Pay-as-Bid Auctions with Budget Constrained Bidders[*†]

Juan Carlos Carbajal[‡] G V A Dharanan[§]

August 2024

Abstract

We study pay-as-bid auctions with budget-constrained bidders with constant marginal valuation (flat demand). We first show that for any bidder, a best-response strategy to a sizable class of strategies necessarily involves the bidder placing flat bids, i.e., equal bids for all the objects. We then establish the existence of equilibrium through a sufficiency condition–close to being necessary– that guarantees the local concavity of the expected payoff function. We subsequently show that the equilibrium breaks down when the number of bidders is sufficiently high. The non-existence result demonstrates that the equilibrium existence in pay-as-bid auctions is sensitive to the number of bidders in the presence of budgets.

Keywords: Pay-as-Bid Auctions, Budget Constraints, Flat-Bid Equilibrium, Share Auctions.

JEL Classification Numbers: D44, C62.

1. Introduction

Economists have long grappled with the problem of selling multiple homogeneous items. A well-accepted solution to this problem is to accomplish this task through auctions. In this context, the pay-as-bid auction– wherein the winner pays her bid for the items she wins– is a commonly used format; for instance, treasury bonds auctions (Hortaçsu and McAdams, 2010) and electricity market auctions (Genc, 2009) use the pay-as-bid auction format.

Previous literature, notably Pycia and Woodward (2023) and Ausubel et al. (2014) has studied pay-as-bid auctions. Absent any financial constraints, these papers find that the equilibrium bid of a bidder reflects her willingness to pay. Nevertheless, in many real-world

[*]Preliminary version; comments welcome.

[†]We thank, in arbitrary order, Carlos Pimienta, Sven Feldmann, Arghya Ghosh, Cheng Liu, Anton Kolotilin, Pei Cheng Yu, Simon Loertscher, Isa Hafalir, Andrzej Skrzypacz, Murali Agastya, Idione Meneghel, Christopher Teh, DJ Thornton, and participants at AETW 2024 and ACE 2024 for valuable comments and feedback. The first author gratefully acknowledges budget support from the Australian Research Council under grant DP190102064.

[‡]School of Economics, UNSW Sydney, Australia. Email: jc.carbajal@unsw.edu.au.

[§]School of Economics, UNSW Sydney, Australia. Email: g.dharanan@student.unsw.edu.au.

scenarios, the bids are also impacted by budget constraints. Indeed, the budget constraints are a salient, "ubiquitous," and practical consideration in a multi-unit auction environment (Bulow et al., 2009, Salant, 1997). However, the literature is largely silent on the issue of budgets in multi-unit auctions.

We study a pay-as-bid auction where the bidders may face budget constraints, limiting their ability to pay. To that end, we work with *hard* budget constraints– no bidder can ever breach her budget. Seminal papers– notably Che and Gale (1998) and Kotowski (2020) have studied the impact of budget constraints on single-unit first-price auctions. Nevertheless, few papers incorporate budget constraints in multi-unit pay-as-bid auctions.

We use the "share auction" framework due to Wilson (1979). The literature has fruitfully used the share auction model to study pay-as-bid auctions, for instance, Back and Zender (1993), Wang and Zender (2002), Ausubel et al. (2014), and Kastl (2011). With hard budget constraints, the tractability of the share auction model comes with a cost– the pay-as-bid auction with budgets becomes a discontinuous game with non-compact, type-dependent action space. Indeed, the budget of a bidder determines the upper bound on the total bid. Thus, our results also contribute to the literature on discontinuous games– we show the existence of an equilibrium in a class of games with non-compact and type-dependent action spaces.

The literature on pay-as-bid auctions has singled out an overarching intuition: in equilibrium, the bidders place a bidding function that is "as flat as possible" (Back and Zender, 1993). When the bidders have a constant marginal valuation for each additional share/object, the equilibrium strategy for each bidder prescribes that the bidder bids an equal amount for each share, i.e., the equilibrium is a flat-bid equilibrium (Ausubel et al., 2014, Kastl, 2011 and Kastl, 2012). Flat-bid equilibria are advantageous; they reduce the multi-unit auction to a single-unit auction.

Nevertheless, the presence of budgets may render the flat-bid intuition infructuous. Indeed, the literature studying multi-unit auctions in the presence of budgets has often drawn a link between multi-unit auctions and Blotto games (Palfrey, 1980 and Ghosh, 2021). We show that the presence of types who may be budget-unconstrained can recover the flat-bid intuition. We present this result using a best-response argument in Theorem 1. The flat-bid intuition in Theorem 1 breaks the link between a Blotto game and a pay-as-bid auction.

The budget constraints break the equivalence between a single-unit first-price auction and a pay-as-bid auction. A bidder's action space is a subset of $\mathbb{R}$ in a single-unit first-price auction, while the bidder's action space is a subset of $W^{1,1}(0,1)$ in a multi-unit auction. Consequently, the budget appears as a constraint in a variational problem rather than a constraint in a single-dimensional optimization problem. Consequently, a flat-bid equilibrium requires additional conditions– necessity and sufficiency– apart from the conditions in Che and Gale (1998) and Kotowski (2020). We establish sufficient and necessary conditions for a symmetric pure flat-bid strategy equilibrium in Theorem 2 and Corollary 1.[1]

The additional conditions in Theorem 2 and Corollary 1 have an intuitive interpretation, viz., the expected payoff function should be locally concave in the bid. In the absence of budgets, the bids reflect the valuations. Therefore, an exogenous log-concave type distribution implies a log-concave bid distribution, leading to a concave expected payoff function. In

[1] $W^{1,1}(0,1)$ is the space of all absolutely continuous functions with domain $[0,1]$.

contrast, budgets create an endogenous distribution of opponent's bids. Therefore, the concavity of the payoff function need not hold. One of our contributions is to provide a sufficient condition so that the payoff function is locally concave near the equilibrium strategy profile. Our equilibrium existence result goes hand-in-hand with the local concavity of the payoff function.

The additional advantage of our equilibrium characterization in Theorem 2 and Corollary 1 is that we uncover an undesirable impact of budgets. These conditions fail when the number of bidders is large. Consequently, the flat-bid equilibrium fails to exist when the number of bidders is large (Theorem 3). As far as we know, the non-existence phenomenon in Theorem 3 is new. Nevertheless, it results from the failure of local concavity. Heuristically, when the number of bidders increases, the expected payoff function for a budget-constrained bidder becomes convex in the bids. Consequently, the flat-bid equilibrium breaks down.

The paper is organized as follows. The next subsection outlines the relevant literature. Section 2 presents the framework. Section 3 establishes Theorem 1. Section 4 presents the existence results (Theorem 2 and Corollary 1). Section 5 presents the non-existence results (Theorem 3), and Section 6 concludes. The formal proofs of the theorems are in the appendix.

1.1. Relevant Literature

The classical share auction model starts from Wilson (1979). Indeed, a part of Wilson (1979)'s motivation to study a share auction model stems from the possible budget constraints of bidders. The share auction model is tractable since it is amenable to a variational approach.

However, the literature on the implications of budget constraints on multi-unit auctions is small. Papers that include budget constraints in multi-unit auctions either restrict to complete information (Palfrey, 1980), two units (Ghosh, 2021), or a limited degree of informational asymmetry (Cole et al., 2022). A complete treatment of the impact of budget constraints with a high degree of informational asymmetry and multiple items is absent; we fill the gap.

The literature on pay-as-bid auctions is underdeveloped compared to the literature on uniform-price auctions. One particular strand of the literature assumes no informational asymmetry between the bidders (See Pycia and Woodward, 2023, Anderson et al., 2013, Wittwer, 2020 etc.). The only source of uncertainty is a stochastic aggregate supply. Nevertheless, as some papers (Holmberg, 2009, Genc, 2009 and Anderson et al., 2013) show, the existence of a pure strategy equilibrium is not guaranteed in this setting.

Another strand of literature (for instance Ausubel et al., 2014, Kastl, 2011, Hortaçsu and McAdams, 2010 and Linnenbrink, 2023) ignores any financial constraints, assumes a private information setting and characterizes the equilibrium. The bidders' optimization problem is a convex optimization problem. Therefore, the first-order conditions are also sufficient. In the case of constant marginal valuations (also called flat or linear demand), the pay-as-bid auctions mimic the single-unit first-price auction. Therefore, the existence of an equilibrium follows the arguments in Vickrey (1961) or Riley and Samuelson (1981).

Another strand of literature studies the impact of budget constraints on the efficiency of final allocations (Dobzinski et al., 2012, Hafalir et al., 2012). Nevertheless, these papers focus on dominant strategy mechanisms instead of Bayesian mechanisms. Consequently, their equilibrium existence results are inapplicable to pay-as-bid auctions.

Palfrey (1980) studies pay-as-bid auctions with budget constraints in the context of complete information, wherein the pay-as-bid auctions are analogous to Blotto games. Ghosh (2021) extends the analogy and studies parallel first-price auctions with private types. Ghosh (2021)'s results can considered a purification of Blotto Games– the presence of private types leads to a pure-strategy equilibrium in pay-as-bid auctions. Nevertheless, Ghosh (2021)'s results preclude the presence of *budget-unconstrained* bidders.

The literature has focused more on single-unit auctions with budget constraints vis-a-vis multi-unit auctions with budget constraints.Che and Gale (1998) is the seminal reference. Kotowski (2020) studies a two-bidder first-price auction with affiliated valuations and independent budgets. Pai and Vohra (2014) characterize the optimal auction for selling a single object to budget-constrained bidders. Single-unit auctions are relatively straightforward to analyze since the first-order conditions suffice for the equilibrium characterization. Indeed, the second-order conditions are always uninformative.

Reny (2011) is the seminal reference for discontinuous Bayesian games and provides sufficient conditions for a monotone pure-strategy equilibrium in discontinuous Bayesian games. Reny (2011)'s conditions stem from a partial order on the type space. Reny (2020) provides a survey of additional results about the existence of Bayes-Nash equilibria in behavioral strategies in a large class of games. Carbonell-Nicolau and McLean (2018) define a condition called "uniformly payoff secure" and shows the existence of Bayes-Nash equilibrium under a weak condition.

2. Setting and Notation

A seller uses a *pay-as-bid (discriminatory) auction* to sell the shares of an infinitely divisible object of unit mass. The seller's opportunity cost is zero. For simplicity, we don't consider reserve prices.

There are N ex-ante symmetric bidders participating in the auction. Each bidder has a valuation $v \in [\underline{v}, \overline{v}] \subseteq \mathbb{R}_+$ for the entire object. We consider bidders with constant marginal valuation for shares; i.e., flat demands. In addition, each bidder has a total budget $w \in [\underline{w}, \overline{w}] \subseteq \mathbb{R}_+$. Both individual valuations and budgets are private information. Thus, the type of any bidder is a random vector (V, W) distributed according to F with support $[\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}]$, and density function f. Note that while types are independent and identically distributed among bidders, each bidder's valuation and budget may be correlated. The common distribution F of the random vector (V, W) induces a marginal distribution F_V over valuations and a marginal distribution F_W over budgets. Denote the respective densities by f_V and f_W, which are positive everywhere in their respective supports.

In a pay-as-bid auction, bidders simultaneously submit individual bidding functions (inverse demand functions). The seller aggregates these bidding functions and awards shares to the highest bidders. In this paper, we restrict attention to absolutely continuous bidding functions to be able to work with the private budget constraints.

Definition 1. *A **bidding function** for bidder $i = 1, 2, \cdots, N$ is a weakly decreasing, absolutely continuous function $b_i \colon [0,1] \to \mathbb{R}_+$ that specifies, for each $s \in [0,1]$, the maximal amount $b_i(s)$ that bidder i is willing to pay to obtain the share indexed s.*

Following the optimal control literature (Vinter, 2010), denote by $\mathcal{B}$ the space of all bidding functions and endow it with the $W^{1,1}$ norm. $\mathcal{B}$ is the action space of each bidder.[2]

As is usual in the literature, for a given profile of bidding functions $\boldsymbol{b} = (b_1, \ldots, b_N)$, the market clearing price P is defined to be the highest rejected bid, i.e.,

$$P \;=\; \min\left\{p : \; \sum_{i=1}^{N} b_i^{-1}(p) \;\leq\; 1\right\}.$$

Bidder i's individual demand function $b_i^{-1}(\cdot)$ in the above expression is constructed by inverting her bidding function and taking its vertical closure —see Ausubel et al. (2014) for details. In the event of a tie, the seller uses a pro-rata tie-breaking rule at the margin.[3]

If bidder i with type realization (v, w) wins a measure of shares $s \in [0,1]$ using the bidding function $b_i(\cdot)$, her payment to the seller is $\int_0^s b_i(t)\,dt$. Bidder i's ex-post payoff Π_i is then

$$\Pi_i \;=\; \int_0^s \big(v \;-\; b_i(t)\big)\,dt$$

as long as $\int_0^s b_i(t)\,dt \;\leq\; w$. If bidder i's total payment $\int_0^s b_i(t)\,dt \;>\; w$, she suffers a large negative payoff. Formally, we impose the hard budget constraints of Che and Gale (1998) in their analysis of first-price auctions (of a single, indivisible object).

An observation before we proceed: the action space of the bidders is a subset of absolutely continuous decreasing functions, that is $\mathcal{B} \subseteq W^{1,1}(0,1)$. Our restriction of action space is stronger than, for instance, Ausubel et al. (2014) and Pycia and Woodward (2023).Ausubel et al. (2014) and Pycia and Woodward (2023) allow for right-continuous bidding functions. We express the individual optimization (best response) problem as a variational problem. The approach necessitates that the derivatives uniquely determine a function. Allowing for discontinuous bidding functions breaks the solution method.

2.1. Assumptions and Strategies

We maintain the next two assumptions throughout the paper.

Assumption 1. *The type space* $[\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}]$ *satisfies:* $\underline{v} \;\leq\; \underline{w} \;\leq\; \overline{v} \;\leq\; \overline{w}$.

Assumption 2. *The joint distribution F over valuations and budgets satisfies the following conditions:*

(i) The marginal densities f_V, f_W and the joint density of F are all differentiable in $\mathrm{int}([\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}])$.

(ii) For any $v \in (\underline{v}, \overline{v})$, the function $w \;\mapsto\; w \;+\; \frac{1}{N-1}\,\frac{F_V(v)+F_W(w)-F(v,w)}{D_w(F_V(v)+F_W(w)-F(v,w))}$ *is strictly increasing.*

[2]The $W^{1,1}$ norm of an absolutely continuous function $b\colon [0,1] \to \mathbb{R}$ is defined as

$$\|b(\cdot)\|_{W^{1,1}} \;=\; \|b(\cdot)\|_1 \;+\; \|b'(\cdot)\|_1 \;=\; \int_0^1 |b(s)|\,ds \;+\; \int_0^1 |b'(s)|\,ds,$$

where $b'(\cdot)$ is the weak derivative of $b(\cdot)$. Note that $b'(\cdot)$ coincides with the standard derivative when $b(\cdot)$ is a differentiable function (Brezis, 2010).

[3]This tie-breaking rule is common in literature —see Back and Zender (1993) and Ausubel et al. (2014), for example.

(iii) The limit $\alpha := \lim_{\substack{v \to \underline{v}^+ \\ w \to \underline{w}^+}} \frac{D_w(F_V(v)+F_W(w)-F(v,w))}{D_v(F_V(v)+F_W(w)-F(v,w))}$ *exists and is finite-valued.*

Assumption 1 rules out issues of non-existence of equilibrium in auctions with budget constraints that have been pointed out in the literature —e.g., Che and Gale (2006), Kotowski (2020). Assumption 2–(ii) is from Che and Gale (1998) and ensures the existence of a unique equilibrium in a single-unit, first price auction under private budgets and private valuations. Assumption 2–(iii) is satisfied by many type distributions and is a regularity assumption.

Definition 2. *A* ***pure strategy*** *for bidder* $i = 1, 2, \cdots, N$ *is a mapping* $\beta_i \colon [\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}] \to \mathcal{B}$*. The image of type* (v, w) *under strategy* β_i *is the bidding function expressed as* $\beta_i(\cdot \mid v, w)$*. A pure strategy* β_i *is* ***budget-feasible*** *for bidder* $i = 1, 2, \cdots, N$ *if it satisfies*

$$\int_0^1 \beta_i(s \mid v, w)\, ds \;\le\; w, \qquad \text{for all } (v, w) \in [\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}].$$

The budget-feasibility condition restricts bidder i's *total bid* $\int_0^1 \beta_i(s \mid v, w)\, ds$ to be less or equal to her budget. Note that a generic type (v, w) of bidder i wins the entire object —and pays her total bid— with a positive probability. Recall that bidder i gets an unbounded negative payoff if she is required to pay above her budget. Therefore, the budget-feasible strategy generating a bidding function $\beta_i(s \mid v, w) \equiv \underline{v}$ weakly dominates every budget-unfeasible strategy. Consequently, the budget-feasibility condition is without any loss of generality. Hereon, we focus on strategy profiles $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$ under which every bidder's strategy is budget-feasible.

Fix a budget-feasible pure strategy β_i for bidder i. Say her type realization (v, w) is *budget-constrained* if $\int_0^1 \beta_i(s \mid v, w) ds = w$ and *budget-unconstrained* if $\int_0^1 \beta_i(s \mid v, w) ds < w$.

Definition 3. *A* ***monotone strategy*** *for bidder* $i = 1, 2, \cdots, N$ *is a budget-feasible pure strategy* β_i *that satisfies following properties:*

(i) β_i *is a continuous mapping (with respect to the* $W^{1,1}$ *topology on* $\mathcal{B}$*).*

(ii) For all $s \in [0, 1]$*, the mapping* $\beta_i(s \mid \cdot, \cdot)$ *is weakly increasing in* (v, w)*, and strictly increasing if both* v *and* w *rise.*

Recall from Assumption 1 that $\underline{v} \le \underline{w}$ and $\overline{v} \le \overline{w}$. Under any monotone strategy, a bidder with valuation realization $\underline{v}$ places a bid of $\underline{v}$ for all shares —such a bidder has no incentive to bid higher than $\underline{v}$ for any share and is never budget-constrained. In particular, $\beta_i(s \mid \underline{v}, \underline{w}) = \underline{v}$ for any bidder i following a monotone bidding strategy. Similarly, a bidder with budget realization $\overline{w}$ never has an incentive to bid above $\overline{w}$, for any share $s \in [0, 1]$. In particular, $\beta_i(s \mid \overline{v}, \overline{w}) \le \overline{w}$. Thus, under monotone strategies, one has $\underline{v} \le \beta_i(s \mid v, w) \le \overline{w}$ for all $s \in [0, 1]$ and all $i = 1, \ldots, N$.

Definition 4. *A* ***flat-bid strategy*** *for bidder* $i = 1, 2, \cdots, N$ *is a monotone strategy* β_i *that generates bidding functions of the form*

$$\beta_i(s \mid v, w) \;=\; \min\{\phi_i(v), w\}, \qquad \text{for each type } (v, w)$$

where $\phi_i(v)$ *is a strictly increasing, non-negative, absolutely continuous function on* $[\underline{v}, \overline{v}]$*.*

In other words, under a flat-bid strategy, the bidding functions are always "flat", i.e. constant in shares. Since we focus on absolutely continuous bidding functions, a flat-bid strategy is always *non-decreasing* as well as *non-increasing*. Note that, by construction, flat-bid strategies are always budget-feasible.

2.2. The Expected Payoff Function and Equilibrium

For strategy profile $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$, the *market clearing price* at type profile $(\boldsymbol{v}, \boldsymbol{w})$ is[4]

$$P(\boldsymbol{v}, \boldsymbol{w}) \;=\; \min\Big\{p : \; \textstyle\sum_{i=1}^{N} \beta_i^{-1}(p \,|\, v_i, w_i) \;\leq\; 1\Big\}.$$

Notice that, at the ex-ante stage, the market-clearing price function is a random variable that depends on the strategy profile $\boldsymbol{\beta}$ — we omit $\boldsymbol{\beta}$ in the expression for the market clearing price to ease the notation. Hereon, $(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i})$ and $\boldsymbol{\beta}_{-i}$ will denote the types and strategy profile of bidder i's opponents.

Recall that the action space of the bidders is a subset of absolutely continuous bidding functions. Therefore, when the market clears at type profile $(\boldsymbol{v}, \boldsymbol{w})$, any bidder i who wins a positive mass of shares $[0, s_i)$ must bid at least $P(\boldsymbol{v}, \boldsymbol{w})$ for any $s \in [0, s_i)$. Thus, bidder i's ex-ante expected payoff under the budget-feasible strategy profile $\boldsymbol{\beta}$ can be expressed as

$$\Pi_i(\boldsymbol{\beta}) \;=\; \int_{\boldsymbol{v},\boldsymbol{w}} \left(\int_0^1 \left(v_i \,-\, \beta_i(s \,|\, v_i, w_i)\right) \mathbf{1}\left\{\beta_i(s \,|\, v_i, w_i) \;>\; P(\boldsymbol{v}, \boldsymbol{w})\right\} ds \right) d \prod_{j=1}^{N} F(v_j, w_j).$$

Bidder i' expected payoff when her type is (v_i, w_i) under the budget-feasible strategy profile $\boldsymbol{\beta}$ can be expressed as

$$\Pi_i(v_i, w_i \,|\, \boldsymbol{\beta}) \;=\; \int_{\boldsymbol{v}_{-i},\boldsymbol{w}_{-i}} \left(\int_0^1 \left(v_i \,-\, \beta_i(s \,|\, v_i, w_i)\right) \mathbf{1}\left\{\beta_i(s \,|\, v_i, w_i) \;>\; P(\boldsymbol{v}, \boldsymbol{w})\right\} ds \right) d \prod_{j \neq i} F(v_j, w_j).$$

A formal definition of the Bayes-Nash equilibrium follows.

Definition 5. *A budget-feasible monotone pure strategy profile* $\boldsymbol{\beta} \equiv \{\beta_j\}_{j=1}^{N}$ *is the* Bayes-Nash equilibrium *of the pay-as-bid auction if for each bidder i, for F-almost every type* (v_i, w_i)*, and for any budget feasible strategy* $\widehat{\beta}_i$*,*

$$\Pi_i(v_i, w_i \,|\, \boldsymbol{\beta}) \geq \Pi_i(v_i, w_i \,|\, \widehat{\beta}_i, \boldsymbol{\beta}_{-i}).$$

Henceforth, we shall use the term equilibrium to refer to the Bayes-Nash equilibrium.

It is more convenient to work with a market clearing price *conditional* on bidder i obtaining the mass of shares $s \in [0, 1]$. We refer to this as the s–price faced by bidder i.

Definition 6. *Fix a strategy profile* $\boldsymbol{\beta}_{-i}$ *for i's opponents. Given type profile* $(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i})$*, the* ***s–price*** *for bidder i is the minimum bid required to win a measure of shares* $0 \leq s \leq 1$*; i.e.,*

$$P_s(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i}) \;:=\; \min\Big\{p : \; \textstyle\sum_{j \neq i} \beta_j^{-1}(p \,|\, v_j, w_j) \;\leq\; 1 - s\Big\}.$$

[4] Recall that $\beta_i^{-1}(p \,|\, v_i, w_i)$ is bidder i's individual demand function and is constructed by inverting her bidding function and taking its vertical closure.

The s–price for bidder i depends on $\boldsymbol{\beta}_{-i}$; we omit this dependency to ease the notation. From bidder i's perspective, $(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i})$ is unknown at the interim stage and therefore P_s is a random variable. Denote its distribution by $G_s(\cdot)$ and its density by $g_s(\cdot)$. If $\boldsymbol{\beta}_{-i}$ is a profile of monotone strategies, the support of $G_s(\cdot)$ is a sub-interval of $[\underline{v}, \overline{w}]$ that includes $\underline{v}$.

We can now use the s–price to express bidder i's expected payoff. Given a generic type profile realization $(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i})$ of her opponents, bidder i wins a measure of shares s using a bidding function $b_i(\cdot)$ if and only if her bid for the share indexed s satisfies $b_i(s) \geq P_s(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i})$. Therefore, for a fixed strategy profile $\boldsymbol{\beta}_{-i}$, bidder i's (interim) expected payoff when her type is (v, w) and she uses a budget-feasible bidding function $b_i(\cdot)$ is

$$\begin{aligned}\Pi_i(b_i \,|\, v, w) &= \int \left(\int_0^1 (v - b_i(s))\, \mathbf{1}\left\{ b_i(s) > P_s(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i}) \right\} ds \right) d \prod_{j \neq i} F(v_j, w_j) \\ &= \int_0^1 (v - b_i(s))\, G_s(b_i(s))\, ds.\end{aligned} \tag{1}$$

Lemma 1 gives an implication of monotonicity; it allows us to focus on a subset of $\mathcal{B}$. In the appendix, we use Lemma 1 to provide a detailed construction of $G_s(\cdot)$.

Lemma 1. *Fix a bidder i and a strategy profile $\boldsymbol{\beta}_{-i}$ of bidder i's opponents. Let β_i be a monotone budget-feasible best response to $\boldsymbol{\beta}_{-i}$. Then the image of any $(v, w) \in [\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}]$ under β_i is a bidding function of the form*

$$\beta_i(s \,|\, v, w) = \min\left\{ \phi_i(s \,|\, v), \gamma_i(s \,|\, w) \right\},$$

where

(i) $\phi_i(\cdot \,|\, v)$ is a weakly decreasing, non-negative, absolutely continuous function on $[0, 1]$;

(ii) $\gamma_i(\cdot \,|\, w)$ is a weakly decreasing, non-negative, absolutely continuous function on $[0, 1]$, that satisfies $\int_0^1 \gamma_i(s \,|\, w)\, ds = w$.

Proof. First, suppose that bidder i of type (v, w) is budget-constrained, i.e., $\int_0^1 \beta_i(s \,|\, v, w)\, ds = w$. Consider another type realization (v', w), where $v' \geq v$. Since β_i is a monotone strategy, it must be that $\beta_i(s \,|\, v', w) \geq \beta_i(s \,|\, v, w)$ for all $s \in [0, 1]$, and therefore $\int_0^1 \beta_i(s \,|\, v', w)\, ds \geq w$. On the other hand, the budget feasibility restriction requires that $\int_0^1 \beta_i(s \,|\, v', w)\, ds \leq w$. Consequently, $\int_0^1 \beta_i(s \,|\, v', w)\, ds = w$. Therefore, for all $v' \geq v$, $\beta_i(s \,|\, v', w) = \beta_i(s \,|\, v, w)$ for each $s \in [0, 1]$, and the bidding function for (v', w) is solely determined by w.

Now, suppose that bidder i of type (v, w) is budget-unconstrained, i.e., $\int_0^1 \beta_i(s \,|\, v, w)\, ds < w$. Recall that β_i is a best response of bidder i to the strategy profile $\boldsymbol{\beta}_{-i}$. Since an increase in budget does not impact bidder i's valuation, it must be that for any $s \in [0, 1]$, bidder i's bid for the share indexed s must not increase under the same valuation. Therefore, for any $\varepsilon > 0$, $\beta_i(s \,|\, v, w + \varepsilon) = \beta_i(s \,|\, v, w)$ for each $s \in [0, 1]$. Consequently, the bidding function for type (v, w) must be determined solely by her valuation. □

3. The Best Response Property of Flat-Bid Strategies

In this section, we show that *any* (pure strategy) best response to a sub-class of monotone strategy profiles must be a flat-bid strategy. Flat-bid strategies are advantageous: when each bidder plays a flat-bid strategy, the pay-as-bid auction broadly mimics a single-unit first-price auction.

Let $0 \leq r < s \leq 1$. When bidder i's allocation moves from r to s, the allocation of shares of at least one of bidder i's opponents strictly decreases. When each bidder $j \neq i$ follows a monotone strategy β_j, it follows that $P_r(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i}) \leq P_s(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i})$ for any type profile realization of bidder i's opponents. Resultantly the distribution of the s–price first-order stochastically dominates the distribution of the r–price.[5]

We require a stronger version of the above observation. To that extent, we focus on the sub-class of monotone strategy profiles that satisfy the next requirement.

Definition 7. *Fix a bidder i and a monotone strategy profile $\boldsymbol{\beta}_{-i}$ of her opponents. The strategy profile $\boldsymbol{\beta}_{-i}$ is **strongly regular** if the family of s–price distributions induced by $\boldsymbol{\beta}_{-i}$, namely $\{G_s \colon s \in [0,1]\}$, satisfies following conditions for all $0 \leq r \leq s \leq 1$:*

(i) $G_s(\cdot)$ weakly dominates $G_r(\cdot)$ in reverse hazard rate order;

(ii) $G_s(\cdot)$ has a decreasing reverse hazard rate.

Strong regularity holds for a sizable class of monotone strategy profiles. For instance, when $N = 2$, and valuations and budgets are independently and uniformly distributed over the unit square, monotone strategies that prescribe affine (in shares) bidding function are strongly regular. In the context of flat-bid strategies with $N \geq 2$, condition (i) in Definition 7 holds trivially, while condition (ii) precludes situations in which the bid increases at a faster rate than the valuation.

Our first result is Theorem 1. Suppose the monotone strategy profile $\boldsymbol{\beta}_{-i}$ is strongly regular. Theorem 1 states that if β_i is a monotone best response to $\boldsymbol{\beta}_{-i}$, then β_i must be a flat-bid strategy. We use this result to justify the search for pure strategy equilibria in the class of flat-bid strategy profiles.

Theorem 1. *Fix any bidder $i = 1, \ldots, N$ and let $\boldsymbol{\beta}_{-i}$ strongly regular. If β_i is a monotone best response strategy to $\boldsymbol{\beta}_{-i}$, then β_i must be a flat-bid strategy.*

Discussion and Proof Sketch of Theorem 1

The intrinsic nature of budget constraints renders many commonly known techniques inapplicable. A tempting– but false– conjecture is to view the pay-as-bid auction as an incomplete information Blotto game; every budget-constrained bidder should bid aggressively for some objects while bidding passively for others. Our point of departure is Theorem 1, which states that a flat-bid strategy is a pure strategy best response to a sizable class of strategy profiles of the opponents.

[5] Recall that the bidding functions are weakly decreasing on $[0, 1]$.

The literature has observed the best response property of flat bids. Pycia and Woodward (2023) show the best response property of flat bids in a setting with a stochastic supply and perfect informational symmetry between bidders. With independent private values and constant marginal valuation, the Euler-Lagrange condition in Wilson (1979), Hortaçsu and McAdams (2010) and Kastl (2012) boils down to flat bids being a necessary condition in the *equilibrium*.

Our framework is different. Since bidders may be budget-constrained, their bids need not reflect their marginal valuation. This disassociation between bids and valuations makes the best-response property of flat bids unapparent. Indeed, we need strong regularity to guarantee the best response property of flat bids. The best response property emerges from a combination of two factors: (i) the monotonicity of the strategies and (ii) the presence of budget-unconstrained bidders. These two factors drive Theorem 1 and sever the link of the pay-as-bid auction with the Blotto game approach of Palfrey (1980) and Ghosh (2021).

We now present the proof sketch of Theorem 1. The formal proof is in Appendix A. We first consider a budget-unconstrained bidder in Lemma 2, which shows that any best response of a budget-unconstrained bidder must be a flat-bid strategy. The argument is as follows. For shares $r < s$, the distribution $G_s(\cdot)$ dominates $G_r(\cdot)$ in reverse hazard rate order. Therefore a budget-unconstrained bidder prefers to place a higher bid for the share indexed s compared to the share indexed r. Since the bidding functions are weakly decreasing, the best that bidder i can do is to place the same bid for the shares indexed s and r. In other words, the best response strategy β_i must generate a flat bidding function for any budget-unconstrained type. Lemma 2 formalizes this argument through the Pontryagin maximum principle.

Lemma 3 ties all the threads together using the of monotonicity of β_i. Formally, for each $w > \underline{v}$, there exists a type $(v(w), w)$ such that for any $\varepsilon > 0$, the type $((v(w) - \varepsilon, w))$ is budget-unconstrained when playing the strategy β_i. By continuity, the type $(v(w), w)$'s bidding function satisfies $\beta_i(s \mid v(w), w) = w$. Bidder i with a type (v, w) is budget-constrained if and only if $v \geq v(w)$. By monotonicity of β_i, $\beta_i(s \mid v, w) = w$. Therefore, every budget-constrained type also places a flat bid.

4. General Existence Result

In this section, we provide the necessary and sufficient conditions for a flat-bid pure strategy equilibrium in pay-as-bid auctions. Under a symmetric flat-bid equilibrium, the multi-unit auction reduces to a single-unit first-price auction. Theorem 2 and Corollary 1 are the main results in this section.

To state the existence result for flat-bid equilibrium, we first rewrite Che and Gale (1998)'s equilibrium strategies in a single-unit auction in a differential form in Equation ϕ for expositional ease. To that end, define the function $H\colon [\underline{v}, \overline{v}] \times [\underline{w}, \overline{w}] \to [0, 1]$ by

$$H(v, w) \;:=\; F_V(v) \;+\; F_W(w) \;-\; F(v, w). \tag{2}$$

In words, $H(v, w)$ represents the probability of the event where either the random valuation attains a value less than or equal to v, or the random budget attains a value less than or equal to w. Denote the partial derivatives of H with respect to its first and second argument respectively as: [6]

$$D_v H(v,w) := f_V(v) - \frac{\partial F(v,w)}{\partial v} \quad \text{and} \quad D_w H(v,w) := f_W(w) - \frac{\partial F(v,w)}{\partial w}.$$

The differential form of the integral equation in Che and Gale (1998) is:

$$\begin{aligned}
\frac{d\phi(v)}{dv} &= \frac{f_V(v)}{\frac{F_V(v)}{(N-1)(v-\phi(v))}} && \text{if } v < v^* \\
\frac{d\phi(v)}{dv} &= \frac{D_v H(v,\phi(v))}{\frac{H(v,\phi(v))}{(N-1)(v-\phi(v))} - D_w H(v,\phi(v))} && \text{if } v > v^* \\
\phi(\underline{v}) &= \underline{v} \quad \text{and} \quad \phi(v^*) = \underline{w}.
\end{aligned} \tag{ϕ}$$

Observe that v^* uniquely solves the following equation:

$$\underline{w} = v^* - \int_{\underline{v}}^{v^*} \frac{F_V(x)^{N-1}}{F_V(v^*)^{N-1}} dx.$$

Assumption 2–(ii) guarantees the existence and uniqueness of the solution to Equation ϕ; not monotonicity. A necessary and sufficient condition for monotonicity is:

$$\lim_{v \to v^{*+}} \frac{d\phi(v)}{dv} = \frac{D_v H(v^*, \underline{w})}{\frac{H(v^*,\underline{w})}{(N-1)(v^*-\underline{w})} - D_w H(v^*, \underline{w})} > 0. \tag{3}$$

Condition 3 is due to Kotowski (2020) and avoids the jump-discontinuity condition in Kotowski (2020). Kotowski (2020)'s non-canonical equilibrium emerges from the failure of condition 3. The function $\phi(\cdot)$ transitions continuously and monotonically from the interval $[\underline{v}, v^*)$ to the interval $(v^*, \overline{v}]$ under condition 3.

Define the function $\lambda : \phi(V) \to V$ as the inverse function of ϕ, i.e. $\lambda(x) := \phi^{-1}(x)$. Observe the that since $\phi(\cdot)$ is C^1 and strictly monotone on the set $(\underline{v}, v^*) \cup (v^*, \overline{v})$, $\frac{d\phi(v)}{dv} > 0$ on $(\underline{v}, v^*) \cup (v^*, \overline{v})$. Therefore

$$\frac{d\lambda(x)}{dx} = \left. \frac{1}{\frac{d\phi(v)}{dv}} \right|_{x=\phi(v)} \qquad x \in (\underline{v}, \underline{w}) \cup (\underline{w}, \phi(\overline{v}))$$

Remark: From here-onward, we use $H(v,\phi(v))$ instead of $F_V(v)$ when $v < v^*$ with the understanding that $D_w H(v,\phi(v)) = 0$, and $D_v H(v,\phi(v)) = f_V(v)$ for any $v < v^*$. We abuse the notation for notational homogeneity and ease of exposition.

Theorem 2. *Let $\phi(\cdot)$ solve the system of ODEs in Equation ϕ and suppose that:*

[6] We adopt the convention that $D_v H(\underline{v}, w)$ and $D_w H(v, \underline{w})$ refer to the right-hand derivatives.

- *the Condition 3 holds, and;*
- *for each $v \in (v^*, \underline{v}]$ and for each $x \in (v^*, v)$, $\frac{d\phi(x)}{dx} < \frac{v-\phi(x)}{2(v-x)}$.*

Then the strategy profile $\boldsymbol{\beta} = \{(\beta_i(s \mid v_i, w_i))\}_{i=1}^N \equiv \{\min\{\phi(v_i), w_i\}\}_{i=1}^N$ constitutes a symmetric pure strategy equilibrium of the auction.

Corollary 1. *Suppose that the strategy profile $\boldsymbol{\beta} = \{(\beta_i(s \mid v_i, w_i))\}_{i=1}^N \equiv \{\min\{\phi(v_i), w_i\}\}_{i=1}^N$ constitutes a symmetric pure strategy equilibrium of the auction, where $\phi(\cdot)$ is strictly monotone and continuous, and solves the system of ODEs in Equation ϕ. Then:*

- *the Condition 3 holds;*
- *for each $v \in (v^*, \underline{v}]$ and for each $x \in (v^*, v)$, $\frac{d\phi(x)}{dx} \leq \frac{v-\phi(x)}{2(v-x)}$.*

Discussion and Proof Sketch of Theorem 2 and Corollary 1

The condition $\frac{d\phi(x)}{dx} < \frac{v-\phi(x)}{2(v-x)}$ results from the interaction of two key factors. The first is the ability to place a non-flat bidding function in response to flat bidding functions. The second is the constraint imposed by budgets. Indeed, this condition emerges as a second-order sufficiency condition. Single-unit auctions do not have this condition. Neither do auctions in the absence of budget constraints. This condition arises only when we combine the presence of budgets with multiple objects.

The mathematical implication of the condition $\frac{d\phi(x)}{dx} < \frac{v-\phi(x)}{2(v-x)}$ is as follows. In the presence of budgets, the expected pay-off function need not be globally concave, hence, Che and Gale (1998)'s conditions are insufficient; we need additional second-order conditions. The second-order conditions in Theorem 2 and Corollary 1 reflect the local concavity of the payoff function. Our proof approach necessitates that the local concavity of the payoff function is in terms of $\|\cdot\|_\infty$ norm. Note that local concavity in $\|\cdot\|_\infty$ norm implies local concavity in $\|\cdot\|_{W^{1,1}}$ norm.

Before we present the proof sketch, we emphasize the inapplicability of some methods used in the literature. Firstly, we cannot use the approach in Ausubel et al. (2014). Ausubel et al. (2014)'s existence result rests upon the ex-post efficiency of a symmetric pay-as-bid auction. With budget constraints, the ex-post allocation may be inefficient with a positive probability.

Since the bidders may be budget-constrained, the payoff security property in Reny (1999) can fail. Reny (2011) is also inapplicable since Reny (2011)'s results bank upon a carefully constructed partial order on the type space. Foreshadowing the result in Theorem 3, such a partial order need not always exist with budget constraints. Even if Reny (2011)'s partial order exists, it may be hard to pin down. Carbonell-Nicolau and McLean (2018) is inapplicable since the sum of the payoff functions for a given type profile need not be upper-semi continuous in the action profile.

Another approach is to use the equilibrium in a dominant strategy Vickrey auction to get the equilibrium in a pay-as-bid auction using the revenue equivalence theorem. Nevertheless, one of Che and Gale (1998)'s insights is the breakdown of revenue equivalence between various auction formats. Consequently, we cannot use Wilson (1979)'s linkage principle approach.

We now present the proof sketch of Theorem 2. The formal proof is in Appendix B. The bidder's optimization problem in the equilibrium is

$$\max_{b(\cdot)\in\mathcal{B}} \int_0^1 (v - b(s))\, H\big(\lambda(b(s)), b(s)\big)^{N-1} ds.$$

We establish in Lemma 4 and Lemma 5 that $\boldsymbol{\beta} \equiv \{\min\{\phi(v_i), w_i\}\}_{i=1}^N$ is only possible flat-bid equilibrium. Lemma 4 uses the Euler-Lagrange condition for a budget-unconstrained type. Recall that $\mathcal{B}$ is the space of all bidding functions and is endowed with the $\|\cdot\|_{W^{1,1}}$, i.e., the $W^{1,1}$ norm. The Euler-Lagrange condition works because we endow $\mathcal{B}$ with $\|\cdot\|_{W^{1,1}}$, i.e., the $W^{1,1}$ norm. Lemma 5 mimics Lemma 3. Remark that the Euler-Lagrange condition only provides the necessity, and not the sufficiency for $\boldsymbol{\beta} \equiv \{\min\{\phi(v_i), w_i\}\}_{i=1}^N$ to be an equilibrium. Therefore, we cannot use Lemma 4 and Lemma 5 to drive a sufficiency argument.

The nature of the problem does not lend itself to an easily verifiable sufficiency condition. Indeed, a bidder's optimization problem is a variational problem with constraints, and a second variation sufficiency condition is out of hand. To get over this issue, we set up and solve a finite-dimensional optimization problem in Lemma 6 to obtain the sufficiency condition. The advantage of the finite-dimensional optimization approach is that it provides intuitive necessity and sufficiency conditions.

We import the second-order conditions from Lemma 6 to the original auction in Lemma 7. To do so, we exploit the fact that the function $t \mapsto (v-t)\,(H(\lambda(t), t)^{N-1}$ is concave in t when $t \in (w - \widehat{\varepsilon}, w + \widehat{\varepsilon})$ for some $\widehat{\varepsilon} > 0$. This concavity helps us to show that $\beta_i(s \,|\, v_i, w_i) = \min\{\phi(v_i), w_i\}$ is a strong ($\|\cdot\|_\infty$) local maximizer of the payoff function.[7]

Since the strategy profile $\boldsymbol{\beta} = \{(\beta_i(s \,|\, v_i, w_i))\}_{i=1}^N$ where $\beta_i(s \,|\, v_i, w_i) = \min\{\phi(v_i), w_i\}$ for each bidder i is the only possible symmetric strategy profile which satisfies Equation ϕ, it is the unique symmetric equilibrium in the pay-as-bid auction.

The condition $\frac{d\phi(x)}{dx} < \frac{v-\phi(x)}{2(v-x)}$ arises from second-order sufficiency condition of KKT Theorem. Therefore, Corollary 1 is immediate from the necessity conditions in Lemma 6. We omit the proof of Corollary 1.

5. Non-Existence Results

Theorem 2 provides sufficient conditions for an equilibrium. The sufficient conditions are also close to necessary (Corollary 1). Nevertheless, the conditions in Corollary 1 may fail when the number of bidders is high. Therefore, the presence of budgets can lead to an undesirable consequence, namely the non-existence of a symmetric pure strategy equilibrium.

We explore the implications of Corollary 1. We first show that the presence of more than two bidders violates the necessity conditions for any type space with $\underline{v} = \underline{w}$. For instance, if the type space is the unit square, and the budgets and valuations are independent and uniformly distributed, the equilibrium in Theorem 2 does not exist when there are more than two bidders.

[7] $\|\cdot\|_{W^{1,1}}$ norm is stronger than $\|\cdot\|_\infty$ norm.

However, the fragility of the equilibrium is far from a knife-edge situation. Indeed, the equilibrium fails to exist when there are a sufficiently large number of bidders unless the marginal density of valuation is steeply decreasing. For instance, if the valuations follow uniform distribution over the interval $[\underline{v}, \overline{v}]$, the equilibrium fails to exist for any type space and type distribution when the number of bidders is sufficiently large.

Theorem 3 states the non-existence result– the presence of budgets has an undesirable theoretical implication. The existence of equilibrium becomes an empirical question that depends upon the number of bidders. In light of Theorem 3, the question of equilibrium existence assumes importance in the choice of auction format.

Theorem 3. *The equilibrium in Theorem 2 is fragile:*

(i) Finite non-existence result: suppose $N > 2$ and $\underline{v} = \underline{w}$. Then a symmetric flat-bid equilibrium does not exist.

(ii) Asymptotic non-existence result: suppose that $\underline{v} < \underline{w}$, and $\sup_{x \le \underline{w}} f_V(x) < 2 f_V(\underline{w})$. Then there exists some N_0 such that a symmetric flat-bid equilibrium does not exist for any $N \ge N_0$.

Discussion and Proof Sketch of Theorem 3

The presence of budgets derails the usual arguments for equilibrium in standard pay-as-bid auctions. Indeed, the second-order conditions arise because of the budgets. The conditions in Theorem 2 guarantee that the expected payoff function of a *budget-constrained* bidder is locally concave near the bidding function $\beta(s \,|\, v, w) \equiv w$. When the number of bidders becomes large, the expected payoff function can become locally convex around the bidding function $\beta(s \,|\, v, w) \equiv w$.[8]

We now present the proof sketch of Theorem 3. The formal proof is in Appendix C. If $\underline{v} = \underline{w}$, a bidder with a generic valuation v is always budget-constrained with positive probability. Consequently, the local concavity of the payoff function always fails when the number of bidders exceeds 2. Indeed, by Corollary 1 the strategy profile $\boldsymbol{\beta} = \{\min\{\phi(v_i), w_i\}\}_{i=1}^N$ is an equilibrium only if each type $(v, \phi(x))$ satisfies $\frac{d\phi(x)}{dx} \le \frac{v - \phi(x)}{2(v-x)}$. If $\underline{v} = \underline{w}$, then $\lim_{v \to \underline{v}^+} \frac{d\phi(v)}{dv} = 1 - \frac{1}{N} < \frac{1}{2}$. Thus, for a type (v, w) with high enough v and a low enough w, the function $t \mapsto (v - t)\left(H(\lambda(t), t)\right)^{N-1}$ is *strictly convex* in t when $t \in (w - \varepsilon, w + \varepsilon)$ for each $\varepsilon > 0$. Consequently, the bidding function $\beta(s \,|\, v, w) \equiv w$ becomes a strict local minimizer of the payoff function– the type (v, w) has a unilateral profitable deviation to a non-flat bidding function.

Now consider the case $\underline{v} < \underline{w}$. Recall that v^* uniquely solves the following equation:

$$\underline{w} = v^* - \int_{\underline{v}}^{v^*} \frac{F_V(x)^{N-1}}{F_V(v^*)^{N-1}} dx.$$

The right-hand derivative $\left.\frac{d\phi(v)}{dv}\right|_{v^*}$ is well defined. When the number of bidders grows, v^* becomes close to $\underline{w}$. Consequently, $\frac{\overline{v} - \underline{w}}{2(\overline{v} - v^*)}$ approaches $\frac{1}{2}$. As N grows, the right-hand derivative $\left.\frac{d\phi(v)}{dv}\right|_{v^*} \le \frac{1}{2}$ only if $D_w H(\underline{w}, \underline{w}) < 0$, which is impossible.

[8] Recall that the notion of local concavity/convexity is in terms of $\|\cdot\|_\infty$.

In both cases, the necessity condition in Corollary 1 that imposes the local concavity of the payoff function fails when N increases.

6. Conclusion

We show necessary and sufficient conditions for equilibrium in a pay-as-bid auction with budget-constrained bidders in the class of flat-bid strategies. We justify our focus on flat-bid strategies through Theorem 1, which states that any best response to a *strongly regular* strategy profile is a flat-bid strategy.

Theorem 2 and Corollary 1 contain the second-order sufficiency and necessity conditions for a symmetric equilibrium. The second-order conditions are new and in addition to conditions in the literature. We subsequently state Theorem 3; the flat-bid equilibrium can break down when the number of bidders is sufficiently high.

Questions emerge. We assume independent types and constant marginal valuation. Are there analogous second-order conditions for settings with decreasing marginal valuation or affiliated types? Does the Non-Existence issue in Theorem 3 go away with decreasing marginal valuation or affiliated types?

Another strand of literature we address is the Bayesian mechanism design literature. With budget constraints, how should a mechanism designer design an optimal or efficient Bayesian auction with a non-fragile equilibrium? Is such a design even possible?

Another open question relates to the choice of auction formats. We study pay-as-bid auctions. Does an analogous second-order condition (Theorem 2) or Non-Existence result (Theorem 3) carry over to Uniform Price auctions? If yes, then for given model primitives, is it possible to rank uniform price and pay-as-bid auctions in terms of the maximal number of bidders each auction format can handle?

We hope the literature addresses these and more such questions in the future.

References

Anderson, E. J., P. Holmberg, and A. B. Philpott (2013): "Mixed strategies in discriminatory divisible-good auctions," *The RAND Journal of Economics*, 44, 1–32.

Ausubel, L. M., P. Cramton, M. Pycia, M. Rostek, and M. Weretka (2014): "Demand Reduction and Inefficiency in Multi-Unit Auctions," *The Review of Economic Studies*, 81, 1366–1400.

Back, K. and J. F. Zender (1993): "Auctions of Divisible Goods: On the Rationale for the Treasury Experiment," *The Review of Financial Studies*, 6, 733–764.

Brezis, H. (2010): *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer New York.

Bulow, J., J. Levin, and P. Milgrom (2009): "Winning Play in Spectrum Auctions," Working Paper 14765, National Bureau of Economic Research.

Carbonell-Nicolau, O. and R. P. McLean (2018): "On the Existence of Nash Equilibrium in Bayesian Games," *Mathematics of Operations Research*, 43, 100–129.

Che, Y.-K. and I. Gale (1998): "Standard Auctions with Financially Constrained Bidders," *The Review of Economic Studies*, 65, 1–21.

——— (2006): "Revenue comparisons for auctions when bidders have arbitrary types," *Theoretical Economics*, 1, 95–118.

Cole, H., D. Neuhann, and G. Ordoñez (2022): "Asymmetric Information and Sovereign Debt: Theory Meets Mexican Data," *Journal of Political Economy*, 130, 2055–2109.

Dobzinski, S., R. Lavi, and N. Nisan (2012): "Multi-unit auctions with budget limits," *Games and Economic Behavior*, 74, 486–503.

Durante, F. and C. Sempi (2015): *Principles of Copula Theory*, Taylor & Francis.

Genc, T. S. (2009): "Discriminatory Versus Uniform-Price Electricity Auctions with Supply Function Equilibrium," *Journal of Optimization Theory and Applications*, 140, 9–31.

Ghosh, G. (2021): "Simultaneous auctions with budgets: Equilibrium existence and characterization," *Games and Economic Behavior*, 126, 75–93.

Hafalir, I. E., R. Ravi, and A. Sayedi (2012): "A near Pareto optimal auction with budget constraints," *Games and Economic Behavior*, 74, 699–708.

Holmberg, P. (2009): "Supply function equilibria of pay-as-bid auctions," *Journal of Regulatory Economics*, 36, 154–177.

Hortaçsu, A. and D. McAdams (2010): "Mechanism Choice and Strategic Bidding in Divisible Good Auctions: An Empirical Analysis of the Turkish Treasury Auction Market," *Journal of Political Economy*, 118, 833–865.

Kastl, J. (2011): "Discrete Bids and Empirical Inference in Divisible Good Auctions," *The Review of Economic Studies*, 78, 974–1014.

——— (2012): "On the properties of equilibria in private value divisible good auctions with constrained bidding," *Journal of Mathematical Economics*, 48, 339–352.

Kotowski, M. H. (2020): "First-price auctions with budget constraints," *Theoretical Economics*, 15, 199–237.

Linnenbrink, D. (2023): "Pay-as-bid Auctions with Private Information," .

Pai, M. M. and R. Vohra (2014): "Optimal auctions with financially constrained buyers," *Journal of Economic Theory*, 150, 383–425.

Palfrey, T. R. (1980): "Multiple-object, discriminatory auctions with bidding constraints: A game-theoretic analysis," *Management Science*, 26, 935–946.

Pycia, M. and K. Woodward (2023): "A Case for Pay-as-Bid Auctions," .

Reny, P. J. (1999): "On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games," *Econometrica*, 67, 1029–1056.

——— (2011): "On the Existence of Monotone Pure-Strategy Equilibria in Bayesian Games," *Econometrica*, 79, 499–553.

——— (2020): "Nash Equilibrium in Discontinuous Games," *Annual Review of Economics*, 12, 439–470.

Riley, J. G. and W. F. Samuelson (1981): "Optimal Auctions," *The American Economic Review*, 71, 381–392.

Salant, D. J. (1997): "Up in the Air: GTE's Experience in the MTA Auction for Personal Communication Services Licenses," *Journal of Economics & Management Strategy*, 6, 549–572.

Vickrey, W. (1961): "Counterspeculation, Auctions, and Competitive Sealed Tenders," *The Journal of Finance*, 16, 8–37.

Vinter, R. (2010): *Optimal Control*, Boston: Birkhäuser Boston.

Wang, J. J. D. and J. F. Zender (2002): "Auctioning Divisible Goods," *Economic Theory*, 19, 673–705.

Wilson, R. (1979): "Auctions of Shares," *The Quarterly Journal of Economics*, 93, 675–689.

Wittwer, M. (2020): "Interconnected pay-as-bid auctions," *Games and Economic Behavior*, 121, 506–530.

APPENDIX

A. Proofs of Results in Section 3

Proof of Theorem 1

Lemma 2. *Let $\boldsymbol{\beta}_{-i}$ be strongly regular and suppose that bidder i's best response prescribes bidding functions of the form $\beta_i(s \mid v, w) = \min\{\phi_i(s \mid v), \gamma_i(s \mid w)\}$. If she is budget-unconstrained under the type realization (v, w), then $\beta_i(s \mid v, w)$ is constant on s.*

Proof. Define $b_0 = \min\{\phi_i(0 \mid v), \gamma_i(0 \mid w)\}$ as bidder i's optimal bid for the share indexed 0. Her best response $\beta_i(\cdot \mid v, w)$ can be obtained as a solution to the following optimal control problem:

$$\max \int_0^1 (v - b(s))\, G_s(b(s))\, ds \qquad \text{subject to} \qquad \begin{aligned} \frac{d\,b(s)}{d\,s} &= z(s), \\ b(0) &= b_0. \end{aligned} \tag{OPT}$$

In the problem OPT, we use $z(s) \leq 0$ as the control variable and $b(s)$ as the state variable. Denote by μ the corresponding co-state variable. Since β_i is a best response to $\boldsymbol{\beta}_{-i}$, the solution to OPT satisfies $z^*(s) = \frac{d\beta_i(s\,|\,v,w)}{ds}$. The corresponding Hamiltonian is

$$\mathcal{H}(b,z,\mu) \;=\; (v\,-\,b)\,G_s(b)\;+\;\mu\,z.$$

Now let $\partial_b\mathcal{H}(b,z,\mu)$ be the limiting sub-differential of $\mathcal{H}$ with respect to b.[9] The adjoint condition, which holds for almost every $s\in[0,1]$, is given by

$$-\frac{d\mu}{ds} \;=\; \partial_b\mathcal{H}(b,z,\mu) \;=\; (v-b)g_s(b)\;-\;G_s(b). \tag{ADJ}$$

Claim 1. *The co-state variable $\mu(\cdot)$ satisfies $\mu(s)\geq 0$, for each $s\in[0,1]$.*

Proof. Note that $\beta_i(s\,|\,\underline{v},\underline{w}) = \underline{v}$ for each $s\in[0,1]$. Therefore, for any $v>\underline{v}$ and $w>\underline{w}$, $\beta_i(s\,|\,v,w)>\underline{v}$ for each $s\in[0,1]$. Since the type (v,w) is budget-unconstrained, the optimal bidding function must follow $b(s)>\underline{v}$ for each $s\in[0,1]$.

We therefore treat OPT as a "free endpoint problem." By the transversality conditions, $\mu(1)=0$. Suppose that there is some $s\in(0,1)$ such that $\mu(s)<0$. Since $\mu(\cdot)$ is absolutely continuous, there exists some $\varepsilon>0$ such that $\mu(\cdot)<0$ on the interval $(s,s+\varepsilon)$.

By the Pontryagin maximum principle, the optimal control variable z^* must maximize the Hamiltonian. Therefore, the optimal control variable satisfies $z^*<0$ on the interval $(s,s+\varepsilon)$. We do not impose any lower bound on z^*, and $z^*<\frac{b_0-\underline{v}}{\varepsilon}$ is possible on the interval $(s,s+\varepsilon)$. Therefore, there exists some $q_\varepsilon\in(s,s+\varepsilon)$ such that $b(\cdot)=\underline{v}$ in the interval $[q_\varepsilon,1]$. This is a contradiction to the monotonicity of β_i. $\square$

We complete the proof of the Lemma by showing that even if $\mu(s)=0$ on some interval, the optimal control satisfies $z^*(s)=0$. Suppose that $\mu(s)=0$ on some interval $[r,q]$ and $\frac{db(s)}{ds}=z^*(s)<0$ almost everywhere on $[r,q]$. Consequently, $\frac{d\mu}{ds}=0$ almost everywhere on $[r,q]$, and $b(r)>b(q)$. It is without loss to assume that $g_r(b(r))$, $g_q(b(r))$, and $g_q\,(b(q))$ are all single-valued. By Equation ADJ,

$$\begin{aligned} b(r)+\frac{G_r(b(r))}{g_r(b(r))} &= b(q)+\frac{G_q\,(b(q))}{g_q\,(b(q))} \\ \Longrightarrow \frac{G_r(b(r))}{g_r(b(r))} &< \frac{G_q\,(b(q))}{g_q\,(b(q))} \leq \frac{G_q\,(b(r))}{g_q\,(b(r))} \leq \frac{G_r(b(r))}{g_r(b(r))}. \end{aligned}$$

The last inequalities follow the fact that $\boldsymbol{\beta}_{-i}$ is strongly regular (see Definition 7, and give us a contradiction. Therefore, the optimal control satisfies

$$z^*(s) \;=\; \frac{db^*(s)}{ds} \;=\; \frac{d\beta_i(s\,|\,v,w)}{ds} \;=\; 0. \qquad \square$$

Lemma 3. *Suppose that bidder i's type realization (v,w) is budget-constrained under the best response strategy β_i. Then type (v,w) necessarily places a flat bid.*

[9] See Vinter (2010), chapter 4 for details.

Proof. From Lemma 1, $\beta_i(s \mid v, w) = \min\{\phi_i(s \mid v), \gamma_i(s \mid w)\}$. Notice that if bidder i's of type (v, w) is budget unconstrained, then $\gamma(0 \mid w) = \gamma(1 \mid w) = w$.

Indeed, suppose that for a given budget w, $\gamma_i(0 \mid w) > \gamma_i(1 \mid w)$. By continuity of β_i in types, there must exist a positive measure of $\tilde{v}$ such that $\gamma_i(0 \mid w) > \phi_i(0 \mid \tilde{v}) \geq \gamma_i(1 \mid w)$. Choose any such $\tilde{v}$ arbitrarily. The type $(\tilde{v}, w)$'s optimal bidding function satisfies

$$\int_0^1 \beta_i(s \mid \tilde{v}, w) ds \;=\; \int_0^1 \min\{\phi_i(s \mid \tilde{v}), \gamma_i(s \mid w)\}\, ds \;<\; \int_0^1 \gamma_i(s \mid w)\, ds \;=\; w.$$

Therefore, the type $(\tilde{v}, w)$ is budget-unconstrained. By Lemma 2, for each such $\tilde{v}$, the type $(\tilde{v}, w)$ must place a flat bid for all the shares. By the continuity of the bidding functions, for each such $\tilde{v}$, there is some $\tilde{s} \in [0, 1)$ such that $\gamma_i(\tilde{s} \mid w) = \phi_i(\tilde{s} \mid \tilde{v})$ and $\gamma_i(s \mid w) \leq \phi_i(s \mid \tilde{v})\; \forall\, s > \tilde{s}$. Observe that $\tilde{s}$ is the last share at which the bidding function is decided by $\phi_i(\cdot \mid \cdot)$. For each $s > \tilde{s}$, the optimal bidding function $\beta_i(s \mid v, w) = \gamma_i(s \mid w)$, and consequently, $\frac{d\gamma_i(s \mid w)}{ds} = 0$.

Since the choice of $\tilde{v}$ is arbitrary in the convex set $\{\tilde{v}\colon \gamma_i(0 \mid w) > \phi_i(0 \mid \tilde{v}) \geq \gamma_i(1 \mid w)\}$, $\tilde{s}$ takes all the values in $(0, 1)$. Therefore, $\frac{d\gamma_i(s \mid w)}{ds} = 0$ for each $s \in (0, 1)$. Consequently, $\gamma_i(0 \mid w) = \gamma_i(1 \mid w)$ and $\gamma_i(s \mid w) = w$ for each $s \in [0, 1]$ and for each w.

For each $w \in [\underline{w}, \overline{w}]$, define:

$$v(w) := \sup\left\{ v\colon \int_0^1 \beta_i(s \mid v, w) ds < w \right\}.$$

Observe that $\int_0^1 \beta_i(s \mid \underline{v}, w) ds = \underline{v} \leq w$ for each $w \in [\underline{w}, \overline{w}]$. Finally, for a given w, $\int_0^1 \beta_i(s \mid x, w) ds$ is a continuous monotone function of x. Therefore, $v(w)$ is well-defined.

By continuity and monotonicity of β_i, the type $(v(w), w)$ places a flat bid of w for all the shares. For each type (v, w) that is budget-constrained, $v \geq v(w)$. By monotonicity of the strategy β_i:

$$\beta_i(s \mid v, w) \geq \beta_i(s \mid v(w), w) = w \;\text{ for each }\; s \in [0, 1].$$

Since $\int_0^1 \beta_i(s \mid v, w) ds = w$, therefore, $\beta_i(s \mid v, w) = w$ for each $s \in [0, 1]$. The proof is complete. □

B. Proofs of Results in Section 4

Proof of Theorem 2

Proof.

Lemma 4. *Suppose that β_i is a monotone best response to $\boldsymbol{\beta}_{-i} = \{(\beta_j(s \mid v_j, w_j))\}_{j \neq i}$ where $\beta_j(s \mid v_j, w_j) = \min\{\phi(v_j), w_j\}$. If bidder i of type (v, w) is budget-unconstrained, then the bidding function $\beta_i(s \mid v, w) = \phi(v)$.*

Proof. By supposition, $\beta_i(\cdot \mid v, w)$ is a solution to bidder i's variational problem:[10]

[10]We purposefully suppress the boundary conditions given by $\beta_i(0 \mid v, w)$ and $\beta_i(1 \mid v, w)$. Since Euler-Lagrange

$$\max \int_0^1 \left(v - b(s)\right) H\big(\lambda(b(s)), b(s)\big)^{N-1} ds \qquad \text{(OPT: Equilibrium)}$$

Since all the densities are differentiable, $\phi(\cdot)$ is C^2 on $(\underline{v}, v^*) \cup (v^*, \overline{v})$. This is so because $\phi(\cdot)$ solves Equation ϕ and is therefore C^1. The right hand side in Equation ϕ is a C^1 function of $\phi(\cdot)$ on the set $(\underline{v}, v^*) \cup (v^*, \overline{v})$, and therefore $\frac{d\phi(v)}{dv}$ must itself be C^1 on the set $(\underline{v}, v^*) \cup (v^*, \overline{v})$. Analogous argument extends to $\lambda(\cdot)$. Therefore, for any $v \neq v^*$, OPT: Equilibrium satisfies all the required conditions for the classical Euler-Lagrange condition. For any $v \neq v^*$, the Euler-Lagrange equation becomes

$$\left(v - b(s)\right) \left(D_v H(\lambda(b(s))\, b(s)) \left. \frac{d\lambda(x)}{dx} \right|_{x=b(s)} + D_w H(\lambda(b(s))\, b(s)) \right) = \frac{H\big(\lambda(b(s))\,, b(s)\big)}{N-1}. \tag{EL}$$

From Equation ϕ, $b(s) \equiv \phi(v)$ is the unique solution to the Equation EL. Therefore, a budget-unconstrained bidder i with a valuation $v \neq v^*$ bids $\phi(v)$ for all the shares. By continuity of β_i, a budget-unconstrained bidder i with a valuation v^* bids $\phi(v^*) = \underline{w}$ for all the shares. In all cases, the bidding function $\beta_i(s \,|\, v, w) = \phi(v)$. □

Lemma 5. *Suppose the type (v, w) is budget-constrained under the equilibrium strategy β_i. Then $\beta_i(s \,|\, v, w) = w$ for each $s \in [0, 1]$.*

Proof. The proof mimics the arguments in Lemma 3 and has been omitted. □

Lemma 6. *Fix $L \in \mathbb{N} \setminus \{1\}$, $v > v^*$, and consider the following optimization problem:*

$$\max_{\boldsymbol{\theta} \in \mathbb{R}^L} \sum_{\ell=1}^{L} \left(\frac{v}{L} - \frac{\theta^\ell}{L} \right) \left(H(\lambda(\theta^\ell), \theta^\ell)^{N-1} \right) \quad \text{s.t.} \sum_{\ell=1}^{L} \theta^\ell - L\phi(x) \leq 0. \tag{Finite OPT}$$

1. *If $x \in (v^*, v)$, then $\boldsymbol{\theta}^* = \big(\phi(x) \cdots, \phi(x)\big)^\top$ is a local solution to Finite OPT.*
2. *If $x \geq v$, then $\boldsymbol{\theta}^* = \big(\phi(v) \cdots, \phi(v)\big)^\top$ is a local solution to Finite OPT.*

Proof. Define the Lagrangian of the problem:

$$\mathcal{L}(v, \phi(x), \boldsymbol{\theta}, \eta) := \sum_{\ell=1}^{L} \left(\frac{v}{L} - \frac{\theta^\ell}{L} \right) \left(H(\lambda(\theta^\ell), \theta^\ell)^{N-1} \right) - \eta \left(\sum_{\ell=1}^{L} \theta^\ell - L\phi(x) \right).$$

If $x \in (v^*, v)$, then $\phi(x) < \phi(v)$ and consequently, the constraint $\sum_{\ell=1}^{L} \theta^\ell - L\phi(x) \leq 0$ is active; $\boldsymbol{\theta}^* = \big(\phi(x) \cdots, \phi(x)\big)^\top$ satisfies the first-order necessary conditions with the Lagrangian multiplier

$$\eta^* := \frac{1}{L} H(x, \phi(x))^{N-1} \left[\frac{v - x}{x - \phi(x)} \right] > 0.$$

condition is a necessity condition, bidder i must optimally choose $\beta_i(0 \,|\, v, w)$ and $\beta_i(1 \,|\, v, w)$.

Note that $\boldsymbol{\theta}^*$ is locally optimal when the second-order sufficient conditions hold. Denote the L-dimensional identity matrix by $\mathbb{I}_{L\times L}$.

$$\nabla^2_{\boldsymbol{\theta\theta}}\mathcal{L}(v,\phi(x),\boldsymbol{\theta}^*,\eta^*) = \frac{1}{L}\frac{H(x,\phi(x))^{N-1}}{(x-\phi(x))^2}\left[2(v-x)-\frac{(v-\phi(x))}{\frac{d\phi(x)}{dx}}\right]\mathbb{I}_{L\times L} \tag{Hessian}$$

The second-order conditions are:

$$\begin{aligned}
&\boldsymbol{y}^\top\nabla^2_{\boldsymbol{\theta\theta}}\mathcal{L}(v,\phi(x),\{\theta_i^\ell\}_{\ell=1}^L,\mu)\boldsymbol{y}\le 0\ :\ \text{necessity}\\
&\boldsymbol{y}^\top\nabla^2_{\boldsymbol{\theta\theta}}\mathcal{L}(v,\phi(x),\{\theta_i^\ell\}_{\ell=1}^L,\mu)\boldsymbol{y}< 0\ :\ \text{sufficiency}\\
&\text{for each }\ \boldsymbol{y}\in\left\{\boldsymbol{t}\in\mathbb{R}^L:\sum_{\ell=1}^L t^\ell=0\right\}.
\end{aligned} \tag{SOC}$$

The second-order conditions (both necessity and sufficiency) hold when $\frac{d\phi(x)}{dx}<\frac{v-\phi(x)}{2(v-x)}$. The proof of the first part of Lemma 6 is complete.

For the second part of Lemma 6, note that by Equation ϕ, $\boldsymbol{\theta}^*=\big(\phi(v)\cdots,\phi(v)\big)^\top$ is the only possible candidate for the solution and the constraint $\sum_{\ell=1}^L\theta^\ell-L\phi(x)\le 0$ is inactive. Further,

$$\nabla^2_{\boldsymbol{\theta\theta}}\mathcal{L}(v,\phi(v),\boldsymbol{\theta}^*,\eta^*) = -\frac{1}{L}\frac{H(v,\phi(v))^{N-1}}{(v-\phi(v))^2}\left[\frac{(v-\phi(v))}{\frac{d\phi(v)}{dv}}\right]\mathbb{I}_{L\times L} \tag{Hessian: Unconstrained}$$

The second-order sufficiency conditions hold since $\nabla^2_{\boldsymbol{\theta\theta}}\mathcal{L}(v,\phi(v),\boldsymbol{\theta}^*,\eta^*)$ is negative definite. The proof of the second part of Lemma 6 is complete. □

Since v is arbitrary in Lemma 6, a budget-unconstrained bidder always faces a convex optimization problem. Therefore, the Euler-Lagrange Equation EL also provides a sufficient condition for $\beta_i(s\,|\,v,w)=\phi(v)$ to be the global maximizer of OPT: Equilibrium.

All that remains to be shown is that any budget-constrained bidder, i.e., a bidder of type (v,w) such that $\phi(v)>w$ also prefers to place a flat bidding function. To that end, we now show that $\beta_i(s\,|\,v,w)=\min\{\phi(v),w\}=w$ is a strong $(\|\cdot\|_\infty)$ local maximizer of OPT: Equilibrium in Lemma 7.

Lemma 7. *Endow the space of bidding functions $\mathcal{B}$ with the $\|\cdot\|_\infty$ norm. Then $\beta_i(s\,|\,v,w)=\min\{\phi(v),w\}$ is a strong $(\|\cdot\|_\infty)$ local maximizer of OPT: Equilibrium.*

Proof. Suppose not. Then for each $\varepsilon>0$, there is a bidding function $\gamma_\varepsilon(\cdot)\in\mathcal{B}$ such that:

$$\begin{aligned}
&(i)\qquad \int_0^1(v-\gamma_\varepsilon(s))H(\lambda(\gamma_\varepsilon(s)),\gamma_\varepsilon(s))^{N-1}ds>(v-\phi(x))H(x,\phi(x))^{N-1}\\
&(ii)\qquad \int_0^1\gamma_\varepsilon(s)ds<w\\
&(iii)\qquad \|\gamma_\varepsilon(\cdot)-\beta_i(\cdot\,|\,v,w)\|_\infty\equiv\|\gamma_\varepsilon(\cdot)-w\|_\infty<\varepsilon.
\end{aligned}$$

We endow $\mathbb{R}^L$ with $\|\cdot\|_\infty$ norm. By Lemma 6, for each L there exists some $\delta(L) > 0$ such that

$$\text{For each } \ \boldsymbol{y} \ \in \mathbb{R}^L; \ \left\| \boldsymbol{y} - (w, w, \cdots, w)^\top \right\|_\infty < \delta(L)$$
$$\implies \sum_{\ell=1}^{L} \left(\frac{v}{L} - \frac{y^\ell}{L} \right) \left(H(\lambda(y^\ell), y^\ell) \right)^{N-1} \leq (v - w) \left(H(\lambda(w), w) \right)^{N-1}$$

For a budget-constrained bidder of type (v, w), $w < \phi(v)$. Recall that $\lambda(\cdot) = \phi^{-1}(\cdot)$. By the conditions in Theorem 2, $2(v - \lambda(w)) - (v - w)\frac{d\lambda(w)}{dw} < 0$, and consequently, $z \mapsto (v - z) \left(H(\lambda(z), z) \right)^{N-1}$ is concave in the interval $(w - \widehat{\varepsilon}, w + \widehat{\varepsilon})$ for some $\widehat{\varepsilon} > 0$. Notice that $\widehat{\varepsilon}$ is independent of L. Therefore, for each $L \in \mathbb{N}$, $\sum\limits_{\ell=1}^{L} \left(\frac{v}{L} - \frac{y^\ell}{L} \right) \left(H(\lambda(y^\ell), y^\ell) \right)^{N-1}$ is concave in $\widehat{\varepsilon}$-neighborhood of $(w, w, \cdots, w)^\top \in \mathbb{R}^L$.

Claim 2 exploits the disassociation between L and $\widehat{\varepsilon}$.

Claim 2. *There exists $\delta > 0$ such that for each $L > 1$, the following implication holds:*

$$\textit{For each } \ \boldsymbol{y} \ \in \mathbb{R}^L; \ \left\| \boldsymbol{y} - (w, w, \cdots, w)^\top \right\|_\infty < \delta$$
$$\implies \sum_{\ell=1}^{L} \left(\frac{v}{L} - \frac{y^\ell}{L} \right) \left(H(\lambda(y^\ell), y^\ell) \right)^{N-1} \leq (v - w) \left(H(\lambda(w), w) \right)^{N-1}$$

Proof. Suppose not. Then 0 is an accumulation point of the sequence $\{\delta(L)\}_{L \geq 1}$. It is without loss to assume that $\{\delta(L)\}_{L \geq 1}$ is a monotonically decreasing sequence and therefore $\lim\limits_{L \to \infty} \delta(L) = 0$. To see this, suppose that $\delta(L+1) > \delta(L)$. Since $(w, w, \cdots, w)^\top \in \mathbb{R}^{L+1}$ solves Finite OPT in $\delta(L+1)$-neighborhood, then $(w, w, \cdots, w)^\top \in \mathbb{R}^{L+1}$ also solves Finite OPT in $\delta(L)$-neighborhood of $(w, w, \cdots, w)^\top \in \mathbb{R}^{L+1}$. Therefore, we can always impose that $\delta(L+1) \leq \delta(L)$.

By supposition, there must exist some L and some $\boldsymbol{y} \in \mathbb{R}_+^{2L}$ such that following conditions hold:

$$(i) \qquad \delta(2L) < \left\| \boldsymbol{y} - (w, w, \cdots, w)^\top \right\|_\infty < \delta(L) < \widehat{\varepsilon};$$
$$(ii) \qquad \sum_{\kappa=1}^{2L} \frac{y^\kappa}{2L} < w;$$
$$(iii) \qquad \sum_{\kappa=1}^{2L} \left(\frac{v}{2L} - \frac{y^\kappa}{2L} \right) \left(H(\lambda(y^\kappa), y^\kappa) \right)^{N-1} > (v - w) \left(H(\lambda(w), w) \right)^{N-1}.$$

Define the vector $\boldsymbol{z} \in \mathbb{R}^L$ with its ℓ^{th} component $z^\ell := \frac{y^\ell + y^{L+\ell}}{2}$ for each $\ell = 1, 2, \cdots, L$. By construction,

$$|z^\ell - w| < \delta(L) < \widehat{\varepsilon} \; \forall \; \ell \in \{1, 2, \cdots, L\} \implies \left\| \boldsymbol{z} - (w, w, \cdots, w)^\top \right\|_\infty < \delta(L) \quad \text{and} \quad \sum_{\ell=1}^{L} \frac{z^\ell}{L} < w.$$

Therefore, z is a feasible point for Finite OPT. Since $t \mapsto (v-t)\,(H(\lambda(t),t)^{N-1}$ is concave in $(w-\widehat{\varepsilon}, w+\widehat{\varepsilon})$, the following inequality holds:

$$\left(\frac{v}{L}-\frac{z^\ell}{L}\right)\left(H(\lambda(z^\ell),z^\ell)\right)^{N-1} \geq \frac{1}{2}\left[\left(\frac{v}{L}-\frac{y^\ell}{L}\right)\left(H(\lambda(y^\ell),y^\ell)\right)^{N-1} + \left(\frac{v}{L}-\frac{y^{L+\ell}}{L}\right)\left(H(\lambda(y^{L+\ell}),y^{L+\ell})\right)^{N-1}\right].$$

Using Lemma 6 and summing up across $\ell = 1, 2, \cdots, L$;

$$(v-w)\,(H(\lambda(w),w))^{N-1} \geq \sum_{\ell=1}^{L}\left(\frac{v}{L}-\frac{z^\ell}{L}\right)\left(H(\lambda(z^\ell),z^\ell)\right)^{N-1}$$

$$\geq \sum_{\ell=1}^{L}\frac{1}{2}\left[\left(\frac{v}{L}-\frac{y^\ell}{L}\right)\left(H(\lambda(y^\ell),y^\ell)\right)^{N-1} + \left(\frac{v}{L}-\frac{y^{L+\ell}}{L}\right)\left(H(\lambda(y^{L+\ell}),y^{L+\ell})\right)^{N-1}\right]$$

$$= \sum_{\kappa=1}^{2L}\left(\frac{v}{2L}-\frac{y^\kappa}{2L}\right)(H(\lambda(y^\kappa),y^\kappa))^{N-1} > (v-w)\,(H(\lambda(w),w))^{N-1}.$$

This is a contradiction. □

Choose δ as in Claim 2. By supposition, there exists some $\gamma_\delta(\cdot) \in \mathcal{B}$ such that:

$$(i) \qquad \int_0^1 (v-\gamma_\delta(s))H(\lambda(\gamma_\delta(s)),\gamma_\delta(s))^{N-1}ds > (v-\phi(x))H(x,\phi(x))^{N-1}$$

$$(ii) \qquad \int_0^1 \gamma_\delta(s)ds < w$$

$$(iii) \qquad \|\gamma_\delta(\cdot) - \beta_i(\cdot\,|\,v,w)\|_\infty = \|\gamma_\delta(\cdot) - w\|_\infty < \delta.$$

The function γ_δ is the point-wise limit of a sequence of step functions $\{\gamma_L\}_{L>1}$, where γ_L has L steps of equal size. Denote by γ_L^ℓ the value of $\gamma_L(s)$ for each $s \in \left[\frac{\ell-1}{L}, \frac{\ell}{L}\right)$. Without loss of generality,

$$\|\gamma_\delta(\cdot) - w\|_\infty < \delta \implies \|\gamma_L(\cdot) - w\|_\infty < \delta \;\; \forall \;\; L > 2.$$

For each L, Since γ_L is a step function with a step size of $\frac{1}{L}$, the following expression holds:

$$\int_0^1 (v-\gamma_L(s))H(\lambda(\gamma_L(s)),\gamma_L(s))^{N-1}ds = \sum_{\ell=1}^{L}\frac{1}{L}\left(v-\gamma_L^\ell\right)\left(H(\lambda(\gamma_L^\ell),\gamma_L^\ell)\right)^{N-1}.$$

Since $\{\gamma_L\}_{L\geq 2} \xrightarrow[p.w.]{} \gamma_\delta$, by the Dominated Convergence theorem:

$$\int_0^1 (v-\gamma_\delta(s))H(\lambda(\gamma_\delta(s)),\gamma_\delta(s))^{N-1}ds = \lim_{L\to\infty}\int_0^1 (v-\gamma_L(s))H(\lambda(\gamma_L(s)),\gamma_L(s))^{N-1}ds.$$

For each L, consider a vector in $\mathbb{R}^L$ with its ℓ^{th} component as γ_L^ℓ. This vector is in the δ-neighborhood of $(\phi(x),\phi(x),\cdots,\phi(x))^\top$ by construction.[11] By Lemma 6,

[11] Recall that $\mathbb{R}^L$ is endowed with $\|\cdot\|_\infty$ norm.

$$\begin{aligned}
&\sum_{\ell=1}^{L}\frac{1}{L}\left(v-\gamma_L^\ell\right)\left(H(\lambda(\gamma_L^\ell),\gamma_L^\ell)\right)^{N-1} && \leq (v-\phi(x))\left(H(x,\phi(x))\right)^{N-1}\\
\Longleftrightarrow &\lim_{L\to\infty}\sum_{\ell=1}^{L}\frac{1}{L}\left(v-\gamma_L^\ell\right)\left(H(\lambda(\gamma_L^\ell),\gamma_L^\ell)\right)^{N-1} && \leq (v-\phi(x))\left(H(x,\phi(x))\right)^{N-1}\\
\Longleftrightarrow &\lim_{L\to\infty}\int_0^1 (v-\gamma_L(s))H(\lambda(\gamma_L(s)),\gamma_L(s))^{N-1}ds && \leq (v-\phi(x))\left(H(x,\phi(x))\right)^{N-1}\\
\Longleftrightarrow &\int_0^1 (v-\gamma_\delta(s))H(\lambda(\gamma_\delta(s)),\gamma_\delta(s))^{N-1}ds && \leq (v-\phi(x))\left(H(x,\phi(x))\right)^{N-1}.
\end{aligned}$$

This is a contradiction. □

By Lemma 7, $\beta_i(s\,|\,v,w)\equiv\min\{\phi(v),w\}=w$ is the strong local maximizer of the payoff function for any budget-constrained type (v,w). Under the conditions in Theorem 2, Problem Finite OPT has no other solution for each $L>1$, and therefore, $(w,w\cdots,w)^\top$ is a global solution to Problem Finite OPT.

Thus, a budget-constrained type (v,w) can never profitably deviate to any other budget-feasible bidding function in OPT: Equilibrium. Consequently, $\beta_i(s\,|\,v,w)=\min\{\phi(v),w\}$ is a global maximizer of the expected payoff function for the type (v,w), i.e., $\beta_i(s\,|\,v,w)=\min\{\phi(v),w\}$ is a global solution to OPT: Equilibrium.

Consequently, $\beta_i(s\,|\,v,w)\equiv\min\{\phi(v),w\}$ is the unique best response to the strategy profile $\boldsymbol{\beta}_{-i}=\{(\beta_j(s\,|\,v_j,w_j))\}_{j\neq i}\equiv\{\min\{\phi(v_j),w_j\}\}_{j\neq i}$. By symmetry, the proof is complete. □

C. Proofs of Results in Section 5

Proof of Theorem 3

Proof. We begin by proving the first part of Theorem 3. By L'Hôpital's rule and Assumption 2–(iii), $\lim_{v\to\underline{v}^+}\frac{d\phi(v)}{dv}=1-\frac{1}{N}$. Therefore, a bidder of type $(\overline{v},\underline{w})$ violates the second-order necessary conditions in Lemma 6, since at $x=\underline{w}=\underline{v}=\phi(\underline{v})$;

$$\frac{\overline{v}-\underline{v}}{2(\overline{v}-\underline{v})}=\frac{1}{2}>\lim_{v\to\underline{v}^+}\frac{d\phi(v)}{dv}=1-\frac{1}{N}.$$

We now show the second statement of Theorem 3. All that we need to show is that when N is large enough, the necessity conditions in Corollary 1 are violated. Hereon, we write $\phi(\cdot)$ as $\phi_N(\cdot)$ to reflect the dependence of $\phi(\cdot)$ on N. Denote by $\lambda_N(\cdot)$ as the inverse of $\phi_N(\cdot)$. Note that $D_vH(\lambda_N(\underline{w}),\underline{w})=f_V(\lambda_N(\underline{w}))$. The conditions in Corollary 1 go through only when for each N,

$$
\begin{aligned}
\frac{f_V(\lambda_N(\underline{w}))}{\frac{F_V(\lambda_N(\underline{w}))}{(N-1)(\lambda_N(\underline{w})-\underline{w})} - D_w H(\lambda_N(\underline{w}),\underline{w})} &\le \frac{\overline{v}-\underline{w}}{2(\overline{v}-\lambda_N(\underline{w}))} \\
\frac{F_V(\lambda_N(\underline{w}))}{(N-1)(\lambda_N(\underline{w})-\underline{w})} - D_w H(\lambda_N(\underline{w}),\underline{w}) &> 0.
\end{aligned}
$$

The following claim handles the limiting arguments.

Claim 3. *Each accumulation point of* $\left\{\frac{F_V(\lambda_N(\underline{w}))}{(N-1)(\lambda_N(\underline{w})-\underline{w})}\right\}_{N\ge 2}$ *belongs to* $\left[\inf_{x\le \underline{w}} f_V(x), \sup_{x\le \underline{w}} f_V(x)\right]$.

Proof. By the Dominated Convergence theorem, the following limit holds:

$$
\lim_{N\to\infty}(\lambda_N(\underline{w})-\underline{w}) = \lim_{N\to\infty}\int_{\underline{v}}^{\lambda_N(\underline{w})}\left(\frac{F_V(x)}{F_V(\lambda_N(\underline{w}))}\right)^{N-1} dx = 0. \tag{4}
$$

Therefore, it suffices to consider the sequence $\{N(\lambda_N(\underline{w})-\underline{w})\}_{N\ge 2}$. Notice that

$$
N(\lambda_N(\underline{w})-\underline{w}) = N\int_{\underline{v}}^{\lambda_N(\underline{w})}\left(\frac{F_V(x)}{F_V(\lambda_N(\underline{w}))}\right)^{N-1} dx = \int_{\underline{v}}^{\lambda_N(\underline{w})}\frac{1}{f_V(x)}\left(\frac{1}{F_V(\lambda_N(\underline{w}))}\right)^{N-1} d\left(F_V(x)^N\right)
$$

$$
\Longrightarrow N(\lambda_N(\underline{w})-\underline{w}) = \int_{\underline{v}}^{\lambda_N(\underline{w})}\frac{1}{f_V(x)}\left(\frac{1}{F_V(\lambda_N(\underline{w}))}\right)^{N-1} d\left(F_V(x)^N\right) \in \left[\frac{F_V(\lambda_N(\underline{w}))}{\sup_{x\le\underline{w}} f_V(x)}, \frac{F_V(\lambda_N(\underline{w}))}{\inf_{x\le\underline{w}} f_V(x)}\right].
$$

Using Equation 4 and rearranging the expression:

$$
\begin{aligned}
\liminf_N \frac{F(\lambda_N(\underline{w}))}{(N-1)(\lambda_N(\underline{w})-\underline{w})} &\in \left[\inf_{x\le\underline{w}} f_V(x), \sup_{x\le\underline{w}} f_V(x)\right] \\
\limsup_N \frac{F(\lambda_N(\underline{w}))}{(N-1)(\lambda_N(\underline{w})-\underline{w})} &\in \left[\inf_{x\le\underline{w}} f_V(x), \sup_{x\le\underline{w}} f_V(x)\right].
\end{aligned}
$$

□

For contradiction, suppose that there is a strictly increasing subsequence of natural numbers $\{N_k\}_{k\ge 1}$ such that for each N_k an equilibrium exists in the auction with N_k bidders. By supposition, the following condition holds for each N_k:

$$
\frac{f_V(\lambda_{N_k}(\underline{w}))}{\frac{F_V(\lambda_{N_k}(\underline{w}))}{(N_k-1)(\lambda_{N_k}(\underline{w})-\underline{w})} - D_w H(\lambda_{N_k}(\underline{w}),\underline{w})} \le \frac{\overline{v}-\underline{w}}{2(\overline{v}-\lambda_{N_k}(\underline{w}))}.
$$

Pass on to a convergent subsequence if necessary and define

$$
\xi := \lim_{k\to\infty}\left\{\frac{F_V(\lambda_{N_k}(\underline{w}))}{(N_k-1)(\lambda_{N_k}(\underline{w})-\underline{w})}\right\}.
$$

At the limit, $\frac{f_V(\underline{w})}{\xi - D_w H(\underline{w},\underline{w})} \leq \frac{1}{2}$. Consequently, $D_w H(\underline{w},\underline{w}) \leq \xi - 2f_V(\underline{w})$.[12] By Claim 3, $\xi \leq \sup\limits_{x \leq \underline{w}} f_V(x)$. Therefore, $\xi - 2f_V(\underline{w}) < 0$ and $D_w H(\underline{w},\underline{w}) < 0$.

By Frechet-Hoeffding bounds (Durante and Sempi, 2015), $F(\underline{w},\underline{w}+\varepsilon) \leq \min\{F_V(\underline{w}), F_W(\underline{w}+\varepsilon)\}$. For $\varepsilon > 0$ small enough, $F(\underline{w},\underline{w}+\varepsilon) \leq F_W(\underline{w}+\varepsilon)$. Therefore, $\lim\limits_{\varepsilon\to 0} D_w F(\underline{w},\underline{w}+\varepsilon) \leq \lim\limits_{\varepsilon\to 0} f_W(\underline{w}+\varepsilon)$ and consequently $D_w H(\underline{w},\underline{w}) \geq 0$. This is a contradiction. The proof is complete. □

D. Distribution of the s–price and the Expected Payoff Function

D.1. Inverse bid functions

We first derive the notion of inverse bid functions. We work with strategies characterized in Lemma 1. For each bidder i and quantity $s \in [0,1]$, we can think of $\phi_i(s\,|\,\cdot)$ as a mapping from $[\underline{v},\overline{v}]$ into $\mathbb{R}_+$. Likewise, we can think of $\gamma_i(s\,|\,\cdot)$ as a function from $[\underline{w},\overline{w}]$ into $\mathbb{R}_+$. Observe that the monotonicity of β_i implies that both $\phi_i(s\,|\,\cdot)$ and $\gamma_i(s\,|\,\cdot)$ are continuous and monotone increasing in their respective domains, and hence invertible. Denote their inverse functions by $\lambda_i(s\,|\,\cdot)$ and $\rho_i(s\,|\,\cdot)$; i.e., $v = \lambda_i(s\,|\,z)$ is bidder i's valuation that generates a bid of z for the share s under β_i, and similarly $w = \rho_i(s\,|\,z)$ is the budget that generates a bid of z for share s under β_i.

Observe that $\lambda_i(s\,|\,\cdot)$ and $\rho_i(s\,|\,\cdot)$ are inverse functions of $\phi_i(s\,|\,\cdot)$ and $\gamma_i(s\,|\,\cdot)$ respectively. Therefore $\lambda_i(s\,|\,\cdot)$ and $\rho_i(s\,|\,\cdot)$ are themselves continuous and strictly monotone, and thus differentiable almost everywhere in their domains. We denote the derivative of $\lambda_i(s\,|\,z)$ with respect to z as $\frac{d\lambda_i(s\,|\,z)}{dz}$, and the derivative of $\rho_i(s\,|\,z)$ with respect to z as $\frac{d\rho_i(s\,|\,z)}{dz}$ wherever the derivatives exist. We adopt the convention that $\frac{d\lambda_i(s\,|\,z)}{dz}$ and $\frac{d\rho_i(s\,|\,z)}{dz}$ refer to the right-hand derivatives at any point of non-differentiability.

Recall from Assumption 1–(i) that $\underline{v} \leq \underline{w}$. In any monotone strategy for bidder i, we impose the restriction $\beta_i(s\,|\,\underline{v},\underline{w}) = \phi_i(s\,|\,\underline{v}) = \underline{v}$, for each $s \in [0,1]$. This restriction states that a bidder with valuation $\underline{v}$ places a bid of $\underline{v}$ for all shares —such a bidder has no incentive to bid higher than $\underline{v}$ for any share and is never budget-constrained. Therefore, the restriction $\phi_i(s\,|\,\underline{v}) = \underline{v}$ is without any loss of generality. We also restrict $\gamma_i(s\,|\,w) \geq \underline{v}$ for all s and w, but notice that $\gamma_i(s\,|\,\underline{w}) < \underline{w}$ is possible for some s.

On the other hand, a bidder i with a budget $\overline{w}$ never has an incentive to bid above $\overline{w}$ for any share $s \in [0,1]$. Therefore, we impose without loss of generality that $\phi_i(s\,|\,v) \leq \overline{w}$ for each share $s \in [0,1]$ and for each valuation $v \in [\underline{v},\overline{v}]$. Similarly, we impose without loss of generality that $\gamma_i(s\,|\,w) \leq \overline{w}$ for each share $s \in [0,1]$ and for each budget $w \in [\underline{w},\overline{w}]$.

Therefore, bidder i's monotone strategy β_i satisfies $\beta_i(s\,|\,v,w) \in [\underline{v},\overline{w}]$, for all $s \in [0,1]$, for all types $(v,w) \in [\underline{v},\overline{v}] \times [\underline{w},\overline{w}]$. For any share $s \in [0,1]$, we can now extend the functions $\lambda_i(s\,|\,\cdot)$ and $\rho_i(s\,|\,\cdot)$ to a common domain $[\underline{v},\overline{w}]$. If for some $z \in [\underline{v},\overline{w}]$, $\phi_i(s\,|\,\overline{v}) < z$, then set $\lambda_i(s\,|\,z) = \overline{v}$ and $\frac{d\lambda_j(q\,|\,z)}{dz} = 0$. Analogously, if for some $z \in [\underline{v},\overline{w}]$ one has $\gamma_i(s\,|\,\overline{w}) < z$, then set $\rho_i(s\,|\,z) = \overline{w}$ and $\frac{d\rho_i(s\,|\,z)}{dz} = 0$.

[12] Observe that if $\xi - D_w H(\underline{w},\underline{w}) < 0$, the flat-bid equilibrium breaks down for large enough N_k anyway. Therefore, we ignore this case.

D.2. Stop-out Price and Expected Payoff Function

The literature uses the distribution of the s–price to define the expected payoff function and establish a necessary condition for the equilibrium in pay-as-bid auctions (See Wilson (1979), Hortaçsu and McAdams (2010), and Kastl (2012)). Nevertheless, as far as we know, the literature misses an explicit formulation of the distribution of the s–price from the model primitives. We provide the distribution of the s–price to avoid the risk of non-existence in the context of budget constrained bidders.

For strategy profile $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_N)$, the *market clearing price* at type profile $(\boldsymbol{v}, \boldsymbol{w})$ is

$$P(\boldsymbol{v}, \boldsymbol{w}) \;=\; \min\Big\{p : \; \textstyle\sum_{i=1}^{N} \beta_i^{-1}(p \,|\, v_i, w_i) \;\leq\; 1\Big\}.$$

Notice that, at the ex ante stage, the market clearing price function is a random variable that depends on the strategy profile $\boldsymbol{\beta}$. We omit $\boldsymbol{\beta}$ to ease the notation.

Recall that we restrict ourselves to continuous bidding functions. Therefore, when the market clears, any bidder i who wins a positive mass of shares $[0, s_i)$ must bid at least $P(\boldsymbol{v}, \boldsymbol{w})$ for any $0 \leq s < s_i$. Thus, bidder i's expected payoff under the budget-feasible strategy profile $\boldsymbol{\beta}$ can be expressed as

$$\Pi_i(v, w \,|\, \boldsymbol{\beta}) \;=\; \int_{([\underline{v},\overline{v}]\times[\underline{w},\overline{w}])^{N-1}} \left(\int_0^1 (v \;-\; \beta_i(s \,|\, v, w))\, \mathbf{1}\left\{\beta_i(s \,|\, v, w) \;>\; p^*(\boldsymbol{v}, \boldsymbol{w})\right\} ds\right) d \prod_{j \neq i} T(v_j, w_j). \tag{5}$$

The expression 5 is not amenable to a variational approach. Therefore, we now derive a more tractable expression for $\Pi_i(v, w \,|\, \boldsymbol{\beta})$. To this end, we compute a conditional distribution of the stop-out price, where the conditioning is on the event that bidder i wins the shares $[0, s)$. This is precisely the notion of s–price.

Define:

$$\Delta_s \;:=\; \Big\{\boldsymbol{q}_{-i} \in [0,1]^{N-1} : \; \sum_{j \neq i} q_j \;=\; 1 \;-\; s\Big\}.$$

For mathematical concreteness, we treat Δ_s as a subset of $\mathbb{R}^{N-2}$ with the usual topology and impose a uniform probability measure $u \colon \mathbb{B}(\Delta_s) \;\to\; [0,1]$ over Δ_s.[13] By an abuse of notation, whenever we write $\int_{\boldsymbol{q}_{-i} \in \Delta_s} l(\boldsymbol{q}_{-i})\, d\boldsymbol{q}_{-i}$ for some real-valued function l, we integrate with respect to the uniform probability measure u. Notice that if $N = 2$, then trivially $\Delta_s = \{1 - s\}$ and Δ_s has a Dirac probability measure concentrated at $1 - s$.

When bidder i's monotone strategy β_i is such that $\beta_i(s \,|\, v_i, w_i) = P_s(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i})$, the resultant stop-out price is $P(\boldsymbol{v}, \boldsymbol{w}) = P_s(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i})$. Further, bidder i wins the shares $[0, s]$ (possibly with the tie). Observe that if bidder i bids strictly above the s–price, bidder i wins a quantity $[0, q)$, where $q > s$. Analogously, if bidder i bids strictly below the s–price, bidder i wins a quantity $[0, r)$, where $r < s$.[14] The s–price is unequal to the stop-out price in both cases. The s–price is identical to the stop-out price if and only if player i wins the shares $[0, s]$, and neither more

[13] $\mathbb{B}(\Delta_s)$ is the Borel σ-field on Δ_s.
[14] If $r = 0$, bidder i wins nothing.

nor less. Therefore, s–price is the stop-out price conditional upon bidder i winning the shares $[0, s)$.

We now compute the distribution of s–price. Fix some $z \in [\underline{v}, \overline{w}]$. Recall that $H(v, w) = F_V(v) + F_W(w) - F(v, w)$. Each opponent j's strategy induces a distribution $H\left(\lambda_j(q_j \,|\, z), \rho_j(q_j \,|\, z)\right)$ on $[\underline{v}, \overline{w}]$. Observe that each type realization $(\boldsymbol{v}_{-i}, \boldsymbol{w}_{-i})$ generates an allocation $\boldsymbol{q}_{-i} \in \Delta_s$. Fix $\boldsymbol{q}_{-i} \in \Delta_s$ and a bidder $j \neq i$ who gets an allocation $[0, q_j]$ under $\boldsymbol{q}_{-i}$. The probability that bidder k bids *at most* z for the share indexed q_j is $H\left(\lambda_j(q_j \,|\, z), \rho_j(q_j \,|\, z)\right)$. Since the types are independent, the probability that the s–price is at most z is:

$$G_s(z) \ := \int\limits_{\Delta_s} \prod_{j \neq i} H\left(\lambda_j(q_j \,|\, z), \rho_j(q_j \,|\, z)\right) \, d\boldsymbol{q}_{-i}$$

For each bidder $j \neq i$:

- $H\left(\lambda_j(q_j \,|\, z), \rho_j(q_j \,|\, z)\right)$ is increasing in z;
- $H\left(\lambda_j(q_j \,|\, \underline{v}), \rho_j(q_j \,|\, \underline{v})\right) = 0$, and;
- $H\left(\lambda_j(q_j \,|\, \overline{w}), \rho_j(q_j \,|\, \overline{w})\right) = 1$.

By construction, $\int\limits_{\Delta_s} 1 \, d\boldsymbol{q}^{-1} = 1$ for each $s \in [0, 1]$. Therefore, $G_s(\cdot)$ is a valid distribution function and its support is a convex subset of $[\underline{v}, \overline{w}]$. By construction, $\underline{v}$ belongs to the support of $G_s(\cdot)$.