TESTABLE RESTRICTIONS OF EQUILIBRIUM OUTCOMES IN STRATEGIC MARKET GAMES

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Abstract. We find testable restrictions of equilibrium outcomes from strategic market games (*a la* Shapley and Shubik). We show restrictions exist for an observable data set from a strategic market game based on an exchange economy with a finite set of traders and commodities. The data set consists of a finite set of traders and the period buying and selling vectors of the economy obtained over discrete time periods. It is market price-endowment rationalizable if, there exist a period price vector, utility functions and initial endowment vector for each traders that can rationalize the data set. We apply our restrictions for a specific class of utility functions.

Keywords: .

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1. Introduction

We use the Shapley-Shubik (1977) strategic market game to find testable restrictions of equilibrium outcomes. An observable market game based on an exchange economy consists of a finite set of traders and the period buying and selling vectors of the economy obtained over discrete time periods. It is market price-endowment rationalizable if, there exist a period price vector, utility functions and initial endowment vector for each traders that can rationalize the data set. We show restrictions exist. We apply our restrictions for a specific class of utility functions.

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1.1. **Related Literature.** Afriat (1967) identified necessary and sufficient conditions (in the form of a linear program) for a finite set of observations on expenditure (based on price vectors and demand bundles) to be consistent with the utility maximization behavior. Forges and Minelli (2009) extended Afriatâ€■s theorem to constraint sets that need not be classical budget sets and need not even be convex. Brown and Matzkin (1996) gave a revealed preference analysis of Walrasian equilibria in an exchange economy and derived observable restrictions on outcomes. Brown and Matzkin (1996) provide the complete set of testable propositions of the pure exchange model on finite observations of the equilibrium manifold and prove that these tests are non-vacuous. Carvajal, Deb, Fense and Quah (2013) considers a set of observations that consists of the price of the good and the output of each firm for an industry that produces a single good and ask whether there are any observable restrictions (that is, restrictions on the data set) implied by the static Cournot equilibrium.

. Our work can easily be placed in the recent literature - different models of oligopoly (Carvajal *et al* 2013, Carvajal *et al* 2014), bargaining situations (Chambers and Echenique, 2014), Nash-bargaining models (Carvajal and González 2014, Cherchye *et al* 2013), finite exchange economies (Bossert and Sprumont 2002, Bachmann 2005) and consumption and exchange with externalities (Deb 2009, Carvajal 2010).

Parallel papers analyzed very similar questions on revealed preferences in Game Theory by Sprumont (2000, 2001) and Ray and Zhou (2001). The central question in this literature is: what are the testable implications of game theoretic solution concepts based on observed outcomes from game forms? Different authors have considered different solution concepts and variations in setting to analyze the above issues in games and in economies (for an earlier survey, see Carvajal *et al* 2004).

Brown and Matzkin provide the complete set of testable propositions of the pure exchange model on finite observations of the equilibrium manifold and prove that these tests are non-vacuous.

Forges and Minelli (2009) extended Afriat's theorem to a class of nonlinear, non-convex budget sets, obtained the analog of the Afriat's inequalities (Afriat 1967) and as a possible application, discussed the test of Nash behavior in strategic market games. Forges and Iehlé (2013) provide the necessary and sufficient condition for the existence of a utility function rationalizing the essential data.

2. The framewrok

Shapley and Shubik (1977) defined a special class of games for exchange economies which they called the strategic market games. Shapley and Shubik (1977) restrict markets as exchange economies without explicit production or consumption processes, in which the commodities are finite in number and perfectly divisible and transferable. In this market the traders, also finite in number, are motivated only by their own final holdings of goods and money given their preferences (represented by their utility functions). Therefore, a market can be denoted by the symbol (N, X, E, U) where $N = \{1, \ldots, n\}$ is a finite set of traders, X is the commodity space and, in particular, $X = (X_1 \times \ldots \times X_l) \times X_{l+1 \equiv m} \in \Re_+^{l+1}$ where the $(l+1)^{\text{th}}$ commodity is money, $E = (e_i = (e_{i1}, \ldots, e_{il}, e_{im}) : i \in N)$ is an indexed collection of points in X representing the endowments of the traders and $U = (U_i : i \in N)$ is an indexed collection of functions from X to $\Re$ representing the utility functions of the traders. The utility functions are assumed to be concave, increasing and continuously differentiable. It is assumed that money as a good is accepted at face value according to the existing conventions of the marketplace, regardless of its intrinsic worth. It is assumed that all exchanges is made with money and hence the set of attainable redistributions of goods will fail to include many redistributions that would be possible if arbitrary transaction-cost free barter were allowed. Therefore, money holding in Shapley and Shubik (1977) is treated as an element of strategy for the traders in the economy. Prices of the commodities depend on the individual trading decisions of the traders. Price of each commodity j is driven upwards by increased buying and downwards by increased selling.[1]

Consider a market (N, X, E, U) and let us imagine l separate trading posts, one for each of the l commodities, where the total supplies $(Q_1, \ldots, Q_l)$, assumed to be all positive, have been deposited for sale "on consignment". Therefore, $Q_j = \sum_{i \in N} q_{ij} > 0$ for all $j \in \{1, \ldots, l\}$ where q_{ij} is the supply of good j by trader i and $q_{ij} \geq 0$. Given any $(Q_1, \ldots, Q_l)$, for each $i \in N$, define $q_i := (q_{i1}, \ldots, q_{il})$. Each trader $i \in N$ makes bids by allocating amounts b_{ij} of his money (that is, the $(l+1)$-th commodity) to trading post j where $j \in \{1, \ldots, l\}$. We shall denote his buying strategy, in the game-theoretic sense, by the vector $b_i = (b_{i1}, \ldots, b_{il})$ where the constraints are (a) $\sum_{j=1}^{l} b_{ij} \leq e_{im}$ and (b) $b_{ij} \geq 0$. The price emerges as a result of the simultaneous bids of all buyers, specifically $p_j = (B_j/Q_j)$ where $B_j := \sum_{i \in N} b_{ij}$. With slight abuse of notation, define $(b, q) := ((b_i, q_i)_{i \in N})$ as an indexed collection of strategies or a strategy profile. Given the strategy $((b_j, q_j)_j \in N \setminus \{i\})$ of all agents $N \setminus \{i\}$, agents i's utility maximization problem (UMP) is to select (b_i, q_i) to maximizes $U_i(x_i(b, q))$ subject to $q_{ij} \in [0, e_{ij}]$, $b_{ij} \geq 0$ for each $j = 1, \ldots, l$, and $\sum_{j=1}^{l} b_{ij} \leq e_{im}$ where $x_i(b, q) = (x_{i1}(b, q), \ldots, x_{il}(b, q), x_{im}(b, q)) \in X$, $x_{ij}(b, q) = e_{ij} - q_{ij} + (b_{ij}/p_j)$ for each $j \in \{1, \ldots, l\}$, and $x_{im}(b, q) = e_{im} - \sum_{j=1}^{l} b_{ij} + \sum_{j=1}^{l} p_j q_{ij}$. Therefore, in general, we have $\frac{\partial x_{ij}}{\partial q_{ij}} = -(1 - \frac{b_{ij}}{B(j)}) \leq 0$, $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$, $\frac{\partial x_{im}}{\partial b_{ij}} = -(1 - \frac{q_{ij}}{Q(j)}) \leq 0$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$.

Definition 1. *Given a market (N, X, E, U), a strategy profile (b, q) is a trading equilibrium if*

(1) *For each good $j \in \{1, \ldots, l\}$, $p_j = \left(\sum_{i \in N} b_{ij} / \sum_{i \in N} q_{ij}\right) > 0$.*

[1]This is in sharp contrast to the classical model that takes prices as given by the indivisible hand which is insensitive to the actions of the sellers at least in the short-run.

(2) *For each* $i \in N$*, given* $((b_j, q_j)_j \in N \setminus \{i\})$ *of all agents* $N \setminus \{i\}$*,* (b_i, q_i) *is a solution to agent* i*'s UMP.*

Definition 2. *Given a market* (N, X, E, U)*, a strategy profile* (b, q) *is a refined trading equilibrium if it is a trading equilibrium such that for each* $i \in N$*,* $e_{im} > \sum_{j=1}^{l} b_{ij}$ *and* $e_{ij} > q_{ij}$ *for all* $j \in \{1, \ldots, l\}$*.*

For any strategy profile $((b_i, q_i)_{i \in N})$, define $B(j) = \sum_{i \in N} b_{ij}$, $Q(j) = \sum_{i \in N} q_{ij}$, and, for any $i \in N$, define $B_{-i}(j) = B(j) - b_{ij}$ and $Q_{-i}(j) = Q(j) - q_{ij}$. Further, for any i and any j, define the ratio $\delta_{ij}(b,q) := [B_{-i}(j) \{\partial U_i(x_i(b,q))/\partial x_{ij}\}] \,/\, [Q_{-i}(j) \{\partial U_i(x_i(b,q))/\partial x_{im}\}]$.

Proposition 1. *Given a market* (N, X, E, U)*, a strategy profile* (b, q) *is a refined trading equilibrium if and only if the following condition hold:*

$$\text{(1)} \qquad For\ any\ j \in \{1, \ldots, l\}, \delta_{ij}(b,q) = p_j^2\ for\ all\ i \in N.$$

The proof of Proposition 1 follows from the standard application of the Kuhn-Tucker conditions and hence is provided in the Appendix. A simplification of condition (1) using $\frac{\partial x_{ij}}{\partial q_{ij}} = -(1 - \frac{b_{ij}}{B(j)})$, $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$, $\frac{\partial x_{im}}{\partial b_{ij}} = -(1 - \frac{q_{ij}}{Q(j)})$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$ gives us the following condition:

$$\text{(2)} \qquad \frac{\partial U_i(x(b,q))}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial q_{ij}} = -\frac{\partial U_i(x(b,q))}{\partial x_{im}} \frac{\partial x_{im}}{\partial q_{ij}}.$$

Condition (2) states that in equilibrium the marginal rise in utility due to a rise in the consumption of x_{ij} caused by an incremental fall in q_{ij} must be equal to the absolute value of marginal fall in utility due to a fall in the consumption of x_{im} caused by this incremental fall in q_{ij}. Using $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$ one can rewrite condition (2) as follows:

$$\text{(3)} \qquad \frac{\partial U_i(x(b,q))}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial b_{ij}} = -\frac{\partial U_i(x(b,q))}{\partial x_{im}} \frac{\partial x_{im}}{\partial b_{ij}}.$$

Like condition (2), condition (3) has a similar interpretation in terms of b_{ij}. The general requirements that $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$ ensures that we have only one refined trade equilibrium condition summarized in condition (1).

Comment: Similar to Carvajal et al Theorem

3. Observable exchange economy

We observe all strategy profiles (and thus the price vectors) in each data point. Specifically, given a set of traders N and the commodity space X for each trader i, we have a sequence of data $\{\mu_1^t\}_{t=1}^T$ obtained over discrete time periods $1, 2, \ldots, T$. For each period t, $\mu^t = (b^t, q^t)$ where $b^t = (b_1^t, \ldots, b_n^t)$ is the period buying of the economy (in terms of money spent) with $b_i^t = (b_{i1}^t, \ldots, b_{il}^t) \in \Re_{++}^l$ for each trader $i \in N$ and $q^t = (q_1^t, \ldots, q_n^t)$ is the period selling of the economy with $q_i^t = (q_{i1}^t, \ldots, q_{il}^t) \in \Re_{++}^l$ for each $i \in N$.

Definition 3. *An observable exchange economy $\mathcal{E} = (N, X, \{\mu^t\}_{t=1}^T)$ consists of a finite set of traders N, the commodity space X and a sequence of data $\{\mu^t\}_{t=1}^T$ obtained over discrete time periods $1, 2, \ldots, T$.*

Given an observable exchange economy $\mathcal{E} = (N, X, \{\mu^t\}_{t=1}^T)$, for each period t, we can automatically observe the price of each commodity $j \in \{1, \ldots, l\}$ since $p_j^t = (B^t(j)/Q^t(j)) > 0$. Comment: This is close to Carvajal et al (less than Forges and Minelli as we just observe the price not all strategies).

Definition 4. *An observable exchange economy $\mathcal{E} = (N, X, \{\mu^t\}_{t=1}^T)$ is said to be market price-endowment rationalizable if, for each $i \in N$, there exists concave, increasing and continuously differentiable utility function $U_i : X \to \Re$, and, for each period t, there exists a vector of initial endowments $e_i^t = (e_{i1}^t, \ldots, e_{il}^t, e_{im}^t) \in X$ for all $i \in N$ such that in the resulting market (N, X, E^t, U), the data $\mu^t = (b^t, q^t)$ is a refined trading equilibrium.*

Example 1. *Consider an economy with two traders ($N = \{1, 2\}$), two goods ($X = X_1 \times X_m$) and two data points $\mu^1 = (b^1, q^1) = ((b^1_{11} = 7/8, q^1_{11} = 7/16), (b^1_{21} = 21/16, q^1_{21} = 7/4))$ and $\mu^2 = (b^2, q^2) = ((b^2_{11} = 9, q^2_{11} = 8), (b^2_{21} = 3, q^2_{21} = 4))$. Therefore, we have an observable exchange economy $\mathcal{E} = (N = \{1, 2\}, X = X_1 \times X_m, \{\mu^t\}^2_{t=1})$. Also assume that the utility function of trader 1 is quasi-linear and is of the form $U_1(x) = (k/(k-1))(x_{11})^{(k-1)/k} + x_{1m}$ where k is any integer greater than 1. Consider the data point μ^1. Observe that $p^1_1 = (b^1_{11} + b^1_{21})/(q^1_{11} + q^1_{21}) = 1$. The equilibrium condition for trader 1 requires $x^1_1 = (x^1_{11}, x^1_{1m}) \in X$ for which $\frac{\partial U_1(x^1_1)}{\partial x_{11}} / \frac{\partial U_1(x^1_1)}{\partial x_{1m}} = 1/(x^1_{11})^{1/k} = (q^1_{21}/b^1_{21})(p^1_1)^2 = (4/3)$ implying $x^1_{11} = (3/4)^k$. To sustain any $x^1_1 = (x^1_{11} = (3/4)^k, x^1_{1m})$ as an equilibrium outcome, it is necessary to find an endowment e^1_{11} of good 1 for trader 1 which is positive. That is, given $x^1_{11} = (3/4)^k = e^1_{11} - q^1_{11} + (b^1_{11}/p^1_1)$, we must have $e^1_{11} = x^1_{11} - b^1_{11} + q^1_{11} = (3/4)^k - (7/16) > 0$. It is easy to verify that $(3/4)^2 > (7/16) > (3/4)^k$ for all $k = 3, 4, 5, \ldots$. Therefore, for the data $\mu^1 = (b^1, q^1) = ((b^1_{11} = 7/8, q^1_{11} = 7/16), (b^1_{21} = 21/16, q^1_{21} = 7/4))$ to be market price-endowment rationalizable with trader 1 having a utility function of the form $U_1(x) = (k/(k-1))(x_{11})^{(k-1)/k} + x_{1m}$ (defined for integers $k > 1$), it is necessary that $k = 2$. Therefore, assume that the utility function of trader 1 is of the form $U_1(x) = 2\sqrt{x_{11}} + x_{1m}$. Consider the data point $\mu^2 = (b^2, q^2) = ((b^2_{11} = 9, q^2_{11} = 8), (b^2_{21} = 3, q^2_{21} = 4))$. Observe that $p^2_1 = (b^2_{11} + b^2_{21})/(q^2_{11} + q^2_{21}) = 1$. Moreover, the equilibrium condition for trader 1 requires that $x^2_1 = (x^2_{11}, x^2_{1m}) \in X$ for which $\frac{\partial U_1(x^2_1)}{\partial x_{11}} / \frac{\partial U_1(x^2_1)}{\partial x_{1m}} = 1/\sqrt{x^2_{11}} = (q^2_{21}/b^2_{21})(p^2_1)^2 = (4/3)$ implying $x^2_{11} = 9/16$. To sustain any $x^2_1 = (x^2_{11} = 9/16, x^2_{1m})$ as an equilibrium outcome, it is necessary to find an endowment $e^2_{11} > 0$ of good 1 for trader 1. But given $x^2_{11} = 9/16 = e^2_{11} - q^2_{11} + (b^2_{11}/p^2_1)$, we have $e^2_{11} = x^2_{11} - b^2_{11} + q^2_{11} = 9/16 - 1 = -7/16 < 0$. Hence, the data point μ^2 is not market price-endowment rationalizable even when trader 1 has a utility function of the form $U_1(x) = 2\sqrt{x_{11}} + x_{1m}$.*

Example 1 suggests that we need restrictions on the indexed collection of concave, increasing and continuously differentiable utility functions $U = (U_i : i \in N)$ if we are interested in market price-endowment rationalizability of *all* observable exchange economies of the form $\mathcal{E} = (N, X, \{\mu^t\}_{t=1}^T)$. Let $E(N, X)$ be the collection of all observable exchange economies. That is, for any finite number of time periods T and any collection of data points $\{\mu^t\}_{t=1}^T$ such that $\mu^t = (b^t, q^t) \in \Re_{++}^{2l\times n}$ for all t, $\mathcal{E} = (N, X, \{\mu^t\}_{t=1}^T) \in E(N, X)$. Let $\mathcal{U}$ be the collection of all concave, increasing and continuously differentiable utility functions.

Definition 5. *The collection of all observable exchange economies $E(N, X)$ is said to be fully market price-endowment rationalizable if, for each $i \in N$, there exists concave, increasing and continuously differentiable utility function $U_i : X \to \Re$ such that any observable exchange economy $\mathcal{E} = (N, X, \{\mu^t\}_{t=1}^T) \in E(N, X)$ is market price-endowment rationalizable.*

For any $i \in N$ and any given concave, increasing and continuously differentiable utility function $U_i : X \to \Re$, define for each $j \in \{1, \ldots, l\}$ the function $g_{ij} : X \cap \Re_{++}^{l+1} \to \Re_{++}$ as $g_{ij}(x) \equiv \frac{\partial U_i(x)}{\partial x_{ij}} / \frac{\partial U_i(x)}{\partial x_{im}}$.

Axiom 1. *The utility function $U_i : X \to \Re$ of any trader $i \in N$ satisfies the regularity condition if for any $z_i \in X \cap \Re_{++}^{l+1}$ and any $y = (y_1, \ldots, y_l) \in \Re_{++}^l$, there exists $x_i \geq z_i$ such that $g_{ij}(x_i) = y_j$ for all $j \in \{1. \ldots, l\}$.*

It is important to note that the regularity condition of Axiom 1 is an ordinal requirement since it is preserved under any increasing transformation of the utility index.

Remark 1. *All the utility functions used in Example 1 fail to satisfy Axiom 1. Consider an economy with two traders ($N = \{1, 2\}$), two goods ($X = X_1 \times X_m$) and, like in Example 1, consider any utility function of trader* 1 *of the form* $U_1(x) = (k/(k-1))(x_{11})^{(k-1)/k} + x_{1m}$

where k is any integer greater than 1. Let $z_1 = (z_{11}, z_{1m})$ be such that $z_{11} > 1$ and $z_{1m} > 0$ and consider any $y_1 > 1$. Axiom 1 requires that there exists $x_1 \in X$ such that $x_1 \geq z_1$ and $g_{11}(x_1) = y_1$. But note that for any $x_1 \in X$ such that $g_{11}(x_1) = \frac{\partial U_1(x_1^1)}{\partial x_{11}} / \frac{\partial U_1(x_1^1)}{\partial x_{1m}} = 1/(x_{11})^{1/k} = y_1 > 1$, we have $x_{11} = 1/(y^k) < 1 < z_{11}$. Since $0 < x_{11} < z_{11}$, it is not possible to find any $x_1 \in X$ such that $x_1 \geq z_1$ and $g(x_1) = y_1$. Hence, we have a violation of Axiom 1.

Given Example 1 and Remark 1, one can ask whether there are well-studied concave, increasing and continuously differentiable utility functions that satisfy Axiom 1.

Example 2. *Consider an economy with two traders, two goods and consider the same data points as in Example 1, that is, $\mu^1 = (b^1, q^1) = ((b^1_{11} = 7/8, q^1_{11} = 7/16), (b^1_{21} = 21/16, q^1_{21} = 7/4))$ and $\mu^2 = (b^2, q^2) = ((b^2_{11} = 9, q^2_{11} = 8), (b^2_{21} = 3, q^2_{21} = 4))$. Therefore, the observable exchange economy is $\mathcal{E} = (N, X, \{\mu^t_1\}^2_{t=1})$. For each $i \in N$, assume that the utility function is Cobb-Douglas and is of the form $U_i(x_i) = A_i(x_{i1})^{\alpha_{i1}}(x_{im})^{1-\alpha_{i1}}$ where $A_i > 0$ and $\alpha_{ij} \in (0, 1)$. For μ^1, we have $p^1_1 = 1$. The equilibrium condition for trader 1 requires $x^1_1 = (x^1_{11}, x^1_{1m}) \in X$ for which $g_{11}(x^1_1) = [\alpha_{11}x^1_{1m}]/[(1-\alpha_{11})x^1_{11}] = (q^1_{21}/b^1_{21})(p^1_1)^2 = (4/3)$ implying (i) $4(1-\alpha_{11})x^1_{11} = 3\alpha_{11}x^1_{1m}$. If $e^1_1 = (e^1_{11} = q^1_{11} + [\alpha_{11}/(1-\alpha_{11})](7/8), e^1_{1m} = b^1_{11} + [(8-3\alpha_{11})/(3\alpha_{11})](7/16))$, then clearly $e^1_{11} > q^1_{11} > 0$, $e^1_{1m} > b^1_{11} > 0$, $x^1_{11} = 7/[8(1-\alpha_{11})]$ and $x^1_{1m} = 7/[6\alpha_{11}]$ and equilibrium condition (i) is satisfied. The equilibrium condition for trader 2 requires $x^1_2 = (x^1_{21}, x^1_{2m}) \in X$ for which $g_{21}(x^1_2) = [\alpha_{21}x^1_{1m}]/[(1-\alpha_{21})x^1_{11}] = (q^1_{11}/b^1_{11})(p^1_1)^2 = (1/2)$ implying (ii) $(1-\alpha_{21})x^1_{21} = 2\alpha_{21}x^1_{2m}$. If $e^1_2 = (e^1_{21} = q^1_{21} + [(8 - 3(1-\alpha_{21}))/(1-\alpha_{21})](7/16), e^1_{2m} = b^1_{21} + [(1-\alpha_{21})/(\alpha_{21})](7/4))$, then clearly $e^1_{21} > q^1_{21} > 0$, $e^1_{2m} > b^1_{21} > 0$, $x^1_{21} = 7/[2(1-\alpha_{21})]$ and $x^1_{2m} = 7/[4\alpha_{21}]$ and equilibrium condition (ii) is satisfied. Therefore, given the Cobb-Douglas utility functions for traders 1 and 2, we have obtained endowments e^1_1, e^1_2 for traders 1 and 2 such that μ^1 is price-endowment rationalizable. Similarly, we can also obtain endowments e^2_1, e^2_2 for traders 1 and 2 such that μ^2 is* *also price-endowment rationalizable. Specifically, it is quite easy to verify that the endowment* $e_1^2 = (e_{11}^2 = q_{11}^2 + [9\alpha_{11}/(1-\alpha_{11})], e_{1m}^2 = b_{11}^2 + [4(3-2\alpha_{11})/\alpha_{11}])$ *for trader 1 and the endowment* $e_2^2 = (e_{21}^2 = q_{21}^2 + [3(3-2(1-\alpha_{21}))]/[2(1-\alpha_{21})], e_{2m}^2 = b_{21}^2 + 4[(1-\alpha_{21})/\alpha_{21}])$ *for trader 2 ensures price-endowment rationalizability of* μ^2. *In general, as shown in the next remark, if traders have Cobb-Douglas utility functions, then we have full price-endowment rationlizability.*

Remark 2. *For any trader* $i \in N$, *consider any Cobb-Douglas utility function, that is, consider* $U_i(x_i) = A_i \left(\prod_{j=1}^{l}(x_{ij})^{\alpha_{ij}}\right)(x_{im})^{\alpha_{im}}$ *where* $A_i > 0$, $\alpha_{im} \in (0,1)$, $\alpha_{ij} \in (0,1)$ *for all* $j \in \{1,\ldots,l\}$ *and* $\sum_{j=1}^{l}\alpha_{ij} + \alpha_{im} = 1$. *Note that for any* $x \in X \cap \Re_{++}^l$ *and any* $j \in \{1,\ldots,l\}$, $g_{ij}(x_i) = [\alpha_{ij}x_{im}]/[\alpha_{im}x_{ij}]$. *Consider any* $z_i \in X \cap \Re_{++}^l$ *and any* $y = (y_1,\ldots,y_l) \in \Re_{++}^l$.

Changed from here... *Let* $\beta_j = \frac{\alpha_{im}}{\alpha_{ij}}y_j$. *Then, we need to show that there exists* $x_i \in X \cap \mathbb{R}_{++}^l$ *with* $x_i \geq z_i$ *such that* $x_{im} = \beta_j x_{ij}$ *for all* $j = 1,\ldots,l$. *Note that* x_{im} *is increasing in* x_{ij} *for all* $j = 1,\ldots,l$. *Take* $x_{im} = \max(\{\beta_j z_{ij}; j = 1,\ldots,l\}, z_{im})$ *and take* $x_{ij} = \frac{x_{im}}{\beta_j}$. *Then,* $x_{im} \geq z_{im}$ *by definition. Also,* $x_{ij} = \frac{x_{im}}{\beta_j} \geq \frac{1}{\beta_j}\max\{\beta_j z_{ij}; j = 1,\ldots,l\} \geq \frac{1}{\beta_j}\beta_j z_{ij} = z_{ij}$. *Hence, the result follows.* ***Changed untill here...Please check once bothe the correctness and presentation...***

For each $j \in \{1,\ldots,l\}$, *define* $a_j(z_i) = y_j/g_{ij}(z_i)$. *Consider that good* $j^* \in \{1,\ldots,l\}$ *such that* $\max\{a_j(z_i)\} = a_{j^*}(z_i) := a_{j^*}$ *(if there are many such goods in* $\{1,\ldots,l\}$, *then pick any one of them). Consider any* $\tau_{j^*} > 0$ *and* $\tau_m > 0$ *such that* $(1+\tau_{j^*})a_{j^*} > 1$ *and* $(1+\tau_{j^*})a_{j^*} = (1+\tau_m)$. *For each* $j \in \{1,\ldots,l\} \setminus \{j^*\}$, *define* τ_j *such that* $(1+\tau_j)a_j(z_i) = (1+\tau_{j^*})a_{j^*} = (1+\tau_m)$. *Since* $a_j(z_i) \leq a_{j^*}$ *for all* j, $\tau_j \geq \tau_{j^*}$ *for all* $j \in \{1,\ldots,l\}$. *Given these* $\tau_j s$*' and* z_i, *consider* $x_i \in X \cap \Re_{++}^l$ *such that* $x_{ij} = (1+\tau_j)z_{ij} \geq z_{ij}$ *for all* $j \in \{1,\ldots,l\}$ *and* $x_{im} = (1+\tau_m)z_m \geq z_{im}$. *Clearly,* $x_i \geq z_i$ *and, more importantly, for any* $j \in \{1,\ldots,l\}$, $g_{ij}(x_i) = [(1+\tau_m)g_{ij}(z)]/(1+\tau_j) = [(1+\tau_m)y_j]/[(1+\tau_j)a_j(z)] = y_j$. *Hence, the result follows.*

Let $\mathcal{U}^*(\subset \mathcal{U})$ be the collection of all utility functions in $\mathcal{U}$ that also satisfies Axiom 1.

Theorem 1. *The collection of all observable exchange economies $E(N,X)$ is fully market price-endowment rationalizable if and only if the indexed collection $U = (U_i : i \in N)$ is such that $U_i \in \mathcal{U}^*$ for each $i \in N$.*

Proof: (If part) Let the utility function of each trader satisfy Axiom 1. We show that any observable exchange economy is price-endowment rationalizable. Consider an observable exchange economy $\mathcal{E} = (N, X, \{\mu^t\}_{t=1}^T)$ where in any period t we have observed $\mu^t = (b^t, q^t)$. We show (b^t, q^t) is a refined trading equilibrium. In particular, we show that (b^t, q^t) satisfies condition (1) of Proposition 1.

Given (b^t, q^t), for each $i \in N$, define $H_{ij}(b^t, q^t) = [Q_{-i}(j)/B_{-i}(j)](p_j^t)^2 > 0$. For any $i \in N$, any $j \in \{1, \ldots, l\}$, define $NT_{ij}^t = (b_{ij}^t/p_j^t - q_{ij}^t)$, $z_{ij}^t = q_{ij}^t + \max\{NT_{ij}^t, 0\}$ and define $z_{im}^t = \sum_{j=1}^l b_{ij}^t + \max\{-\sum_{j=1}^l p_j^t NT_{ij}^t, 0\}$. Observe that for each $i \in N$, $z_i^t = (z_{i1}^t, \ldots, z_{il}^t, z_{im}^t) \in X$. By Axiom 1, for each $i \in N$, there exists $x_i^t \geq z_i^t$ such that $g_{ij}(x_i^t) = H_{ij}(b^t, q^t)$ for each $j \in \{1, \ldots, j\}$. Since $g_{ij}(x_i^t) = H_{ij}(b^t, q^t)$ for each $i \in N$ and $j \in \{1, \ldots, l\}$, we have $\delta_{ij}(b^t, q^t) = (p_j)^2$ for all $i \in N$ and $j \in \{1, \ldots, l\}$, and hence (b^t, q^t) satisfies condition (1) of Proposition 1. Moreover, for each $i \in N$, the endowment vector $e_i^t = (e_{i1}^t, \ldots, e_{il}^t, e_{im}^t)$ is such that $e_{ij}^t = x_{ij}^t - NT_{ij}^t \geq q_{ij}^t + \max\{NT_{ij}^t, 0\} - NT_{ij}^t \geq q_{ij}^t > 0$ for all $j \in \{1, \ldots, l\}$ and $e_{im}^t = x_{im}^t + \sum_{j=1}^l p_j^t NT_{ij}^t \geq \sum_{j=1}^l b_{ij}^t + \max\{-\sum_{j=1}^l p_j^t NT_{ij}^t, 0\} + \sum_{j=1}^l p_j^t NT_{ij}^t \geq \sum_{j=1}^l b_{ij}^t > 0$. Therefore, the restrictions $e_{ij}^t \geq q_{ij}^t$ and the restrictions $e_{im}^t \geq \sum_{j=1}^l b_{ij}^t > 0$ for all i and all j are also satisfied. This completes the proof of the if part.

(Only-if part) For each $i \in N$, let U_i be the associated utility function which is concave, increasing and continuously differentiable. Suppose that any observable exchange economy $\mathcal{E}^1 = (N, X, \{\mu^t\}_{t=1}^T)$ is market price-endowment rationalizable with respect to this indexed collection of utility functions $(U_i)_{i \in N}$. Then, we show that the utility function U_i of each trader i must satisfy Axiom 1. Suppose not. Then, there exist $i \in N$,

$j \in \{1, \ldots, l\} \cup \{m\}$, $y_j \in \Re_{++}$ and $z \in X$ such that for all $x \geq z$, $g_{ij}(x) \neq y_j$. We construct an observable exchange economy $\mathcal{E}^1 = (N, X, \{\mu^t\}_{t=1}^T)$ that is not price-endowment rationalizable. Take some $t \in T$ and consider $\mu^t = (b^t, q^t)$ such that

(i) $\frac{B^t_{-i}(j)}{Q^t_{-i}(j)} = y_j$,
(ii) $b^t_{ij} = z_{ij} + c$ for some large enough $c > 0$,
(iii) $q^t_{ij} = b^t_{ij} - (1 - y_j)Q^t_{-i}(j)$,
(iv) $p^t_k = 1$ and $b^t_{ik} > z_{ik}$, for all $k \in \{1, \ldots, l\} \setminus \{j\}$ and
(v) $\sum_{k=1}^l q^t_{ik} > z_{im}$.

First, we argue that such a choice of (b^t, q^t) is possible. To see this, note that (i), (ii) and (iii) are not related to (iv) and (v), and, (iii) depends only on (i) and (ii). Further note that for all $k \in \{1, \ldots, l\} \setminus \{j\}$, b^t_{ik} and q^t_{ik} can be chosen arbitrarily large maintaining $p^t_k = 1$. Therefore, (iv) and (v) can be met by choosing b^t_{ik} and q^t_{ik} sufficient large for all $k \in \{1, \ldots, l\} \setminus \{j\}$. Finally, by choosing the constant c large enough in (ii), we can make sure that b^t_{ij} and q^t_{ij} are positive.

Now, we show that $\mathcal{E}$ is not price-endowment rationalizable. First, note that $p^t_j = 1$. This is because, by (i), $B^t_{-i}(j) = y_j Q^t_{-i}(j)$, and hence by (iii), $Q^t(j) = q^t_{ij} + Q^t_{-i}(j) = b^t_{ij} + B^t_{-i}(j) = B^t(j)$ implying $p^t_j = 1$. This, together with (iv), implies that $p^t_k = 1$ for all $k \in \{1, \ldots, l\}$. Assume for contradiction that $\mathcal{E} = (N, X, \{\mu^t\}_{t=1}^T)$ is price-endowment rationalizable. Then, (b^t, q^t) must be a refined trading equilibrium with some endowment vector $e^t = (e^t_1, \ldots, e^t_n) \in X^n$. Since $p^t_j = 1$, by (S1) we have $g_{ij}(x_i(b^t, q^t)) = y_j$. Since $g_{ij}(x) \neq y_j$ for all $x \geq z$, $g_{ij}(x(b^t, q^t)) = y_j$ implies that $x_i(b^t, q^t) \ngeq z$. Therefore, there exists $k \in \{1, \ldots, l\} \cup \{m\}$ such that $x_{ik}(b^t, q^t) < z_{ik}$. Suppose $x_{ik}(b^t, q^t) < z_{ik}$ for some $k \in \{1, \ldots, l\}$. This means $e^t_{ik} - \frac{q^t_{ik}}{p^t_k} + b^t_{ik} < z_{ik}$. Since from (iv) we have $p^t_k = 1$ and $b^t_{ik} > z_{ik}$, it follows that $e^t_{ik} < q^t_{ik}$, which violates the restriction $e^t_{ik} \geq q^t_{ik}$. Now suppose $x^t_{im} < z_{im}$. Then, $e^t_{im} + \sum_{k=1}^l q^t_{ik} p^t_k - \sum_{k=1}^l b^t_{ik} < z_{im}$. Since $p^t_k = 1$ for all $k \in \{1, \ldots, l\}$ and since (v) requires that $\sum_{k=1}^l q^t_{ik} > z_{im}$, we have $e^t_{im} < \sum_{k=1}^l b^t_{ik}$, which violates the restriction $e^t_{im} \geq \sum_{k=1}^{l} b^t_{ik}$. Therefore, (b^t, q^t) cannot be a refined trading equilibrium. This completes the proof of the only-if part. □

Remark 3. *Can one generalize the result in Remark 2 to include all homothetic preferences, that is, all utility functions that are homogeneous of degree one? The answer is no. Suppose that there are* N *traders and three goods* ($X = X_1 \times X_2 \times X_m$) *and assume that the utility function of a trader* $i \in N$ *is homogeneous of degree one and is of the form* $U_i(x) = 2\sqrt{x_{i1}x_{i2}} + x_{im}$. *Consider any* $z_i = (z_{i1}, z_{i2}, z_{im}) \in X \cap \Re^3_{++}$ *and any* $y = (y_1 = \sqrt{1+\eta}, y_2 = \sqrt{1-\eta})$ *with* $\eta \in (0,1)$. *For Axiom 1 to hold it is necessary to find* $x_i = (x_{i1}, x_{i2}, x_{im}) \in X$ *such that* $x_i \geq z_i$ *and the following conditions hold: (a)* $g_{i1}(x_i) = \sqrt{(x_{i2}/x_{i1})} = \sqrt{1+\eta}$ *and (b)* $g_{i2}(x_i) = \sqrt{(x_{i1}/x_{i2})} = \sqrt{1-\eta}$. *Condition (a) implies (c)* $x_{i2} = (\sqrt{1+\eta})x_{i1}$ *and condition (b) implies (d)* $x_{i1} = (\sqrt{1-\eta})x_{i2}$ *and we have* $x_{i1} = x_{i2} = 0$. *Given* y, *the* x_i *for which* $g_{i1}(x_i) = y_1 = \sqrt{1+\eta}$ *and* $g_{i2}(x_i) = y_2 = \sqrt{1-\eta}$ *is such that* $x_i = (x_{i1} = 0, x_{i2} = 0, x_{im})$. *Given* $z_{i1} > 0$, $z_{i2} > 0$ *and* $x_{i1} = x_{i2} = 0$, *there does not exists any* $x_i \geq z_i$ *such that conditions (a) and (b) hold.*

The log-transformation of any Cobb-Douglas utility function is additive.[2] One can show that a family of additive utility functions satisfy Axiom 1. Let $\mathcal{F} = \{(F_{ij}(.)\}_{j=1}^{l}, F_{im}(,))\}$ be any collection of increasing and strictly concave single variable functions defined from $\Re_{++}$ to $\Re$ and also assume that the range of the first derivative $F'_{ik}(.)$ is $\Re_{++}$ for all $k \in \{1, \ldots, l\} \cup \{m\}$. Given any such collection $\mathcal{F}$, define the utility function as follows:

$$U_i(x_i) = A_i + \sum_{j=1}^{l} F_{ij}(x_{ij}) + F_{im}(x_{im}), \; A_i \in \Re. \tag{4}$$

Proposition 2. *Any additive utility function given by (4) satisfies Axiom 1.*

[2]Suppose $U_i(x_i) = A_i \left(\prod_{j=1}^{l}(x_{ij})^{\alpha_{ij}}\right)(x_{im})^{\alpha_{im}}$ where $A_i > 0$, $\alpha_{im} \in (0,1)$, $\alpha_{ij} \in (0,1)$ for all $j \in \{1, \ldots, l\}$ and $\sum_{j=1}^{l} \alpha_{ij} + \alpha_{im} = 1$. Then $\bar{U}_i(x_i) = \ln U_i(x_i) = \ln A_i + \sum_{j=1}^{l} \alpha_{ij} \ln x_{ij} + \alpha_{im} \ln x_{im}$.

The proof of Proposition 2 is similar to the proof of the Cobb-Douglas result (Remark 2) and hence is provided in the Appendix.

4. Rationalizability under Specific Utility Functions

In this section, we discuss rationalizability under specific utility functions. Throughout this section, we assume that for all $j \in \{1, \ldots, l\} \cup \{m\}$,

(1) $\lim_{x_{ij} \to \infty} F_j(x_{ij}) = \kappa_j$, and
(2) $F'_j(x_{ij})$ is continuous, decreasing and $\lim_{x_{ij} \to \infty} F'_j(x_{ij}) = \kappa'_j$.

In our subsequent analysis, for simplicity of our presentation, whenever $\kappa'_j = 0$ for some j, by $\frac{1}{\kappa'_j}$ we denote infinity.

4.1. **Additive Utility Function.** A utility function $U_i(x_i)$ of trader i is called additive if

$$U_i(x_i) = A_i + \sum_{j=1}^{l} F_{ij}(x_{ij}) + F_{im}(x_{im}), \;\; A_i \in \Re.$$

Suppose trader i has additive utility function U_i. Then, a data set (b, q) is rationalizable for trader i (at any time t) if and only if for any $j \in \{1, \ldots, l\}$ there exist $x_{ij} \geq \frac{b_{ij}}{p(j)} - q_{ij}$ and $x_{im} \geq \sum_{j=1}^{l} p_j q_{ij} - \sum_{j=1}^{l} b_{ij}$ such that $p_j^2 \frac{Q_{-i}(j)}{B_{-i}(j)} = \frac{F'_j(x_{ij})}{F'_m(x_m)}$. This means a data set (b, q) is rationalizable if and only if

$$\frac{\kappa'_j}{F'_m(\sum_{j=1}^{l} p_j q_{ij} - \sum_{j=1}^{l} b_{ij})} \leq p_j^2 \frac{Q_{-i}(j)}{B_{-i}(j)} \leq \frac{F'_j(\frac{b_{ij}}{p(j)} - q_{ij})}{\kappa'_m}$$

for all $j = 1, \ldots, l$.

Note that if $\kappa'_j = 0$ for all $j \in \{1, \ldots, l\} \cup \{m\}$, then the above equation is satisfied for every data set. This gives us the following corollary.

Corollary 1. *Suppose the utility function of each individual i is additive, where for all $j \in \{1, \ldots, l\} \cup \{m\}$, F'_{ij} is continuous, decreasing and positive with $\lim_{x_{ij}} F'_{ij} = 0$. Then, every data set (b, q) is rationalizable.*

4.1.1. *Quasi-linear Utility Functions.* If U_i is quasi-linear, then $F'_m(x_m) = 1$. Therefore, a data set (b, q) is rationalizable for trader i (at any time t) if and only if $\kappa'_j \leq p_j^2 \frac{Q_{-i}(j)}{B_{-i}(j)} \leq F'_j(\frac{b_{ij}}{p(j)} - q_{ij})$ for all j.

4.2. **Multiplicative Utility Functions.** A utility function $U_i(x_i)$ of trader i is called multiplicative if

$$U_i(x_i) = A_i \prod_{j=1}^{l} (F_{ij}(x_{ij})(F_{im}(x_{im})), A_i > 0.$$

Suppose trader i has multiplicative utility function U_i. Then, a data set (b, q) is rationalizable for trader i (at any time t) if for any $j \in \{1, \ldots, l\}$, there exist $x_{ij} \geq \frac{b_{ij}}{p(j)} - q_{ij}$ and $x_{im} \geq \sum_{j=1}^{l} p_j q_{ij} - \sum_{j=1}^{l} b_{ij}$ such that $p_j^2 \frac{Q_{-i}(j)}{B_{-i}(j)} = \frac{F'_j(x_{ij})}{F_j(x_{ij})} \frac{F_m(x_{im})}{F'_m(x_m)}$.

Therefore, a data set (b, q) is rationalizable for trader i if and only if for any $j \in \{1, \ldots, l\}$,

$$\frac{\kappa'_j}{\kappa_j} \frac{F_m(\sum_{j=1}^{l} p_j q_{ij} - \sum_{j=1}^{l} b_{ij})}{F'_m(\sum_{j=1}^{l} p_j q_{ij} - \sum_{j=1}^{l} b_{ij})} \leq p_j^2 \frac{Q_{-i}(j)}{B_{-i}(j)} \leq \frac{F'_j(\frac{b_{ij}}{p(j)} - q_{ij})}{F_j(\frac{b_{ij}}{p(j)} - q_{ij})} \frac{\kappa_m}{\kappa'_m}. \tag{5}$$

Note that for all $j \in \{1, \ldots, l\} \cup \{m\}$, if $\kappa_j = \infty$ or $\kappa'_j = 0$, then the left hand side of Equation (5) is 0 and the right hand side of that is ∞. This means if $\kappa_j = \infty$ or $\kappa'_j = 0$, then Equation (5) is satisfied for every data set (b, q). Now, recall that F'_j is decreasing, positive and continuous for all $j \in \{1, \ldots, l\} \cup \{m\}$. This means for all $j \in \{1, \ldots, l\} \cup \{m\}$, either $\kappa_j = \infty$ or $\kappa'_j = 0$. This gives us the following corollary.

Corollary 2. *Suppose the utility function of each individual i is multiplicative, where F'_j is continuous, decreasing and positive. Then, every data set (b, q) is rationalizable.*

5. Appendix

Proof of Proposition 1: Given a market (N, X, E, U), if (b, q) is a trading equilibrium, then for each $i \in N$, (b_i, q_i) (given $((b_j, q_j)_{j \in N \setminus \{i\}})$ maximizes $U_i(x_i(b, q))$ subject to $q_{ij} \in (0, e_{ij}]$, $b_{ij} \geq 0$ for $j = 1, \ldots, l$, and $\sum_{j=1}^{l} b_{ij} \leq e_{im}$, where $x_i(b, q) = (x_{i1}(b, q), \ldots, x_{il}(b, q), x_{im}(b, q)) \in X$, $x_{ij}(b, q) = e_{ij} - q_{ij} + (b_{ij}/p_j)$ for each $j \in \{1, \ldots, l\}$, and $x_{im}(b, q) = e_{im} - \sum_{j=1}^{l} b_{ij} + \sum_{j=1}^{l} p_j q_{ij}$. Therefore, $\frac{\partial x_{ij}}{\partial q_{ij}} = -(1 - \frac{b_{ij}}{B(j)}) \leq 0$, $p_j \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial x_{ij}}{\partial q_{ij}} = 0$, $\frac{\partial x_{im}}{\partial b_{ij}} = -(1 - \frac{q_{ij}}{Q(j)}) \leq 0$ and $p_j \frac{\partial x_{im}}{\partial b_{ij}} + \frac{\partial x_{im}}{\partial q_{ij}} = 0$.

Define $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{il}) \in \Re^l_+$, $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{il}) \in \Re^l_+$, $\beta_i = (\beta_{i1}, \ldots, \beta_{il}) \in \Re^l_+$ and $\delta_i \in \Re_+$. The Lagrangian function for the optimization problem of traders $i \in N$ is the following:

$$L(b, q, \lambda_i, \gamma_i, \beta_i, \delta_i) = U_i(x_i(b, q)) + \sum_{j=1}^{l} \lambda_{ij} q_{ij} + \sum_{j=1}^{l} \gamma_{ij} (e_{ij} - q_{ij}) + \sum_{j=1}^{l} \beta_{ij} b_{ij} + \delta_i \left(e_{im} - \sum_{j=1}^{l} b_{ij} \right).$$

The Kuhn-Tucker conditions are the following:

$$\frac{\partial L}{\partial q_{ij}} = \frac{\partial U_i}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial q_{ij}} + \frac{\partial U_i}{\partial x_{im}} \frac{\partial x_{im}}{\partial q_{ij}} + \lambda_{ij} - \gamma_{ij} \leq 0 \textit{ and } q_{ij} \frac{\partial L}{\partial q_{ij}} = 0 \textit{ for each good } j, \tag{6}$$

$$\lambda_{ij} \geq 0, q_{ij} \geq 0 \textit{ and } \lambda_{ij} q_{ij} = 0 \textit{ for each multiplier } \lambda_{ij} \textit{ given } i, \tag{7}$$

$$\gamma_{ij} \geq 0, e_{ij} \geq q_{ij} \textit{ and } \gamma_{ij}(e_{ij} - q_{ij}) = 0 \textit{ for each multiplier } \gamma_{ij} \textit{ given } i, \tag{8}$$

$$\frac{\partial L}{\partial b_{ij}} = \frac{\partial U_i}{\partial x_{ij}} \frac{\partial x_{ij}}{\partial b_{ij}} + \frac{\partial U_i}{\partial x_{im}} \frac{\partial x_{im}}{\partial b_{ij}} + \beta_{ij} - \delta_i \leq 0 \textit{ and } b_{ij} \frac{\partial L}{\partial b_{ij}} = 0 \textit{ for each good } j, \tag{9}$$

$$\beta_{ij} \geq 0, b_{ij} \geq 0 \textit{ and } \beta_{ij} b_{ij} = 0 \textit{ for each multiplier } \beta_{ij} \textit{ given } i, \textit{ and}, \tag{10}$$

$$\delta_i \geq 0,\ e_{im} \geq \sum_{j=1}^{l} b_{ij} \textit{ and } \delta_i \left(e_{im} - \sum_{j=1}^{l} b_{ij} \right) = 0 \textit{ for the multiplier } \delta_i. \tag{11}$$

The utility function of each trader i is concave and continuously differentiable and all the constraints are linear and hence the Kuhn-Tucker conditions are necessary and sufficient for this optimization problem. From the first part of both conditions (6) and (9) it follows that for each good j,

$$-\left(\frac{B_{-i}(j)}{B(j)}\right)\frac{\partial U_i}{\partial x_{ij}} + p_j\left(\frac{Q_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} + \lambda_{ij} - \gamma_{ij} \leq 0, \textit{ and} \tag{12}$$

$$\frac{1}{p_j}\left(\frac{B_{-i}(j)}{B(j)}\right)\frac{\partial U_i}{\partial x_{ij}} - \left(\frac{Q_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} + \beta_{ij} - \delta_i \leq 0. \tag{13}$$

Multiplying p_j to condition (13) gives

$$\left(\frac{B_{-i}(j)}{B(j)}\right)\frac{\partial U_i}{\partial x_{ij}} - p_j\left(\frac{Q_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} + p_j\beta_{ij} - p_j\delta_i \leq 0. \tag{14}$$

Adding conditions (12) and (14) we get

$$\lambda_{ij} + p_j\beta_{ij} - \gamma_{ij} - p_j\delta_i \leq 0 \tag{15}$$

If (b, q) is a refined trading equilibrium, then $\gamma_{ij} = \delta_i = 0$ and from (15) we get $\lambda_{ij} + p_j\beta_{ij} \leq 0$. Since $\lambda_{ij} \geq 0$, $p_j > 0$ and $\beta_{ij} \geq 0$ we have $\lambda_{ij} = \beta_{ij} = 0$. Therefore, for any refined trading equilibrium (b, q), we have $\gamma_{ij} = \delta_i = \lambda_{ij} = \beta_{ij} = 0$ and from conditions (12) and (14) we have only one equilibrium condition given by

$$-\left(\frac{B_{-i}(j)}{B(j)}\right)\frac{\partial U_i}{\partial x_{ij}} + p_j\left(\frac{Q_{-i}(j)}{Q(j)}\right)\frac{\partial U_i}{\partial x_{im}} = 0. \tag{16}$$

By substituting $B(j) = p_j Q(j)$ in condition (16) and then simplifying it we get

$$\left(\frac{B_{-i}(j)}{Q_{-i}(j)}\right) \frac{\left(\frac{\partial U_i}{\partial x_{ij}}\right)}{\left(\frac{\partial U_i}{\partial x_{im}}\right)} = p_j^2. \tag{17}$$

Given $\delta_{ij}(b,q) := [B_{-i}(j) \{\partial U_i(x_i(b,q))/\partial x_{ij}\}] \,/\, [Q_{-i}(j) \{\partial U_i(x_i(b,q))/\partial x_{im}\}]$, from condition (17) we get $\delta_{ij}(b,q) = p_j^2$ and the result follows. $\square$

Proof of Proposition 2: For any trader $i \in N$, consider any utility function given by (4). Note that for any $x \in X \cap \Re^{l+1}_{++}$ and any $j \in \{1,\ldots,l\}$, $g_{ij}(x_i) = [F'_{ij}(x_{ij})]/[F'_{im}(x_{im})]$. Consider any $z_i \in X \cap \Re^{l+1}_{++}$ and any $y = (y_1,\ldots,y_l) \in \Re^l_{++}$. For each $j \in \{1,\ldots,l\}$, define $b_j(z_i) = F'_{ij}(z_{ij})/y_j$. Consider that good $j^* \in \{1,\ldots,l\}$ such that $\min\{b_j(z_i)\} = b_{j^*}(z_i) := b_{j^*}$ (if there are many such goods in $\{1,\ldots,l\}$, then pick any one of them). Using the fact that the range of both the first derivatives $F'_{ij^*}(.)$ and $F'_{im}(.)$ is $\Re_{++}$, consider any $\tau_{j^*} > 0$ and $\tau_m > 0$ such that $F'_{ij^*}((1+\tau_{j^*})z_{ij})/y_{j^*} = F'_{im}((1+\tau_m)z_{im})$. For each $j \in \{1,\ldots,l\} \setminus \{j^*\}$, define τ_j such that $F'_{ij}((1+\tau_j)z_{ij})/y_j = F'_{ij^*}((1+\tau_{j^*})z_{ij^*})/y_{j^*} = F'_{im}((1+\tau_m)z_{im})$. Since $b_j(z_i) \le b_{j^*}$ for all j, by strict concavity $\tau_j > 0$ for all $j \in \{1,\ldots,l\}$. Given these τ_js' and z_i, consider $x_i \in X \cap \Re^{l+1}_{++}$ such that $x_{ij} = (1+\tau_j)z_{ij} \ge z_{ij}$ for all $j \in \{1,\ldots,l\}$ and $x_{im} = (1+\tau_m)z_m \ge z_{im}$. Clearly, $x_i \ge z_i$ and, more importantly, for any $j \in \{1,\ldots,l\}$, $g_{ij}(x_i) = F'_{ij}((1+\tau_j)z_{ij})/F'_{im}((1+\tau_m)z_{im}) = y_j$. Hence, the result follows. $\square$

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