Learning a Network

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Abstract

This paper investigates how rational agents make decisions in networks with incomplete information about the overall network structure. Using an infinitely repeated local interaction game model, we show that Bayesian Nash Equilibrium (BNE) under incomplete information is coincide to the Nash Equilibrium (NE) of a complete information game, implying that, despite incomplete knowledge, agents' behavior converges to the equilibrium predicted under full information over time.. Our results highlight the continued importance of Katz-Bonacich centrality in incomplete networks and identify conditions under which agents can learn the network structure, with learning time scaling linearly with the number of players. These findings enhance our understanding of decision-making and information diffusion in complex networks and offer practical insights for policymakers and researchers.

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1 Introduction

1.1 Overview

Networks play a crucial role as sources of information. Individuals communicate with those they are connected to, and employ this information towards forming opinions and making decisions. For example, a firm's decision to adopt a new technology is influenced by competitor or collaborator firms decisions to do so, and adolescents' consumption of tobacco or alcohol is affected by their peers' consumption choices. While information and influence can flow to an agent only through those it is directly connected to, indirect connections also implicitly feed into its chosen behavior. Those who influce the agent are themselves influcence by others, and hence individual behavior in a network inevibly depends on its entire architecture.

When faced with a decision problem, a rational agent emdedded in a network can internalize this fact. That is, it may rationalize that the influce being exerted upon it is the result of not only the shape of its local neighborhood, but of the enrie network architecture. For instance, when confronted with the choice of believing a rumor in a socal network, knowing which actors (or sources) swayed the opinions of an agents neighbors would better inform whether or not the agent itself would be swayed by the opinions of its own neighbors. Therefore, as long as a rational agent has access to information on how the network is connected beyond its local neighborhood, it can use this information to better inform its decision.

One of the major challenges in employing the full network topology in exerting rational action, rests with the reality of incomplete information. Participant of real word social and economic networks rarely if ever have access to information regarding their entire architectures. For example, firms know who their own suppliers are, but may not now who their suppliers suppliers are. Similarly, social media users know who their platform friends are, but do not typically know who their friends' friends are. The prevelence of incomplete network information in social netwoks has also been demonstrated empirically. For instance, by conducting surveys on Indian villages, Breaza et al. (2018) show that when asked to name who their firends' firends are, individuals do a poor job in identifying them.

These stylized facts give rise to the following set of questions which are the primary focus of this paper. If network participants.. Assuming that information regarding the structure of the network is encoded in agents behavior, what do the learning dyamics looks like when agents are allowed to observe their neighbors actions over time?

We consider an environment in which a set of myopic but rational agents play an infinetly repeated local interaction game of incomplete information. Our research focuses on scenarios with local complementarities between players, building upon the linear-quadratic network game model proposed by Ballester et al. (2006).

Our findings yield several important insights: We first show that the Bayesian Nash Equilibrium (BNE) for arbitrary ex-ante distributions over graphs is exactly the same as the Nash Equilibrium (NE) under a complete information game. This discovery holds significant implications for predicting players' actions under incomplete information, suggesting that over time, play converges towards the Nash Equilibrium predicted by complete information. Additionally, this result serves as a bridge between incomplete and complete network games. Under rational Bayesian learning, after learning ends, a player will act as if they know the true network. This result aligns with some game theory papers(Jordan (1995) and Kalai and Lehrer (1995), as both our paper and theirs show the coincidence of BNE and NE. Furthermore, this result also shows that $Katz - Bonacich(KB)$ centrality is still important in an incomplete network game. This shows that our paper bridges between an incomplete network game and a complete game.

This research contributes to the growing body of literature on Bayesian learning in networks, an area that has received comparatively less attention than DeGroot learning models. Additionally, our work aligns with and complements the existing literature. By providing a comprehensive analysis of decision-making and learning processes in a simple and intuitive way, our work offers both theoretical insights and practical implications for understanding complex social and economic systems. Our findings not only advance the theoretical underpinnings of network game theory but also provide valuable insights for policymakers, business leaders, and researchers seeking to understand and influence behavior in interconnected systems.

1.2 Related Literature

Bayesian learning within decision-making contexts has been extensively explored in the literature. Much of the research focuses on analyzing the circumstances under which repeated pairwise communication among a finite group of individuals leads to consensus. Geanakoplos and Polemarchakis (1982), Parikh and Krasucki (1990), Aumann et al. (1995), and Gale and Kariv (2003) have investigated how two players adjust their posterior beliefs regarding the true state after observing each other's actions, demonstrating the convergence of posterior beliefs and equilibrium actions. However, these studies typically assume perfect monitoring of actions, whereas our model accounts for the limited observation capabilities of agents.

Considering Bayesian learning within a network presents added complexity. When agents can only observe the actions of their neighbors, perfect monitoring becomes impractical due to the limited information available. Agents must consider the network's connectivity and update their beliefs accordingly, rendering Bayesian learning within networks challenging. Consequently, much of the literature on learning and the evolution of behavior and opinions in social networks assumes agents with bounded rationality or non-Bayesian settings(Bala and Goyal (1998), Golub and Jackson (2010), Acemoglu et al. (2010), Jadbabaie et al. (2012)).

Mueller-Frank (2013) provides relevant analysis of Bayesian learning under incomplete information within a repeated game framework. Their study reveals local indifference between connected agents, implying that after learning ends, any action an agent selects becomes optimal for all their neighbors. However, this paper assumes identical utility functions for all players, thereby overlooking potential positive or negative externalities in decision-making. In contrast, our Bayesian model incorporates such externalities using a linear-quadratic utility function (Ballester et al. (2006)), yielding distinct results from local indifference due to varying optimal responses among agents based on connectivity.

Our equilibrium concept in a stage game is closely related to the work of Chaudhuri et al. (2024), who examine Bayesian Equilibrium when players can only observe their neighbors' actions under a linear-quadratic utility function (Ballester et al. (2006)). Similar to their study, we identify the Bayesian Equilibrium at each stage and investigate how it evolves over repeated games, thereby extending the findings of Chaudhuri et al. (2024).

Our primary finding demonstrates that after the learning process ends, the Bayesian Nash Equilibrium (BNE) aligns precisely with the Nash Equilibrium under complete information. Jordan (1995) illustrates that Bayesian Nash Equilibria in incomplete repeated games asymptotically converge to the set of Nash Equilibria for complete repeated games, with further extensions by Kalai and Lehrer (1993) and Kalai and Lehrer (1995) relaxing the assumption of identical prior beliefs among players.

However, our study has different points from previous work in several respects. Firstly, while prior studies assume perfect monitoring, we account for imperfect monitoring inherent in network settings where players can only observe their neighbors. Secondly, prior studies rely on the martingale convergence theorem, whereas we establish that after a specific time, $t*$, the BNE at any stage $t \geq t^*$ precisely matches the NE without invoking the limit theorem.

Furthermore, by examining repeated games with incomplete information and imperfect monitoring, Linial (1994), Renault and Tomala (2004) and Li and Tan (2020) have analyzed network structures conducive to players acquiring knowledge of the true state. In contrast, we identify another sufficient condition for perfect learning, focusing on Katz-Bonacich centrality rather than network structure.

The subsequent sections of this paper are organized as follows: Section 2 introduces network theory tools and defines the game setup. In Section 3, we characterize the BNE and present our main result that the BNE coincides with the NE after learning concludes. Section 4 discusses conditions for perfect learning and the rate of convergence. Finally, Section 5 provides concluding remarks. All proofs, as well as additional discussion on certain aspects of our model, are relegated to the appendix.

2 Model

2.1 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ denote the set of players. Letting $i \sim j$ denote a link between i and j, a network (or graph) g is the collection of all pairwise links that exist between players. The links are undirected such that $i \sim j \in g$ implies $j \sim i \in g$. The network can be represented by its adjacency matrix which is also denoted as $g = [g_{ij}]$, where $g_{ij} = 1$ if a link exists between players i and j, and $g_{ij} = 0$ otherwise. There are no self-loops so that $g_{ii} = 0$ for all $i \in N$.

The neighborhood of player i under a network g is the set of players with whom i is linked and is denoted by $N_i(g) = \{j : g_{ij} = 1\}$. The size of this set is i's degree which counts the player's direct connections: $d_i(g) \equiv |N_i(g)|$.

A walk of length s from a node i to a node j_s is a sequence of links in the network $i \sim j_1, j_1 \sim j_2, \ldots, j_{s-1} \sim j_s$. It is denoted by $ij_1j_2 \ldots j_s$. Given two nodes i and j_s, there may exist more than one such walk. Using the adjacency representation, the number of walks of length s from node i to node j_s can be computed by the ij_s element of the adjacency matrix raised to the s^{th} power, g^s.

Finally, let $g^0 = I$ denote the identity matrix. Then, for a sufficiently small $\lambda > 0$, the following influence matrix $M(g, \lambda) = [m_{ij}(g)]$ is well-defined and non-negative:

$$M(g, \lambda) \equiv [I - \lambda g]^{-1} = \sum_{s=0}^{\infty} \lambda^s g^s.$$

Each element $m_{ij}(g)$ measures the total number of walks of all lengths from player i to player j. Given $M(g, \lambda)$, the *Katz-Bonacich* (KB) centrality of player i, $KB_i(g, \lambda)$, is the i^{th}-component of the vector $KB(g, \lambda) = M(g, \lambda)\mathbf{1}_n$, where $\mathbf{1}_n$ is the n-dimensional column vector of ones. It measures the total number of discounted walks of all lengths originating from i to all the other players where longer paths are discounted more.

2.2 The Game

We study an infinitely repeated variant of the incomplete information network game of Chaudhuri et al. (2024) in which agents information about the network is restricted to the identity of their immediate neighbors only.

In the first stage, Nature, a non-strategic player, chooses a network out of a set containing networks on the number of vertices equal to the number of agents. This set may contain all possible networks on a given vertex set, or any subset of them. The chosen network is drawn from an ex-ante distribution that is common knowledge among all agents. Following Nature's draw, players realize their direct connections (they can see the agents with whom they are linked) but do not know the network's architecture beyond that. Using the information on their direct connections and Bayes' rule, agents update their beliefs about the network chosen by Nature and simultaneously exert action to maximize interim linear quadratic payoffs.

Following action exertion in the first stage, players observe the action levels chosen by their neighbors. With actions being informative about the linking profile of agents, observing neighbors actions allows agents to further update their beliefs about the true architecture of the network. With these revised beliefs player again proceed to simultaneously exert action in the second stage, with this process of belief updating and action exertion repeating ad infinitum.

It is assumed that players are myopic but rational. That is, we seek to characterize the learning process induced by the sequence of stage game equilibria. We proceed to formally describe the game.

2.2.1 Time

Time is discrete and indexed by $t \in \{1, 2, ...\}$.

2.2.2 Ex-Ante Beliefs

Denote by $\mathcal{G}$ the set of networks that Nature selects from whose cardinality is denoted by $|\mathcal{G}|$. For example, if this set contains all possible unweighted and undirected networks with n players, then its cardinality would be given by $|\mathcal{G}| = 2^{\frac{n(n-1)}{2}}$.

Let $p \in \Delta(\mathcal{G})$ denote a probability distribution over $\mathcal{G}$, with $\Delta(\mathcal{G})$ denoting the set of all probability distributions over $\mathcal{G}$. Nature's singular role in our game is to choose a specific network $g \in \mathcal{G}$ in the beginning of the first stage. This choice is made according to some $p \in \Delta(\mathcal{G})$, and these ex-ante beliefs are assumed to be common knowledge among players.

2.2.3 Types

We assume that agents can only identify the players with who they are directly connected to. Stated differently, they can only observe their own row of the adjacency matrix corresponding to the network drawn by Nature. Moreover, as the game moves forward in time, by observing their neighbors behavior, agents will gain additional information about the true architecture of the network. Since players information change from period to period, we define types (private information) in a dynamic fashion. In particular, the type of a player in a particular period t is taken to be a set consisting of all those networks that rationalize all the information available to it. This is defined via the following notion of an indistinguishable set of networks.

Definition 1. In the counterfactual scenario in which Nature has selected network g, we denote by $T^t_{i(g)}$ the set of all networks that player i would not be able to distinguish from in period t conditional on all its available information.

If g was selected by Nature, $T^t_{i(g)}$ consists of all those networks that player i would not be able to disqualify as having been drawn given the player's available information. In an arbitrary period $t > 1$ the formal definition of $T^t_{i(g)}$ is provided in a later section as it is defined dynamically and employs equilibrium actions. However, during the first stage $t = 1$ no prior actions have been exerted, and hence the only source of information available to players is their realized neighborhood following Nature's draw. Recalling that $N_i(g)$ denotes the neighborhood set of i under a graph g, then the set of networks that are indistinguishable from g for player i is given by:

$$T^1_{i(g)} = \{g' \in \mathcal{G} | N_i(g') = N_i(g)\}$$

$T^t_{i(g)}$ may consist of a single network if the agent learns the true network drawn by Nature. It may also consist of all possible networks in $\mathcal{G}$ if it cannot distinguish between any of them. More importantly, it is also possible that $T^t_{i(g)} = T^t_{i(g')}$ even if $g \neq g'$. We will demonstrate this possibility in the example below.

In our game, these indistinguishable sets of networks define player types. In particular, the type set of player i in period t is given by

$$T^t_i = \{T^t_{i(g)} | g \in \mathcal{G}\}.$$

Note that T^t_i is a collection of subsets of graphs since a type, $T^t_{i(g)}$, is itself set of graphs. Thus, $T^t_i \subseteq \mathcal{P}(\mathcal{G})/\{\emptyset\}$, where $\mathcal{P}(\mathcal{G})$ is the power set of $\mathcal{G}$. Moreover, since $\mathcal{P}(\mathcal{G})/\{\emptyset\}$ is a finite collection, T^t_i is also finite for any $i \in N$ and any t. The period t type space of the game is

$$T^t = \times_{i \in N} T_i^t.$$

To distinguish between a possible type of a player with its realized type, we use notation I_i^t to denote the latter. For example, if Nature chooses $g' \in \mathcal{G}$, then $I_i^t = T_{i(g')}^t$. Even though $I_i^t \in T_i^t$ and player i knows its private information I_i^t , this does not imply that it knows what g induces I_i^t. As mentioned above, this is because it could be the case that $T_{i(g)}^t = T_{i(g')}^t$ with $g \neq g'$. All players can construct the type sets of others and assign probabilities to elements of them being realized, but realized types I_i^t for all $i \in N$ are private information.

To illustrate the construction of type sets, suppose $t = 1$ in a 4-player game with $\mathcal{G} = \{g_a, g_b, g_c, g_d\}$ as shown in Figure 1.

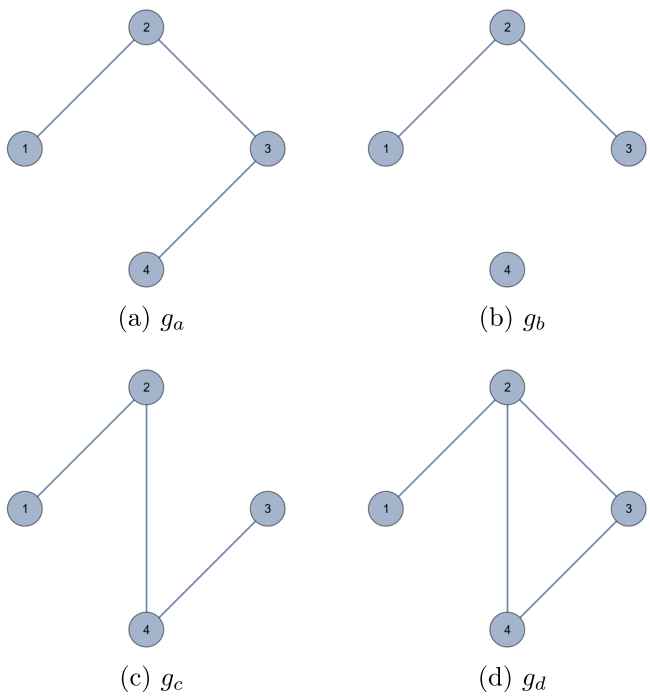


Figure 1: A four player game with $\mathcal{G} = \{g_a, g_b, g_c, g_d\}$

Consider player 1. In Figure 1, we have that $N_1(g_a) = N_1(g_b) = N_1(g_d) = N_1(g_d) = \{2\}$. Therefore, player 1's realized neighborhood, and hence the information it is able to extract from any of these networks is identical. This implies all $g \in \mathcal{G} = \{g_a, g_b, g_c, g_d\}$ are indistinguishable to player 1 regardless of which one is realized by Nature, i.e., $T_{1(g_a)}^1 = T_{1(g_b)}^1 = T_{1(g_c)}^1 = T_{1(g_d)}^1 = \{g_a, g_b, g_c, g_d\}$. This shows that $T_{i(g)}^t$ can be the same to $T_{i(g')}^t$ even if $g \neq g'$.

Now consider player 2. We have $N_2(g_a) = N_2(g_b) = \{1, 3\}$, $N_2(g_c) = \{1, 4\}$, and $N_2(g_d) =$

$\{1,3,4\}$. This implies that $T^1_{2(g_a)} = T^1_{2(g_b)} = \{g_a, g_b\}$, $T^1_{2(g_c)} = \{g_c\}$, and $T^1_{2(g_d)} = \{g_d\}$. Consequently, player 2 will not be able to distinguish between g_a and g_b if either of them are realized. On the other hand, if either g_c or g_d are realized then player 2 would know the entire network since its neighborhood sets under both realizations are unique. Similar constructions can be made for players' 3 and 4 type sets.

It is important to note that all players can construct the type sets of all others. Consider for example the type $T^1_{2(g_a)} = \{g_a, g_b\}$. Since all players know that Nature selects a network from $\mathcal{G} = \{g_a, g_b, g_c, g_d\}$, and that in the first stage the only information available to players stems from their realized neighborhood, any player can conduct the counterfactual that if g_a is realized, then player 2 will be connected to players 1 and 3 and thus will not be able to distinguish between graphs g_a and g_b.

With regard to realized types, suppose Nature chooses g_a. In this case, the realized type of player 1 is $I^1_1 = \{g_a, g_b, g_c, g_d\} = T^1_{1(g_a)}$. Note, however, that $T^1_{1(g_a)} = T^1_{1(g_b)} = T^1_{1(g_c)} = T^1_{1(g_d)}$. Thus, player 1 cannot know which graph induced its realized type. Similarly, when g_a is the true graph, $I^1_2 = T^1_{2(g_a)} = T^1_{2(g_b)} = \{g_a, g_b\}$, $I^1_3 = T^1_{3(g_a)} = T^1_{2(g_d)} = \{g_a, g_d\}$, and $I^1_4 = T^1_{4(g_a)} = \{g_a\}$ which implies that players 2 cannot distinguish between g_a and g_b, player 3 cannot distinguish between g_a and g_d, but player 4 knows the true network.

In summary, with $\mathcal{G} = \{g_a, g_b, g_c, g_d\}$ as shown in Figure 1, all players can construct their own as well as other players period 1 type sets as follows:

$$\begin{aligned}
T^1_1 &= \{T^1_{1(g_a)}, T^1_{1(g_b)}, T^1_{1(g_c)}, T^1_{1(g_d)}\} = \{T^1_{1(g_a)}\} &&= \{\{g_a, g_b, g_c, g_d\}\}.\\
T^1_2 &= \{T^1_{2(g_a)}, T^1_{2(g_b)}, T^1_{2(g_c)}, T^1_{2(g_d)}\} = \{T^1_{2(g_a)}, T^1_{2(g_c)}, T^1_{2(g_d)}\} &&= \{\{g_a, g_b\}, \{g_c\}, \{g_d\}\}.\\
T^1_3 &= \{T^1_{3(g_a)}, T^1_{3(g_b)}, T^1_{3(g_c)}, T^1_{3(g_d)}\} = \{T^1_{3(g_a)}, T^1_{3(g_b)}, T^1_{3(g_c)}\} &&= \{\{g_a, g_d\}, \{g_b\}, \{g_c\}\}.\\
T^1_4 &= \{T^1_{4(g_a)}, T^1_{4(g_b)}, T^1_{4(g_c)}, T^1_{4(g_d)}\} = \{T^1_{4(g_a)}, T^1_{4(g_b)}, T^1_{4(g_c)}\} &&= \{\{g_a\}, \{g_b\}, \{g_c, g_d\}\}.
\end{aligned}$$

2.2.4 Belief Updating

Given prior beliefs and a realized type $I^t_i = T^t_{i(g)}$, player i will assign a probability to player j being of realized type $T^t_{j(g')}$ according to Baye's rule:

$$p(I^t_j = T^t_{j(g')} | I^t_i = T^t_{i(g)}) = \frac{\sum_{g'' \in T^t_{i(g)} \cap T^t_{j(g')}} p(g'')}{\sum_{g'' \in T^t_{i(g)}} p(g'')}.$$

The denominator gives the total probability mass that Nature selects graphs in $T^t_{i(g)}$, while the numerator is the probability of all those graphs in the intersection $T^t_{i(g)} \cap T^t_{j(g')}$. As an example, consider Figure 1 and suppose that Nature selects a graph according to the distribution $(p(g_a), p(g_b), p(g_c), p(g_d)) = (\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})$. Moreover, suppose g_a is realized so that $I_1^1 = T^1_{1(g_a)} = \{g_a, g_b, g_c, g_d\}$. Then, to player 2 being of realized type $I_2^1 = T^1_{2(g_a)} = \{g_a, g_b\}$ player 1 will assign:

$$p(I_2^1 = T^1_{2(g_a)} | I_1^1 = T^1_{1(g_a)}) = \frac{p(g_a) + p(g_b)}{p(g_a) + p(g_b) + p(g_c) + p(g_d)} = \frac{3}{10}.$$

2.2.5 Expected Payoff and Equilibrium

We assume that players ignore the future effects of their decisions. That is, they myopically exert the actions today based on current beliefs without regarding the effects of their actions on other players or future information availability. This can be the result of players heavily discounting the future.

At each stage t, all players have the same action set $\mathbf{A} \equiv \mathbb{R}_+$ and simultaneously exert action to maximize interim linear-quadratic payoffs. Assuming player i's realized type is $T^t_{i(g)}$, these are given by:

$$\begin{aligned} E[u_i | I_i^t = T^t_{i(g)}] = a_i^t(I_i^t = T^t_{i(g)}) - \frac{1}{2}(a_i^t(I_i^t = T^t_{i(g)}))^2 \\ + \lambda a_i^t(I_i = T^t_{i(g)}) \sum_{j=1}^{n} (g_{ij}^{I_i^t = T^t_{i(g)}}) \sum_{T^t_{j(g)} \in T^t_j} p(I_j^t = T^t_{j(g)} | I_i^t = T^t_{i(g)}) a_j^t(I_j^t = T^t_{j(g)}) \end{aligned}$$

where $g_{ij}^{I_i^t = T^t_{i(g)}}$ captures whether i is connected to j in the network $g \in T^t_{i(g)}$, and a_i^t is the action of i in stage t. Note that for any player i, all graphs in an arbitrary type $T^t_{i(g)}$ induce the same neighborhood sets. That is, if $g, g' \in T^t_{i(g)}$, then $N_i(g) = N_i(g'), \forall i \in N$ and $\forall t = 1, 2, \ldots$. We will show this formally in following subsection.

The utility function is an extension of the Ballester et al. (2006) utility function incorporating player types. The first two terms in the utility function capture the direct benefit and cost to player i from exerting its own action. The third term captures local complementarities with those agents that the player is connected to, with λ measuring the strength of this complementarity. Note, however, that unlike the complete information model of Ballester et al. (2006) agents need to form beliefs about the actions of other players.

For each player i in stage t, a pure strategy σ_i^t maps each possible type to an action. That is,

$$\sigma_i^t = (a_i^t(I_i^t = T^t_{i(g)}))_{T^t_{i(g)} \in T_i^t}.$$

This is a simultaneous move game of incomplete information so we use Bayes-Nash as the equilibrium notion.

Definition 2. The pure strategy profile $\sigma^{*t} = (\sigma_i^{*t}, \sigma_{-i}^{*t})$, where $\sigma_i^{*t} = (a_i^{*t}(I_i^t = T_{i(g)}^t))_{T_{i(g)}^t \in T_i^t}$ is a Bayesian-Nash equilibrium (BNE) of the t^{th} stage game if:

$$a_i^{*t}(I_i^t = T_{i(g)}^t) \in \underset{a_i^t(I_i^t = T_{i(g')}^t)}{argmax} \ E[u_i | I_i^t = T_{i(g)}^t], \forall i \in N, \, \forall T_{i(g)}^t \in T_i^t.$$

With type sets being common knowledge, all player can compute BNE action profile of every stage game.

2.2.6 Dynamic Type Updating

We assume that players who are connected can perfectly observe each others actions at the end of each period. Since actions are type dependent, this implies that by observing adjacent agents behavior, players can extract additional information about each others types and hence the true architecture of the network.

Definition 3. Suppose g has been realized by Nature so that, $I_j^t = T_{j(g)}^t$. For player i, the set of types that rationalize player $j's$ actions is given by

$$B^t(a_j^t(I_j^t = T_{j(g)}^t)) = \{T_{j(g')}^t \in T_j^t | a_j^t(I_j^t = T_{j(g')}^t) \in argmax_{a_j^t \in A} E[u_j | I_j^t = T_{j(g)}^t]\}.$$

Moreover, let $B_f^t(a_j^t(I_j^t = T_{j(g)}^t))$ denote the set where we merge all the elements of all subsets of $B^t(a_j^t(I_j^t = T_{j(g)}^t))$:[1]

$$B_f^t(a_j^t(I_j^t = T_{j(g)}^t)) = \bigcup_{T_{j(g')}^t \in B^t(a_j^t(I_j^t = T_{j(g)}^t))} T_{j(g')}^t$$

Note that what player i observes is the action, not the type, and there may be more that one type that induces the action observed by the player. To state this formally, let $g, g' \in T_{i(g)}^t$ and suppose that $T_{j(g)}^t \neq T_{j(g')}^t$. If player i observes $a_j^t(I_j^t = T_{j(g)}^t)$ and it holds that $a_j^t(I_j^t = T_{j(g)}^t) = a_j^t(I_j^t = T_{j(g')}^t)$, then $T_{j(g)}^t, T_{j(g')}^t \in B^t(a_j^t)$. In this case, player i cannot know for certain the realized type of player j since both $T_{j(g)}^t$ and $T_{j(g')}^t$ would induce the same equilibrium action of player j.

Consider again our example in Figure 1 with $(p(g_a), p(g_b), p(g_c), p(g_d)) = (\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})$ and

[1] For example, if $B^t(a_j^t(I_j^t = T_{j(g_a)}^t)) = \{T_{j(g_a)}^t, T_{j\{g_c\}}^t\}$ where $T_{j(g_a)}^t = \{g_a, g_b\}$ and $T_{j(g_c)}^t = \{g_c, g_d\}$, then $B_f^t(a_j^t) = \{g_a, g_b, g_c, g_d\}$.

$\lambda = 1/4$. In this case it can be shown that

$$a_2^1(I_2^1 = T_{2(g_a)}^1) = a_2^1(I_2^1 = T_{2(g_b)}^1) = 1.789$$
$$a_2^1(I_2^1 = T_{2(g_c)}^1) = 1.877$$
$$a_2^1(I_2^1 = T_{2(g_d)}^1) = 2.387.$$

Now suppose that g_a has been realized by Nature so that $I_2^1 = T_{2(g_a)}^1 = \{g_a, g_b\}$ and $I_3^1 = T_{3(g_a)}^1 = \{g_a, g_d\}$. Since players 2 and 3 are connected, player 3 observes that 2 exerts $a_2^1 = 1.789$. Therefore, for player 3, the set of types that rationalize player 2's action is $B^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{T_{2(g_a)}^1, T_{2(g_b)}^1\} = \{\{g_a, g_b\}\}$ and hence $B_f^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{g_a, g_b\}$. Consequently, after observing the action, player 3 knows that the set of networks that are possible from player 2's perspective are either g_a or g_b . In the second period $t = 2$, player 3 can update its belief about the true network by combining its original information I_3^1 with the information extracted from player 2's action, i.e., $I_3^2 = I_3^1 \cap B_f^1(a_2^1) = \{g_a, g_d\} \cap \{g_a, g_b\} = \{g_a\}$.

Since players will observe the actions of all their neighbors, and can can compute the full BNE profile of its stage game, the preceding idea leads to the following type updating rule, which also applies to how the type space of the game evolves over time.

Definition 4. Player i whose realized type in period t is I_i^t updates its type according to:

$$I_i^t = I_i^{t-1} \cap \left(\bigcap_{j \in N_i(g^{I_i^{t-1}})} (B_f^{t-1}(a_j^{t-1}(I_j^{t-1} = T_{j(g^*)}^{t-1}))) \right),$$

where $g^*$is the realized graph.

Similarly, the updating rule for an arbitrary $T_{i(g)}^t$ is given by:

$$T_{i(g)}^t = \begin{cases} \{g' \in \mathcal{G} | N_i(g') = N_i(g)\} & t = 1 \\ T_{i(g)}^{t-1} \cap \left(\bigcap_{j \in N_i(g^{T_{i(g)}^{t-1}})} (B_f^{t-1}(a_j^{t-1}(I_j^{t-1} = T_{j(g)}^{t-1}))) \right) & t = 2, 3, 4, \ldots \end{cases}$$

where

$$N_i(g^{T_{i(g)}^{t-1}}) = \left\{ j \in N | g'_{ij} = 1 \text{ and } g' \in T_{i(g)}^{t-1} \right\}$$

With the formal definition of types in hand, we proceed to discuss some of their properties.

Remark 1. $g \in T_{i(g)}^t$, for any $i \in N$, any $g \in \mathcal{G}$ and any $t = 1, 2, \ldots$.

This follows directly form the definition of $T_{i(g)}^t$. From player $i's$ perspective, $T_{i(g)}^t$ consists of all those graphs that are indistinguishable from g in period t. Thus, the graph g itself should lie in $T^t_{i(g)}$. This implies that $T^t_{i(g)}$ is nonempty for any i, t, and g.

Remark 2. For any $g, g' \in T^t_{i(g)}$, $N_i(g) = N_i(g'), \forall i \in N, \forall t = 1, 2, 3, \dots$

Suppose $g, g' \in T^t_{i(g)}$. Since by construction $T^t_{i(g)}$ is non-increasing in t, it then follows that $g, g' \in T^1_{i(g)}$. Recalling that $T^1_{i(g)} = \{g' \in \mathcal{G} | N_i(g') = N_i(g)\}$, then the neighbors of player i under g and g' must be same. Thus, we can write $g_{ij}^{I_i^t = T^t_{i(g)}} = g_{ij} = g'_{ij}$ so that $N_i(g^{T^t_{i(g)}}) = N_i(g) = N_i(g')$ for any $g, g' \in T^t_{i(g)}$. The type updating rule with $t \geq 2$ can thus be rewritten as

$$T^t_{i(g)} = T^{t-1}_{i(g)} \cap \left(\bigcap_{j \in N_i(g)} (B_f^{t-1}(a_j^{t-1}(I_j^{t-1} = T^{t-1}_{j(g)}))) \right)$$

Recall that $T^t_{i(g)}$ is non-increasing in t. Furthermore, since $\mathcal{G}$ is finite, $T^t_{i(g)}$ is also finite. Therefore, $T^t_{i(g)}$ is a nonempty, non-increasing and finite set for any $i \in N$, for any $g \in \mathcal{G}$, and for any $t = 1, 2, 3, \dots$.

Lastly, recall that I_i^t denotes the realized type of player i. As mentioned previously, even though the player knows its realized type, it may not know what graph induced this type. This is because there may be $g, g' \in \mathcal{G}$ such that $I_i^t = T^t_{i(g)} = T^t_{i(g')}$. The following lemma and remark show this formally.

Lemma 1. *For any $g, g' \in \mathcal{G}$, either $T^t_{i(g)} = T^t_{i(g')}$ or $T^t_{i(g)} \cap T^t_{i(g')} = \emptyset$, for any $i \in N$ and any $t = 1, 2, 3, \dots$.*

Remark 3. For any $i \in N$, if $g, g' \in T^t_{i(g)}$, then $T^t_{i(g)} = T^t_{i(g')}$ for any $t = 1, 2, 3, \dots$.

Suppose $g \neq g'$. When $g' \in T^t_{i(g)}$,then $g' \in T^t_{i(g)} \cap T^t_{i(g')}$,implying that $T^t_{i(g)} \cap T^t_{i(g')} \neq \emptyset$. Thus, by lemma 1, $T^t_{i(g)} = T^t_{i(g')}$. As an example, suppose that $I_i^t = \{g_a, g_b, g_c\}$. If g_a was selected by Nature, then this realized type was generated by $I_i^t = T^t_{i(g_a)}$. However, since $g_a, g_b, g_c \in T^t_{i(g_a)}$, then $T^t_{i(g_a)} = T^t_{i(g_b)} = T^t_{i(g_c)}$. Thus, player i cannot know which network induced its private information I_i^t. Nonetheless, and as is stated in the following remark, the network selected by Nature must always belongs to players realized types.

Remark 4. Let g^* be the realized graph. Then, $g^* \in I_i^t, \forall t = 1, 2, 3, \dots$ and $\forall i \in N$.

2.2.7 Example of the Learning Process

We now illustrate the learning process using our example from Figure 1 where we set $(p(g_a), p(g_b), p(g_c), p(g_d)) = (\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})$, $\lambda = \frac{1}{4}$, and where it is assumed that g_a has been selected by Nature.

The BNE action profile of player 2 is given by:

$$
\begin{aligned}
a_2^1(I_2^1 = T_{2(g_a)}^1) &= a_2^1(I_2^1 = T_{2(g_b)}^1) = 1.789 \\
a_2^1(I_2^1 = T_{2(g_c)}^1) &= 1.877 \\
a_2^1(I_2^1 = T_{2(g_d)}^1) &= 2.387
\end{aligned}
$$

Since g_a has been realized, player 1 will observe its sole neighbor's, player 2's, action to be $a_2^1(I_2^1 = T_{2(g_a)}^1) = 1.789$. Thus, the set of player 2 types that rationalize player 1's observation are $B^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{T_{2(g_a)}^1, T_{2(g_b)}^1\}$, with a corresponding set of networks $B_f^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{g_a, g_b\}$. Using this information in period $t = 2$, player 1 updates its beliefs regarding the true network realized by Nature according to

$$
I_1^2 = I_1^1 \cap B_f^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{g_a, g_b, g_c, g_d\} \cap \{g_a, g_b\} = \{g_a, g_b\}.
$$

Even though player 1 can only observe the action of player 2, it can still compute the BNE action profile of all players. This allows it to consistently construct what the type sets of all players in period $t = 2$ will be. To see this, first note that with its updated realized type $I_1^2 = \{g_a, g_b\}$, player 1 knows that the network selected by Nature is either g_a or g_b. Note further that in both of these networks, player 2 is connected to player 1 and player 3. Therefore, player 1 knows that player 2 will observe 1's and 3's actions. The BNE action profile of player 3 is given by:

$$
\begin{aligned}
a_3^1(I_3^1 = T_{3(g_a)}^1) &= a_3^1(I_3^1 = T_{3(g_d)}^1) = 2.041 \\
a_3^1(I_3^1 = T_{3(g_b)}^1) &= 1.447 \\
a_3^1(I_3^1 = T_{3(g_c)}^1) &= 1.498
\end{aligned}
$$

Recall that from player 1's perspective, either g_a or g_b are the true networks. In the scenario in which g_a is the true network, player 1 knows that player 2 will observe player 3's action to be $a_3^1(I_3^1 = T_{3(g_a)}^1) = 2.041$. In this case, player 1 knows that the corresponding type of player 2, $T_{2(g_a)}$ will be updated in stage $t = 2$ as follows:

$$
\begin{aligned}
T_{2(g_a)}^2 &= T_{2(g_a)}^1 \cap B_f^1(a_3^1(I_3^1 = T_{3(g_a)}^1)) \cap B_f^1(a_1^1(I_1^1 = T_{1(g_a)}^1)) \\
&= \{g_a, g_b\} \cap \{g_a, g_d\} \cap \{g_a, g_b, g_c, g_d\} = \{g_a\}.
\end{aligned}
$$

On the other hand, in the scenario in which g_b is the true network, player 1 knows that player 2 will observe $a_3^1(I_3^1 = T_{3(g_b)}^1) = 1.447$, and not $a_3^1(I_3^1 = T_{3(g_a)}^1) = 2.041$. The set of networks that would rationalize player 3's action in this case would therefore be $B_f^1(a_3^1(I_3^1 = T_{3(g_b)}^1)) = \{g_b\}$. Thus, player 1 can infer that player 2's type $T_{2(g_b)}$, will be updated as follows:

$$\begin{aligned} T_{2(g_b)}^2 &= T_{2(g_b)}^1 \cap B_f^1(a_3^1(I_3^1 = T_{3(g_b)}^1)) \cap B_f^1(a_1^1(I_1^1 = T_{1(g_b)}^1)) \\ &= \{g_a, g_b\} \cap \{g_b\} \cap \{g_a, g_b, g_c, g_d\} = \{g_b\} \end{aligned}$$

In this fashion, player 1 can determine how all of its neighbor's period $t = 1$ types will be updated next period. This process, however, can be performed even if players are not connected. To see this, consider player 4's BNE action profile:

$$\begin{aligned} &a_4^1(I_4^1 = T_{4(g_a)}^1) = 1.51 \\ &a_4^1(I_4^1 = T_{4(g_b)}^1) = 1 \\ &a_4^1(I_4^1 = T_{4(g_c)}^1) = a_4^1(I_4^1 = T_{4(g_d)}^1) = 1.994 \end{aligned}$$

Recall that $I_1^2 = T_{1(g_a)}^2 = \{g_a, g_b\}$. In the scenario in which g_a is the true network, player 3 will be connected to 2 and 4. Therefore, player 1 knows that player 3 will observe $a_4^1 = 1.51$ and $a_2^1 = 1.789$. Player 1 can thus infer that player 3 will update its information according to:

$$\begin{aligned} T_{3(g_a)}^2 &= T_{3(g_a)}^1 \cap B_f^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) \cap B_f^1(a_4^1(I_4^1 = T_{4(g_a)}^1)) \\ &= \{g_a, g_d\} \cap \{g_a, g_b\} \cap \{g_a\} = \{g_a\} \end{aligned}$$

A similar calculation holds for $T_{3(g_b)}^2$ in the scenario in which g_b is the true network. Thus, even if players are not connected, each player can infer any other player's type updating rule by fixing networks in their own realized type. This is because they know what each player will observe in the reference frames of these fixed networks, and consequently how types will be updated. In sum, the type sets of each player at the second stage are common knowledge and are given by: [2]

$$\begin{aligned} &T_1^2 = \{T_{1(g_a)}^2, T_{1(g_b)}^2, T_{1(g_c)}^2, T_{1(g_d)}^2\} = \{T_{1(g_a)}^2, T_{1(g_c)}^2, T_{1(g_d)}^2\} = \{\{g_a, g_b\}, \{g_c\}, \{g_d\}\}. \\ &T_2^2 = \{T_{2(g_a)}^2, T_{2(g_b)}^2, T_{2(g_c)}^2, T_{2(g_d)}^2\} = \{\{g_a\}, \{g_b\}, \{g_c\}, \{g_d\}\}. \\ &T_3^2 = \{T_{3(g_a)}^2, T_{3(g_b)}^2, T_{3(g_c)}^2, T_{3(g_d)}^2\} = \{\{g_a\}, \{g_b\}, \{g_c\}, \{g_d\}\}. \\ &T_4^2 = \{T_{4(g_a)}^2, T_{4(g_b)}^2, T_{4(g_c)}^2, T_{4(g_d)}^2\} = \{\{g_a\}, \{g_b\}, \{g_c\}, \{g_d\}\}. \end{aligned}$$

Furthermore, since we are assuming that g_a has been selected by Nature, the realized types

[2] Note that $T_{1(g_a)}^2 = T_{1(g_b)}^2$.

of each player are $I_1^2 = T^2_{1(g_a)} = \{g_a, g_b\}$, $I_2^2 = T^2_{2(g_a)} = \{g_a\}$, $I_3^2 = T^2_{3(g_a)} = \{g_a\}$, and $I_4^2 = T^2_{4(g_a)} = \{g_a\}$. This implies that at the beginning of the second stage, all players except for player 1 know the true network.

At this point, it is important to note that even if a player knows the true network, this does not imply that will necessarily exert the complete information action. This is because, it still internalizes the uncertainty of others. To see this, recall that g_a is the true network so that player 3's realized type at stage 2 is $I_3^2 = T^2_{3(g_a)} = \{g_a\}$. Player 3 knows that since g_a is the true network, player 1 will observe player 2's action so that its type will be updated according to:

$$T^2_{1(g_a)} = T^1_{1(g_a)} \cap B^1_f(a^1_2(I^1_2 = T^1_{2(g_a)})) = \{g_a, g_b, g_c, g_d\} \cap \{g_a, g_b\} = \{g_a, g_b\}.$$

Hence, even if it knows the true network, player 3 knows that player 1 is still uncertain about player 3's types. This is because under g_a, $T^2_{3(g_a)} = \{g_a\}$, while and under g_b, $T^2_{3(g_b)} = \{g_b\}$. Thus, both g_a and g_b are still possible from player 1's perspective and player 3 knows this. The fact that individuals internalize the uncertainty of others will be reflected in equilibrium actions.

Lastly, with the type space in period 2 being common knowledge, all players can compute the corresponding BNE. The BNE action profile of player 2 in the second stage is given by:

$$\begin{aligned}
a^2_2(I^2_2 = T^2_{2(g_a)}) &= 1.813 \\
a^2_2(I^2_2 = T^2_{2(g_b)}) &= 1.716 \\
a^2_2(I^2_2 = T^2_{2(g_c)}) &= 1.818 \\
a^2_2(I^2_2 = T^2_{2(g_c)}) &= 2.486
\end{aligned}$$

Player 1 being connected to player 2 observes that $a^2_2(I^2_2 = T^2_{2(g_a)}) = 1.813$. With the information stemming from its neighbor's period $t = 2$ actions, player 1 can can further update its type as follows:

$$\begin{aligned}
I^3_1 = T^3_{1(g_a)} &= T^2_{1(g_a)} \cap B^2_f(a^2_2(I^2_2 = T^2_{2(g_a)})) \\
&= \{g_a, g_b\} \cap \{g_a\} = \{g_a\}.
\end{aligned}$$

Thus, at beginning of the third stage, all players know the true network.

3 Equilibrium and Convergence

Recall that we are interested in the learning dynamics induced by the myopic sequence of stage game equilibria.

3.1 Stage Game Equilibrium

Given expected utility, the best response of player i in stage $t$whose realized type is $I_i^t = T_{i(g)}^t$ is given by:

$$a_i^t(I_i^t = T_{i(g)}^t) = 1 + \lambda \sum_{j=1}^{n} g_{ij}^{I_i^t = T_{i(g)}^t} \sum_{T_{j(g)}^t \in T_j^t} p(I_j^t = T_{j(g)}^t | I_i^t = T_{i(g)}^t) a_j^t(I_j^t = T_{j(g)}^t),$$

where $g_{ij}^{I_i^t = T_{i(g)}^t} = \{g_{ij} | g \in T_{i(g)}^t\}$.[3] Let η_i^t be the cardinality of player $i's$ type set at time t, and $\eta^t = \sum_{i \in N} \eta_i^t$. Then, the system characterizing best responses for all players can be written in vector notation as follows:

$$\mathbf{a}^t = \mathbf{1}_{\eta^t} + \lambda \mathbb{B}^t \mathbf{a}^t,$$

where $\mathbf{1}_{\eta^t}$ is the η^t-dimensional column vector of 1's, $\mathbf{a}^t = [\mathbf{a}_i^t]_{i \in N}$, $\mathbf{a}_i^t = (a_i^t(I_i^t = T_{i(g)}^t))_{T_{i(g)}^t \in T_i^t}$, and $\mathbb{B}^t$ is a block matrix that assumes the following form:

$$\mathbb{B}^t = \begin{pmatrix} \mathbf{0} & G_{1\sim 2} & \dots & G_{1\sim n} \\ G_{2\sim 1} & \mathbf{0} & \dots & G_{2\sim n} \\ \dots & \dots & \dots & \dots \\ G_{n\sim 1} & G_{n\sim 2} & \dots & \mathbf{0} \end{pmatrix}$$

with

$$[G_{i\sim j}]_{T_{i(g)}^t, T_{j(g)}^t} = g_{ij}^{I_i^t = T_{i(g)}^t} p(I_j^t = T_{j(g)}^t | I_i^t = T_{i(g)}^t), \forall T_{i(g)}^t \in T_i^t, T_{j(g)}^t \in T_j^t$$

As illustrated by Chaudhuri et al. (2024), the matrix $\mathbb{B}^t$ may be interpreted as a weighted in directed network connecting player types. For instance, consider agent i and the block $[G_{i\sim j}]_{T_{i(g)}^t, T_{j(g)}^t}$ whose elements are of the form $g_{ij}^{I_i^t = T_{i(g)}^t} p(I_j^t = T_{j(g)}^t | I_i^t = T_{i(g)}^t)$. The first term $g_{ij}^{I_i^t = T_{i(g)}^t}$ identifies whether player i, whose type in period t is $T_{i(g)}^t$, is connected to player j. When multiplied by the second term $p(I_j^t = T_{j(g)}^t | I_i^t = T_{i(g)}^t)$, $g_{ij}^{I_i^t = T_{i(g)}^t} p(I_j^t = T_{j(g)}^t | I_i^t = T_{i(g)}^t)$ states that if player i is connected to player j, it will assign a probability to player j being

[3] As we mentioned above, any graph contained in a specific type induces the same neighborhood for the player..

of type $T^t_{j(g)}$ equal to $p(I^t_j = T^t_{j(g)}|I^t_i = T^t_{i(g)})$. In this sense, the $g_{ij}^{I^t_i=T^t_{i(g)}} p(I^t_j = T^t_{j(g)}|I^t_i = T^t_{i(g)})$ can be understood as a weighted and directed link from $T^t_{i(g)}$to$T^t_{j(g)}$ representing the probability that agent i with type $T^t_{i(g)}$ assigns to j being a type $T^t_{j(g)}$.

Proposition 1. *For any stage t, there exist a unique pure strategy BNE for $\lambda \in [0, \frac{1}{n-1})$. Moreover, for any $s \in \mathbb{N}_+$, let $j_1, j_2, \ldots, j_s$ denote an arbitrary collection of s indices. Then, for any prior distribution, any realized network $g \in \mathcal{G}$ and any $t = 1, 2, 3, \ldots$, the equilibrium actions of agents are given by:*

$$a_i^{t^*}(I^t_i = T^t_{i(g)}) = \sum_{s=0}^{\infty} \lambda^s \beta^{(s)}_{i,T^t_{i(g)}}, \forall i \in N, \forall T^t_{i(g)} \in T^t_i,$$

where

$$\beta^{(s)}_{i,T^t_{i(g)}} = \sum_{j_1,\cdots,j_s=1}^{n} \sum_{T^t_{j_1(g)} \in T^t_{j_1}} \cdots \sum_{T^t_{j_s(g)} \in T^t_{j_s}} g_{ij_1}^{I^t=T^t_{i(g)}} g_{j_1 j_2}^{I^t_{j_1}=T^t_{j_1(g)}} \cdots g_{j_{s-1}j_s}^{I^t_{s-1}=T^t_{j_{s-1}(g)}}$$
$$p(I^t_s = T^t_{j_s(g)}|I^t_{s-1} = T^t_{j_{s-1}(g)}) \cdots p(I^t_{j_1} = T^t_{j_1(g)}|I^t_i = T^t_{i(g)}),$$

and $g_{ij_1}^{I^t=T^t_{i(g)}}$ represents the connectivity pattern induced by the type $T^t_{i(g)}$.

Proposition 1 is similar to the BNE characterization of Chaudhuri et al. (2024). They show that in a static game of incomplete information, when agents are endowed with linear quadratic preferences and with types being defined by rows of the adjacency matrix, they will exert an equilibrium action equal to the expected complementarity they are able to extract from the network steaming from walks of different lengths. That is, agents will form beliefs about the expected number of discounted walks of all lengths they have in the network, and will exert an action equaling their sum. While type sets are different in our model, being defined by sets of indistinguishable networks each period, our proposition shows that agents will perform an identical calculation in exerting their equilibrium effort each period. The difference lies in the type-object over which agents belief are defined and hence with the probabilities assigned to these walks.

Remark 5. Let $a^c_i(g)$ denote agent $i's$ complete information action. Then:

$$a^c_i(g) = \sum_{s=0}^{\infty} \lambda^s \left[\sum_{j_1,\cdots,j_s=1}^{n} g_{ij_1} g_{j_1 j_2} \ldots g_{j_{s-1}j_s} \right] \equiv \sum_{s=0}^{\infty} \lambda^s d^{(s)}_i = KB_i(g, \lambda).$$

That is, in complete information version of the game where all agents observe the architecture of the entire network, equilibrium actions of agents reduce to the actual number of discounted walks agents have in the network, i.e., their KB centrality.

3.2 Convergence of Equilibrium Actions

In this section, we examine the convergence properties induced by the myopic sequence of stage game Bayes-Nash equilibria. This convergence bridges the gap between incomplete and complete information behavior, offering insights into how beliefs and actions stabilize over time. We start by formally defining the end of the learning process.

Definition 5. We say that learning ends if there exists a t^* such that for any $t \geq t^*$, $T^t_{j(g)} = T^{t^*}_{j(g)}$, for all $j \in N$ and $g \in \mathcal{G}$.

In other words, learning ends if there is no player that can update any type beyond stage t^*. This includes players realized types $I^t_i = T^t_{i(g^*)}$, where $g^* \in \mathcal{G}$, is the graph selected by Nature. The following proposition guarantees that learning will end at a finite stage of the game.

Proposition 2. *There exists a finite t^* such that learning ends.*

Recall that each stage game BNE can be computed by all players, implying that all players can consistently update the type space of the game every period. As a consequence of this, learning ending is common knowledge for all players and all players know when no other player can update its beliefs further. Next, we state an important property of BNE actions once learning has ended.

Lemma 2. *Suppose that learning ends at t^*. For any $t \geq t^*$, and for any $g', g'' \in T^t_{i(g)}$, $a^t_j(I^t_j = T^t_{j(g')}) = a^t_j(I^t_j = T^t_{j(g'')})$, $\forall\ i \in N$, $\forall\ j \in N_i(g)$, and $\forall\ g \in \mathcal{G}$.*

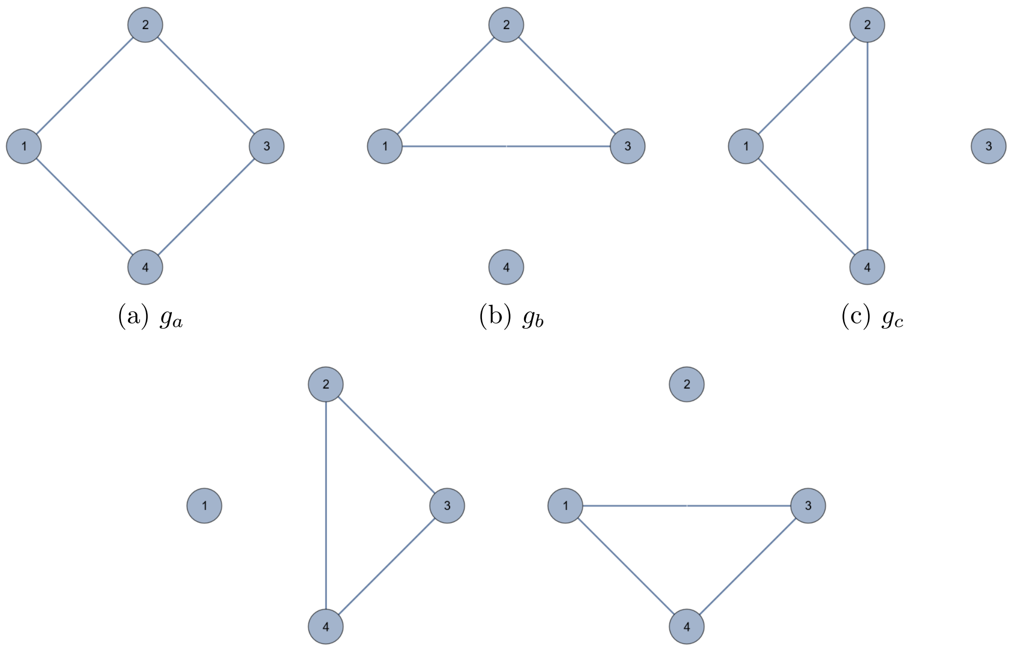
Suppose learning ends at t^* and $t \geq t^*$. If $g', g'' \in T^t_{i(g)}$, then networks g' and g'' are indistinguishable to g for player i after learning has ended. For those networks networks, there exist associated types of player j, $T^t_{j(g')}$ and $T^t_{j(g'')}$ such that $p(I^t_j = T^t_{j(g')}|I^t_i = T^t_{i(g)}) > 0$ and $p(I^t_j = T^t_{j(g'')}|I^t_i = T^t_{i(g)}) > 0$.[4] That is, from player $i's$ perspective whose type includes networks g' and g'', both $T^t_{j(g')}$ and $T^t_{j(g'')}$ are possible types for player j. The lemma states that if the actions associated with types $T^t_{j(g')}$ and $T^t_{j(g'')}$ were different, then player i could still learn. Indeed, if $a^t_j(I^t_j = T^t_{j(g')}) \neq a^t_j(I^t_j = T^t_{j(g'')})$ and $a^t_j(I^t_j = T^t_{j(g')})$ is observed, player i would be able to infer that g'' can not induce player $j's$ realized type. This would imply that $g'' \notin T^{t+1}_{i(g)}$, leading to further learning.

It is important to note that the lemma does not imply that all actions for all types of a particular player are identical after learning ends. Instead, it is a restriction on the equilibrium

[4] This is because $g' \in T^t_{j(g')} \cap T^t_{i(g)}$ and $g'' \in T^t_{j(g'')} \cap T^t_{i(g)}$.

actions of player j contingent on types that are induced by networks in a specific type of its neighbor i after learning has ended. That is, a neighbor j's types that remain possible from player i's perspective must lead to the same action. If they didn't, player i would continue learning, contradicting the assumption that learning has ended.

We illustrate Lemma 2 and its consequences via the example shown in Figure 2, where it is assumed that $p(g) = \frac{1}{5}$ for all $g \in \mathcal{G} = \{g_a, g_b, g_c, g_d, g_e\}$ and $\lambda = \frac{1}{4}$.



(a) g_a (b) g_b (c) g_c

(d) g_d (e) g_e

Figure 2: A four player game with $\mathcal{G} = \{g_a, g_b, g_c, g_d, g_e\}$

Period $t = 1$ type sets for each player are given by

$$T_1^1 = \{T_{1(g_a)}^1, T_{1(g_b)}^1, T_{1(g_c)}^1, T_{1(g_d)}^1, T_{1(g_e)}^1\} = \{\{g_a, g_b\}, \{g_c\}, \{g_d\}, \{g_e\}\}$$
$$T_2^1 = \{T_{2(g_a)}^1, T_{2(g_b)}^1, T_{2(g_c)}^1, T_{2(g_d)}^1, T_{2(g_e)}^1\} = \{\{g_a, g_c\}, \{g_b\}, \{g_d\}, \{g_e\}\}$$
$$T_3^1 = \{T_{3(g_a)}^1, T_{3(g_b)}^1, T_{3(g_c)}^1, T_{3(g_d)}^1, T_{3(g_e)}^1\} = \{\{g_a, g_d\}, \{g_b\}, \{g_c\}, \{g_e\}\}$$
$$T_4^1 = \{T_{4(g_a)}^1, T_{4(g_b)}^1, T_{4(g_c)}^1, T_{4(g_d)}^1, T_{4(g_e)}^1\} = \{\{g_a, g_e\}, \{g_b\}, \{g_c\}, \{g_d\}\}$$

with corresponding stage $t = 1$ BNE profile

$$
\begin{aligned}
&a_1^1(I_1^1 = T_{1(g_a)}^1) = a_1^1(I_1^1 = T_{1(g_b)}^1) = a_1^1(I_1^1 = T_{1(g_c)}^1) = a_1^1(I_1^1 = T_{1(g_e)}^1) = 2, a_1^1(I_1^1 = T_{1(g_d)}^1) = 1 \\
&a_2^1(I_2^1 = T_{2(g_a)}^1) = a_2^1(I_2^1 = T_{2(g_b)}^1) = a_2^1(I_2^1 = T_{2(g_c)}^1) = a_2^1(I_2^1 = T_{2(g_d)}^1) = 2, a_2^1(I_2^1 = T_{2(g_e)}^1) = 1 \\
&a_3^1(I_3^1 = T_{3(g_a)}^1) = a_3^1(I_3^1 = T_{3(g_c)}^1) = a_3^1(I_3^1 = T_{3(g_d)}^1) = a_3^1(I_3^1 = T_{3(g_e)}^1) = 2, a_3^1(I_3^1 = T_{3(g_b)}^1) = 1 \\
&a_4^1(I_4^1 = T_{4(g_a)}^1) = a_4^1(I_4^1 = T_{4(g_b)}^1) = a_4^1(I_4^1 = T_{4(g_d)}^1) = a_4^1(I_4^1 = T_{4(g_e)}^1) = 2, a_4^1(I_4^1 = T_{4(g_c)}^1) = 1.
\end{aligned}
$$

Observe that in this example, learning ends in the first period. To see this, consider player 1 and suppose that network g_a is realized. In this case, we have $I_1^1 = \{g_a, g_b\}, I_2^1 = \{g_a, g_c\}, I_3^1 = \{g_a, g_d\}$ and $I_4^1 = \{g_a, g_e\}$. Player 1 will observe the actions of players 2 and 4 to be $a_2^1(I_2^1 = T_{2(g_a)}^1) = 2$ and $a_4^1(I_4^1 = T_{4(g_a)}^1) = 2$. Given the BNE action profile, it follows that $B^1(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{T_{2(g_a)}^1, T_{2(g_b)}^1, T_{2(g_c)}^1, T_{2(g_d)}^1\}$, $B^t(a_4^1(I_4^1 = T_{4(g_a)}^1)) = \{T_{2(g_a)}^1, T_{2(g_b)}^1, T_{2(g_d)}^1, T_{2(g_e)}^1\}$, and hence $B_f^t(a_2^1(I_2^1 = T_{2(g_a)}^1)) = \{g_a, g_b, g_c, g_d\}$, $B_f^t(a_4^1(I_4^1 = T_{4(g_a)}^1)) = \{g_a, g_b, g_d, g_e\}$. Thus,

$$
\begin{aligned}
T_{1(g_a)}^2 &= T_{1(g_a)}^1 \cap \left(B_f^t(a_2^1(I_2^1 = T_{2(g_a)}^1)) \cap B_f^t(a_4^1(I_4^1 = T_{4(g_a)}^1))\right) \\
&= \{g_a, g_b\} \cap (\{g_a, g_b, g_c, g_d\} \cap \{g_a, g_b, g_d, g_e\}) = \{g_a, g_b\} \\
&= T_{1(g_a)}^1
\end{aligned}
$$

implying that type $T_{1(g_a)}^1$ can not be updated further. A similar argument holds for all other players, and all of their types. To see Lemma 2, observe that $g_a, g_b \in T_{1(g_a)}^1$ and the corresponding types of player 2, $T_{2(g_a)}^1 = \{g_a, g_c\}$ and $T_{2(g_b)}^1 = \{g_b\}$ lead to the same action. The same holds for player 1's second neighbor, player 4. Thus, after observing the actions of its neighbors, player 1 realizes that all of its neighbors types induced by the networks in $T_{1(g_a)}^1$ lead to the same action.

The consequence of this observation is that player 1 can use this information to reconstruct its best response. Noting that $p(I_2^1 = T_{2(g_a)}^1 | I_1^1 = T_{1(g_a)}^1) = p(I_2^1 = T_{2(g_b)}^1 | I_1^1 = T_{1(g_a)}^1) = p(I_4^1 = T_{4(g_a)}^1 | I_1^1 = T_{1(g_a)}^1) = p(I_4^1 = T_{2(g_b)}^1 | I_1^1 = T_{1(g_a)}^1) = 1/2$ player 1's best response if given by:

$$
\begin{aligned}
a_1^1(I_1^1 = T_{1(g_a)}^1) &= 1 + \lambda \sum_{j=1}^{4} g_{1j}^{I_1^1 = T_{1(g_a)}^1} \sum_{T_{j(g)}^1 \in T_j^1} p(I_j^1 = T_{j(g)}^1 | I_1^1 = T_{1(g_a)}^1) a_j^1(I_j^1 = T_{j(g)}^1) \\
&= 1 + \lambda(\frac{1}{2} a_2^1(I_2^1 = T_{2(g_a)}^1) + \frac{1}{2} a_2^1(I_2^1 = T_{2(g_b)}^1)) + (\frac{1}{2} a_4^1(I_4^1 = T_{4(g_a)}^1) + \frac{1}{2} a_4^1(I_4^1 = T_{4(g_b)}^1)).
\end{aligned}
$$

Since $a_2^1(I_2^1 = T_{2(g_a)}^1) = a_2^1(I_2^1 = T_{2(g_b)}^1)$ and $a_4^1(I_4^1 = T_{4(g_a)}^1) = a_4^1(I_4^1 = T_{4(g_b)}^1)$, we then have:

$$a_1^t(I_1^1 = T_{1(g_a)}^1) = 1 + \lambda(a_2^1(I_2^1 = T_{2(g_a)}^1) + a_4^1(I_4^1 = T_{4(g_a)}^1)) = 1 + \lambda \sum_{j=1}^{4} g_{1j}^{I_1^1 = T_{1(g_a)}^1} a_j^1(I_j^1 = T_{j(g_a)}^1).$$

This best response, however, can be rewritten further on the basis that learning has ended. Under the type $T_{2(g_a)}^1$, player 1 knows that player 2 will observe player 1's and 3's actions. Since learning has ended, all the types corresponding to the graphs in $T_{2(g_a)}^1 = \{g_a, g_c\}$ lead to the same action, $a_1^1(I_1^1 = T_{1(g_a)}^1) = a_1^1(I_1^1 = T_{1(g_c)}^1) = 2$ and $a_3^1(I_3^1 = T_{3(g_a)}^1) = a_3^1(I_3^1 = T_{3(g_c)}^1) = 2$. Moreover since learning ending is common knowledge, player 1 knows that player 2's best-response function can also be rewritten in a similar fashion as its own:

$$\begin{aligned} a_2^1(I_2^1 = T_{2(g_a)}^1) &= 1 + \lambda \sum_{k=1}^{4} g_{2k}^{I_2^1 = T_{2(g_a)}^1} \sum_{T_{k(g)}^1 \in T_k^1} p(I_k^1 = T_{k(g)}^1 | I_2^1 = T_{2(g_a)}^1) a_k^1(I_k^1 = T_{k(g)}^1) \\ &= 1 + \lambda(a_1^t(I_1^1 = T_{1(g_a)}^1) + a_3^1(I_3^1 = T_{3(g_a)}^1)) \\ &= 1 + \lambda \sum_{k=1}^{4} g_{2k}^{I_2^1 = T_{2(g_a)}^1} a_k^1(I_k^1 = T_{k(g_a)}^1). \end{aligned}$$

Player 1 can perform a similar calculation by considering the type of its other neighbor $T_{4(g_a)}^1$, leading to

$$a_1^1(I_1^1 = T_{1(g_a)}^1) = 1 + \lambda \sum_{j=1}^{4} g_{1j}^{I_1^1 = T_{1(g_a)}^1} \left(1 + \lambda \sum_{k=1}^{4} g_{jk}^{I_j^1 = T_{2(g_a)}^1} a_k^1(I_k^1 = T_{k(g_a)}^1). \right).$$

Invoking the fact that learning has ended, this reduction of best responses can be performed about any other player, with repeated substitution yielding:

$$a_1^1(I_1^1 = T_{1(g_a)}^1) = 1 + \lambda \sum_{j=1}^{4} g_{1j}^{I_1^1 = T_{1(g_a)}^1} + \lambda^2 \sum_{j=1}^{4} g_{1j}^{I_1^1 = T_{1(g_a)}^1} \sum_{k=1}^{4} g_{jk}^{I_j^1 = T_{j(g_a)}^1} + \dots .$$

Consequently, player 1 can deduce that after learning has ended, its action is identical to its KB centrality in g_a. This process can be generalized to any player and any type, leading to the following theorem.

Theorem 1. *Suppose that Nature has selected g^*. Then, once learning ends, $a_i^t(I_i^t = T_{i(g^*)}^t) = a_i^c(g^*), \forall i \in N$, where $a_i^c(g^*)$ is the equilibrium action of player i under the complete information network game played over g^*.*

As discussed earlier, after learning ends, the possible types of player j from player i's perspective result in identical actions, preventing further learning. Thus, even if player i does not know player j's exact type, the identical actions across possible types from i's view resolve the uncertainty. Consequently, the incomplete network game behaves as if it were a complete information game.

Note that Theorem 1 does not require all agents to know the true network perfectly. That is, players realized types need not be singleton sets after learning ends. What it does show, however, is that a player's realized type must consist of all those networks that (i) admit the player's observed neighborhood, and (ii) give the same KB centrality. The question of when do agents learn the network perfectly is addressed in the following section.

4 Concluding Remarks

In this paper, we have explored the dynamics of learning and decision-making in networked environments with incomplete information. By studying the linear-quadratic network game model with local complementarities, we have addressed key questions regarding equilibrium dynamics, the conditions under which agents can learn the true network structure, and the identification of key players in the learning process.

Our findings indicate that even under incomplete information, the Bayesian Nash Equilibrium (BNE) coincides with the Nash Equilibrium (NE) of a complete information game. This result bridges the gap between incomplete and complete network games, suggesting that rational agents, through repeated interactions, eventually act as if they possess complete knowledge of the network. The persistence of Katz-Bonacich centrality in these settings underscores its importance, even when agents lack full information, further solidifying the connection between our model and the broader game theory literature. Also, this result can help to predict individuals' behaviors under incomplete information.

Our research contributes to the understanding of learning in networks, offering both theoretical advancements and practical implications. By providing a rigorous analysis of the interplay between network structure, information, and behavior, we offer insights that are valuable for academics, policymakers, and practitioners alike. Future research could expand upon our work by exploring scenarios in which individuals consider the future consequences of their decisions. If agents place greater emphasis on future payoffs, the equilibrium outcomes may differ, as suggested by classical repeated game theory (Friedman (1971)).

Appendix A: Proofs

Proof of Lemma 1

We prove the result by induction. Start by recalling the definition of $T^t_{i(g)}$

$$T^t_{i(g)} = \begin{cases} \{g' \in \mathcal{G} | N_i(g') = N_i(g)\} & t = 1 \\ T^{t-1}_{i(g)} \cap \left(\bigcap_{j \in N_i(g)} (B^{t-1}_f(a^{t-1}_j(I^{t-1}_j = T^{t-1}_{j(g)}))) \right) & t = 2, 3, 4, \ldots \end{cases}$$

First, consider $T^1_{i(g)}$ and $T^1_{i(g')}$. Note that, at stage 1, types are determined by neighborhoods. That is, if $g, g' \in T^1_{i(g)}$, then, $N_i(g) = N_i(g')$. Suppose that $T^1_{i(g)} \neq T^1_{i(g')}$ and there exists g'' such that $g'' \in T^1_{i(g)} \cap T^1_{i(g')}$. Then, it has to be the case that $N_i(g'') = N_i(g_a)$ for any $g_a \in T^1_{i(g)}$ since $g'' \in T^1_{i(g)}$. Similarly, $N_i(g'') = N_i(g_b)$ for any $g_b \in T^1_{i(g')}$. This implies that, $N_i(g'') = N_i(g_a) = N_i(g_b), \forall g_a \in T^1_{i(g)}, \forall g_b \in T^1_{i(g')}$. Thus, if $g_a \in T^1_{i(g)}$, then $g_a \in T^1_{i(g')}$ since g_a induces the same neighborhood as any other graph in $T^1_{i(g')}$. Similarly, if $g_b \in T^1_{i(g')}$, then $g_b \in T^1_{i(g)}$, which implies $T^1_{i(g)} = T^1_{i(g')}$. And this is a contradiction.

Now, assume that either $T^t_{i(g)} = T^t_{i(g')}$ or $T^t_{i(g)} \cap T^t_{i(g')} = \emptyset, \forall i \in N$. If $T^t_{i(g)} \cap T^t_{i(g')} = \emptyset$, then the intersection of $T^{t+1}_{i(g)}$ and $T^{t+1}_{i(g')}$ is also empty since $T^t_{i(g)}$ is non-increasing in $t, \forall g \in \mathcal{G}$.

Now suppose that $T^t_{i(g)} = T^t_{i(g')}$ and that there exists a g'' such that $g'' \in T^{t+1}_{i(g)} \cap T^{t+1}_{i(g')}$ and $T^{t+1}_{i(g)} \neq T^{t+1}_{i(g')}$. Note that the neighborhood of i under type $T^t_{i(g)}$ must be the same as its neighborhood under $T^t_{i(g')}$ since $T^t_{i(g)} = T^t_{i(g')}$.

Let i and j be connected and suppose that the graph g has be realized by Nature. Note that for any $g_a \in T^{t+1}_{i(g)}$, it has to be the case that $a^t_j(I^t_j = T^t_{j(g_a)}) = a^t_j(I^t_j = T^t_{j(g)})$. If not, such a g_a cannot be in $T^{t+1}_{i(g)}$. Similarly, for any $g_b \in T^{t+1}_{i(g')}$, $a^t_j(T^t_{j(g_b)}) = a^t_j(T^t_{j(g')})$. Since $g'' \in T^{t+1}_{i(g)} \cap T^{t+1}_{i(g')}$, $a^t_j(I^t_j = T^t_{j(g)}) = a^t_j(I^t_j = T^t_{j(g'')}) = a^t_j(I^t_j = T^t_{i(g')})$. Thus, for any $g_a \in T^{t+1}_{i(g)}$, $g_a \in T^{t+1}_{i(g')}$ since $g_a \in T^t_{i(g)} = T^t_{i(g')}$ and it induces the same action as type $T^{t+1}_{i(g')}$. Similarly, if $g_b \in T^{t+1}_{i(g')}$, then $g_b \in T^{t+1}_{i(g)}$, which implies $T^{t+1}_{i(g)} = T^{t+1}_{i(g')}$ and it is a contradiction.

■

Proof of Proposition 1

To show existence, define a map $P : \mathbb{R}^{\eta^t} \to \mathbb{R}^{\eta^t}$, such that

$$P(\mathbf{a}^t) = \mathbf{1}_{\eta^t} + \lambda \mathbb{B}^t \mathbf{a}^t$$

with $\mathbb{B}^t$ as defined in section 3.1. Let $(\mathbb{R}^{\eta^t}, \|\cdot\|_\infty)$ be the vector space with $\|\cdot\|_\infty$ being the sup-norm on $\mathbb{R}^{\eta^t}$. Hence, we can write:

$$\|P(\mathbf{x}) - P(\mathbf{y})\|_\infty = |\lambda| \|\mathbb{B}^t(\mathbf{x} - \mathbf{y})\|_\infty \leq \lambda(n-1)\|\mathbf{x} - \mathbf{y}\|_\infty = r\|\mathbf{x} - \mathbf{y}\|_\infty$$

where $r = \lambda(n-1)$ and the first inequality results from the fact that $\|\mathbb{B}^t \mathbf{a}\|_\infty \leq (n-1)\|\mathbf{a}\|_\infty$ since the rows of $\mathbb{B}^t$ sum to $n-1$. Thus, we get that P is a contraction on $\mathbb{R}^{\eta^t}$ as long as $r \in [0, 1)$. This holds as long as $\lambda < \frac{1}{n-1}$. Thus, if $0 \leq \lambda < \frac{1}{n-1}$, P is a contraction on $\mathbb{R}^{\eta^t}$ and $(\mathbb{R}^{\eta^t}, \|\cdot\|_\infty)$ is a complete vector space. Therefore, by the Banach fixed point theorem, there exists a unique $\mathbf{a}^{t^*} \in \mathbb{R}^{\eta^t}$ such that

$$P(\mathbf{a}^{t^*}) = \mathbf{a}^{t^*} \Rightarrow \mathbf{a}^{t^*} = \mathbf{1}_{\eta^t} + \lambda \mathbb{B}^t \mathbf{a}^{t^*}.$$

Consequently, there exists a unique pure strategy BNE for the game whenever $\lambda \in [0, \frac{1}{n-1})$ for any $t = 1, 2, 3, \ldots$.To establish the walk based characterization, recall the best responses for all players, which are written by in a vector form:

$$\mathbf{a}^t = \mathbf{1}_{\eta^t} + \lambda \mathbb{B}^t \mathbf{a}^t.$$

Then, the equilibrium actions at t for $\lambda \in [0, \frac{1}{n-1})$ can be written in the form

$$\begin{aligned}\mathbf{a}^{t^*} &= (\mathbf{I}_{\eta^t \times \eta^t} - \lambda \mathbb{B}^t)^{-1} \cdot \mathbf{1}_{\eta^t} \\ &= \mathbf{1}_{\eta^t} + \lambda \mathbb{B}^t \cdot \mathbf{1}_{\eta^t} + \lambda^2 (\mathbb{B}^t)^2 \cdot \mathbf{1}_{\eta^t} + \ldots\end{aligned}$$

for an agent $i \in N$ with a type $T^t_{i(g)}$ at t, the equilibrium action is given by

$$a_i^{t^*}(I_i^t = T^t_{i(g)}) = 1 + \lambda[\mathbb{B}^t \cdot \mathbf{1}]_{i, T^t_{i(g)}} + \lambda^2[(\mathbb{B}^t)^2 \cdot \mathbf{1}]_{i, T^t_{i(g)}} + \ldots$$

Expanding each term $[(\mathbb{B}^t)^s \cdot \mathbf{1}]_{i, T^t_{i(g)}}$ for all $s \in \mathbb{N}_+$ gives the equilibrium characterization.

■

Proof of Proposition 2

Recall that for any $i \in N$, any $g \in \mathcal{G}$, and any t we have $g \in T^t_{i(g)}$ so that $T^t_{i(g)}$ is non-empty. Next, from the definition of type updating we have $T^{t+1}_{i(g)} = T^t_{i(g)} \cap \left(\bigcap_{j \in N_i(g)} B^t_f(a^t_j(I^t_j = T^t_{j(g)})) \right)$ implying that $T^t_{i(g)}$ is non-increasing in t. Moreover, note that since $\mathcal{G}$ is finite, $T^t_{i(g)}$ is also finite. To prove the result, suppose that such a t^* does not exist and consider the infinite sequence defined by player i's dynamic type updates: $\{T^1_{i(g)}, T^2_{i(g)}, \ldots, \}$. Since there is no t^* such that for all $t \geq t^*, T^t_{i(g)} = T^{t^*}_{i(g)}$, and $T^t_{i(g)}$ is non-increasing in t, then there is a strictly decreasing subsequence $\{T^{t_1}_{i(g)}, T^{t_2}_{i(g)}, \ldots\}$ satisfying $T^{t_1}_{i(g)} \supset T^{t_2}_{i(g)} \supset \ldots$. However, since the cardinality of $\mathcal{G}$ is finite, $T^{t_k}_{i(g)}$ must also be finite and we know that $T^{t_k}_{i(g)}$ is nonempty for any $k \in \mathbb{N}$. Hence, such a subsequence $\{T^{t_k}_{i(g)}\}_{k=1}^{\infty}$ satisfying the decreasing property cannot exist, so there must exist a t^* such that for any $t \geq t^*, T^t_{i(g)} = T^{t^*}_{i(g)}, \forall i \in N, \forall g \in \mathcal{G}$. The same argument applies for players realized types I^t_i.

■

Proof of Lemma 2

Let $g', g'' \in T^t_{i(g)}$ for some $t \geq t^*$ with $g' \neq g''$. Note that the neighborhood of i is the same under both g' and g'' since $g', g'' \in T^t_{i(g)}$. Assume that $T^t_{j(g')} \neq T^t_{j(g'')}$, $g_{ij} = 1$, and $a^t_j(I^t_j = T^t_{j(g')}) \neq a^t_j(I^t_j = T^t_{j(g'')})$. Recall that that player i can only observe its neighbor's j action, without knowing its true type at t. Without loss of generality, suppose that player i observes $a^t_j(I^t_j = T^t_{j(g'')})$. This implies that $T^t_{j(g')} \notin B^t(a^t_j(I^t_j = T^t_{j(g'')}))$, and hence $g' \notin B^t_f(a^t_j(I^t_j = T^t_{j(g'')}))$ as $g' \in T^t_{j(g')}$. By the definition of $T^{t+1}_{i(g)} = T^t_{i(g)} \cap \left(\bigcap_{j \in N_i(g)}(B_f(a^t_j(I^t_j = T^t_{j(g)})))\right)$, it follows that $g' \notin T^{t+1}_{i(g)}$. Moreover, since $g' \in T^t_{i(g)}$, then learning can happen, and hence we arrive at a contradiction. On the other hand, if $T^t_{j(g')} = T^t_{j(g'')}$, then $a^t_j(I^t_j = T^t_{j(g')}) = a^t_j(I^t_j = T^t_{j(g'')})$.

■

Proof of Theorem 1

Suppose learning ends at t^* and $t \geq t^*$. By lemma 2 we can rewrite

$$
\begin{aligned}
a^t_i(I^t_i = T^t_{i(g)}) &= 1 + \lambda \sum_{j=1}^{n} g_{ij}^{I^t_i = T^t_{i(g)}} a^t_j(I^t_j = T^t_{j(g)}) \\
&= 1 + \lambda \sum_{j=1}^{n} g_{ij}^{I^t_i = T^t_{i(g)}} (1 + \lambda \sum_{k=1}^{n} g_{jk}^{I^t_j = T^t_{j(g)}} a^t_k(I^t_k = T^t_{k(g)})) \\
&= 1 + \lambda \sum_{j=1}^{n} g_{ij}^{I^t_i = T^t_{i(g)}} + \lambda \sum_{j=1}^{n} g_{ij}^{I^t_i = T^t_{i(g)}} \lambda \sum_{k=1}^{n} g_{jk}^{I^t_j = T^t_{j(g)}} a^t_k(I^t_k = T^t_{k(g)}) \\
&= 1 + \lambda \sum_{j=1}^{n} g_{ij}^{I^t_i = T^t_{i(g)}} + \lambda^2 \sum_{j=1}^{n} g_{ij}^{I^t_i = T^t_{i(g)}} \sum_{k=1}^{n} g_{jk}^{I^t_j = T^t_{j(g)}} + \ldots .
\end{aligned}
$$

This equation implies that $a^t_i(I^t_i = T^t_{i(g)})$ is the KB-centrality of player i under the graph g after learning ends.

Let g^* be the realized graph. Then, $I^t_i = T^t_{i(g^*)}$. Assume $g \in T^t_{i(g^*)}$, implying $T^t_{i(g)} = T^t_{i(g^*)}$. Thus, $a^t_i(I^t_i = T^t_{i(g)}) = a^t_i(I^t_i = T^t_{i(g^*)})$. So, the KB-centrality of player i is the same under any graph $g \in T^t_{i(g^*)}$. Because player i knows one of the graphs in the realized type is the true graph g^*, $a^t_i(I^t_i = T^t_{i(g^*)}) = a^c_i(g^*)$.

■

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