{\sigma_i^{JA}(\theta_i', \theta_{-i})} \theta_{ik}' - (-\sum_{k=1}^{\sigma_i^{JA}(\theta_i, \theta_{-i})} \theta_{ik}') \geq -\sum_{k=1}^{\sigma_i^{JA}(\theta_i', \theta_{-i})} \theta_{ik} - (-\sum_{k=1}^{\sigma_i^{JA}(\theta_i, \theta_{-i})} \theta_{ik}) \tag{15}$$

$$\sum_{k=\sigma_i^{JA}(\theta_i', \theta_{-i})+1}^{\sigma_i^{JA}(\theta_i, \theta_{-i})} \theta_{ik}' \geq \sum_{k=\sigma_i^{JA}(\theta_i', \theta_{-i})+1}^{\sigma_i^{JA}(\theta_i, \theta_{-i})} \theta_{ik} \tag{16}$$

Without loss of generality, let $\sigma_i^{JA}(\theta_i', \theta_{-i}) < \sigma_i^{JA}(\theta_i, \theta_{-i})$, then condition 16 must hold $\forall k \in \{\sigma_i^{JA}(\theta_i', \theta_{-i}) + 1, \sigma_i^{JA}(\theta_i, \theta_{-i})\}$. It is necessary that this be true for $k = \sigma_i^{JA}(\theta_i', \theta_{-i}) + 1 = \sigma_i^{JA}(\theta_i, \theta_{-i})$. That is $\theta_{ik}' \geq \theta_{ik}$. If it holds for all such $k \in \{2, \ldots, n\}$, then it is straightforward to show that condition 16 must be satisfied. Notice that the W-Mon condition does not include transfers, and especially in the case of quasi-linear utilities, all types of agents evaluate every equal difference in transfer exactly the same.

[6] If for some period some agent has unit waiting cost zero, this does not hold, but such indifference must hold for all types of agents to contradict complete ordering, which is not the case.

The Bikhchandani et al (2006) result tells us the restriction of types for which σ^{JA} is implementable but does not tell us anything about the transfer. Since the result must hold for all profiles θ_{-i}, we can always construct profiles for which $\sigma_i^{JA}(\theta_i, \theta_{-i})$ can take any value from $\{2, \ldots, n\}$, whenever $\sigma_i^{JA}(\theta_i', \theta_{-i})+1 = \sigma_i^{JA}(\theta_i, \theta_{-i})$,we must have $\theta_{ik}' \geq \theta_{ik}, \forall k \in \{2, \ldots, n\}$. These restrictions do not apply to agents' reports for the first period. Hence, we let the agents be multidimensional but restrain the private-information to first-period waiting cost only. In subsection 4.1, we propose a sufficient restriction on their admissible types, which allows for the existence of deterministic DSIC mechanisms.

4.1 One-Dimensional Private-Information : Necessary and Sufficient Condition

Consider the set of agents $N = \{1, \ldots, n\}$. The agents can report their one-dimensional type $\theta_i \in \Theta_i \subseteq \mathbb{R}_{++} \setminus \{\infty\}$ and cost-function $f_i(k, \theta_i)$, where k is the period for which cost is being reported. If $f_i(\cdot, \theta_i)$ is unrestricted; an agent can simply report $f_i(k, \theta_i) = \theta_{ik}$ as in the preceding discussion. We allow different agents to have different cost functions, but these are assumed to be public-information and hence not a part of agents' strategic reports.

Proposition 2 For a queueing problem $\mathcal{Q}^D$, with set of agents N each with a type $\theta_i \in \Theta_i \subseteq \mathbb{R}_{++} \setminus \{\infty\}$ and cost functions $f_i(k, \theta_i)$ where $k \in \{1, \ldots, n\}$, the queueing rule $\sigma^{JA}(\theta)$ is implementable in Dominant Strategies if and only if, $\forall i \in N, k \in \{1, \ldots, n\}, \theta_{-i} \in \prod_{j \neq i} \Theta_j$ and $\forall \theta_i, \theta_i' \in \Theta_i$, the functions $f_i(k, \theta_i) : \{1, \ldots, n\} \times \Theta_i \to \mathbb{R}_{++} \setminus \{\infty\}$ satisfy:

$$\sigma_i^{JA}(\theta_i, \theta_{-i}) > \sigma_i^{JA}(\theta_i', \theta_{-i}) \implies \sum_{k=\sigma_i^{JA}(\theta_i', \theta_{-i})+1}^{\sigma_i^{JA}(\theta_i, \theta_{-i})} f_i(k, \theta_i') \geq \sum_{k=\sigma_i^{JA}(\theta_i', \theta_{-i})+1}^{\sigma_i^{JA}(\theta_i, \theta_{-i})} f_i(k, \theta_i) \quad (17)$$

Proof From the restriction $\theta_i \in \Theta_i \subseteq \mathbb{R}_{++} \setminus \{\infty\}$, the domain of types is bounded and complete, so the necessity and sufficiency of *W-Mon* follows from the (Theorem 2) result of Bikhchandani et al (2006). The sufficiency of *W-Mon* also follows from the result of Saks and Yu (2005) since our domain is convex, as already noted in Mitra (2001) for the unrestricted domain. It only remains to prove that the queueing rule σ^{JA}, which is deterministic, satisfies

W-Mon if condition 17 holds. Suppose the antecedent $\sigma_i^{JA}(\theta_i,\theta_{-i}) > \sigma_i^{JA}(\theta_i',\theta_{-i})$ is true, then $\sum_{k=1}^{\sigma_i^{JA}(\theta_i,\theta_{-i})} f_i(k,\theta_i) \leq \sum_{k=1}^{\sigma_i^{JA}(\theta_i,\theta_{-i})} f_i(k,\theta_i')$, in accordance with the algorithm. If condition 17 holds, then we have:

$$\sum_{k=\sigma_i^{JA}(\theta_i',\theta_{-i})+1}^{\sigma_i^{JA}(\theta_i,\theta_{-i})} f_i(k,\theta_i') \geq \sum_{k=\sigma_i^{JA}(\theta_i',\theta_{-i})+1}^{\sigma_i^{JA}(\theta_i,\theta_{-i})} f_i(k,\theta_i)$$

$$-\sum_{k=1}^{\sigma_i^{JA}(\theta_i',\theta_{-i})} f_i(k,\theta_i') + \sum_{k=1}^{\sigma_i^{JA}(\theta_i,\theta_{-i})} f_i(k,\theta_i') \geq -\sum_{k=1}^{\sigma_i^{JA}(\theta_i',\theta_{-i})} f_i(k,\theta_i) + \sum_{k=1}^{\sigma_i^{JA}(\theta_i,\theta_{-i})} f_i(k,\theta_i)$$

$$U_i(\sigma_i^{JA}(\theta_i',\theta_{-i});\theta_i') - U_i(\sigma_i^{JA}(\theta_i,\theta_{-i});\theta_i') \geq U_i(\sigma_i^{JA}(\theta_i',\theta_{-i});\theta_i) - U_i(\sigma_i^{JA}(\theta_i,\theta_{-i});\theta_i)$$

In the last step of calculation, we add the transfer terms $\tau_i(\theta_i',\theta_{-i}) - \tau_i(\theta_i,\theta_{-i})$ to both sides. Irrespective of the true type of agent-i, this transfer difference is evaluated as the same difference in utility by any agent type. If condition 17 holds, then the queueing rule σ^{JA} satisfies *W-Mon*. This completes the proof.

□

The necessary restrictions on domain obtained by us are not easy to use in search of mechanisms. More structure over the domain is needed to be able to identify mechanisms that are DSIC and implement the rule σ^{JA}. Section 5 furthers the discussion in this regard.

5 Domain Restriction: One-Dimensional Private-Information

When per-period waiting costs are constants i.e., for all agents $i \in \mathcal{N}$, $\theta_{ik} = \theta_i \in \mathbb{R}_+$ for all $k \in \{1, 2, \ldots, n\}$, then Rawlsian queueing rule (coincides with aggregate cost minimising queueing rule) can be implemented by DSIC mechanisms (see Mitra (2001), Chun (2006a), Hashimoto and Saitoh (2012) etc.).

5.1 Domain Restriction

We restrict the domain to one-dimensional private-information setting but not constant per-period costs. We use the notation $f_i(k,\theta_i) > 0$ to denote the k^{th}-period waiting cost of agent-i

of one-dimensional private-type $\theta_i \in \mathbb{R}_+$. $F_i^k(\theta_i) = \sum_{l=1}^{k} f_i(l, \theta_i)$ denotes the total waiting cost of agent-i when he waits for $k \in \{1, \ldots, n\}$ periods. We put the following restrictions:

(public-information) The functions $f_i(k, \cdot)$ are public-information for all periods $k \in \{1, \ldots, n\}$ and all agents $i \in \mathcal{N}$. In general, $f_i(k, \theta_i) \neq f_j(k, \theta_i)$ for two distinct agents i and j [7].

(private-information) The only private-information is the first period waiting cost for all agents, i.e., $f_i(1, \theta_i) = \theta_i$ for all agents-$i \in \mathcal{N}$.

(per-period costs) The functions $f_i(k, \cdot)$ are continuous and non-decreasing in their second argument.

(last-period cost) The functions $F_i^n(\theta_i) = \sum_{l=1}^{n} f_i(l, \theta_i) : \mathbb{R}_+ \to \mathbb{R}_+$ have a full range. It is unbounded on upper-side, there exists $\theta_i \in \mathbb{R}_+ : F_i^n(\theta_i) = 0$. By definition, it is continuous and increasing in θ_i since $f_i(k, \theta_i)$ are non-decreasing for all $k \in \{2, \ldots, n\}$ and $f_i(1, \theta_i) = \theta_i$ is increasing in θ_i.

The last-period cost assumption of full range is for ease of exposition and to ensure that every agent can get served in the last position for some report of θ_i at every fixed θ_{-i}. Otherwise, fix any θ_{-i} and $f_j(k, \theta_j)$ for all $j \neq i$, and let $f_i(k, \theta_i) = a_k \theta_i + b_k$ where $a_k, b_k > 0$ for all $k \in \{2, \ldots, n\}$. Then, $F_i^n(\theta_i) = \sum_{k=1}^{n}(a_k \theta_i + b_k)$. The number $\sum_{k=1}^{n} b_k$ may be such that $\sigma_i^{JA}(\theta_i, \theta_{-i}) \neq n$ for any report $\theta_i \in \mathbb{R}_+$.

5.2 Domain Restriction: Implications

The class of queueing problems with the restricted domain is $\mathcal{Q}^D = < \mathcal{N}, \{f_i(k, \cdot)\}_{i \in \mathcal{N}} >$. Given our domain restriction, the Just Algorithm works as follows: $\sigma_i^{JA}(\theta_i, \theta_{-i}) = n$ if $i = \arg\min_{j \in \mathcal{N}} F_j^n(\theta_j)$, where tie(s) are assumed to be resolved. Then, $\sigma_k^{JA}(\theta_i, \theta_{-i}) = n - 1$ if $k = \arg\min_{j \in \mathcal{N} \setminus \{i\}} F_j^n(\theta_j)$, where tie(s) are assumed to be resolved. By looking at the allocation of positions, we cannot decide the order between $F_i^{n-1}(\theta_i)$ and $F_k^{n-1}(\theta_k)$. Suppose agent-i reports a very high type $\bar{\theta}_i$ such that for some fixed θ_{-i}, $\sigma_i(\bar{\theta}_i, \theta_{-i}) = 1$. Such $\bar{\theta}_i$ exists because $F_i^k(\theta_i)$ are increasing functions of θ_i for all periods $k \in \{1, \ldots, n\}$ and θ_{-i} is fixed. Similarly, since $F_i^n(\theta_i)$ has full range in $\mathbb{R}_+$, for some arbitrarily small $\underline{\theta}_i$, $\sigma_i^{JA}(\underline{\theta}_i, \theta_{-i}) = n$. However, unlike queueing problem with constant per-period costs, such domain restriction

[7] Since $f_i(k, \cdot) \neq f_j(k, \cdot)$ in general, the functions also specify a type for each agent but this is public-information.

does not guarantee, for a fixed θ_{-i}, that agent-i can obtain any queue position by reporting some waiting cost. Example 2 illustrates a case where agent-3 can never get queue position-2.

Example 2 An Illustration of Limited Accessibility to Queue Positions Consider three agents $\mathcal{N} = \{1, 2, 3\}$. Let the tie-breaking rule be $1 \succ_{TB} 2 \succ_{TB} 3$. Fix $\theta_1 = 5$ and $\theta_2 = 7$. The cost functions are given by:

- Agent 1: $f_1(2, \theta_1) = \theta_1,\ f_1(3, \theta_1) = 18\theta_1$.
- Agent 2: $f_2(2, \theta_2) = 2\theta_2,\ f_2(3, \theta_2) = 11\theta_2$.
- Agent 3: $f_3(2, \theta_3) = 3\theta_3,\ f_3(3, \theta_3) = 3\theta_3$.

We examine how agent 3's reported type θ_3 affects their position in the queue.

$$\theta = \begin{bmatrix} 5 \\ 7 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 90 \\ 7 & 14 & 77 \\ \theta_3 & 3\theta_3 & 3\theta_3 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 10 & 100 \\ 7 & 21 & 98 \\ \theta_3 & 4\theta_3 & 7\theta_3 \end{bmatrix}$$

If agent-3 reports his cost of waiting for three periods more than 98 - only then will he not be served in the third position. If he not served third, then agent-2 will be served third. Agent-3 cannot be served in the second position if reports waiting cost for two periods more than 10. But as agent-3 changes his reports from zero to any arbitrarily large number, he crosses the threshold waiting cost of 10 for position-2 before he can cross the threshold waiting cost of 98 for position-3. An agent can be served in an earlier position only if he reports his total waiting cost for all later positions more than the respective threshold waiting costs. This example demonstrates that agent 3 cannot access position 2 regardless of their reported type. The structure of the cost functions and the agents' reported types result in agent 3 being assigned either to position 3 (when $\theta_3 \leq 14$) or position 1 (when $\theta_3 > 14$), but never to position 2. This concludes the example.

Example 2 illustrates that, under our domain restriction, agents may be constrained in the queue positions they can obtain due to the interplay between their cost functions and those of other agents. It highlights that even with one-dimensional private information, the heterogeneity of agents' cost evolutions can prevent certain queue positions from being accessible.

This underscores the importance of carefully designing mechanisms that account for these limitations while striving to implement the Rawlsian queue.

Consider the profile $(\bar{\theta}_i, \theta_{-i})$ with $\sigma_i^{JA}(\bar{\theta}_i, \theta_{-i}) = 1$. For every agent $k \neq i$, let $\sigma_k^{JA}(\bar{\theta}_i, \theta_{-i}) = \hat{k}$. The cost cut-off(s) of agent-i for all positions $\hat{k} \in \{2, \ldots, n\}$ are defined as the costs of the agent getting served in position-$\hat{k}$ ($=F_k^{\hat{k}}(\theta_k)$) when agent-i is served first. Since θ_i is fixed, we suppress the dependence of cut-off on θ_{-i} for ease of notation.

Definition 5 (Cost Cut-off of agent-i for position $\hat{k}$) For a given θ_{-i} and per-period cost functions $f_j(\hat{k}, \theta_k)$ for all agents $j \in \mathcal{N}$ and all positions $\hat{k} \in \{2, \ldots, n\}$, the cost cut-off of agent-i for position-$\hat{k}$ is $(F_k^{\hat{k}}(\theta_k))$.

For every position $\hat{k} \in \{2, \ldots, n\}$, we can calculate agent-i's type cut-off as the highest type that agent-i should have been so that he could obtain position $\hat{k}$ in the sequence or the lowest type that he should have been to obtain a position earlier than $\hat{k}$. This type is found by equating agent-i's cost of waiting for $\hat{k}$ periods to the cost cut-off for that position.

Definition 6 (Type Cut-off of agent-i for position $\hat{k}$) For a given θ_{-i} and per-period cost functions $f_j(\hat{k}, \theta_k)$ for all agents $j \in \mathcal{N}$ and all positions $\hat{k} \in \{2, \ldots, n\}$, the type cut-off of agent-i for position-$\hat{k}$ is $\theta_i^{\hat{k}} = (F_i^{\hat{k}})^{-1}(F_k^{\hat{k}}(\theta_k))$.

Agent-i can obtain a position earlier than $\hat{k}$ only if his reported type $\theta_i \geq \theta_i^{\hat{k}}$ [8]. However, this is not sufficient. Because of the way that the Just Algorithm works, an agent cannot get a position $\hat{k} - 1$ before passing the cost cut-off(s) for all positions $\hat{k}, \ldots, n$. If agent-i reporting θ_i obtains a position $\hat{k} - 1$, then it must be the case that $\theta_i \geq \theta_i^p$ for all $p \in \{\hat{k}, \ldots, n\}$. Given θ_{-i}, the set of type cut-off of agent-i for all positions is the set $= \{\theta_i^n, \ldots, \theta_i^2\}$. But there is no position-based ordering of the cut-off(s). For any report, $\theta_i \in [0, \theta_i^{\hat{k}})$, agent-$i$ cannot get a position earlier that position $\hat{k}$, which means that if $\theta_i^{\hat{k}} \leq \theta_i^n$ then as agent-i's report increases from zero to θ_i^n, his position continues to be position-n, and if his report increases any further, he has already crossed the cost cut-off for period $\hat{k}$. Therefore, he never obtains position $\hat{k}$

[8] In case of a tie, the tie-breaking rule decides the position of the agent. But the type cut-off(s) can be calculated without considering explicitly how ties are resolved.

for any of his possible reports. This is why the assumption on last-period costs is crucial. It dictates that every agent-i, for any θ_{-i} should be able to obtain the last position. Agent-i may obtain position $\hat{k}$ for his report $\theta_i \in [0, \theta_i^{\hat{k}}]$ but the agent only obtains the last position for all of his reports $\theta_i \in [0, \theta_i^n]$. If $\theta_i^{\hat{k}} \leq \theta_i^n$, then $[0, \theta_i^{\hat{k}}] \subseteq [0, \theta_i^n]$. Every type cut-off of agent-i for position $\hat{k}$ satisfying $\theta_i^{\hat{k}} \leq \theta_i^n$ is irrelevant. Since $F_i^k(\theta_i) = \sum_{l=1}^{k} f_i(l, \theta_i) = \theta_i + \sum_{l=1}^{k} f_i(l, \theta_i)$ is increasing in θ_i, every agent-i, for any θ_{-i} can also obtain the first position in the sequence selected by Just Algorithm. Consider the set of type cut-off(s) of agent-i for all positions $:= \{\theta_i^n, \ldots, \theta_i^2\}$. We order this set in decreasing order of sequence positions to obtain the vector $(\theta_i^n, \ldots, \theta_i^2)$. From this vector, we delete all irrelevant type cut-off(s) $\theta_i^{\hat{k}} \leq \theta_i^n$ to obtain the reduced vector $(\theta_i^{m_0} = \theta_i^n, \ldots, \theta_i^s)$ for some $s \in \{2, \ldots, n-1\}$ where the elements are ordered in decreasing order of sequence positions. Let $\theta_i^{m_1}$ be the second element in the reduced vector $(\theta_i^{m_0} = \theta_i^n, \theta_i^{m_1}, \ldots, \theta_i^s)$. From this reduced vector, we preserve $\theta_i^{m_0} = \theta_i^n$ and delete all irrelevant type cut-off(s) $\theta_i^{\hat{k}} \leq \theta_i^{m_1}$ to obtain the reduced vector $(\theta_i^n, \ldots, \theta_i^s)$ for some $s \in \{2, \ldots, n-1\}$ where the elements are ordered in decreasing order of sequence positions. We continue such reduction iteratively until we get a vector$(\theta_i^n = \theta_i^{m_0}, \theta_i^{m_1}, \ldots, \theta_i^{m_{M(i)}})$ for some $M(i) \in \{0, \ldots, n-2\}$ where the elements are ordered in decreasing order of sequence positions and $\theta_i^{m_l} < \theta_i^{m_{l+1}}$ for all $l \in \{0, \ldots, M(i)-1\}$. This is the type cut-off vector for agent-i.

Definition 7 (Type Cut-off vector of agent-i) For all agents $j \in \mathcal{N}$, a given θ_{-i}, per-period cost functions $f_j(\hat{k}, \theta_k)$ and all positions $\hat{k} \in \{2, \ldots, n\}$, agent-$i$'s type cut-off vector is defined as $\theta_{\mathbf{i}}^{\mathbf{cfs}} := (\theta_i^n = \theta_i^{m_0}, \theta_i^{m_1}, \ldots, \theta_i^{m_{M(i)}})$ where every $\theta_i^{m_l}$ is a type cut-off of agent-i for some position $\hat{m}_l \in \{2, \ldots, n\}$ satisfying $\hat{m}_l > \hat{m}_{l+1}$ and $\theta_i^{m_l} < \theta_i^{m_{l+1}}$ for all $l \in \{0, \ldots, M(i)-1\}$.

Every agent can obtain the first and the last position for some report. The number of positions that agent-i can obtain by varying his reports is equal to $M(i)+1$. If agent-i's report $\theta_i \in [0, \theta_i^n]$ [9], he is served last. If $\theta_i \in (\theta_i^{m_l}, \theta_i^{m_{l+1}})$, then $\sigma_i^{JA}(\theta_i, \theta_{-i}) = \hat{m}_{l+1}$ because agent-i has more than the minimum cost in all positions after $\hat{m}_{l+1}$ and he has the minimum cost

[9] If his reported cost is tied with any cut-off, the tie-breaking rule $\succ_{TB}$ allocates the position to agent-i.

for that position.

We define the transfer rule τ^{JA} below.

Definition 8 The transfer rule $\tau^{JA}(\theta_i, \theta_{-i})$ for any profile $(\theta_i, \theta_{-i}) \in \mathbb{R}^n_+$, every agent-$i \in \mathcal{N}$ with cut-off(s) vector $\theta_{\mathbf{i}}^{\mathbf{cfs}} := (\theta_i^n = \theta_i^{m_0}, \ldots, \theta_i^{m_{M(i)}})$, and arbitrary $h_i(\theta_{-i}) : \mathbb{R}^{n-1}_+ \to \mathbb{R}$ is defined as:

$$\tau_i^{JA}(\theta) = \begin{cases} h_i(\theta_{-i}) & \text{if } \sigma_i^{JA}(\theta) = n \\ h_i(\theta_{-i}) - \sum_{r=1}^{l} \sum_{j=\hat{m}_r+1}^{\hat{m}_{r-1}} f_i(j, \theta_i^{m_{r-1}}) & \text{if } \sigma_i^{JA}(\theta) = \hat{m}_l \end{cases} \tag{18}$$

The transfer of agent-i according the rule τ^{JA} is the following:

- If the agent is served last, he gets an arbitrary amount $h(\theta_{-i})$.
- If his position (say position-$k = \hat{m}_r$) is not the last position then for each position $k+1, k+2, \ldots, \hat{m}_{r-1}$ where $\theta_i^{m_{r-1}}$ is the lowest type for which agent-i could get position $\hat{m}_r$, he pays the cost $\sum_{j=k+1}^{\hat{m}_r} f_i(j, \theta_i^{m_{r-1}})$, for all positions $\hat{m}_{r-1}+1, \hat{m}_{r-1}+2, \ldots, \hat{m}_{r-2}$, the lowest type he should have been to be served in position $\hat{m}_{r-1}$ is the cut-off $\theta_i^{m_{r-2}}$, so he pays the cost $\sum_{j=\hat{m}_{r-1}+1}^{\hat{m}_{r-2}} f_i(j, \theta_i^{m_{r-2}})$, and so on.

We state our main result as Theorem 2.

Theorem 2 For any $\mathcal{Q}^D =< \mathcal{N}, \{f_i(k,\cdot)\}_{i\in\mathcal{N}} >$ and any profile $\theta \in \mathbb{R}^n_+$, the mechanism $\mu^{JA} = (\sigma^{JA}, \tau)$ is DSIC if and only if the transfer rule is τ^{JA}.

Proof For any arbitrary agent-i, fix any $\theta_{-i} \in \mathbb{R}^{n-1}_+$. Consider any mechanism $\mu = (\sigma^{JA}, \tau)$. Let $\mu_i(\theta) = (\sigma_i^{JA}(\theta), \tau_i(\theta))$ denote agent-i's bundle under the mechanism μ when profile θ is reported. Let $u_i(\mu_i(\theta_i', \theta_{-i}); \theta_i)$ denote the utility of agent-i from the bundle $\mu_i(\theta_i', \theta_{-i})$ when his true type is θ_i and he reports θ_i'.

For any $k \in \{1, \ldots, n\}$, let $\theta_i^k, \hat{\theta}_i^k \in \mathbb{R}_+$ be any two reports of agent-i such that $\sigma_i^{JA}(\theta_i^k, \theta_{-i})$

$=\sigma_i^{JA}(\hat{\theta}_i^k, \theta_{-i}) = k$ [10].

$$u_i(\mu_i(\theta_i^k, \theta_{-i}); \theta_i^k) = -F_i^k(\theta_i^k) + \tau_i(\theta_i^k, \theta_{-i}) \tag{19}$$

$$u_i(\mu_i(\hat{\theta}_i^k, \theta_{-i}); \theta_i^k) = -F_i^k(\theta_i^k) + \tau_i(\hat{\theta}_i^k, \theta_{-i}) \tag{20}$$

$$u_i(\mu_i(\hat{\theta}_i^k, \theta_{-i}); \hat{\theta}_i^k) = -F_i^k(\hat{\theta}_i^k) + \tau_i(\hat{\theta}_i^k, \theta_{-i}) \tag{21}$$

$$u_i(\mu_i(\theta_i^k, \theta_{-i}); \hat{\theta}_i^k) = -F_i^k(\hat{\theta}_i^k) + \tau_i(\theta_i^k, \theta_{-i}) \tag{22}$$

$$\tau_i(\theta_i^k, \theta_{-i}) = \tau_i(\hat{\theta}_i^k, \theta_{-i}) \tag{23}$$

If the mechanism μ implements the queueing rule σ^{JA} in Dominant Strategies, then 19 $\geq$ 20 and 21 $\geq$ 22. The transfer of any agent must be independent of his own report if his position in the queue does not change, i.e., condition 23 is necessary.

We now consider an agent's reports when the reports lead to different queue positions. Since θ_{-i} is fixed, there is some agent-j satisfying : $j = \arg\min_{l \in \mathcal{N} \setminus \{i\}} F_l^n(\theta_l)$ [11]. Let θ_i^n be the highest report $\theta_i \in \mathbb{R}_+$ such that $i \in \arg\min_{l \in \mathcal{N}} F_l^n(\theta_l)$ [12]. Notice that θ_i^n is the lowest report for which agent-i can obtain a better position than the last position if the tie breaking rule favours him. Therefore $F_i^n(\theta_i^n) = F_j^n(\theta_j) = \min_{l \in \mathcal{N}} F_l^n(\theta_l)$, and hence $\theta_i^n = F_i^{n^{-1}}(F_j^n(\theta_j)$ - the cut-off for agent-i for position-n. Implementation in Dominant Strategies would demand that the utility of agent-i be the same no matter how the tie is resolved i.e. the mechanism be Essentially Single Valued. If this is not true, then agent-i can misreport to be in a tie (or not in a tie) to get the advantage (or avoid disadvantage) of the tie-breaking rule. Hence we can calculate his utility i at position-n, and the position he would get if tie is resolved differently. The tie-breaking rule is the same, but with arbitrary choice of agent-i, arbitrary θ_{-i} all cases need consideration. Let $\hat{m}_1$ denote the position of agent-i if tie is resolved in his favour. This demands that the utilities in equation 24 and in 25 be equal. Let $\theta_i^{m_1}$ denote the highest report fo which $\sigma^{JA}(\theta_i, \theta_{-i})$ may be $\hat{m}_1$.

$$u_i(\mu_i(\theta_i^n, \theta_{-i}); \theta_i^n) = -F_i^n(\theta_i^n) + \tau_i(\theta_i^n, \theta_{-i}) \tag{24}$$

$$u_i(\mu_i(\theta_i^n, \theta_{-i}); \theta_i^n) = -F_i^{\hat{m}_1}(\theta_i^n) + \tilde{\tau}_i(\theta_i^n, \theta_{-i}) \tag{25}$$

[10] It can be verified that $\sigma^{JA}(\theta_i^k, \theta_{-i}) = \sigma^{JA}(\hat{\theta}_i^k, \theta_{-i})$

[11] If there are more than one such agents, consider any such agent arbitrarily.

[12] The functions $F_i^k(\theta_i)$ are increasing functions of θ_i. Hence, we can find a unique θ_i corresponding to any value of $F_i^k(\theta_i)$ for all periods, all agents, and all reports θ_i.

Thus, another necessary condition for the transfer rule is condition 26.

$$\tau_i(\theta_i^n, \theta_{-i}) - \tilde{\tau}_i(\theta_i^n, \theta_{-i}) = F_i^n(\theta_i^n) - F_i^{\hat{m}_1}(\theta_i^n) = \sum_{l=\hat{m}_1+1}^{n} f_i(l, \theta_i^n) \tag{26}$$

Suppose $\theta_i^{m_{l-1}}$ is the lowest type of agent-i so that he may obtain position $\hat{m}_l \in \{1, \ldots, n-1\}$ and the highest type so that he can obtain position $\hat{m}_{l-1} \in \{2, \ldots, n\}$. Clearly, $\hat{m}_l < \hat{m}_{l-1}$ and $\theta_i^{m_{l-1}}$ is the type cut-off of agent-i for position $\hat{m}_{l-1}$. Since DSIC demands Essentially Single-Valued-ness, we need the utilities in equations 27 and 28 to be equal.

$$u_i(\mu_i(\theta_i^{m_{l-1}}, \theta_{-i}); \theta_i^{m_{l-1}}) = -F_i^{\hat{m}_{l-1}}(\theta_i^{m_{l-1}}) + \tau_i(\theta_i^{m_{l-1}}, \theta_{-i}) \tag{27}$$

$$u_i(\mu_i(\theta_i^{m_{l-1}}, \theta_{-i}); \theta_i^{m_{l-1}}) = -F_i^{\hat{m}_l}(\theta_i^{m_{l-1}}) + \tilde{\tau}_i(\theta_i^{m_{l-1}}, \theta_{-i}) \tag{28}$$

Thus, another necessary condition for the transfer rule is condition 29 for all positions $\hat{m}_l$ and $\hat{m}_{l-1}$ obtainable by agent-i.

$$\tau_i(\theta_i^{m_{l-1}}, \theta_{-i}) - \tilde{\tau}_i(\theta_i^{m_{l-1}}, \theta_{-i}) = F_i^{\hat{m}_{l-1}}(\theta_i^{m_{l-1}}) - F_i^{\hat{m}_l}(\theta_i^{m_{l-1}}) = \sum_{l=\hat{m}_l+1}^{\hat{m}_{l-1}} f_i(l, \theta_i^{m_{l-1}}) \tag{29}$$

From conditions 23 and 29, if $\theta_i \in \mathbb{R}_+$[13] is any report such that $\sigma^{JA}(\theta_i, \theta_{-i}) = \hat{m}_l$, then for all obtainable positions $\hat{m}_l \in \{1, \ldots, n-1\}$, equation 30 is necessary.

$$\tau_i(\theta_i, \theta_{-i}) = \tilde{\tau}_i(\theta_i^{m_{l-1}}, \theta_{-i}) = \tau_i(\theta_i^{m_{l-1}}, \theta_{-i}) - \sum_{l=\hat{m}_l+1}^{\hat{m}_{l-1}} f_i(l, \theta_i^{m_{l-1}}) \tag{30}$$

From equation 30, we have $\tilde{\tau}_i(\theta_i^{m_{l-1}}, \theta_{-i}) = \tau_i(\theta_i^{m_l}, \theta_{-i})$. Suppose $\bar{\theta}_i$ is such that $\sigma_i^{JA}(\bar{\theta}_i, \theta_{-i}) = \hat{m}_r$ for some $r \in \{0, \ldots, M(i)\}$. From 30, $\tau_i(\bar{\theta}_i, \theta_{-i}) = \tilde{\tau}_i(\theta_i^{m_{r-1}}, \theta_{-i}) = \tau_i(\theta_i^{m_r}, \theta_{-i})$. Let $\hat{m}_r > \hat{m}_l$, without loss of generality.

$$u_i(\mu_i(\theta_i, \theta_{-i}); \theta_i) = -F_i^{\hat{m}_l}(\theta_i) + \tau_i(\theta_i, \theta_{-i}) = -F_i^{\hat{m}_l}(\theta_i) + \tau_i(\theta_i^{m_l}, \theta_{-i}) \tag{31}$$

$$u_i(\mu_i(\bar{\theta}_i, \theta_{-i}); \bar{\theta}_i) = -F_i^{\hat{m}_r}(\bar{\theta}_i) + \tau_i(\bar{\theta}_i, \theta_{-i}) = -F_i^{\hat{m}_r}(\bar{\theta}_i) + \tau_i(\theta_i^{m_r}, \theta_{-i}) \tag{32}$$

$$u_i(\mu_i(\bar{\theta}_i, \theta_{-i}); \theta_i) = -F_i^{\hat{m}_r}(\theta_i) + \tau_i(\bar{\theta}_i, \theta_{-i}) = -F_i^{\hat{m}_r}(\theta_i) + \tau_i(\theta_i^{m_r}, \theta_{-i}) \tag{33}$$

$$u_i(\mu_i(\theta_i, \theta_{-i}); \bar{\theta}_i) = -F_i^{\hat{m}_l}(\bar{\theta}_i) + \tau_i(\theta_i, \theta_{-i}) = -F_i^{\hat{m}_l}(\bar{\theta}_i) + \tau_i(\theta_i^{m_l}, \theta_{-i}) \tag{34}$$

[13] We know from the way the Just Algorithm works that such $\theta_i \in [\theta_i^{m_{l-1}}, \theta_i^{m_l}]$

The DSIC condition requires that 24 $\geq$ 33 and 25 $\geq$ 34 which together demand condition 35.

$$\sum_{l=\hat{m}_l+1}^{\hat{m}_r} f_i(l,\theta_i) \geq \tau_i(\theta_i^{m_r},\theta_{-i}) - \tau_i(\theta_i^{m_r},\theta_{-i}) \geq \sum_{l=\hat{m}_l+1}^{\hat{m}_r} f_i(l,\bar{\theta}_i) \tag{35}$$

Proof Let $\{1,\dots,\hat{m}_l,\dots,\hat{m}_r,\dots,n\}$ be obtainable positions for agent-i and $(\theta_i^n,\dots,\theta_i^{m_r},\dots,\theta_i^{m_l},\dots,\theta_i^{m_{M(i)}})$ be the type cut-off(s) vector. Let $l = r+t$ for some $1 \leq t \leq M(i)-1$. Then $\theta_i^{m_r} \leq \theta_i^{m_{r+t}} \leq \theta_i^{m_l}$ for all t. The cost functions $f_i(k,\theta_i)$ are non-decreasing for all $k \in \{2,\dots,n\}$. Hence, the inequality 35 is always valid. Moreover, if the necessary conditions 23, 29 and 30 hold then condition 35 always holds, and is thus not a binding condition. This means that if the mechanism is DSIC for reports that change obtainable positions locally then the mechanism is also DSIC for reports that change the agent's position globally. Adding condition 26 to other necessary conditions, we get $\tau = \tau^{JA}$. This completes the only-if part of the proof. It is easy to verify that the transfer rule τ^{JA} satisfies conditions 23, 26 29 and 30. The verification is left to the reader. This completes the proof.

$\square$

Remark 3 In the transfer rule τ^{JA}, notice that if for every agent-$i \in N$ and $\forall\theta_{-i} \in \mathbb{R}_{++}$, we let $h_i(\theta_{-i}) = 0$, then the sum of transfers is negative. Therefore the identified class of mechanism includes *feasible* mechanisms.

We end this section with a demonstration of the proposed mechanism. We take the same values as in Example 2 and demonstrate that agent-2 cannot gainfully misreport.

Example 3 Consider three agents $\mathcal{N} = \{1,2,3\}$. Let the tie-breaking rule be : $1 \succ_{TB} 2 \succ_{TB} 3$. Let $\theta_1 = 5$, $\theta_2 = 7$ and $\theta_3 = 15$. The cost matrix is given below.

$$\theta = \begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = \begin{bmatrix}\theta_1 & \theta_1 & 18\theta_1\\ \theta_2 & 2\theta_2 & 11\theta_2\\ \theta_3 & 3\theta_3 & 3\theta_3\end{bmatrix} \rightarrow \begin{bmatrix}\theta_1 & 2\theta_1 & 20\theta_1\\ \theta_2 & 3\theta_2 & 14\theta_2\\ \theta_3 & 4\theta_3 & 7\theta_3\end{bmatrix} = \begin{bmatrix}5 & 10 & 100\\ \theta_2 & 3\theta_2 & 14\theta_2\\ 15 & 60 & 105\end{bmatrix}$$

If agent-2 reports truthfully, then he is served in position 3, and obtains a transfer of $h_2(\theta_{-2})$. His total utility is $u_2(\sigma_2^{JA}(\theta),\tau_2^{JA}(\theta)) = h_2(\theta_{-2}) - 98$.

If agent-2 reports his type $\theta'_2 \in (0, \frac{100}{14}]$, then $\sigma_2^{JA}(\theta'_2, \theta_{-2}) = 3$, and $\tau_2^{JA}(\theta'_2, \theta_{-2}) = h_2(\theta_{-2})$. Therefore, $u_2(\sigma_2^{JA}(\theta'_2, \theta_{-2}), \tau_2^{JA}(\theta'_2, \theta_{-2}); \theta_2) = -98 + h_2(\theta_{-2}) = u_2(\sigma_2^{JA}(\theta), \tau_2^{JA}(\theta))$.
If he reports his type $\hat{\theta}_2 \in (\frac{100}{14}, 20]$, then $\sigma_2^{JA}(\hat{\theta}_2, \theta_{-2}) = 2$ and $\tau_2^{JA}(\hat{\theta}_2, \theta_{-2}) = h_2(\theta_{-2}) - 11(\frac{100}{14})$. Therefore, $u_2(\sigma_2^{JA}(\hat{\theta}_2, \theta_{-2}), \tau_2^{JA}(\hat{\theta}_2, \theta_{-2}); \theta_2) = -21 + h_2(\theta_{-2}) - 11(\frac{100}{14}) \approx -99.57 + h_2(\theta_{-2}) < u_2(\sigma_2^{JA}(\theta), \tau_2^{JA}(\theta))$.
Similarly, for any report $\bar{\theta}_2 \in (20, \infty)$, $\sigma_2^{JA}(\bar{\theta}_2, \theta_{-2}) = 1$ and $\tau_2^{JA}(\bar{\theta}_2, \theta_{-2}) = h_2(\theta_{-2}) - 11(\frac{100}{14}) - 2(20)$. Therefore, $u_2(\sigma_2^{JA}(\bar{\theta}_2, \theta_{-2}), \tau_2^{JA}(\bar{\theta}_2, \theta_{-2}); \theta_2) = -7 + h_2(\theta_{-2}) - 11(\frac{100}{14}) - 2(20) \approx -125.57 + h_2(\theta_{-2}) < u_2(\sigma_2^{JA}(\theta), \tau_2^{JA}(\theta))$. Hence, with the transfer rule τ^{JA}, agent-2 can never gainfully manipulate. The same may be verified for other agents and considering other true types of agent-2. This completes the example.

Example **??**

6 Conclusion

In this paper, we examined the challenge of implementing a Rawlsian queueing rule in queueing problems where agents have heterogeneous per-period waiting costs. We introduced the *Just Algorithm*, a straightforward method that consistently selects a Rawlsian queue under complete information by minimising the maximum individual waiting cost among all agents. Our primary objective was to design mechanisms that implement the Rawlsian queue selected by the *Just Algorithm* in Dominant Strategies, thereby ensuring Rawlsian fairness even when agents act strategically.
We found that within the unrestricted domain of agents' types, where agents possess multidimensional private information, no Dominant Strategy Incentive-Compatible (DSIC) mechanism exists that can implement the Rawlsian queue selected by the *Just Algorithm*. Furthermore, the Rawlsian queue selected by the *Just Algorithm* is not even ex-post implementable in our setting. This negative result underscores the inherent challenges of designing fair mechanisms in multidimensional environments, even under quasilinear preferences.
To address this impossibility, we introduced a domain restriction to *one-dimensional private-information*. Specifically, while agents differ in how their per-period waiting costs evolve over periods, this aspect is public-information. Their private-information is confined

to their first-period waiting cost. This restriction is non-trivial because it does not allow us to identify a Rawlsian queue solely by ordering agents' private-information, contrasting with the achievements in seeking First-Best mechanisms as discussed in Mitra (2001). If agent's differ only in the private type and the publicly known aspect is identical for all agents, then the aggregate cost minimising queue is also a Rawlsian queue and results from Mitra (2001) would apply. But we did not impose any such restriction.

Within the restricted domain, we identified a class of DSIC mechanisms that implement the Rawlsian queueing rule in Dominant Strategies. An interesting observation is that while the cut-off(s) approach is well studied in the Mechanism Design literature, the same approach applied to our frameworks yields different number of cut-off(s) for different agents. Further, one agent may be pivotal to determining cut-off(s) for multiple queue positions for another agent, and not all queue positions might be accessible for an agent given other's types. The origin of this novel feature lies in the functions determining how agents' costs evolve with queue positions. The DSIC mechanism we present is robust in the sense that none of this lies beyond the scope of our mechanism.

These findings contributes to the broader investigation of implementing fair social choice or public decision rules in quasi-linear environments. It highlights the complexities involved in mechanism design when dealing with *multidimensional private-information* and the pursuit of fairness. While our domain restriction is sufficient for the existence of DSIC mechanisms implementing the Rawlsian queue, it may not be necessary. This observation opens avenues for future research to explore the necessary conditions for the existence of such mechanisms. Notably, these conditions align with the *Weak-Monotonicity* condition identified in Bikhchandani et al (2006), but characterising such mechanisms remains an open question of interest.

By addressing these challenges, our work lays the groundwork for further exploration into fair mechanism design in queueing problems with heterogeneous waiting costs. We hope that this research stimulates additional studies aimed at understanding and overcoming the obstacles inherent in implementing fairness in complex economic settings.

A Example: Rawlsian Queue

Example 4 Consider a three-agent case, $N = \{i, j, k\}$. Let the reported waiting cost vectors be $\theta_i = (2, 11, 1), \theta_j = (3, 4, 1),$ and $, \theta_k = (5, 9, 1)$. The profile is given by:

$$\theta = \begin{bmatrix} \theta_i \\ \theta_j \\ \theta_k \end{bmatrix} = \begin{bmatrix} 2 & 11 & 1 \\ 3 & 4 & 1 \\ 5 & 9 & 1 \end{bmatrix} \tag{36}$$

Table 4 summarises the problem. There are a total of six possible queues. Queue **ijk** means

Queue(s)→	**ijk**	**ikj**	**jik**	**jki**	**kij**	**kji**
$\sum_{h=1}^{\sigma_i} \theta_{ih}$	2	2	13	14	13	14
$\sum_{h=1}^{\sigma_j} \theta_{jh}$	7	8	3	3	8	7
$\sum_{h=1}^{\sigma_k} \theta_{kh}$	15	14	15	14	5	5
$\max_{l \in N} \sum_{h=1}^{\sigma_l} \theta_{lh}$	15	14	15	14	13	14
$\sum_{l \in N} \sum_{h=1}^{\sigma_l} \theta_{lh}$	24	24	31	31	26	26

Table 1 Individual costs, Aggregate Costs, and Maximum Individual costs in all possible queues for the given θ.

that agent-i is served first, followed by agent-j in the second position, and agent-k in the third position. Whenever agent-i is served in the first position(in queues **ijk** and **ikj**), the cost incurred is equal to the first column entry in row-i of profile θ, i.e 2, whenever agent-i is served in the second position(in queues **jik** and **kij**), the cost incurred is equal to the sum of the first column and second column entry in row-i of profile θ, i.e 2+11=13, and whenever agent-i is served in the third position(in queues **jki** and **kji**), the cost incurred is equal to the sum of the entries in the first three columns in row-i of profile θ, i.e 2+11+1=14. The cost for other agents and queues is calculated similarly. Table 4 lists all possible queues in columns and the costs incurred by each of the agents in that queue. For each of the six queues, we calculate $\max_{l \in N} \sum_{h=1}^{\sigma_l} \theta_{lh}$, which is the maximum individual cost incurred by any agent in that queue in corresponding rows. For instance, in the queue **kji**, agent-i incurs a cost of 14, which is the maximum individual cost in that queue. There is only one Rawlsian queue:

$\mathbf{kij} \in \arg\min_{\sigma\in\Sigma(N)} \max_{l\in N} \sum_{h=1}^{\sigma_l} \theta_{lh}$. Note that there are two efficient [14] queues: **ijk** and $\mathbf{kij} \in \arg\min_{\sigma\in\Sigma(N)} \sum_{l\in N} \sum_{h=1}^{\sigma_l} \theta_{lh}$, but they are not Rawlsian.

Example 4 demonstrates the distinction between efficient and Rawlsian queues.

To demonstrate the impossibility of designing a Dominant Strategy Incentive-Compatible (DSIC) mechanism under unrestricted types, consider the following two-agent example.

Example 5 Consider a two agent case, $N = \{1, 2\}$ with reported waiting cost vectors $\theta_1 = (8, 3)$ and $\theta_2 = (7, 3)$. The Just Algorithm for perceptive agents assigns $\sigma_1^{JA}(\theta) = 1$. The utility of agent-1 is $u_1(\sigma^{JA}(\theta), \tau(\theta)) = -(8) + \tau((8, 3), (7, 3))$. Suppose agent-1 were to misreport the waiting cost vector as $\tilde{\theta_1} = (5, 4)$. $\sigma_1^{JA}(\tilde{\theta}_1, \theta_2) = 2$. Therefore, $u_1(\sigma^{JA}(\tilde{\theta}_1, \theta_2), \tau(\tilde{\theta}_1, \theta_2)) = -(8 + 3) + \tau(\tilde{\theta}_1, \theta_2)$. Implementation in Dominant Strategies demands: $u_1(\sigma^{JA}(\theta), \tau(\theta)) \geq u_1(\sigma^{JA}(\tilde{\theta}_1, \theta_2), \tau(\tilde{\theta}_1, \theta_2)) \equiv 3 \geq \tau(\tilde{\theta}_1, \theta_2) - \tau((8, 3), (7, 3))$. If the true waiting cost vector of agent-1 is $(5, 4)$ and the misreport is $(8, 3)$, then Implementation in Dominant Strategies demands: $u_1(\sigma^{JA}(\tilde{\theta}_1, \theta_2), \tau(\tilde{\theta}_1, \theta_2)) \geq u_1(\sigma^{JA}(\theta), \tau(\theta)) \equiv \tau(\tilde{\theta}_1, \theta_2) - \tau((8, 3), (7, 3)) \geq 4$. One and only one of the two Implementation can hold, and therefore, it is impossible to find any transfer rules $\tau(\theta), \tau(\tilde{\theta}_1, \theta_2))$ satisfying both conditions simultaneously.

Example 5 confirms that, even in a simple two-agent scenario, no transfer rule can satisfy the conditions required for DSIC implementation when agents have unrestricted types. It highlights the challenges of achieving fairness in multi-dimensional settings and motivates the need for domain restrictions.

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