Quantity Regulation and Welfare with Asymmetric Information*

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Abstract

Competition regulators worldwide attempt to protect consumer interests as well as small producers by penalizing any large firm which behaves in a non-competitive manner. As expected, large firms in such markets try to avoid being identified as dominant to preempt any legal scrutiny (penalty or divestment) or any other interference in operational matters by the regulator.

We present a simple model encapsulating this scenario and investigate the effect of quantity regulation designed to restrain large firms on (primarily consumer) welfare. We consider a model where one large firm coexists with several identical small firms in the same market while selling a differentiated product. In an asymmetric information setting - where the small firms are unsure of the costs of the large firm but their common costs are public knowledge - we characterize the unique Bayes-Nash equilibria with and without regulation while assuming that each firm voluntarily chooses to abide by market caps imposed by the regulator. We find the effect of regulation on welfare, even in this idealized setting, to be ambiguous in general. For a special case of uniformly distributed private information, we find that no-regulation is the best regulation.

1 Introduction

Competition regulators all over the world attempt to further the twin objectives of protecting consumer welfare and shielding small producers in a market. For example, the European Commission explicitly states in Article 81(1) of European Community (EC) Treaty that it seeks to "*enhance consumer welfare*" by aiding efficient allocation of resources and protecting competition. The same Commission also states in Article 87 and 88 that it would like to "*aid*" small and medium-sized enterprises with less than 250 employees.[1] However, in many markets where large firms operate, these two objectives are mutually exclusive. The regulators in such markets often end up restrain-

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[1]See here.

ing large firms through stringent quantity regulations despite the loss of economic efficiency that could have accrued from economies of scale and scope.[2]

The large firms, on the other hand, are wary of being labeled as a 'dominant firm' or 'complex monopoly' since that may lead to structural interventions like "treble damages" (as per the Clayton Act of 1914 in the United States of America (USA)), fixing of price controls along with other business practices (provided for in the Competition Act of 1998 in the United Kingdom (UK)), merger & acquisition restrictions (as per the Hart-Scott-Rodino Act of 1976 in USA), and sometimes the extreme step of divestment of firm.[3] Hence, in many markets, the large firms try to limit their size so as to not invite structural interventions by the regulator. For example, Indian telecommunications company 'Jio' voluntarily restricted its size by increasing per-minute prices (from zero) to avoid being identified as a dominant firm in terms of market share. It did so voluntarily to avoid the possibility of prosecution for predatory pricing by the Competition Commission of India (CCI) as per section 27(b) of the Competition Act 2002.[4] Similarly, in 2022, 'Shopee', which was an e-commerce platform like Amazon in India, managed to extricate itself from predatory pricing charges despite making thousands of daily sales at extremely low prices; by cleverly ensuring that CCI found its market share too low to ascribe any 'significant market power.'[5] In another such instance, Google-backed 'GPay' and Walmart-backed 'PhonePe' in the Unified Payments Interface (UPI) platform market of India, have agreed to limit their market share below 30% (without the market regulator National Payments Corporation of India (NPCI) would revoke their business license).[6][7]

To model this social reality, we: (i) consider a market where a large firm with cost advantage co-exists with several smaller identical small firms while selling a differentiated product, and (ii) make a behavioural assumption that the large firm voluntarily abides by any given quantity regulation. In addition, we assume that cost structure of the small firms is public knowledge, while that of the large firm is not. Thus, our model takes the most optimistic view of the consequence of quantity regulation in an asymmetric information setting - and asks the following research question: *does such regulation enhance consumer welfare? If so, then what is the best regulation?*

[2]As noted by Motta (2004) (see page 17), the European Commission employs the *de minimis* principle to favour small and medium-sized enterprises ostensibly, to avoid job losses in European Union. This suggests that the Commission is primarily interested in curbing of anti-competitive practices of large firms through different kinds of structural interventions. In fact the German Supreme Court in 2002 forbade supermarket chains, which were construed to be large enough, from selling of basic goods like milk, margarine and sugar below purchase price to protect smaller sellers (see the bulletin by WilmerHale here).

[3]This extreme step was take in early twentieth century for Standard Oil and American Tobacco, where both firms were split up to protect consumer interests under the aegis of Sherman Act in USA. More recently, in 2000, as noted by Motta (2004) (page xvii), the Department of Justice (DoJ) of the USA asked for the splitting of Microsoft into two smaller companies in a court of law. Currently, Johnson, B. in Vanderbilt JETLaw blog (here) notes that there is a substantial debate about splitting of "Big Tech " firms (Alphabet, Amazon, Apple, Microsoft, and Meta).

[4]See article here.

[5]See article here.

[6]See article here.

[7]One may use the definition provided by [Coglianese and Mendelson (Oxford Handbook of Regulation)] to refer to these instances as "*meta-regulation*", where the regulator mentions an industry-wide cap leaving individual firms 'considerable discretion' on how to achieve it (see the Environmental Protection Agency's 33/50 program discussion on page 22).

Note that we predominantly focus on consumer welfare instead total welfare. This is because most competition legislations explicitly state that their primary objective is to protect consumer interests. For example, the EU competition legislation states in the following articles (bold emphasis added);[8]

> *"Article 81(1): The objective of Article 81 is to protect competition on the market as a means of* ***enhancing consumer welfare*** *and of ensuring an efficient allocation of resources.*
>
> *Article 81(3): The agreement must contribute to improving the production or distribution of goods or contribute to promoting technical or economic progress.* ***Consumers must receive a fair share of the resulting benefits****."*

Similarly, the Indian Competition Act 2002 states that:[9];

> *"An Act to provide, keeping in view of the economic development of the country, for the establishment of a Commission to prevent practices having adverse effects on competition, to promote and sustain competition in markets,* ***to protect the interests of consumers*** *and to ensure freedom of trade carried on by other participants in markets, in India, and for matters connected therewith or incidental thereto."*

Thus, our model studies the impact of quantity regulation (unlike setting of a price floor or support) brought about through legislations, on consumer welfare that these bills were passed to increase. We are able to characterize the unique Bayes-Nash equilibrium in our model with and without such regulation. We identify the maximum possible consumer welfare with regulatory caps, and then attempt to compare it with the consumer welfare without regulation. In the general case, we are unable to obtain an unambiguous result. However, for a special case, where the costs of large firm are distributed uniformly and the extent of private information is substantial enough - we find that no-regulation consumer welfare is always greater than welfare with regulation. This leads us to infer that in this special case, no-regulation is the best regulation.[10]

2 Literature Review

This paper adds to the previous work on regulation of a dominant large firm. Two of the earliest works on regulating a dominant firm in an oligopolistic setting are Caillaud (1990) and Holmes (1996). In terms of motivation, Caillaud (1990) is the closest to ours. Caillaud (1990) considers a strategic large low-cost dominant firm along with a fringe of smaller high-cost non-strategic perfectly competitive firms that produce a homogeneous good, and studies the impact of regulation by taxation of customers who buy from the large firm. Caillaud (1990) finds that presence of fringe

[8] See here.

[9] See here.

[10] Note that we are unable to preclude possibilities where the distributional assumption is such that a certain level of quantity regulation is optimal and better than no-regulation.

firms enhances social welfare. Holmes (1996), too, considers a dominant large firm with a non-strategic competitive fringe producing homogeneous goods, and finds that quantity regulations like market/production caps and market share caps reduce consumer welfare. Both these papers use a complete information setting. In contrast, we consider a dominant large strategic smaller firms with multiple strategic small firms producing differentiated goods, and a different asymmetric information structure. Such market structures can be found in quite a few industries like search engines, operating systems, e-commerce, etc.

There are quite a few papers that analyze regulation of a dominant firm in the *spatial* duopoly market structure in an incomplete information setting (with the two firms located at the extreme ends of a linear market). Unlike the aforementioned papers, these papers consider the unregulated (and typically smaller) firms to be strategic in nature. One of the earliest such papers is Biglaiser and Ma (1995), which considers a Stackelberg competition market where the dominant firm is the leader, and it is the only firm on whom price-regulation via two-part-tariffs apply. Biglaiser and Ma (1995) use an incomplete information setting where the customer valuation differential between the two firms at any given location, is unknown to the regulator but public knowledge among the two firms, and then, presents an optimal regulation policy.

Wolinsky (1997) uses a similar structure but allows firms to compete both in quality as well as quantity of sale. Wolinsky (1997) disregards any possibility of regulation in the quality dimension, and compares social impacts of 'only-price-regulation' and 'both-price-quantity-regulation' without identifying any specific firm as dominant *a priori*. Like Biglaiser and Ma (1995), Wolinsky (1997) too assumes that firms know each other's private information (in this case, cost functions), but these are unknown to the regulator. In contrast to these paper, our differentiated oligopoly model involves multiple small firms with an explicitly identified large firm at whom the regulations are directed, and the private information of the large firm in our model is unknown to the small firms.[11] Further, as noted in Wang (2000), the aforementioned linear market papers implicitly impose an exogenous location restriction on the firms. Our model does not invoke such restrictions and can be generalized to multiple small firms.

3 Model

Consider a market where a large low-cost firm and $N := \{1, ..., n\}$ identical small firms engage in quantity competition. All identical small firms produce the same good, which is different in quality from the good produced by the low-cost large firm. Thus, each small firm competes with other small firms as well as the large low-cost firm. We assume that each small firm $i \in N$ is publicly

[11] Anton and Gertler (2004), too, use this duopoly spatial market structure to focus, like Biglaiser and Ma (1995), on an exogenously fixed quality differentiated market and obtain an optimal regulatory policy. They use a symmetric incomplete information setting about firms' costs, and consider regulatory policies that tax as well as assign prices to be paid and quantities to be bought to each customer on the basis of her location. In contrast, Wang (2000) adopts a mechanism design approach where multiple firms producing a homogeneous good report their private costs to the regulator, who in turn prescribes the quantities that each firm should sell and the prices that they should charge. Both these papers are different from ours in their modelling of the market regulator as well as the information structure in which firm compete.

known to have constant marginal cost c_H, and the large firm has constant marginal cost c. The latter is a private information of the large firm and is publicly known to be distributed over $[\underline{c}, \bar{c}]$ with a smooth distribution function $F(\cdot)$, where $\underline{c} \geq 0$ and $\bar{c} < c_H$. Thus, the lowest possible type of the large firm must have a positive marginal cost, and the highest possible cost type of the large firm must have a marginal cost that is less than that of the small firm. Let $\tau := \int_{\underline{c}}^{\bar{c}} cf(c)\, dc$ be the expected marginal cost of the large firm. All firms are assumed to have zero fixed cost of production.

We denote the total output produced by the large and small firms by Q_L and Q_H, respectively. Let q_H^i be the output of ith small firm so that $Q_H = \sum_{i=1}^n q_H^i$. As in Singh and Vives [1984], we consider a representative consumer with a utility function

$$U(Q_L, Q_H, M) = \alpha_L Q_L + \alpha_H Q_H - \frac{1}{2}(Q_L^2 + Q_H^2 + 2\gamma Q_L Q_H) + M, \quad \alpha_L, \alpha_H, M > 0, \tag{1}$$

which is a quadratic function of two differentiated goods Q_L and Q_H and a linear function of money, M. The parameter $\gamma \in [0, 1]$ represents the extent of product differentiation in a manner such that the two types of goods are considered independent (not affecting the each other's demand functions) when $\gamma = 0$; and the goods are identical when $\gamma = 1$. In our model, we restrict $\gamma \in (0, 1)$ to capture a non-trivial degree of product differentiation, and obtain the following demand functions from our from (1):

$$p_H^i = \alpha_H - Q_H - \gamma Q_L, \; \forall\, i \in N, \tag{2}$$
$$P_L = \alpha_L - Q_L - \gamma Q_H. \tag{3}$$

In the demand functions (2) and (3), α_L and α_H are the intercepts of the linear demand curve for the large and the ith small firms, respectively. Similarly, P_L and p_H^i are the prices charged by the large and ith small firm, respectively.

To keep our analysis tractable and ensure that our equilibrium expressions are positive, we assume the following regularity conditions. We assume that (**C1**) $\alpha_L > \alpha_H$, which signifies that there is a large enough market demand for the large firm's product vis-a-vis the small firms' products. Next, we assume that (**C2**) $\alpha_L > \bar{c} and \alpha_H > c_H$, without which it would not make sense for any firm to operate. Finally, we consider that the degree of quality differential between the products of the small and large firms - is substantial enough so that

$$(\mathbf{C3})\; \gamma \in \left(0, min\left\{1, \frac{2(\alpha_H - c_H)}{\alpha_L - \underline{c}}\right\}\right).$$

Thus, our paper presents a Bayesian game of market competition with differentiated commodities. We use the standard solution concept of Bayes Nash Equilibrium (BNE). We present characterizations of the equilibrium with and without imposition of regulatory cap, and focus on how the corresponding expected equilibrium consumer surplus changes with market regulation.

4 Result

4.1 No-regulation

In this section, we assume that there are no regulatory restrictions on the quantity of output produced by the low-cost large firm. We characterize the unique BNE for this case in the following result.

Theorem 1 *When there is no market regulation on the large firm, the unique equilibrium is* $\left(\bar{Q}_L(.), \{\bar{q}_H^i\}_{i=1}^n\right)$ *where;*

- $\bar{Q}_L(c) = \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1) - \gamma^2 n]} - \frac{c}{2}$, *for all* $c \in [\underline{c}, \bar{c}]$,
- $\bar{q}_H^i = \frac{2(\alpha_H - c_H) + \gamma(\tau - \alpha_L)}{2(n+1) - \gamma^2 n} \qquad \forall\ i = 1, ..., n.$

Proof: We present the proofs of necessity and sufficiency separately below.
Necessity. Fix any equilibrium $(\bar{Q}_L, \{\bar{q}_H^i\}_{i \in N})$. Note that the large firm has complete information about its marginal cost as well as small firm's marginal cost. So, any large firm type $c \in [\underline{c}, \bar{c}]$ must maximize its profit by solving following problem;

$$\max_{Q_L > 0} \Pi_L(Q_L, \{\bar{q}_H^i\}_{i=1}^n, c),$$

where $\Pi_L(Q_L, \{\bar{q}_H^i\}_{i=1}^n, c) := \left(\alpha_L - Q_L - \gamma \sum_{i=1}^n \bar{q}_H^i - c\right) Q_L$. The first-order necessary condition (F.O.N.C) for this problem is;[12]

$$\frac{\partial \Pi_L}{\partial Q_L} = \alpha_L - 2Q_L - \gamma \sum_{i=1}^n \bar{q}_H^i - c = 0,$$

which gives the following reaction function of the large firm of type c;

$$\bar{Q}_L(c) = \frac{\alpha_L - \gamma \sum_{i=1}^n \bar{q}_H^i - c}{2}. \tag{4}$$

On the other hand, any small firm $i \in N$ has incomplete information about the cost of the large firm. It must maximize its expected profit given the distribution of cost of the large firm by solving the following problem;

$$\max_{q_H^i > 0} \Pi_H^i(q_H^i, \bar{Q}_L(c), \{\bar{q}_H^j\}_{i \neq j}, c_H),$$

where $\Pi_H^i(q_H^i, \bar{Q}_L(c), \{\bar{q}_H^j\}_{i \neq j}, c) := \int_{\underline{c}}^{\bar{c}} q_H^i \left(\alpha_H - q_H^i - \sum_{j \neq i} \bar{q}_H^j - \gamma \bar{Q}_L(c) - c_H\right) f(c)\, dc$. The first-order necessary condition (F.O.N.C) for this problem is;

$$\frac{\partial \Pi_H^i}{\partial q_H^i} = \int_{\underline{c}}^{\bar{c}} \left(\alpha_H - 2q_H^i - \sum_{j \neq i} \bar{q}_H^j - \gamma \bar{Q}_L(c) - c_H\right) f(c)\, dc = 0.$$

[12] The second order sufficiency condition is satisfied : $\frac{\partial^2 \Pi_L}{\partial Q_L^2} = -2 < 0$.

Therefore, we get that:

$$
\begin{aligned}
\bar{q}_H^i &= \int_{\underline{c}}^{\bar{c}} \left(\alpha_H - \sum_{j \in N} \bar{q}_H^j - \gamma \bar{Q}_L(c) - c_H \right) f(c)\, dc \\
\implies \bar{q}_H^i &= \left(\alpha_H - \sum_{j \in N} \bar{q}_H^j - c_H \right) \int_{\underline{c}}^{\bar{c}} f(c)\, dc - \gamma \int_{\underline{c}}^{\bar{c}} \bar{Q}_L(c) f(c)\, dc \\
\implies \bar{q}_H^i &= \left(\alpha_H - \sum_{j \in N} \bar{q}_H^j - c_H \right) - \gamma \int_{\underline{c}}^{\bar{c}} \bar{Q}_L(c) f(c)\, dc. \qquad (5)
\end{aligned}
$$

Summing for all $i \in N$, we get that;

$$
\begin{aligned}
\sum_{j \in N} \bar{q}_H^i &= n \left(\alpha_H - \sum_{j \in N} \bar{q}_H^j - c_H \right) - n\gamma \int_{\underline{c}}^{\bar{c}} \bar{Q}_L(c) f(c)\, dc \\
\implies \sum_{j \in N} \bar{q}_H^i &= \frac{n}{n+1} (\alpha_H - c_H) - \frac{n\gamma}{n+1} \int_{\underline{c}}^{\bar{c}} \bar{Q}_L(c) f(c)\, dc \qquad (6)
\end{aligned}
$$

Substituting (6) into (5) we get:

$$
\bar{q}_H^i = \frac{1}{n+1} \left[(\alpha_H - c_H) - \gamma \int_{\underline{c}}^{\bar{c}} \bar{Q}_L(c) f(c)\, dc \right]. \qquad (7)
$$

From (7), we can see that all the small firms will produce the same level of output. We can now get the equilibrium output of ith small firm by substituting (4) into (7);

$$
\begin{aligned}
\bar{q}_H^i &= \frac{1}{n+1} \left[(\alpha_H - c_H) - \gamma \int_{\underline{c}}^{\bar{c}} \frac{\alpha_L - \gamma n \bar{q}_H^i - c}{2} f(c)\, dc \right] \\
\implies \bar{q}_H^i &= \frac{1}{n+1} \left[(\alpha_H - c_H) - \frac{\gamma}{2} \left\{ (\alpha_L - \gamma n \bar{q}_H^i) \int_{\underline{c}}^{\bar{c}} f(c)\, dc - \int_{\underline{c}}^{\bar{c}} c f(c)\, dc \right\} \right] \\
\implies \bar{q}_H^i &= \frac{1}{2(n+1)} \left[2(\alpha_H - c_H) - \gamma(\alpha_L - \gamma n \bar{q}_H^i) + \gamma \int_{\underline{c}}^{\bar{c}} c f(c)\, dc \right] \\
\implies \bar{q}_H^i &= \frac{2(\alpha_H - c_H) - \gamma \alpha_L + \gamma \int_{\underline{c}}^{\bar{c}} c f(c)\, dc}{2(n+1) - \gamma^2 n} \\
\implies \bar{q}_H^i &= \frac{2(\alpha_H - c_H) + \gamma(\tau - \alpha_L)}{2(n+1) - \gamma^2 n}, \qquad (8)
\end{aligned}
$$

where, as defined earlier, $\tau = \int_{\underline{c}}^{\bar{c}} c f(c)\, dc$ is the expected cost of large firm. Note that the denominator of the right hand side of (8) is always positive as $\gamma \in (0, 1)$. Further, by construction, $\tau > \int_{\underline{c}}^{\bar{c}} \underline{c} f(c) dc = \underline{c}$, and by our regularity condition **(C3)**, $2(\alpha_H - c_H) - \gamma(\alpha_L - \underline{c}) > 0$. Therefore, we can infer that for any $i \in N$, $\bar{q}_H^i$ is positive, and hence, well-defined.

Finally, by substituting (8) in (4), we get the equilibrium strategy of the large firm as the following:

$$\bar{Q}_L(c) = \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1) - \gamma^2 n]} - \frac{c}{2}, \forall\, c \in [\underline{c}, \bar{c}]. \tag{9}$$

Now, by (9), for all $c \in [\underline{c}, \bar{c}]$, $\bar{Q}_L(c) \geq \bar{Q}_L(\bar{c})$. Further,

$$\begin{aligned} \bar{Q}_L(\bar{c}) &= \frac{2(n+1)(\alpha_L - \bar{c}) - 2\gamma n(\alpha_H - c_H) + \gamma^2 n(\bar{c} - \int_{\underline{c}}^{\bar{c}} cf(c)\, dc)}{4(n+1) - 2\gamma^2 n} \\ &= \frac{2(n+1)(\alpha_L - \bar{c}) - 2\gamma n(\alpha_H - c_H) + \gamma^2 n(\int_{\underline{c}}^{\bar{c}} (\bar{c} - c)f(c)\, dc)}{4(n+1) - 2\gamma^2 n}. \end{aligned}$$

Now, by construction, $c_H > \bar{c}$, and by regularity condition **(C1)**, $\alpha_L > \alpha_H$. Therefore, we get that $\bar{Q}_L(\bar{c}) > 0$ implying that $\bar{Q}_L(.)$ function has strictly positive images at all points. Hence, from (8) and (9), the proof of necessity follows.

Sufficiency. Consider a BNE where each small firm i produces $\bar{q}_H^i$ that is given by (8). The best response of the large firm of any type $c \in [\underline{c}, \bar{c}]$ must be computed by solving the maximization problem

$$\max_{Q_L > 0} \Pi_L(Q_L, \{\bar{q}_H^i\}_{i=1}^n, c)$$

mentioned in the proof of necessity above. As noted in footnote 12, the objective function of this problem is strictly concave in Q_L irrespective of the values of other arguments as $\frac{\partial^2 \Pi_L}{\partial Q_L^2} = -2 < 0$. Therefore, the unique best response $\bar{Q}_L(c)$ must be characterized by (4), and so, as argued in the proof of necessity we can get $\bar{Q}_L(c)$ to be the same expression as in (9).
Arguing similarly, if the large firm plays $\bar{Q}_L(.)$, and all small firms $j \neq i$ produce $\bar{q}_H^j$; then the best response of firm i would be to solve,

$$\max_{q_H^i > 0} \Pi_H^i(q_H^i, \bar{Q}_L(c), \{\bar{q}_H^j\}_{i \neq j}, c_H).$$

It is easy to see that $\frac{\partial^2 \Pi_H^i}{\partial {q_H^i}^2} = -2$, which implies that the objective function is strictly concave in q_H^i irrespective of the value of the other arguments, and so, the unique best response of the small firm i, $\bar{q}_H^i$, must be same as that in (4).
Thus, the proof of sufficiency follows. □

Theorem 1 shows that our model of market competition is a unique equilibrium where the informed party, in this case, the large firm, plays a linear strategy that is decreasing in its type. The intuition behind such a strategy is as follows. Higher cost of the large firm requires it keep market price higher so as to recover good margins on the units sold. It achieves this by reducing output in response to realization of higher marginal costs of production.

It is interesting to note Theorem 1 describes the unique equilibrium where firms produce positive outputs. We present below the equilibrium prices conditional on the marginal cost realization of the large low-cost firm.

Proposition 1 *Whenever the large firm realizes a marginal cost $c \in [\underline{c}, \bar{c}]$, the corresponding equilibrium market prices $\bar{P}_L(c), \{\bar{p}^i_H(c)\}_{i=1}^n$ are follows;*

- $\bar{P}_L(c) = \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1)-\gamma^2 n]} + \frac{c}{2}$,
- $\bar{p}^i_H(c) = \frac{2(2\alpha_H - \gamma\alpha_L) + n(2-\gamma^2)(2c_H - \gamma\tau)}{2[2(n+1)-\gamma^2 n]} + \frac{\gamma c}{2}, \qquad \forall\ i = 1, ..., n.$

Proof: Fix any $c \in [\underline{c}, \bar{c}]$, and note that by Theorem 1 and demand equation (3), the equilibrium price charged by the large firm of type c is as follows:

$$\begin{aligned}
\bar{P}_L(c) &= \alpha_L - \bar{Q}_L(c) - \gamma n \bar{q}^i_H \\
&= \alpha_L - \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1)-\gamma^2 n]} + \frac{c}{2} - \gamma n\left\{\frac{2(\alpha_H - c_H) + \gamma(\tau - \alpha_L)}{2(n+1) - n\gamma^2}\right\} \\
&= \frac{2\alpha_L[2(n+1)-\gamma^2 n] - 2(n+1)\alpha_L + 2\gamma n(\alpha_H - c_H) + \gamma^2 n\tau - 4\gamma n(\alpha_H - c_H) - 2\gamma^2 n(\tau - \alpha_L)}{2[2(n+1)-\gamma^2 n]} + \frac{c}{2} \\
&= \frac{4(n+1)\alpha_L - 2\gamma^2 n\alpha_L - 2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau + 2\gamma^2 n\alpha_L}{2[2(n+1)-\gamma^2 n]} + \frac{c}{2} \\
&= \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1)-\gamma^2 n]} + \frac{c}{2}.
\end{aligned} \tag{10}$$

Thus, we get that $\bar{P}_L(c) = \bar{Q}_L(c) + c$, which implies that $\bar{P}_L(c) > 0$, and so, is well-defined. Arguing similarly, we compute the equilibrium price charged by any small firm i when the large firm is of type c as follows:

$$\begin{aligned}
\bar{p}^i_H(c) &= \alpha_H - n\bar{q}^i_H - \gamma\bar{Q}_L(c) \\
&= \alpha_H - n\left\{\frac{2(\alpha_H - c_H) + \gamma(\tau - \alpha_L)}{2(n+1) - n\gamma^2}\right\} - \gamma\left\{\frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1)-\gamma^2 n]} - \frac{c}{2}\right\} \\
&= \frac{2\alpha_H[2(n+1)-\gamma^2 n] - 4n(\alpha_H - c_H) - 2\gamma n(\tau - \alpha_L) - 2\gamma(n+1)\alpha_L + 2\gamma^2 n(\alpha_H - c_H) + \gamma^3 n\tau}{2[2(n+1)-\gamma^2 n]} + \frac{\gamma c}{2} \\
&= \frac{4n\alpha_H + 4\alpha_H - 2\gamma^2 n\alpha_H - 4n\alpha_H + 4nc_H - 2\gamma n\tau + 2\gamma n\alpha_L - 2\gamma n\alpha_L - 2\gamma\alpha_L + 2\gamma^2 n\alpha_H - 2\gamma^2 nc_H + \gamma^3 n\tau}{2[2(n+1)-\gamma^2 n]} + \frac{\gamma c}{2} \\
&= \frac{4\alpha_H + 2n(2-\gamma^2)c_H - 2\gamma\alpha_L - \gamma n(2-\gamma^2)\tau}{2[2(n+1)-\gamma^2 n]} + \frac{\gamma c}{2} \\
&= \frac{2(2\alpha_H - \gamma\alpha_L) + n(2-\gamma^2)(2c_H - \gamma\tau)}{2[2(n+1)-\gamma^2 n]} + \frac{\gamma c}{2}
\end{aligned}$$

As noted in Theorem 1, $[2(n+1) - \gamma^2 n] > 0$. Further, as argued in proof of Theorem 1, $c_H > \bar{c} \implies c_H > \tau$, and so, $\gamma < 1$ implies that $n(2-\gamma^2)(2c_H - \gamma\tau) > 0$. Now, by regularity condition **(C3)**, $2\alpha_H - \gamma\alpha_L > 2\alpha_H - [2(\alpha_H - c_H) + \gamma\underline{c}] = 2c_H - \gamma\underline{c} > c_H - \bar{c} > 0$. Thus, we can infer that $\bar{p}^i_H(c) > 0$, and so, is well-defined for each small firm i. Hence, the result follows. $\square$

Now, as mentioned earlier, we are interested in enquiring about impact of regulation in such markets through changes in consumer welfare.[13] However, given the asymmetric information setting of our problem, we can only compute expected values for consumer welfare as our equilibrium can

[13] We focus on consumer welfare since many real-life competition policy legislations are conspicuously aspire to protect consumer interests.

only predict the expected values for equilibrium outcome (prices and outputs). We present below the expected equilibrium output of the large firm;

$$\begin{aligned}
\mathbb{E}[\bar{Q}_L] &= \int_{\underline{c}}^{\bar{c}} \left(\frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{4(n+1) - 2\gamma^2 n} - \frac{c}{2} \right) f(c)\, dc \\
&= \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1) - \gamma^2 n]} \int_{\underline{c}}^{\bar{c}} f(c)\, dc - \frac{1}{2} \int_{\underline{c}}^{\bar{c}} c f(c)\, dc \\
&= \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1) - \gamma^2 n]} - \frac{\tau}{2} \\
&= \frac{(n+1)(\alpha_L - \tau) - \gamma n(\alpha_H - c_H)}{2(n+1) - \gamma^2 n}.
\end{aligned}$$

Now, the expected equilibrium price of the good sold by the large firm can easily be computed by noting (10) shown in Proposition 1, which states that $\bar{P_L}(c) = \bar{Q}_L(c) + c$ for all $c \in [\underline{c}, \bar{c}]$. This implies that;

$$\begin{aligned}
\mathbb{E}[\bar{P}_L] &= \mathbb{E}[\bar{Q}_L] + \tau \\
&= \frac{(n+1)(\alpha_L + \tau) - \gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2(n+1) - \gamma^2 n}. \qquad (11)
\end{aligned}$$

Similarly, we can compute common expected equilibrium price of the goods sold by the small firms as follows:

$$\begin{aligned}
\mathbb{E}[\bar{p}_H^i] &= \int_{\underline{c}}^{\bar{c}} \left(\frac{2(2\alpha_H - \gamma\alpha_L) + n(2-\gamma^2)(2c_H - \gamma\tau)}{2[2(n+1) - \gamma^2 n]} + \frac{\gamma c}{2} \right) f(c)\, dc \\
&= \left(\frac{2(2\alpha_H - \gamma\alpha_L) + n(2-\gamma^2)(2c_H - \gamma\tau)}{2[2(n+1) - \gamma^2 n]} \right) \int_{\underline{c}}^{\bar{c}} f(c)\, dc + \frac{\gamma}{2} \int_{\underline{c}}^{\bar{c}} c f(c)\, dc \\
&= \frac{2(2\alpha_H - \gamma\alpha_L) + n(2-\gamma^2)(2c_H - \gamma\tau)}{2[2(n+1) - \gamma^2 n]} + \frac{\gamma\tau}{2} \\
&= \frac{2(2\alpha_H - \gamma\alpha_L) + n(2-\gamma^2)(2c_H - \gamma\tau) + \gamma\tau[2(n+1) - \gamma^2 n]}{2[2(n+1) - \gamma^2 n]} \\
&= \frac{2(2\alpha_H - \gamma\alpha_L) + 2(2-\gamma^2)nc_H - 2n\gamma\tau + \gamma^3 n\tau + 2n\gamma\tau + 2\gamma\tau - \gamma^3 n\tau}{2[2(n+1) - \gamma^2 n]} \\
&= \frac{2\alpha_H - \gamma\alpha_L + (2-\gamma^2)nc_H + \gamma\tau}{[2(n+1) - \gamma^2 n]}. \qquad (12)
\end{aligned}$$

We mention in the following corollary, two interesting features of the unique equilibrium described by Theorem 1. They are: (i) The expected output by the large firm decreases as the number of small firms increases, and (ii) the expected equilibrium price charged by the large firm decreases as the number of small firms increases.

Corollary 1 *The equilibrium expected output produced and price charged by the large firm,* $\mathbb{E}[\bar{Q}_L]$ *and* $\mathbb{E}[\bar{P}_L]$*, are decreasing in the number of small firms* n*.*

Proof: To accomplish our proof, we define a valued functions $D : \mathbb{R}_{++} \mapsto \mathbb{R}$ such that for any $z > 0$, $D(z) := \frac{(z+1)(\alpha_L-\tau)-\gamma z(\alpha_H-c_H)}{2(z+1)-\gamma^2 z}$. Thus, defining $z' := \frac{1}{z}$, we can write that:[14]

$$\begin{aligned}
D(z) &= \frac{(1+z')(\alpha_L-\tau)-\gamma(\alpha_H-c_H)}{2(1+z')-\gamma^2} \\
\implies \frac{dD(z)}{dz} &= \frac{\partial\left[\frac{(1+z')(\alpha_L-\tau)-\gamma(\alpha_H-c_H)}{2(1+z')-\gamma^2}\right]}{\partial z'} \times \frac{dz'}{dz} \\
&= \frac{[2(1+z')-\gamma^2](\alpha_L-\tau)-2[(1+z')(\alpha_L-\tau)-\gamma(\alpha_H-c_H)]}{[2(1+z')-\gamma^2]^2} \times \left(-\frac{1}{z^2}\right) \\
&= \frac{2(1+z')(\alpha_L-\tau)-\gamma^2(\alpha_L-\tau)-2(1+z')(\alpha_L-\tau)+2\gamma(\alpha_H-c_H)}{[2(1+z')-\gamma^2]^2} \times \left(-\frac{1}{z^2}\right) \\
&= -\frac{\gamma[2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)]}{z[2(1+z')-\gamma^2]^2}
\end{aligned}$$

It is easy to see that the denominator of the expression above is strictly positive. Further, as argued in proof of necessity of Theorem 1, regularity condition **(C3)** implies that $[2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)] > 0$. Thus, we can infer that $\frac{dD(z)}{dz} < 0$. Since, by construction for any $n \in \mathbb{N}$, $\mathbb{E}[\bar{Q}_L] = D(n)$, the result for equilibrium expected output follows. Further, by (11), $\mathbb{E}[\bar{P}_L] = \mathbb{E}[\bar{Q}_L] + \tau$, and so the result for the equilibrium expected price also follows. □

Corollary 1 presents an intuitive result where increase in competition, albeit in slightly differentiated product category, leads to a contraction of the equilibrium output and price of the low-cost large firm - *even in an asymmetric information setting*. One might expect the same to be true for any individual small firm. However, as the following result shows: whenever the number of small firms increases, the aggregate equilibrium output by small firms *increases*, while the equilibrium output of each individual firm decreases.

Corollary 2

- *The equilibrium output of each small firm is decreasing in n, but the aggregate output of the small firms is increasing in n.*
- *The equilibrium expected price charged by small firms is decreasing in n.*

Proof: To prove that the aggregate equilibrium output of the small firms is increasing in n, note that by Theorem 1, each small firm i produces the same output $\frac{2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)}{2(n+1)-\gamma^2 n}$. Therefore, the aggregate equilibrium output by all small firms taken together is

$$\frac{n[2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)]}{2(n+1)-n\gamma^2} = \frac{2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)}{2\left(1+\frac{1}{n}\right)-\gamma^2},$$

which can easily be seen to be increasing in n.

[14]Note that the rational functions like $D(.)$ are differentiable wherever the denominator polynomial function has a non-zero image.

To establish the inverse relationship between individual equilibrium output of any small firm i with respect to the number n of small firms, as in Corollary 1, we define $\hat{D}:\mathbb{R}_{++}\mapsto\mathbb{R}$ such that for all $z>0$, $\hat{D}(z):=\frac{2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)}{2(z+1)-z\gamma^2}$, and note that for all $z>0$, by regularity condition **(C3)**, $\frac{d\hat{D}(z)}{dz}=-\frac{[2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)](2-\gamma^2)}{[2(z+1)-z\gamma^2]^2}<0$. Hence, the result follows.
Finally, to show that the equilibrium expected price is decreasing in n, define $\bar{D}:\mathbb{R}_{++}\mapsto\mathbb{R}$, such that for all $z>0$, $\bar{D}(z):=\frac{(2\alpha_H-\gamma\alpha_L)+zc_H(2-\gamma^2)+\gamma\tau}{[2(z+1)-\gamma^2 z]}$, and note that for all $z>0$,

$$
\begin{aligned}
\frac{d\bar{D}(z)}{dz} &= \frac{[2(z+1)-\gamma^2 z]c_H(2-\gamma^2)-(2-\gamma^2)[(2\alpha_H-\gamma\alpha_L)+zc_H(2-\gamma^2)+\gamma\tau]}{[2(z+1)-\gamma^2 z]^2}\\
&= \frac{[2(z+1)-\gamma^2 z]c_H(2-\gamma^2)-(2-\gamma^2)[(2\alpha_H-\gamma\alpha_L)+2zc_H-\gamma^2 zc_H+\gamma\tau]}{[2(z+1)-\gamma^2 z]^2}\\
&= \frac{2(z+1)c_H(2-\gamma^2)-\gamma^2 zc_H(2-\gamma^2)-(2-\gamma^2)\{(2\alpha_H-\gamma\alpha_L)+\gamma\tau\}-2zc_H(2-\gamma^2)+\gamma^2 zc_H(2-\gamma^2)}{[2(z+1)-\gamma^2 z]^2}\\
&= \frac{(2-\gamma^2)\{2(z+1)c_H-(2\alpha_H-\gamma\alpha_L)-2zc_H-\gamma\tau\}}{[2(z+1)-\gamma^2 z]^2}\\
&= \frac{(2-\gamma^2)\{2c_H-2\alpha_H+\gamma\alpha_L-\gamma\tau\}}{[2(z+1)-\gamma^2 z]^2}\\
&= -\frac{(2-\gamma^2)\{2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)\}}{[2(z+1)-\gamma^2 z]^2}\\
&= \frac{d\hat{D}(z)}{dz}
\end{aligned}
$$

Now, as shown earlier, $\frac{d\hat{D}(z)}{dz}<0$ always, and so, $\frac{d\bar{D}(z)}{dz}<0$, and hence, the result follows. □

Now we compute the equilibrium expected consumer welfare that will serve as a no-regulation benchmark with respect to which we shall later compare welfare levels obtained by imposing quantity cap market regulation. We first compute the equilibrium consumer surplus conditional on given marginal cost value $c\in[\underline{c},\bar{c}]$ below;

$$
\begin{aligned}
\bar{CS}(c) &= \alpha_L\bar{Q}_L(c)+n\alpha_H\bar{q}_H^i-\frac{1}{2}\left\{\bar{Q}_L^2(c)+(\bar{q}_H^i)^2+2\gamma n\bar{q}_H^i\bar{Q}_L(c)\right\}-\bar{p}_L(c)\bar{Q}_L(c)-\bar{p}_H^i(c)\bar{q}_H^i\\
&= \{\alpha_L-\bar{p}_L(c)\}\bar{Q}_L(c)+n\{\alpha_H-\bar{p}_H^i(c)\}\bar{q}_H^i-\frac{\bar{Q}_L^2(c)}{2}-\frac{(n\bar{q}_H^i)^2}{2}-\gamma n\bar{q}_H^i\bar{Q}_L(c) \qquad (13)
\end{aligned}
$$

Now, we substitute the values of $\bar{Q}_L(c),\bar{q}_H^i,\bar{p}_L(c)$ and $\bar{p}_H^i(c)$ from Theorem 1 and Proposition 1 in above expression:

$$
\begin{aligned}
\bar{CS}(c) = &\left\{\alpha_L-\frac{2(n+1)\alpha_L-2\gamma n(\alpha_H-c_H)-\gamma^2 n\tau}{2[2(n+1)-\gamma^2 n]}-\frac{c}{2}\right\}\left\{\frac{2(n+1)\alpha_L-2\gamma n(\alpha_H-c_H)-\gamma^2 n\tau}{2[2(n+1)-\gamma^2 n]}-\frac{c}{2}\right\}\\
+ \;& n\left\{\frac{2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)}{2(n+1)-n\gamma^2}\right\}\left\{\alpha_H-\frac{2(2\alpha_H-\gamma\alpha_L)+n(2-\gamma^2)(2c_H-\gamma\tau)}{2[2(n+1)-\gamma^2 n]}-\frac{\gamma c}{2}\right\}\\
- \;& \frac{1}{2}\left\{\frac{2(n+1)\alpha_L-2\gamma n(\alpha_H-c_H)-\gamma^2 n\tau}{2[2(n+1)-\gamma^2 n]}-\frac{c}{2}\right\}^2-\frac{n^2}{2}\left\{\frac{2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)}{2(n+1)-n\gamma^2}\right\}^2\\
- \;& \gamma n\left\{\frac{2(n+1)\alpha_L-2\gamma n(\alpha_H-c_H)-\gamma^2 n\tau}{2[2(n+1)-\gamma^2 n]}-\frac{c}{2}\right\}\left\{\frac{2(\alpha_H-c_H)+\gamma(\tau-\alpha_L)}{2(n+1)-n\gamma^2}\right\},
\end{aligned}
$$

which implies that for any $c \in [\underline{c}, \bar{c}]$;

$$
\begin{aligned}
\bar{CS}(c) = & \left\{ \frac{2(n+1)\alpha_L + 2\gamma n(\alpha_H - c_H) - \gamma^2 n(2\alpha_L - \tau)}{2[2(n+1) - \gamma^2 n]} - \frac{c}{2} \right\} \left\{ \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1) - \gamma^2 n]} - \frac{c}{2} \right\} \\
+ & \quad n \left\{ \frac{2(\alpha_H - c_H) + \gamma(\tau - \alpha_L)}{2(n+1) - n\gamma^2} \right\} \left\{ \frac{2\gamma\alpha_L + n(2-\gamma^2)\{2(\alpha_H - c_H) + \gamma\tau\}}{2[2(n+1) - \gamma^2 n]} - \frac{\gamma c}{2} \right\} \\
- & \quad \frac{1}{2} \left[\left\{ \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1) - \gamma^2 n]} \right\}^2 + \frac{c^2}{4} - c \left\{ \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1) - \gamma^2 n]} \right\} \right] \\
- & \quad \frac{n^2}{2} \left\{ \frac{2(\alpha_H - c_H) + \gamma(\tau - \alpha_L)}{2(n+1) - n\gamma^2} \right\}^2 \\
- & \quad \gamma n \left\{ \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1) - \gamma^2 n]} - \frac{c}{2} \right\} \left\{ \frac{2(\alpha_H - c_H) + \gamma(\tau - \alpha_L)}{2(n+1) - n\gamma^2} \right\}.
\end{aligned}
$$

In order to simplify the expression obtained above, we introduce certain notations to denote the terms independent of c in the following manner:

$$
\begin{aligned}
R_1 := & \quad \frac{2(n+1)\alpha_L + 2\gamma n(\alpha_H - c_H) - \gamma^2 n(2\alpha_L - \tau)}{2[2(n+1) - \gamma^2 n]}, \\
R_2 := & \quad \frac{2(n+1)\alpha_L - 2\gamma n(\alpha_H - c_H) - \gamma^2 n\tau}{2[2(n+1) - \gamma^2 n]}, \\
R_3 := & \quad \bar{q}_H^i = \frac{2(\alpha_H - c_H) + \gamma(\tau - \alpha_L)}{2(n+1) - n\gamma^2}, \\
R_4 := & \quad \frac{2\gamma\alpha_L + n(2-\gamma^2)\{2(\alpha_H - c_H) + \gamma\tau\}}{2[2(n+1) - \gamma^2 n]}.
\end{aligned}
$$

By substituting these notations we get a simpler exposition of $\bar{CS}(c)$ as follows;

$$
\begin{aligned}
= & \quad \left\{ R_1 - \frac{c}{2} \right\} \left\{ R_2 - \frac{c}{2} \right\} + nR_3 \left\{ R_4 - \frac{\gamma c}{2} \right\} - \frac{1}{2} \left[R_2^2 + \frac{c^2}{4} - cR_2 \right] - \frac{n^2}{2} R_3^2 - \gamma n R_3 \left\{ R_2 - \frac{c}{2} \right\} \\
= & \quad R_1 R_2 - \frac{c}{2} R_1 - \frac{c}{2} R_2 + \frac{c^2}{4} + nR_3 R_4 - \frac{n\gamma c R_3}{2} - \frac{R_2^2}{2} - \frac{c^2}{8} + \frac{cR_2}{2} - \frac{n^2}{2} R_3^2 - \gamma n R_3 R_2 + \frac{\gamma n c R_3}{2} \\
= & \quad R_1 R_2 + nR_3 R_4 - \gamma n R_3 R_2 - \frac{c}{2} R_1 - \frac{R_2^2}{2} - \frac{n^2}{2} R_3^2 + \frac{c^2}{8}. \qquad (14)
\end{aligned}
$$

The equation (14) above allows us to obtain a simple expression for the expected equilibrium consumer surplus $\mathbb{E}[CS]$ without any market regulation below:

$$\begin{aligned}\mathbb{E}[\bar{CS}] &= \int_{\underline{c}}^{\bar{c}} \left(R_1R_2 + nR_3R_4 - \gamma nR_3R_2 - \frac{c}{2}R_1 - \frac{R_2^2}{2} - \frac{n^2}{2}R_3^2 + \frac{c^2}{8}\right) f(c)\,dc \\ &= \left(R_1R_2 + nR_3R_4 - \gamma nR_3R_2 - \frac{R_2^2}{2} - \frac{n^2}{2}R_3^2\right)\int_{\underline{c}}^{\bar{c}} f(c)\,dc - \frac{R_1}{2}\int_{\underline{c}}^{\bar{c}} c\, f(c)\,dc + \frac{1}{8}\int_{\underline{c}}^{\bar{c}} c^2\, f(c)\,dc \\ &= R_1R_2 + nR_3R_4 - \gamma nR_3R_2 - \frac{R_2^2}{2} - \frac{n^2}{2}R_3^2 - \frac{\tau R_1}{2} + \frac{1}{8}\int_{\underline{c}}^{\bar{c}} c^2 f(c)\,dc. \qquad (15)\end{aligned}$$

Thus, we find that without any market regulatory cap on the firms in our model, the consumer surplus in our model is linear function of Riemann integral $\int_{\underline{c}}^{\bar{c}} c^2 f(c)\,dc$ which would follow from the exact nature of the apriori distribution of private information. In Section 4.3, we present results for uniform distribution where $\tau = S := \frac{\bar{c}+\underline{c}}{2}$, and $\int_{\underline{c}}^{\bar{c}} c^2 f(c)\,dc = \frac{\bar{c}^2+\underline{c}^2+\bar{c}\underline{c}}{3}$.

4.2 With regulation

In this section, we characterize market equilibria under a regulatory production cap designed to protect small firms. We assume that the market regulator has announced a quantity cap of K such that any firm selling more than K will be deemed to have abused market dominance and, thereby, is liable to face government scrutiny and possible punitive actions affecting its operations.[15]

In the following proposition, we show that for any market regulatory cap, K, it can never bind on some small firm but not on any type of low-cost large firm.

Proposition 2 *For any $K \geq 0$, there exists no equilibrium $\langle \hat{Q}_L(.), \{\hat{q}_H^j\}_{j\in N}\rangle$ such that there exist a small firm $i \in N$ with $\hat{q}_H^i = K$ and*

$$\hat{Q}_L(c) < K, \forall\, c \in [\underline{c}, \bar{c}].$$

Proof: Fix any K, any $i \in N$, and suppose that there exists an equilibrium $\langle \hat{Q}_L(.), \{\hat{q}_H^j\}_{j\in N}\rangle$ with $\hat{q}_i = K$. Note that by symmetry of the small firms, in equilibrium, $\hat{q}_H^j = K$ for all $j \in N$. Therefore, for any c, $\hat{Q}_L(c)$ must solve:

$$\max_{Q_L > 0} \Pi_L(Q_L, \{\hat{q}_H^j\}_{j=1}^n, c)$$

It is easy to see that first order necessary condition requires that $\hat{Q}_L(c) = \frac{\alpha_L - \gamma nK - c}{2}$.[16] Further, by supposition, $\hat{Q}_L(c) < K$ for all c, and so, we get that $\frac{\alpha_L - \gamma nK - \underline{c}}{2} < K$, which implies that **(a)** $\frac{\alpha_L - \underline{c}}{2+\gamma n} < K$.

Finally, by supposition, we can infer that for any small firm j, j's best response to all other small firms producing K, and the low-cost firm producing as per $\hat{Q}_L(.)$ must be greater than or equal to

[15] Some examples of such punitive actions have been noted in the introduction.

[16] It is easy to see that the second order sufficiency holds.

K (or else production cap K would not bind). Now this best response of j must solve the following problem;

$$\max_{q>0} \int_{\underline{c}}^{\bar{c}} \left\{ \alpha_H - (n-1)K - \gamma \hat{Q}_L(c) - q - c_H \right\} q f(c) dc.$$

As before, it is easy to see that the first order necessary condition is as follows:

$$\begin{aligned} \alpha_H - (n-1)K - 2q - c_H - \frac{\gamma}{2} \int_{\underline{c}}^{\bar{c}} \{\alpha_L - \gamma n K - c\} f(c) dc &= 0 \\ \alpha_H - (n-1)K - c_H - \frac{\gamma(\alpha_L - \gamma n K)}{2} + \frac{\gamma}{2} \int_{\underline{c}}^{\bar{c}} c f(c) dc &= 2q \\ q_j^* := \frac{2(\alpha_H - c_H) - 2(n-1)K - \gamma(\alpha_L - \tau) + \gamma^2 n K}{4}. \end{aligned}$$[17]

Now, as argued above $q_j^* \geq K$, which implies that

$$\begin{aligned} \frac{2(\alpha_H - c_H) - \gamma(\alpha_L - \tau)}{2(n-1) - \gamma^2 n + 4} &\geq K \\ \Longleftrightarrow \frac{2(\alpha_H - c_H) - \gamma(\alpha_L - \tau)}{2(n+1) - \gamma^2 n} &\geq K. \end{aligned}$$

Thus, using inequality **(a)** obtained earlier, we now get that

$$(\mathbf{b})\ \frac{\alpha_L - \underline{c}}{2 + \gamma n} < K \leq \frac{2(\alpha_H - c_H) - \gamma(\alpha_L - \tau)}{2(n+1) - \gamma^2 n}.$$

However, inequality **(b)** implies that:

$$\begin{aligned} \frac{\alpha_L - \underline{c}}{2 + \gamma n} - \frac{2(\alpha_H - c_H) - \gamma(\alpha_L - \tau)}{2(n+1) - \gamma^2 n} &< 0 \\ \Longrightarrow \frac{2(n+1)(\alpha_L - \underline{c}) - (4 + 2\gamma n)(\alpha_H - c_H) + 2\gamma(\alpha_L - \tau)}{(2 + \gamma n)[2(n+1) - \gamma^2 n]} &< 0 \\ \Longrightarrow \frac{2(n+1)(\alpha_L - \underline{c}) - (4 + 2\gamma n)(\alpha_H - c_H) + 2\gamma(\alpha_H - c_H)}{(2 + \gamma n)[2(n+1) - \gamma^2 n]} &< 0, \end{aligned}$$

which implies that $\gamma > \frac{(n+1)(\alpha_L - \underline{c}) - 2(\alpha_H - c_H)}{(n-1)(\alpha_H - c_H)} > 1$, which is a contradiction. $\square$

Note that Proposition 2 presents an intuitive result which rules out an equilibrium where a market cap applies on small firms but never on low-cost large firm, even with quality differentiation. We present below a result that considers another extreme possibility that the market cap binds on all types of large firms as well as the small firms. We show that unlike Proposition 2, there is a possibility of such an equilibrium if the market cap is set too low.

Proposition 3 *An equilibrium $\langle Q_L^*(.), \{q_H^{*^i}\}_{i \in N} \rangle$ where the market cap binds on all types of the large firm as well as any one small firm exists if and only if $K \leq \frac{\alpha_H - c_H}{n+1+\gamma}$.*

Proof:

Necessity: Fix any $K > 0$, and any equilibrium $\langle Q_L^*(.), \{q_H^{*^i}\}_{i \in N}\rangle$ where there exists a small firm that produces K, while all types of the low-cost large firm produce K. As argued earlier, by the symmetry of the small firms, we can infer that all small firms produce the same output K in equilibrium. Therefore, it must be that the unique solution q^* to the problem $\max_{q>0}\{(\alpha_H - q - (n-1)K - \gamma K - c_H)q\}$ must be no less than K, that is, $q^* \geq K$. Now, it is easy to see that

$$\frac{\partial\{(\alpha_H - q^* - (n-1)K - \gamma K - c_H)q^*\}}{\partial q} = \alpha_H - (n-1)K - \gamma K - c_H - 2q^* = 0,$$

which implies that $q^* = \frac{\alpha_H - (n-1+\gamma)K - c_H}{2} \geq K \iff K \leq \frac{\alpha_H - c_H}{n+1+\gamma}$. Similarly, by supposition, for any $c \in [\underline{c}, \bar{c}]$, the unique solution $Q^*(c)$ to $\max_{Q>0}\{(\alpha_L - Q - \gamma nK - c)Q\}$ must not be less than K, that is, $Q^*(c) \geq K, \forall\, c$. It is easy to see this implies that

$$Q^*(c) = \frac{\alpha_L - \gamma nK - c}{2}, \forall\, c,$$

and so, $Q^*(\bar{c}) \geq K \iff K \leq \frac{\alpha_L - \bar{c}}{2+\gamma n}$. Hence, the proof of necessity follows.

Sufficiency: The sufficiency easily follows by noting:(i) the uniqueness of the solutions to the maximization problems in the proof of necessity above, and (ii) the inequality $2 + \gamma n < n + 1 + \gamma$ which means that $\frac{\alpha_H - c_H}{n+1+\gamma} < \frac{\alpha_L - \bar{c}}{2+\gamma n}$. □

In Propositions 2 and 3 above, we consider the possibilities where market cap binds for any one small firm, while either binding on all types of low-cost firm or on no type of low-cost firm. In Proposition 5 later, we will consider the intermediate possibility that market cap binds for some small firm, and only a few types of the large firm; after which, in Theorem 2, we will analyze the most likely situation where the market cap does not bind on any small firm, but binds on low-cost firm with positive probability.

But, first, we need to present the following Lemma 1, which shows that in any equilibrium, if market cap binds for one type of low-cost large firm c^*, then it binds for all types of low-cost firms that have a lower marginal cost than c^*, and then build on this result to present Proposition 5.

Lemma 1 *For any equilibrium, if there exists type $c^* \in [\underline{c}, \bar{c}]$ such that its equilibrium output is K, then equilibrium output of every type $c \in [\underline{c}, c^*]$ is K.*

Proof: Fix any equilibrium $\langle \tilde{Q}_L(c), \{\tilde{q}_H\}_{i=1}^n\rangle$ such that the ith small firm produces output $\tilde{q}_H^i$ and the large firm of type c produces $\tilde{Q}_L(c)$. Further, suppose that there exists a large firm type

$c^* \in [\underline{c}, \bar{c}]$ such that $\tilde{Q}_L(c^*) = K$. This implies that $\forall\, Q_L \leq K$:

$$\begin{aligned}
\left(\alpha_L - K - \gamma \sum_{i=1}^{n} \bar{q}_H^i - c^*\right) K &\geq \left(\alpha_L - Q_L - \gamma \sum_{i=1}^{n} \bar{q}_H^i - c^*\right) Q_L \\
\Longleftrightarrow \alpha_L(K - Q_L) - (K^2 - Q_L^2) &- \gamma \sum_{i=1}^{n} \bar{q}_H^i (K - Q_L) - c^*(K - Q_L) \geq 0 \\
\Longleftrightarrow \alpha_L - (K + Q_L) - \gamma \sum_{i=1}^{n} \bar{q}_H^i &> c^*.
\end{aligned}$$

Therefore, we get that for all $c \in [\underline{c}, c^*]$, $\alpha_L - (K + Q_L) - \gamma \sum_{i=1}^{n} \bar{q}_H^i > c$, which implies that $\tilde{\Pi}_L(c, K) > \tilde{\Pi}_L(c, Q_L)$ for all $Q_L < K$, and so, $\tilde{Q}_L(c) = K \quad \forall\, c \in [\underline{c}, c^*]$. □

Now, we consider equilibria where the market cap does not bind on any small firm, but may bind on some types of low cost firms. Note that by Lemma 1, in such equilibria, there must exist a $c^*(K) \in [\underline{c}, \bar{c}]$ such that all types of large firm with cost less than or equal to $c^*(K)$ would find it optimal to produce K units, while other types will find it optimal to produce outputs that are strictly less than K. Now, if $c^*(K) = \underline{c}$, then the market cap does not bind with positive probability (that is never binds on the low-cost firm), and the equilibrium obtained in such a scenario would be the same as the one obtained in Theorem 1 without any market regulation.

On the other extreme, if $c^*(K) = \bar{c}$, then the market cap binds on all types meaning that the asymmetry of information ceases to matter, all small firms best respond identically to the low-cost large firm producing the publicly known market cap quantity K. We characterize such an equilibrium below.

Proposition 4 *For any $K > 0$, there exists an equilibrium $\langle \hat{Q}_L(.), \{\hat{q}_H^j\}_{j \in N} \rangle$ where $\forall\, c \in [\underline{c}, \bar{c}]$, $\hat{Q}_L(c) = K$ and for all $j \in N$, $\hat{q}_H^j = \frac{\alpha_H - \gamma K - c_H}{n+1} < K$ if and only if,*

$$K \in \left(\frac{\alpha_H - c_H}{n + 1 + \gamma}, \frac{(\alpha_L - \bar{c})(n+1) - \gamma n(\alpha_H - c_H)}{2(n+1) - \gamma^2 n} \right].$$

Proof:
Necessity: Fix any $K > 0$ and any equilibrium $\langle \hat{Q}_L(.), \{\hat{q}_H^j\}_{j \in N} \rangle$ such that $\hat{Q}_L(c) = K, \forall\, c \in [\underline{c}, \bar{c}]$. Therefore, for any high cost firm $i \in N$, $\hat{q}_H^i$ must solve;

$$\max_{q_H^i > 0} \left(\alpha_H - \sum_{j \neq i} \hat{q}_H^j - q_H^i - \gamma K - c_H \right) q_H^i.$$

It is easy to see that the first order necessary condition is $\alpha_H - \sum_{j \neq i} \hat{q}_H^j - 2q_H^i - \gamma K - c_H = 0$, which implies that $\hat{q}_H^i = \alpha_H - \sum_{j \in N} q_H^j - \gamma K - c_H$. This further implies that $\sum_{i \in N} q_H^i = n\alpha_H -$

$n\sum_{j\in N} q_H^j - n\gamma K - nc_H$, which means that $\sum_{j\in N}\hat{q}_H^i = \frac{n(\alpha_H-\gamma K-c_H)}{n+1}$. Thus, we can infer that,

$$\hat{q}_H^i = \{\alpha_H - \gamma K - c_H\} - \frac{n(\alpha_H - \gamma K - c_H)}{n+1} = \frac{\alpha_H - \gamma K - c_H}{n+1}, \forall\, i \in N.$$

Note that by supposition, K does not bind on any small firm, and so, $\hat{q}_H^i = \frac{\alpha_H-\gamma K-c_H}{n+1} < K \iff \frac{\alpha_H-c_H}{(n+1+\gamma)} < K$.

Now, for any large low-cost firm of type $c \in [\underline{c}, \bar{c}]$ would find it optimal to best respond to all small firms producing $\hat{q}_H^i = \frac{\alpha_H-\gamma K-c_H}{n+1}$, by choosing to produce a $\hat{Q}_L(c)$ which solves;

$$\max_{Q_L>0}(\alpha_L - \gamma n\hat{q}_H^i - Q_L - c)Q_L.$$

The first order necessary condition would be $\alpha_L - \gamma n\hat{q}_H^i - 2\hat{Q}_L - c = 0$, which implies that $\hat{Q}_L(c) = \frac{\alpha_L-\gamma n\hat{q}_H^i-c}{2}$. As before, by supposition $\hat{Q}_L(c) \geq K$ for all $c \in [\underline{c}, \bar{c}]$, which implies $\hat{Q}_L(\bar{c}) \geq K$. Therefore, $\frac{\alpha_L-\gamma n\hat{q}_H^i-\bar{c}}{2} \geq K \iff K \leq \left(\frac{\alpha_L-\bar{c}}{2} - \frac{\gamma n\{\alpha_H-\gamma K-c_H\}}{2(n+1)}\right)$, and so, $2(n+1)K \leq (n+1)(\alpha_L - \bar{c}) - \gamma n(\alpha_H - c_H) + \gamma^2 nK$, which reduces to $K \leq \frac{(n+1)(\alpha_L-\bar{c})-\gamma n(\alpha_H-c_H)}{2(n+1)-\gamma^2 n}$.

Finally, note that $\frac{n+1+\gamma}{2+\gamma n} \geq 1$ for all $n \in \mathbb{N}$ and by construction $\frac{(\alpha_H-c_H)}{(\alpha_L-\bar{c})} < 1$. So

$$\begin{aligned}
&\tfrac{(\alpha_H-c_H)}{(\alpha_L-\bar{c})} \leq \tfrac{n+1+\gamma}{2+\gamma n}\\
\iff& \tfrac{(\alpha_H-c_H)(2+\gamma n)}{n+1+\gamma} \leq (\alpha_L - \bar{c}),\\
\iff& (\alpha_H - c_H)\left\{\tfrac{2(n+1)+\gamma n(n+1)}{(n+1+\gamma)}\right\} \leq (n+1)(\alpha_L - \bar{c}),\\
\iff& (\alpha_H - c_H)\left\{\tfrac{2(n+1)-\gamma^2 n+\gamma n^2+\gamma n+\gamma^2 n}{(n+1+\gamma)(2(n+1)-\gamma^2 n)}\right\} \leq \tfrac{(n+1)(\alpha_L-\bar{c})}{2(n+1)-\gamma^2 n},\\
\iff& (\alpha_H - c_H)\left\{\tfrac{1}{n+1+\gamma} + \tfrac{\gamma n}{2(n+1)-\gamma^2 n}\right\} \leq \tfrac{(n+1)(\alpha_L-\bar{c})}{2(n+1)-\gamma^2 n},\\
\iff& \tfrac{\alpha_H-c_H}{(n+1+\gamma)} \leq \tfrac{(n+1)(\alpha_L-\bar{c})-\gamma n(\alpha_H-c_H)}{2(n+1)-\gamma^2 n}.
\end{aligned}$$

Hence, the proof of necessity follows.

Sufficiency. To establish proof of sufficiency, we need to show that if $\frac{\alpha_H-c_H}{(n+1+\gamma)} \leq \frac{(n+1)(\alpha_L-\bar{c})-\gamma n(\alpha_H-c_H)}{2(n+1)-\gamma^2 n}$, then there exists an equilibrium $\langle\hat{Q}_L(.), \{\hat{q}_H^j\}_{j\in N}\rangle$ where $\forall\, c \in [\underline{c}, \bar{c}]$, $\hat{Q}_L(c) = K$ and for all $j \in N$, $\hat{q}_H^j \leq K$. It is easy to see that the existence of such an equilibrium follows from the equilibrium obtained in the proof of necessity – since the objective function for the maximization problem of the small firms is strictly concave and the relevant boundary arguments consist of equivalence relations. □

There exists a counter-intuitive possibility where all the small firms are rationed by the market cap with positive probability, but the low-cost firm is not rationed at all. The following result shows that this can never be an equilibrium outcome.

Proposition 5 *For any $K > 0$, there does not exist an equilibrium $\langle \hat{Q}_L(.), \{\hat{q}_H^i\}_{i\in N}\rangle$ such that $\hat{q}_H^j = K$ for some $j \in N$, and there exists a measurable subset of types $S \subseteq [\underline{c}, \bar{c}]$ such that*

$$\hat{Q}_L(c) < K, \forall\, c \in S.$$

Proof: Fix any $K > 0$, and any equilibrium $\langle \hat{Q}_L(.), \{\hat{q}_H^i\}_{i\in N}\rangle$ such that there exists a small firm $i \in N$ with $\hat{q}_H^i = K$, and there exists a type $c' \in (\underline{c}.\bar{c})$ such that $\hat{Q}_L(c) = K \quad \forall\, c \in [\underline{c}, c']$ and $\hat{Q}_L(c) < K \quad \forall\, c \in (c', \bar{c}]$. As argued earlier, by symmetry of small firms, we can infer that $\hat{q}_H^j = K$ for all $\forall\, j \in N$. Therefore, $\forall\, c \in [\underline{c}, \bar{c}]$, $\hat{Q}_L(c)$ must solve;

$$\max_{Q_L > 0} (\alpha_L - Q_L - \gamma n K - c) Q_L.$$

The first order necessary condition is $\alpha_L - \gamma n K - c - 2Q_L = 0$, which implies that

$$\hat{Q}_L(c) = \begin{cases} \frac{\alpha_L - \gamma n K - c}{2} & \forall\, c \in (c', \bar{c}] \\ K & \forall\, c \in [\underline{c}, c']. \end{cases} \tag{16}$$

Thus, we can infer that for all $c \in [\underline{c}, c']$, $\frac{\alpha_L - \gamma nK - c}{2} \geq K$ implies that in limit $\frac{\alpha_L - \gamma nK - c'}{2} \geq K$. Similarly, by construction, $K \geq \frac{\alpha_L - \gamma nK - c}{2}$ for all $c \in (c', \bar{c}]$, and so, in limit $K \geq \frac{\alpha_L - \gamma nK - c'}{2}$. Hence, it follows that $c' = \alpha_L - K(2 + \gamma n)$, and so, by construction, $\underline{c} < \alpha_L - K(2 + \gamma n) < \bar{c}$. This condition implies that (**a**) $K \in \left(\frac{\alpha_L - \bar{c}}{2 + \gamma n}, \frac{\alpha_L - \underline{c}}{2 + \gamma n} \right)$.

Now, consider the best response of the uninformed small firms. Note that $\forall\, i \in N$, $\hat{q}_H^i$ must solve;

$$\max_{q_H > 0} \left[\int_{\underline{c}}^{c'} q_H \left(\alpha_H - q_H - (n-1)K - \gamma K - c_H \right) f(c) dc + \int_{c'}^{\bar{c}} q_H \left(\alpha_H - q_H - (n-1)K - \gamma \hat{Q}_L(c) - c_H \right) f(c) dc \right]$$
$$\Longleftrightarrow \max_{q_H > 0} \left[\{ q_H (A - \gamma K - c_H) \} \int_{\underline{c}}^{c'} f(c) dc + \{ q_H (A - c_H) \} \int_{c'}^{\bar{c}} f(c) dc \; - \gamma q_H \int_{c'}^{\bar{c}} \hat{Q}_L(c) f(c) dc \right],$$

where $A := \alpha_H - q_H - (n-1)K$. Therefore, by (16), the objective function of this problem reduces to;

$$\{ q_H (A - \gamma K - c_H) \} F(c') + \{ q_H (A - c_H) \} [1 - F(c')] - \gamma q_H \int_{c'}^{\bar{c}} \frac{\alpha_L - \gamma n K - c}{2} f(c) dc$$
$$\Longleftrightarrow \alpha_H q_H - q_H^2 - (n-1) K q_H - c_H q_H - \gamma K q_H F(c') - \gamma q_H \left\{ \frac{(\alpha_L - \gamma n K)[1 - F(c')]}{2} - \frac{1}{2} \int_{c'}^{\bar{c}} c f(c) dc \right\}.$$

It is easy to see that the first order necessary condition to this maximization problem is:

$$\alpha_H - 2q_H - (n-1)K - c_H - \gamma K F(c') - \frac{\gamma}{2} \left\{ (\alpha_L - \gamma n K)[1 - F(c')] - \int_{c'}^{\bar{c}} c f(c) dc \right\} = 0,$$

which implies that:

$$\hat{q}_H^i = \frac{\alpha_H - c_H - K\{(n-1) + \gamma F(c')\}}{2} - \frac{\gamma \{\alpha_L - \gamma n K\}[1 - F(c')]}{4} + \frac{1}{4} \int_{c'}^{\bar{c}} c f(c) dc, \forall\, i \in N.$$

Further, note that by the supposition, for any small firm i the market cap binds, that is $\hat{q}_H^i > K$, which implies that:

$$
\begin{aligned}
& \frac{\alpha_H - c_H - K\{(n-1) + \gamma F(c')\}}{2} - \frac{\gamma\{\alpha_L - \gamma n K\}[1 - F(c')]}{4} + \frac{1}{4}\int_{c'}^{\bar{c}} c f(c) dc \geq K, \\
\iff \quad & 2(\alpha_H - c_H) - 2K\{(n-1) + \gamma F(c')\} - \gamma\{\alpha_L - \gamma n K\}[1 - F(c')] + \gamma\left\{\bar{c}F(\bar{c}) - c'F(c') - \int_{c'}^{\bar{c}} F(c)\, dc\right\} \geq 4K, \\
\iff \quad & 2(\alpha_H - c_H) - \gamma\alpha_L[1 - F(c')] + \gamma\bar{c} \geq 4K + 2K\{(n-1) + \gamma F(c')\} - \gamma^2 n K[1 - F(c')] + c'F(c') + \int_{c'}^{\bar{c}} F(c)\, dc, \\
\iff \quad & 2(\alpha_H - c_H) - \gamma(\alpha_L - \bar{c}) \geq K\{2(n+1) + 2\gamma F(c')\} - \gamma^2 n K[1 - F(c')] - \gamma F(c')\{\alpha_L - c'\} + \int_{c'}^{\bar{c}} F(c)\, dc.
\end{aligned}
$$

Now, by construction, $\alpha_L - c' = K(2 + \gamma n)$, and so, we get that:

$$
\begin{gathered}
2(\alpha_H - c_H) - \gamma(\alpha_L - \bar{c}) > K\{2(n+1) + 2\gamma F(c') - \gamma^2 n[1 - F(c')]\} - \gamma K(2 + \gamma n)F(c') + \textstyle\int_{c'}^{\bar{c}} F(c)\, dc \\
\iff 2(\alpha_H - c_H) - \gamma(\alpha_L - \bar{c}) > K\{2(n+1) - \gamma^2 n\} + \textstyle\int_{c'}^{\bar{c}} F(c)\, dc \\
\iff K + \frac{\int_{c'}^{\bar{c}} F(c)\, dc}{2(n+1) - \gamma^2 n} < \frac{2(\alpha_H - c_H) - \gamma(\alpha_L - \bar{c})}{2(n+1) - \gamma^2 n},
\end{gathered}
$$

and so, by condition **(a)**, we get that

$$
\begin{aligned}
& \frac{2(\alpha_H - c_H) - \gamma(\alpha_L - \bar{c})}{2(n+1) - \gamma^2 n} - \frac{\alpha_L - \bar{c}}{2 + \gamma n} && > 0 \\
\implies \quad & \frac{4(\alpha_H - c_H) + 2\gamma n(\alpha_H - c_H) - 2\gamma(\alpha_L - \bar{c}) - \gamma^2 n(\alpha_L - \bar{c}) - 2(n+1)(\alpha_L - \bar{c}) + \gamma^2 n(\alpha_L - \bar{c})}{(2(n+1) - \gamma^2 n)(2 + \gamma n)} && > 0 \\
\implies \quad & \frac{2[(2 + \gamma n)(\alpha_H - c_H) - (\alpha_L - \bar{c})(n + 1 + \gamma)]}{(2(n+1) - \gamma^2 n)(2 + \gamma n)} && > 0.
\end{aligned}
$$

Since, $(\alpha_H - c_H) < (\alpha_L - \bar{c})$ by our regularity conditions **(C1)** and **(C2)**, this inequality implies that $2 + \gamma n > n + 1 + \gamma \iff \gamma(n-1) > n - 1$, which implies that $\gamma > 1$, which is a contradiction.
$\square$

Finally, we come to the most *typical* possibility in terms of empirical observation, where the quantity market cap binds for some types, and not for other types of low-cost firms, while not binding on any small firms.

Theorem 2 *For any $K > 0$, let $c^*(K)$ be such that:*

$$
K = \frac{(n+1)(\alpha_L - \hat{c}^*(K)) - \gamma n(\alpha_H - c_H) - \frac{\gamma^2 n}{2}\int_{\hat{c}^*(K)}^{\bar{c}} (c - c^*(K)) f(c) dc}{2(n+1) - \gamma^2 n}.
$$

Then for any $K > 0$, in equilibrium, the market caps binds on the low-cost firm with positive probability less than 1, while not binding on at least one small firm, if and only if

$$
K \in \left(\max\left\{ \frac{(n+1)(\alpha_L - \bar{c}) - \gamma n(\alpha_H - c_H)}{2(n+1) - \gamma^2 n}, \frac{2(\alpha_H - c_H) - \gamma\alpha_L(1 - F(c^*(K))) + \gamma\theta(K)}{2(n+1) - \gamma^2 n(1 - F(c^*(K))) + \gamma F(c^*(K))} \right\},\ \bar{Q}_L(\underline{c}) \right).
$$

The unique equilibrium $\langle \tilde{Q}_L(.), \{\tilde{q}_H^j\}_{j \in N} \rangle$ in this case satisfies the following:

- $\tilde{Q}_L(c) = \begin{cases} K & \textit{for } c \in [\underline{c}, c^*(K)] \\ \frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 n K F(c^*(K)) - \frac{\gamma^2 n \theta(K)}{2}}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]} - \frac{c}{2} & \textit{for } c \in (c^*(K), \bar{c}] \end{cases}$
- $\tilde{q}_H^j = \frac{2(\alpha_H - c_H) - 2\gamma K F(c^*(K)) - \gamma\alpha_L[1 - F(c^*(K))] + \gamma\theta(K)}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]}$ *for all $j \in N$,*

where $\theta(K) := \int_{\hat{c}^*(K)}^{\bar{c}} cf(c)dc$.

Proof:

Necessity: Fix any K, and any equilibrium $\langle \tilde{Q}_L(.), \{\tilde{q}_H^j\}_{j\in N}\rangle$ such that the $\tilde{q}_H^i < K$ for some $i \in N$, and there exists a strict measurable subset of types S of low-cost firm such that for all $c \in S$, $\tilde{Q}_L(c) = K$. Since all the small firms are identical, we can infer that $\tilde{q}_H^j < K$ for all $j \in N$. Further, by Lemma 1, there exists a $c^*(K) \in [\underline{c}, \bar{c}]$ such that;

$$\tilde{Q}_L(c) = \begin{cases} K & \text{for } c \in [\underline{c}, c^*(K)] \\ \tilde{Q}_L(c) & \text{for } c \in (c^*(K), \bar{c}], \end{cases}$$

and so, any low-cost large firm type having cost $c \in [c^*(K), \bar{c}]$ must maximize its profit by solving following problem;

$$\max_{Q_L > 0} \Pi_L(Q_L, \{\tilde{q}_H^i\}_{i=1}^n; c),$$

where for any $Q_L > 0$ and any $c \in [\underline{c}, \bar{c}]$, $\Pi_L(Q_L, \{\tilde{q}_H^i\}_{i=1}^n; c) := \left(\alpha_L - Q_L - \gamma \sum_{i=1}^n \tilde{q}_H^i - c\right) Q_L$. The first-order necessary condition (F.O.N.C) for this problem is $\frac{\partial \Pi_L}{\partial Q_L} = \alpha_L - 2Q_L - \gamma \sum_{i=1}^n \tilde{q}_H^i - c = 0$,[18] which gives the following reaction function for the low-cost large firm of type c;

$$\tilde{Q}_L(c) = \frac{\alpha_L - \gamma \sum_{i=1}^n \tilde{q}_H^i - c}{2}. \tag{17}$$

Therefore, any small firm i must maximize its profit by solving the following problem,

$$\max_{q_H^i > 0} \Pi_H^i(q_H^i, \tilde{Q}_L(c), \{\tilde{q}_H^j\}_{i\neq j}, c),$$

where for any $q_H^i > 0$,

$$\begin{aligned}
&\Pi_H^i(q_H^i, \tilde{Q}_L(c), \{\tilde{q}_H^j\}_{i\neq j}, c) := \int_{\underline{c}}^{\bar{c}} q_H^i \left(\alpha_H - q_H^i - \sum_{i\neq j}^{n-1} \tilde{q}_H^j - \gamma \tilde{Q}_L(c) - c_H\right) f(c)\, dc \\
= &\int_{\underline{c}}^{c^*(K)} q_H^i \left(\alpha_H - q_H^i - \sum_{i\neq j}^{n-1} \tilde{q}_H^j - \gamma K - c_H\right) f(c)\, dc + \int_{c^*(K)}^{\bar{c}} q_H^i \left(\alpha_H - q_H^i - \sum_{i\neq j}^{n-1} \tilde{q}_H^j - \gamma \tilde{Q}_L(c) - c_H\right) f(c) dc \\
= &\; q_H^i \left(\alpha_H - q_H^i - \sum_{i\neq j}^{n-1} \tilde{q}_H^j - \gamma K - c_H\right) \int_{\underline{c}}^{c^*(K)} f(c)\, dc + \int_{c^*(K)}^{\bar{c}} q_H^i \left(\alpha_H - q_H^i - \sum_{i\neq j}^{n-1} \tilde{q}_H^j - \gamma \tilde{Q}_L(c) - c_H\right) f(c) dc \\
= &\; q_H^i \left(\alpha_H - q_H^i - \sum_{i\neq j}^{n-1} \tilde{q}_H^j - \gamma K - c_H\right) F(c^*(K)) + \int_{c^*(K)}^{\bar{c}} q_H^i \left(\alpha_H - q_H^i - \sum_{i\neq j}^{n-1} \tilde{q}_H^j - \gamma \tilde{Q}_L(c) - c_H\right) f(c) dc.
\end{aligned}$$

[18] The second order sufficiency condition is satisfied : $\frac{\partial^2 \Pi_L}{\partial Q_L^2} = -2 < 0$.

The first-order necessary condition (F.O.N.C) for this problem is as follows.[19]

$$
\begin{aligned}
&\frac{\partial \Pi_H^i}{\partial q_H^i} = \left(\alpha_H - 2q_H^i - \sum_{j\neq i}^{n-1} \tilde{q}_H^j - \gamma K - c_H\right) F(c^*(K)) \;+ \int_{c^*(K)}^{\bar{c}} \left(\alpha_H - 2q_H^i - \sum_{j\neq i}^{n-1} \tilde{q}_H^j - \gamma \tilde{Q}_L(c) - c_H\right) f(c)\,dc = 0 \\
\Longleftrightarrow\quad & \left(\alpha_H - 2q_H^i - \sum_{j\neq i}^{n-1} \tilde{q}_H^j - \gamma K - c_H\right) F(c^*(K)) \;+ \left(\alpha_H - 2q_H^i - \sum_{j\neq i}^{n-1} \tilde{q}_H^j - c_H\right) \int_{c^*(K)}^{\bar{c}} f(c)\,dc \;- \gamma \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc = 0 \\
\Longleftrightarrow\quad & \left(\alpha_H - 2q_H^i - \sum_{j\neq i}^{n-1} \tilde{q}_H^j - \gamma K - c_H\right) F(c^*(K)) \;+ \left(\alpha_H - 2q_H^i - \sum_{j\neq i}^{n-1} \tilde{q}_H^j - c_H\right) [1 - F(c^*(K))] \;- \gamma \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc = 0,
\end{aligned}
$$

and so we get the following reaction function,

$$
\tilde{q}_H^i = \alpha_H - \sum_{j\in n} \tilde{q}_H^j - c_H - \gamma \left\{ KF(c^*(K)) + \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc \right\}. \tag{18}
$$

Summing (18) for all i, we get;

$$
\begin{aligned}
& \sum_{j\in n} \tilde{q}_H^j = n\alpha_H - n \sum_{j\in n} \tilde{q}_H^j - nc_H - n\gamma \left\{ KF(c^*(K)) + \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc \right\} \\
\Longleftrightarrow\quad & \sum_{j\in n} \tilde{q}_H^j = \frac{n}{n+1} \left[\alpha_H - c_H - \gamma \left\{ KF(c^*(K)) + \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc \right\} \right], \qquad (19)
\end{aligned}
$$

and so, by substituting (19) into (18), we get that;

$$
\begin{aligned}
\tilde{q}_H^i &= (\alpha_H - c_H) - \frac{n}{n+1} \left[\alpha_H - c_H - \gamma \left\{ KF(c^*(K)) + \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc \right\} \right] - \gamma \left\{ KF(c^*(K)) + \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc \right\} \\
&= \frac{1}{n+1} \left[(\alpha_H - c_H) - \gamma \left\{ KF(c^*(K)) + \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc \right\} \right]. \qquad (20)
\end{aligned}
$$

Thus, (20) represents the reaction function of any small firm to the strategy of the low-cost firm types. So, by substituting (20) into the reaction function of any type c of low-cost firm, that is (17); we can infer that,

$$
2\tilde{Q}_L(c) = (\alpha_L - c) - \frac{\gamma n}{n+1} \left[(\alpha_H - c_H) - \gamma \left\{ KF(c^*(K)) + \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc \right\} \right].
$$

Since, by construction, $\tilde{Q}_L(.)$ is measurable, we get that

$$
\begin{aligned}
& 2\int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc = \int_{c^*(K)}^{\bar{c}} \left([\alpha_L - c] - \frac{\gamma n}{n+1} \left[(\alpha_H - c_H) - \gamma \left\{ KF(c^*(K)) + \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc \right\} \right] \right) f(c)\,dc \\
\Longleftrightarrow\quad & 2\int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc = \left[\alpha_L - \frac{\gamma n}{n+1} \left\{ (\alpha_H - c_H) - \gamma \left(KF(c^*(K)) + \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc \right) \right\} \right] \int_{c^*(K)}^{\bar{c}} f(c)\,dc \;- \int_{c^*(K)}^{\bar{c}} cf(c)\,dc \\
\Longleftrightarrow\quad & 2\int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc = \left[\alpha_L - \frac{\gamma n(\alpha_H - c_H)}{n+1} + \frac{\gamma^2 nKF(c^*(K))}{n+1} \right] \{1 - F(c^*(K))\} + \left[\frac{\gamma^2 n}{n+1} \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc \right] \{1 - F(c^*)\} - \int_{c^*(K)}^{\bar{c}} cf(c)\,dc \\
\Longleftrightarrow\quad & \int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c)\,dc = \frac{[\alpha_L(n+1) - \gamma n(\alpha_H - c_H) + \gamma^2 nKF(c^*(K))][1 - F(c^*(K))] - \theta(K)(n+1)}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]} \qquad (21)
\end{aligned}
$$

[19] The second order sufficiency condition is satisfied : $\frac{\partial^2 \Pi_H^i}{\partial Q_L^2} = -2 < 0$.

where, $\theta(K) := \int_{c^*(K)}^{\bar{c}} cf(c)\,dc$. Substituting (21) into (20), we get the equilibrium output that must be produced by any small firm i;

$$\tilde{q}_H^i = \frac{2(\alpha_H - c_H) - 2\gamma K F(c^*(K)) - \gamma\alpha_L[1 - F(c^*(K))] + \gamma\theta(K)}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]}.$$

By supposition, $\tilde{q}_H^i < K$ which implies that

$$(\mathbf{a}) \;\; \frac{2(\alpha_H - c_H) - \gamma\alpha_L[1 - F(c^*(K))] + \gamma\theta(K)}{2(n+1) - \gamma^2 n[1 - F(c^*(K))] + 2\gamma F(c^*(K))} < K.$$

Further, substituting the value of $\tilde{q}_H^i$ into (17), we get the equilibrium output produced by the large low-cost firm of type $c \in (c^*(K), \bar{c}]$;

$$(\mathbf{b})\; \tilde{Q}_L(c) = \frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 n K F(c^*(K)) - \frac{\gamma^2 n\theta(K)}{2}}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]} - \frac{c}{2} < K,$$

and, as argued in proof of Proposition 5, for any type of low-cost firm of type $c \in [\underline{c}, c^*(K)]$, by construction,

$$(\mathbf{c}) \;\; \left[\frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 n K F(c^*(K)) - \frac{\gamma^2 n\theta(K)}{2}}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]} - \frac{c}{2}\right] \geq K.$$

Together, condition **(b)** and **(c)** imply that:

$$\begin{aligned}
& K = \frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 n K F(c^*(K)) - \frac{\gamma^2 n\theta(K)}{2}}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]} - \frac{c^*(K)}{2} \\
\iff \;& (n+1)\alpha_L - \gamma n(\alpha_H - c_H) - K\{2(n+1) - \gamma^2 n\} = (n+1)c^*(K) + \frac{\gamma^2 n}{2}[\theta(K) - c^*(K)\{1 - F(c^*(K))\}] \\
\iff \;& K = \frac{(n+1)(\alpha_L - c^*) - \gamma n(\alpha_H - c_H) - \frac{\gamma^2 n}{2}\int_{c^*}^{\bar{c}}(c - c^*(K))f(c)dc}{2(n+1) - \gamma^2 n},
\end{aligned}$$

Note that the function for any $x \in [\underline{c}, \bar{c}]$, $g(x) := \frac{(n+1)(\alpha_L - x) - \gamma n(\alpha_H - c_H) - \frac{\gamma^2 n}{2}\int_x^{\bar{c}}(c-x)f(c)dc}{2(n+1) - \gamma^2 n}$ is differentiable. This is because $\int_x^{\bar{c}}(c - x)f(c)dc = -\int_{\bar{c}}^{x} cf(c)dc - x\int_x^{\bar{c}} f(c)dc = -\int_{\bar{c}}^{x} cf(c)dc - x[1 - F(x)]$, and so, by fundamental theorem of calculus, $\frac{d[\int_x^{\bar{c}}(c-x)f(c)dc]}{dx}$ exists and is equal to $-xf(x) - [1 - F(x)] + xf(x) = -[1 - F(x)]$. Thus, for any $x \in (\underline{c}, \bar{c})$, $g'(x) = \frac{-(n+1) + \frac{\gamma^2 n}{2}[1 - F(x)]}{2(n+1) - \gamma^2 n} = \frac{-1 - n\left(1 - \frac{\gamma^2}{2}[1 - F(x)]\right)}{2(n+1) - \gamma^2 n} < 0$. Since, by construction $c^*(K) \in (\underline{c}, \bar{c})$, we can infer that $K \in (g(\bar{c}), g(\underline{c}))$. Further, as $g(\underline{c}) = \bar{Q}_L(\underline{c})$, we get that,

$$(\mathbf{d}) \;\; K \in \left(\frac{(n+1)(\alpha_L - \bar{c}) - \gamma n(\alpha_H - c_H)}{2(n+1) - \gamma^2 n}, \bar{Q}_L(\underline{c})\right).$$

Hence, the proof of necessity follows from the conditions **(a)** and **(d)**.

Sufficiency: To establish sufficiency, we need to show that $\langle \tilde{Q}_L(.), \{\tilde{q}_H^j\}_{j\in N}\rangle$ constitutes an equilibrium if

$$K \in \left(\max\left\{ \frac{(n+1)(\alpha_L - \bar{c}) - \gamma n(\alpha_H - c_H)}{2(n+1) - \gamma^2 n}, \frac{2(\alpha_H - c_H) - \gamma\alpha_L(1 - F(c^*(K))) + \gamma\theta(K)}{2(n+1) - \gamma^2 n(1 - F(c^*(K))) + 2\gamma F(c^*(K))} \right\},\ \bar{Q}_L(\underline{c}) \right).$$

As noted in the proof of necessity, such a value of K implies that $c^*(K) \in (\underline{c}, \bar{c})$. Further, as shown in the proof of necessity - where for any c, $\tilde{Q}_L(c)$ was obtained after substituting the value of $\tilde{q}_H^i$ in the reaction function (17) - the best response of any low-cost firm of type c to every small firm playing $\tilde{q}_H^i$: is greater than K if $c \leq c^*(K)$, or else is less than or equal to K. Finally, note that

$$\begin{aligned}
\int_{c^*(K)}^{\bar{c}} \tilde{Q}_L(c) f(c) dc &= \left\{ \frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 nKF(c^*(K)) - \frac{\gamma^2 n\theta(K)}{2}}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]} \right\} [1 - F(c^*(K))] - \frac{\int_{c^*(K)}^{\bar{c}} cf(c)dc}{2} \\
&= \left\{ \frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 nKF(c^*(K))}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]} \right\} [1 - F(c^*(K))] \\
&\quad - \frac{\gamma^2 n\theta(K)[1 - F(c^*(K))]}{2\{2(n+1) - \gamma^2 n[1 - F(c^*(K))]\}} - \frac{\theta(K)}{2} \\
&= \left\{ \frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 nKF(c^*(K))}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]} \right\} [1 - F(c^*(K))] \\
&\quad - \frac{\gamma^2 n\theta(K)[1 - F(c^*(K))] + \{2(n+1) - \gamma^2 n[1 - F(c^*(K))]\}\theta(K)}{2\{2(n+1) - \gamma^2 n[1 - F(c^*(K))]\}} \\
&= \left\{ \frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 nKF(c^*(K))}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]} \right\} [1 - F(c^*(K))] - \frac{(n+1)\theta(K)}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]} \\
&= \frac{[\alpha_L(n+1) - \gamma n(\alpha_H - c_H) + \gamma^2 nKF(c^*(K))][1 - F(c^*(K))] - \theta(K)(n+1)}{2(n+1) - \gamma^2 n[1 - F(c^*(K))]}.
\end{aligned}$$

Note that the right hand side of the equation above is same as that of (21) in proof of necessity, and so, we can infer from the proof of necessity that the best response of any small firm i to the low-cost firm types playing $\tilde{Q}_L(.)$ is to play $\tilde{q}_H^i$. Thus, the proof of sufficiency follows. ☐

4.3 A distributional assumption

It is easy to see that equilibrium expected consumer surplus depends on the market cap, whenever it binds. However, as can be seen from Theorem 2, it is very difficult to compute changes in expected surplus using the most general form of the prior distributional assumption over the marginal cost of the low-cost large firm. Hence, in this subsection, we assume a uniform distribution so that $F(x) = x/(\bar{c} - \underline{c})$ for any $x \in [\bar{c}, \underline{c}]$, and then present results on how the expected consumer surplus varies with respect to level of market cap.[20] While these results would be valid to only this specific case, a careful estimation of this distribution can easily inform market regulation policy using Theorem 2. To solve this further, we will make a distributional assumption for the marginal cost of the large firm. Let, the marginal cost $c \in (\underline{c}, \bar{c})$ be uniformly distributed. So, for any $c \in (\underline{c}, \bar{c})$:

$$F(c) = \frac{c}{(\bar{c} - \underline{c})}; \quad f(c) = \frac{1}{(\bar{c} - \underline{c})}.$$

[20]With this distributional assumption, we get that $\int_{c^*(K)}^{\bar{c}} \frac{c}{(\bar{c}-\underline{c})}\, dc = \frac{\bar{c}^2 - c^*(K)^2}{2(\bar{c}-\underline{c})}$.

The following result shows that if there is sufficient technological advantage of large firms over small firms, then there cannot be an equilibrium where the market cap binds with non-degenerate probability.

Proposition 6 *Suppose that the marginal cost of large firm is distributed uniformly over* $[\bar{c}, \underline{c}]$ *and* $\bar{c} > 2\underline{c}$*. For any* $K > 0$*, either the market cap* K *binds for all types of large firms or else it does not bind for any type.*

Proof: Fix any $K > 0$, and suppose that there exists an equilibrium where the market cap binds on large firm with non-degenerate probability. By Proposition 5, and Theorem 2, it follows from our supposition that there must exist an equilibrium $\langle \tilde{Q}_L(.), \{\tilde{q}_H^j\}_{j\in N}\rangle$ such that the $\tilde{q}_H^i < K$ for all $i \in N$, and there exists a $c^*(K) \in [\underline{c}, \bar{c}]$ such that;

$$\tilde{Q}_L(c) = \begin{cases} K & \text{for } c \in [\underline{c}, c^*(K)] \\ \frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 nKF(c^*(K)) - \frac{\gamma^2 n\theta(K)}{2}}{2(n+1) - \gamma^2 n[1-F(c^*(K))]} - \frac{c}{2} & \text{for } c \in (c^*(K), \bar{c}]. \end{cases}$$

Further, we know from Theorem 2 that $\tilde{Q}_L(c^*(K)) = K$. Thus, uniform distribution of c over $[\underline{c}, \bar{c}]$ implies that:

$$\begin{aligned}
& \frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 nKF(c^*(K)) - \frac{\gamma^2 n\theta(K)}{2}}{2(n+1) - \gamma^2 n[1-F(c^*(K))]} - \frac{c^*(K)}{2} &&= K \\
\implies & \frac{(n+1)\alpha_L - \gamma n(\alpha_H - c_H) + \gamma^2 nKF(c^*(K)) - \frac{\gamma^2 n\theta(K)}{2} - K\{2(n+1) - \gamma^2 n[1-F(c^*(K))]\}}{2(n+1) - \gamma^2 n[1-F(c^*(K))]} &&= \frac{c^*(K)}{2} \\
\implies & (n+1)\alpha_L - \gamma n(\alpha_H - c_H) - \frac{\gamma^2 n\theta(K)}{2} - K\{2(n+1) - \gamma^2 n\} &&= \frac{c^*(K)\{2(n+1) - \gamma^2 n[1-F(c^*(K))]\}}{2} \\
\implies & (n+1)\alpha_L - \gamma n(\alpha_H - c_H) - K\{2(n+1) - \gamma^2 n\} &&= \frac{c^*(K)\{2(n+1) - \gamma^2 n[1-F(c^*(K))]\}}{2} + \frac{\gamma^2 n\theta(K)}{2} \\
\implies & (n+1)\alpha_L - \gamma n(\alpha_H - c_H) - K\{2(n+1) - \gamma^2 n\} &&= (n+1)c^*(K) + \frac{\gamma^2 n}{2}\{\theta(K) - c^*(K)[1-F(c^*(K))]\}.
\end{aligned}$$

We know, $\theta(K) := \int_{c^*(K)}^{\bar{c}} cf(c)\,dc$, and so, by our distributional assumption, $\theta(K) = \frac{\bar{c}^2 - c^*(K)^2}{2(\bar{c} - \underline{c})}$. Substituting this in the equation above we get;

$$\begin{aligned}
(n+1)\alpha_L - \gamma n(\alpha_H - c_H) - K\{2(n+1) - \gamma^2 n\} &= (n+1)c^*(K) + \frac{\gamma^2 n}{2}\left\{\frac{\bar{c}^2 - c^*(K)^2}{2(\bar{c} - \underline{c})} - \frac{c^*(K)(\bar{c} - c^*(K))}{\bar{c} - \underline{c}}\right\} \\
\implies (n+1)\alpha_L - \gamma n(\alpha_H - c_H) - K\{2(n+1) - \gamma^2 n\} &= (n+1)c^*(K) + \frac{\gamma^2 n}{2}\left\{\frac{\bar{c}^2 + c^*(K)^2 - 2\bar{c}c^*(K)}{2(\bar{c} - \underline{c})}\right\} \\
\implies (n+1)\alpha_L - \gamma n(\alpha_H - c_H) - K\{2(n+1) - \gamma^2 n\} &= (n+1)c^*(K) + \frac{\gamma^2 n\bar{c}^2}{4(\bar{c} - \underline{c})} + \frac{\gamma^2 nc^*(K)^2}{4(\bar{c} - \underline{c})} - \frac{\gamma^2 n\bar{c}c^*(K)}{2(\bar{c} - \underline{c})} \\
\implies (n+1)\alpha_L - \gamma n(\alpha_H - c_H) - K\{2(n+1) - \gamma^2 n\} &= \frac{\gamma^2 n\bar{c}^2}{4(\bar{c} - \underline{c})} + \frac{\gamma^2 nc^*(K)^2}{4(\bar{c} - \underline{c})} - c^*(K)\left\{(n+1) - \frac{\gamma^2 n\bar{c}}{2(\bar{c} - \underline{c})}\right\},
\end{aligned}$$

we get the following quadratic equation;

$$c^*(K)^2\left(\frac{\gamma^2 n}{4(\bar{c} - \underline{c})}\right) + c^*(K)\left\{(n+1) - \frac{\gamma^2 n\bar{c}}{2(\bar{c} - \underline{c})}\right\} + \frac{\gamma^2 n\bar{c}^2}{4(\bar{c} - \underline{c})} - (n+1)\alpha_L + \gamma n(\alpha_H - c_H) + K\{2(n+1) - \gamma^2 n\} = 0.$$

Simplifying the quadratic equation obtained above further, we get that:

$$c^*(K)^2\gamma^2 n+c^*(K)\left\{4(n+1)(\bar{c}-\underline{c})-2\gamma^2 n\bar{c}\right\}+\gamma^2 n\bar{c}^2-4(n+1)\alpha_L(\bar{c}-\underline{c})+4\gamma n(\alpha_H-c_H)(\bar{c}-\underline{c})+4K(\bar{c}-\underline{c})\{2(n+1)-\gamma^2 n\}=0,$$

which implies that

$$c^*(K)=\frac{2\gamma^2 n\bar{c}-4(n+1)(\bar{c}-\underline{c})}{2\gamma^2 n} \pm\frac{\sqrt{\{4(n+1)(\bar{c}-\underline{c})-2\gamma^2 n\bar{c}\}^2-4\gamma^2 n\left[\gamma^2 n\bar{c}^2-4(\bar{c}-\underline{c})\{(n+1)\alpha_L-\gamma n(\alpha_H-c_H)-K\{2(n+1)-\gamma^2 n\}\}\right]}}{2\gamma^2 n} \tag{22}$$

Now, observe that for all $n\geq 1$, $\bar{c}>2\underline{c}\implies 2\gamma^2 n\bar{c}-4(n+1)(\bar{c}-\underline{c})<0$, and so, to get a positive root, we need the discriminant of this quadratic equation to be strictly positive.[21] Let us denote this discriminant by Δ. Therefore,

$$\begin{aligned}\Delta &:= 16(\bar{c}-\underline{c})^2(n+1)^2+4\gamma^4 n^2\bar{c}^2-16(\bar{c}-\underline{c})(n+1)\gamma^2 n\bar{c}-4\gamma^4 n^2\bar{c}^2\\ &\quad +16\gamma^2 n(n+1)\alpha_L(\bar{c}-\underline{c})-16\gamma^3 n^2(\alpha_H-c_H)(\bar{c}-\underline{c})-16\gamma^2 nK(\bar{c}-\underline{c})\{2(n+1)-\gamma^2 n\}\\ &= 16(\bar{c}-\underline{c})^2(n+1)^2-16(\bar{c}-\underline{c})(n+1)\gamma^2 n\bar{c}+16\gamma^2 n(n+1)\alpha_L(\bar{c}-\underline{c})\\ &\quad -16\gamma^3 n^2(\alpha_H-c_H)(\bar{c}-\underline{c})-16\gamma^2 nK(\bar{c}-\underline{c})\{2(n+1)-\gamma^2 n\}.\end{aligned} \tag{23}$$

To simplify this expression further, define $\delta:=\bar{c}-\underline{c}$. Thus from (23), we get that:

$$\Delta=16\{\delta^2(n+1)^2-\delta(n+1)\gamma^2 n\bar{c}+\gamma^2 n(n+1)\alpha_L\delta-\gamma^3 n^2(\alpha_H-c_H)\delta-\gamma^2 nK\delta\{2(n+1)-\gamma^2 n\}\}.$$

Now, to obtain a positive root of (22), we require that:

$$\begin{aligned}&\Delta > \{2\gamma^2 n\bar{c}-4(n+1)(\bar{c}-\underline{c})\}^2=\{2\gamma^2 n\bar{c}-4(n+1)\delta\}^2\\ \implies\ &16\{\delta^2(n+1)^2-\delta(n+1)\gamma^2 n\bar{c}+\gamma^2 n(n+1)\alpha_L\delta-\gamma^3 n^2(\alpha_H-c_H)\delta-\gamma^2 nK\delta\{2(n+1)-\gamma^2 n\}\}\\ &> 16\left\{\frac{\gamma^4 n^2\bar{c}^2}{4}+\delta^2(n+1)^2-\gamma^2 n\delta\bar{c}(n+1)\right\}\\ \implies\ &\gamma^2 n(n+1)\alpha_L\delta-\gamma^3 n^2(\alpha_H-c_H)\delta-\gamma^2 nK\delta\{2(n+1)-\gamma^2 n\}>\frac{\gamma^4 n^2\bar{c}^2}{4}\\ \implies\ &(n+1)\alpha_L\delta>\gamma n(\alpha_H-c_H)\delta+K\delta\{2(n+1)-\gamma^2 n\}+\frac{\gamma^2 n\bar{c}^2}{4}.\end{aligned} \tag{24}$$

Note that a market cap value K can never bind if $K\geq\alpha_L$ (that is, if the cap is greater than the maximum possible demand for the low-cost firm, then it will always exceed any equilibrium output

[21] Note we cannot consider a negative root since then $c^*(K)<0$ which contradicts $\underline{c}\geq 0$.

by any type of low-cost firm). Hence, we can infer that $K < \alpha_L$, and so, from (24), we get that:

$$
\begin{aligned}
& (n+1)\alpha_L\delta > \gamma n(\alpha_H - c_H)\delta + \alpha_L\delta\{2(n+1) - \gamma^2 n\} + \frac{\gamma^2 n\bar{c}^2}{4} \\
\iff\ & \alpha_L\delta\{(n+1) - 2(n+1) + \gamma^2 n\} > \gamma n(\alpha_H - c_H)\delta + \frac{\gamma^2 n\bar{c}^2}{4} \\
\iff\ & \alpha_L\delta\{n(\gamma^2 - 1) - 1\} > \gamma n(\alpha_H - c_H)\delta + \frac{\gamma^2 n}{4}. \qquad (25)
\end{aligned}
$$

Now, observe that the expression $n(\gamma^2-1)-1 < 0\ \forall n \in \mathbb{N}$ because $\gamma \in (0,1)$. Since by construction $\delta, \gamma, n, \alpha_L > 0$ and by regularity condition **(C2)**, $\alpha_H > c_H$; we can infer that (25) is a contradiction. And so, we can infer that there can be no equilibrium where the market cap with non-degenerate probability marginal cost $c \in (\underline{c}, \bar{c})$ is uniformly distributed and $\bar{c} > 2\underline{c}$. □

Proposition 6 above establishes that when there is sufficient gap between the technologies of the large and the small firms (that is, $\bar{c} > \underline{c}$), and the types of large firms are distributed uniformly - there can be only two types of equilibrium where market cap binds. One possible type of such equilibrium is one where market cap does not bind on any type of large firm, and so, by Proposition 2, the market cap does not bind on any small firm. Thus, this equilibrium is same as the equilibrium described in Theorem 1 where the market cap does not matter to any firm of any type.

The other possible type of equilibrium emanating from Proposition 6, is one where market cap binds on all types of large firm. By Propositions 3 and 4, there can be only two possible equilibria of this nature: (i) where market cap binds on all small firms, and (ii) where market does not bind on any small firm. Thus, we can infer that whenever large firm types are distributed uniformly, and $\bar{c} > 2\underline{c}$; the equilibrium $\langle \hat{\hat{Q}}_L(.), \{\hat{\hat{q}}^i_H\}_{i\in N}\rangle$,where a market cap K binds must be such that $\hat{\hat{Q}}_L(c) = K, \forall\, c$, and for all $i \in N$,

$$
\hat{\hat{q}}^i_H = \begin{cases} K & \text{if } K \in \left(0, \frac{\alpha_H - c_H}{n+1+\gamma}\right] \\ \frac{\alpha_H - \gamma K - c_H}{n+1} < K & \text{if } K \in \left(\frac{\alpha_H - c_H}{n+1+\gamma}, \frac{(\alpha_L - \bar{c})(n+1) - \gamma n(\alpha_H - c_H)}{2(n+1) - \gamma^2 n}\right]. \end{cases}
$$

Note that under our additional distributional assumptions, if the market cap binds, then it binds for all types. This means that whenever a market cap K binds, the equilibrium output of the large firm is K irrespective of its private marginal cost. This means we can predict the equilibrium outputs and price without any uncertainty, and so, also specify the exact equilibrium consumer surplus in each case.

In the corollary below, we present the different *exact* consumer surplus values corresponding to the two possible equilibria (Propositions 3 and 4) where a market cap K binds.

Corollary 3 *Let $\bar{c} > 2\underline{c}$ and the types of large firm are distributed uniformly over $[\underline{c}, \bar{c}]$. For any*

$K > 0$, *if the market cap binds then the exact consumer surplus is:*

$$\hat{\hat{CS}}(K) = \begin{cases} \frac{K^2(1+n^2+2\gamma n)}{2} & \textit{if } K \in \left(0, \frac{\alpha_H - c_H}{n+1+\gamma}\right] \\ \frac{(n+1)(n+1-2n\gamma^2)K^2+n^2(\alpha_H-c_H-\gamma K)^2+2n(n+1)(\alpha_H-c_H)\gamma k}{2(n+1)^2} & \textit{if } K \in \left(\frac{\alpha_H-c_H}{n+1+\gamma}, \frac{(n+1)(\alpha_L-\bar{c})-\gamma n(\alpha_H-c_H)}{2(n+1)-\gamma^2 n}\right]. \end{cases}$$

Proof: Fix any $K \in \left(0, \frac{\alpha_H-c_H}{n+1+\gamma}\right]$. By Proposition 3, we know that equilibrium production of each type of large firm, as well as each small firm is K. This means that there is no uncertainty in terms of output produced by the large firm. Hence, by (2) and (3), we can predict the *sure* prices of produce of the large firm $p_L^* = \alpha_L - (1+\gamma n)K$ and the produce of any small firm i, ${p_H^*}^i = \alpha_H - (n+\gamma)K$. By (1), the expected consumer surplus is:

$$\begin{aligned} \hat{\hat{CS}}(K) &= \alpha_L Q_L^* + n\alpha_H {q_H^*}^i - \frac{1}{2}\{{Q_L^*}^2 + (n{q_H^*}^i)^2 + 2\gamma n Q_L^* {q_H^*}^i\} - p_L^* Q_L^* - n{p_H^*}^i {q_H^*}^i \\ &= \alpha_L K + \alpha_H nK - \frac{1}{2}\left\{K^2 + n^2K^2 + 2\gamma nK^2\right\} - K\{\alpha_L - (1+\gamma n)K\} - nK\{\alpha_H - (n+\gamma)K\} \\ &= \alpha_L K + \alpha_H nK - \frac{K^2}{2}\left\{1+n^2+2\gamma n\right\} - \alpha_L K + (1+\gamma n)K^2 - \alpha_H nK + n(n+\gamma)K^2 \\ &= -\frac{K^2}{2} - \frac{n^2K^2}{2} - \gamma nK^2 + K^2 + \gamma nK^2 + n^2K^2 + \gamma nK^2 \\ &= \frac{K^2}{2}\left\{1+n^2+2\gamma n\right\}. \end{aligned} \tag{26}$$

Similarly, fix any $K \in \left(\frac{\alpha_H-c_H}{n+1+\gamma}, \frac{(n+1)(\alpha_L-\bar{c})-\gamma n(\alpha_H-c_H)}{2(n+1)-\gamma^2 n}\right]$. By Proposition 4, $\hat{Q}_L(c) = K, \forall\, c \in [\underline{c}, \bar{c}]$, and $\hat{q}_H^i = \frac{\alpha_H - c_H - \gamma K}{n+1}, \forall\, i \in N$. Again, by (2) and (3), the price of the produce of the large firm is $\hat{p}_L(K) = \frac{(n+1)\alpha_L - K(n+1-\gamma^2 n) - \gamma n(\alpha_H - c_H)}{n+1}$, and any small firm i is $\hat{p}_H^i(K) = \frac{\alpha_H + nc_H - \gamma K}{n+1}$. Therefore, by Proposition 4, the consumer surplus in this case turns out to be;

$$
\begin{aligned}
\hat{\hat{CS}}(K) &= \alpha_L \hat{Q}_L + n\alpha_H \hat{q}_H^i - \frac{1}{2}\{\hat{Q}_L^2 + (n\hat{q}_H^i)^2 + 2\gamma n \hat{Q}_L \hat{q}_H^i\} - \hat{p}_L \hat{Q}_L - n\hat{p}_H^i \hat{q}_H^i \\
&= \hat{Q}_L(\alpha_L - \hat{p}_L) + n(\alpha_H - \hat{p}_H^i)\hat{q}_H^i - \frac{1}{2}\{\hat{Q}_L^2 + (n\hat{q}_H^i)^2 + 2\gamma n \hat{Q}_L \hat{q}_H^i\} \\
&= K\left\{\alpha_L - \frac{(n+1)\alpha_L - K(n+1-\gamma^2 n) - \gamma n(\alpha_H - c_H)}{n+1}\right\} + n\left\{\alpha_H - \frac{\alpha_H + nc_H - \gamma K}{n+1}\right\}\left\{\frac{\alpha_H - c_H - \gamma K}{n+1}\right\} \\
&\quad - \frac{K^2}{2} - \frac{n^2\{\alpha_H - c_H - \gamma K\}^2}{2(n+1)^2} - \frac{\gamma n K\{\alpha_H - c_H - \gamma K\}}{(n+1)} \\
&= \frac{K\{K(n+1-\gamma^2 n) + \gamma n(\alpha_H - c_H)\}}{n+1} + \frac{n\{n\alpha_H - nc_H + \gamma K\}\{\alpha_H - c_H - \gamma K\}}{(n+1)^2} - \frac{K^2}{2} - \frac{n^2\{\alpha_H - c_H - \gamma K\}^2}{2(n+1)^2} \\
&\quad - \frac{\gamma n K\{\alpha_H - c_H - \gamma K\}}{(n+1)} \\
&= \frac{1}{2(n+1)^2}\Big[2(n+1)K\{K(n+1-\gamma^2 n) + \gamma n(\alpha_H - c_H)\} + 2n(n\alpha_H - nc_H + \gamma K)(\alpha_H - c_H - \gamma K) - (n+1)^2 K \\
&\qquad - n^2(\alpha_H - c_H - \gamma K)^2 - 2\gamma n(n+1)K(\alpha_H - c_H - \gamma K)\Big] \\
&= \frac{1}{2(n+1)^2}\Big[2(n+1)(n+1-\gamma^2 n)K^2 + 2\gamma n(n+1)(\alpha_H - c_H)K + 2n(\alpha_H - c_H - \gamma K)(n\alpha_H - nc_H + \gamma K - n\gamma K - \gamma K) \\
&\qquad - (n+1)^2 K - n^2(\alpha_H - c_H - \gamma K)^2\Big] \\
&= \frac{(n+1)K^2\{2(n+1) - 2\gamma^2 - (n+1)\} + 2\gamma n(n+1)(\alpha_H - c_H)K + 2n^2(\alpha_H - c_H - \gamma K)^2 - n^2(\alpha_H - c_H - \gamma K)^2}{2(n+1)^2} \\
&= \frac{(n+1)(n+1-2n\gamma^2)K^2 + n^2(\alpha_H - c_H - \gamma K)^2 + 2n(n+1)(\alpha_H - c_H)\gamma k}{2(n+1)^2}.
\end{aligned} \tag{27}
$$

Thus, the result follows from (26) and (27). □

In the Corollary 3, we have obtained the expression of consumer surplus as a function of market cap (K) for the two possible equilibria under uniform distribution, under the assumption that them market cap binds. Now, we show below that: if market penetration of the large firm is high enough, and the degree of quality differentiation among the products is high enough, then the best regulation is no-regulation.

Theorem 3 *Let $\bar{c} > 2\underline{c}$ and the types of large firm are distributed uniformly over $[\underline{c}, \bar{c}]$.*

(A) *The consumer surplus when the market cap binds, $\hat{\hat{CS}}(.)$ is continuous, and has a global maximum at $K^*(\gamma) := \frac{(n+1)(\alpha_L - \bar{c}) - \gamma n(\alpha_H - c_H)}{2(n+1) - \gamma^2 n}$.*

(B) *There exists a $\gamma^* \in (0, 1)$, such that for all $\gamma \in \left(0, \min\left\{\gamma^*, \frac{2(\alpha_H - c_H)}{\alpha_L - \underline{c}}\right\}\right)$,*

$$\hat{\hat{CS}}(K^*) \leq \mathbb{E}(\bar{CS}).$$

Proof: We present proof of each assertion below separately.

Proof of (A). To check how consumer surplus changes due to change in market cap, we note that both the consumer surplus functions obtained in Corollary 3 are increasing in K. That is, for all $K \in$

$\left(0, \frac{\alpha_H - c_H}{n+1+\gamma}\right]$, $\frac{\delta \hat{\hat{CS}}(K)}{\delta K} = K(1+n^2+2\gamma n) > 0$. Further, for all $K \in \left(\frac{\alpha_H - c_H}{n+1+\gamma}, \frac{(n+1)(\alpha_L - \bar{c}) - \gamma n(\alpha_H - c_H)}{2(n+1)-\gamma^2 n}\right]$, $\frac{\delta \hat{\hat{CS}}}{\delta K} = \frac{K\{1+(1-\gamma^2)(n^2+2n)\}+\gamma n(\alpha_H - c_H)}{(n+1)^2} > 0$. Thus, the $\hat{\hat{CS}}(.)$ function is rising over the region $\left(0, \frac{(n+1)(\alpha_L - \bar{c}) - \gamma n(\alpha_H - c_H)}{2(n+1)-\gamma^2 n}\right]$. In addition, as shown in subsection 6.1 of Appendix, $\hat{\hat{CS}}\left(\frac{\alpha_H - c_H}{n+1+\gamma}\right) = \lim_{K \to \frac{\alpha_H - c_H}{n+1+\gamma}+} \hat{\hat{CS}}(K)$, which implies that $\hat{\hat{CS}}(.)$ is continuous. Thus the $\hat{\hat{CS}}(.)$ function gets maximized at $K^*(\gamma) := \frac{(n+1)(\alpha_L - \bar{c}) - \gamma n(\alpha_H - c_H)}{2(n+1)-\gamma^2 n}$.

Proof of (B). The maximum consumer surplus at $K^*(\gamma)$ is shown in subsection 6.2 of Appendix to be the following:

$$\hat{\hat{CS}}(K^*(\gamma)) = \frac{n^2(4-3\gamma^2)(\alpha_H - c_H)^2 + \{1+(1-\gamma^2)(n^2+2n)\}(\alpha_L - \bar{c})^2 + 2\gamma n(1+\gamma^2 n - n)(\alpha_L - \bar{c})(\alpha_H - c_H)}{2[2(n+1)-\gamma^2 n]^2}. \tag{28}$$

Let Δ be the difference between the *no*-regulation expected consumer surplus $\mathbb{E}(\bar{CS})$, and the maximum possible consumer surplus with regulatory cap, $\hat{\hat{CS}}(K^*(\gamma))$. That is, $\Delta(\gamma) := \mathbb{E}(\bar{CS}) - \hat{\hat{CS}}(K^*(\gamma))$, which is a well-defined rational function of γ. Hence, $\Delta(.)$ is continuous in γ, and we show in subsection 6.3 of Appendix that:

$$\lim_{\gamma \to 0} \Delta(\gamma) > 0.$$

Therefore, we can infer that there exists a small enough positive $\gamma^* \in (0,1)$, such that for all $\gamma \in \left(0, \min\left\{\gamma^*, \frac{2(\alpha_H - c_H)}{\alpha_L - \underline{c}}\right\}\right)$, $\Delta(\gamma) > 0$ implying that no-regulation expected consumer welfare is greater than the maximum possible consumer welfare with regulation.

□

Remark 1 Thus, we find that under the assumption of uniform distribution, if the extent of private information about (the marginal cost of) the large firm is substantial enough (so that $\bar{c} > 2\underline{c}$); then the best policy for a market regulator who wants to protect consumer interests is to *not* impose any market cap. That is, *no regulation is the best regulation in this scenario.* Of course, one cannot preclude the possibility that may arise for other distributional assumptions where market cap is enhance consumer welfare.

5 Conclusion

This paper studies the impact of quantity regulation on a large firm when its constant marginal cost is not common knowledge in a market where many other smaller, but strategic firms operate. The smaller firms sell an identical product, but this product is differentiated with respect to the product sold by the large firm. We consider a quantity competition setting under the assumption that the large firm does not want to get labeled as a '*dominant firm*' for fear of loss of operational autonomy, and so, would voluntarily abide by any market cap imposed by a market regulator.[22] We

[22]We do not consider the case of price competition, because it would yield a trivial equilibrium where the small firms sell at the common constant marginal cost (that is, $p_H^* = c_H$), thereby making the private information about

characterize the unique Bayes-Nash equilibria with and without quantity regulation and present the equilibrium consumer surplus as a function of the market cap imposed (in the former case).

Finally, we present a special case where the private information about the large firm is substantial enough. We show that under uniform distribution, consumer welfare is maximized when no market cap is imposed. That is, in this case, no regulation is the best regulation.

6 Appendix

6.1 Proof of $\hat{\hat{CS}}\left(\frac{\alpha_H - c_H}{n+1+\gamma}\right) = \lim_{K \to \tilde{K}+} \hat{\hat{CS}}(K)$, where $\tilde{K} := \frac{\alpha_H - c_H}{n+1+\gamma}$

By Corollary 1, we have $\hat{\hat{CS}}(K) = \frac{(n+1)(n+1-2n\gamma^2)K^2 + n^2(\alpha_H - c_H - \gamma K)^2 + 2n(n+1)(\alpha_H - c_H)\gamma k}{2(n+1)^2}$ for any $K > \frac{\alpha_H - c_H}{n+1+\gamma}$. Therefore,

$$
\begin{aligned}
\lim_{K \to \tilde{K}+} \hat{\hat{CS}}(\tilde{K}) &= \frac{(n+1)(n+1-2n\gamma^2)\left\{\frac{\alpha_H - c_H}{n+1+\gamma}\right\}^2 + n^2\left\{(\alpha_H - c_H) - \gamma\left(\frac{\alpha_H - c_H}{n+1+\gamma}\right)\right\}^2 + 2\gamma n(n+1)(\alpha_H - c_H)\left(\frac{\alpha_H - c_H}{n+1+\gamma}\right)}{2(n+1)^2} \\
&= \frac{1}{2(n+1)^2}\left\{\frac{(n+1)(n+1-2n\gamma^2)(\alpha_H - c_H)^2}{(n+1+\gamma)^2} + \frac{n^2\{(\alpha_H - c_H)(n+1+\gamma) - \gamma(\alpha_H - c_H)\}^2}{(n+1+\gamma)^2} + \frac{2\gamma n(n+1)(\alpha_H - c_H)^2}{n+1+\gamma}\right\} \\
&= \frac{1}{2(n+1)^2}\left\{\frac{(n+1)(n+1-2n\gamma^2)(\alpha_H - c_H)^2}{(n+1+\gamma)^2} + \frac{n^2\{(\alpha_H - c_H)(n+1)\}^2}{(n+1+\gamma)^2} + \frac{2\gamma n(n+1)(\alpha_H - c_H)^2}{n+1+\gamma}\right\} \\
&= \frac{(n+1)(\alpha_H - c_H)^2\left\{(n+1-2n\gamma^2) + n^2(n+1) + 2\gamma n(n+1+\gamma)\right\}}{2(n+1)^2(n+1+\gamma)^2} \\
&= \frac{(\alpha_H - c_H)^2\left\{n+1-2n\gamma^2 + n^2(n+1) + 2\gamma n^2 + 2\gamma n + 2\gamma^2 n\right\}}{2(n+1)(n+1+\gamma)^2} \\
&= \frac{(\alpha_H - c_H)^2\left\{(n+1) + n^2(n+1) + 2\gamma n(n+1)\right\}}{2(n+1)(n+1+\gamma)^2} \\
&= \frac{(\alpha_H - c_H)^2\left\{(n+1)(1+n^2+2\gamma n)\right\}}{2(n+1)(n+1+\gamma)^2} \\
&= \frac{(\alpha_H - c_H)^2(1+n^2+2\gamma n)}{2(n+1+\gamma)^2}.
\end{aligned}
$$

Note that from (26), we know $\hat{\hat{CS}}(K) = \frac{K^2(1+n^2+2\gamma n)}{2}$ $\forall\ K \in \left(0, \frac{\alpha_H - c_H}{n+1+\gamma}\right]$. Substituting the value of $\tilde{K}$, we get $\hat{\hat{CS}}(\tilde{K}) = \frac{(\alpha_H - c_H)^2(1+n^2+2\gamma n)}{2(n+1+\gamma)^2}$. □

the marginal cost of large firm redundant.

6.2 Proof of $\hat{\hat{CS}}(K^*) = \frac{n^2(4-3\gamma^2)(\alpha_H - c_H)^2 + \{1+(1-\gamma^2)(n^2+2n)\}(\alpha_L-\bar{c})^2 + 2\gamma n(1+\gamma^2 n - n)(\alpha_L-\bar{c})(\alpha_H-c_H)}{2[2(n+1)-\gamma^2 n]^2}$.

Substituting the value of $K^* = \frac{(n+1)(\alpha_L-\bar{c})-\gamma n(\alpha_H-c_H)}{2(n+1)-\gamma^2 n}$ in (27), we get

$$
\begin{aligned}
\hat{\hat{CS}}(K^*) = \frac{1}{2(n+1)^2}\Bigg[&(n+1)(n+1-2n\gamma^2)\left\{\frac{(n+1)(\alpha_L-\bar{c})-\gamma n(\alpha_H-c_H)}{2(n+1)-\gamma^2 n}\right\}^2 \\
&+ n^2\left\{(\alpha_H-c_H) - \frac{\gamma\{(n+1)(\alpha_L-\bar{c})-\gamma n(\alpha_H-c_H)\}}{2(n+1)-\gamma^2 n}\right\}^2 \\
&+ 2\gamma n(n+1)(\alpha_H-c_H)\left\{\frac{(n+1)(\alpha_L-\bar{c})-\gamma n(\alpha_H-c_H)}{2(n+1)-\gamma^2 n}\right\}\Bigg] \\
= \frac{1}{2(n+1)^2}\Bigg[&\frac{(n+1)(n+1-2n\gamma^2)\{(n+1)^2(\alpha_L-\bar{c})^2+\gamma^2 n^2(\alpha_H-c_H)^2-2\gamma n(n+1)(\alpha_L-\bar{c})(\alpha_H-c_H)\}}{\{2(n+1)-\gamma^2 n\}^2} \\
&+ n^2\left[\frac{\{2(n+1)-\gamma^2 n\}(\alpha_H-c_H)-\gamma(n+1)(\alpha_L-\bar{c})+\gamma^2 n(\alpha_H-c_H)}{2(n+1)-\gamma^2 n}\right]^2 \\
&+ \frac{2\gamma n(n+1)^2(\alpha_H-c_H)(\alpha_L-\bar{c})-2\gamma^2 n^2(n+1)(\alpha_H-c_H)^2}{2(n+1)-\gamma^2 n}\Bigg] \\
= \frac{1}{2(n+1)^2}\Bigg[&\frac{(n+1)^3(n+1-2n\gamma^2)(\alpha_L-\bar{c})^2+\gamma^2 n^2(n+1)(n+1-2n\gamma^2)(\alpha_H-c_H)^2}{\{2(n+1)-\gamma^2 n\}^2} \\
&- \frac{2\gamma n(n+1)^2(n+1-2n\gamma^2)(\alpha_L-\bar{c})(\alpha_H-c_H)}{\{2(n+1)-\gamma^2 n\}^2} \\
&+ n^2\left\{\frac{4(n+1)^2(\alpha_H-c_H)^2+\gamma^2(n+1)^2(\alpha_L-\bar{c})^2-4\gamma(n+1)^2(\alpha_L-\bar{c})(\alpha_H-c_H)}{\{2(n+1)-\gamma^2 n\}^2}\right\} \\
&+ \frac{2\gamma n(n+1)^2(\alpha_H-c_H)(\alpha_L-\bar{c})-2\gamma^2 n^2(n+1)(\alpha_H-c_H)^2}{2(n+1)-\gamma^2 n}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{2(n+1)^2\{2(n+1)-\gamma^2 n\}^2}\Big[&(n+1)^3(n+1-2n\gamma^2)(\alpha_L-\bar{c})^2+\gamma^2 n^2(n+1)(n+1-2n\gamma^2)(\alpha_H-c_H)^2 \\
&- 2\gamma n(n+1)^2(n+1-2n\gamma^2)(\alpha_L-\bar{c})(\alpha_H-c_H)+4n^2(n+1)^2(\alpha_H-c_H)^2 \\
&+ \gamma^2 n^2(n+1)^2(\alpha_L-\bar{c})^2-4\gamma n^2(n+1)^2(\alpha_L-\bar{c})(\alpha_H-c_H) \\
&+ \{2(n+1)-\gamma^2 n\}\{2\gamma n(n+1)^2(\alpha_H-c_H)(\alpha_L-\bar{c})-2\gamma^2 n^2(n+1)(\alpha_H-c_H)^2\}\Big] \\
= \frac{1}{2(n+1)^2\{2(n+1)-\gamma^2 n\}^2}\Big[&(n+1)^2(\alpha_L-\bar{c})^2\{(n+1)(n+1-2\gamma^2 n)+\gamma^2 n^2\} \\
&+ (n+1)(\alpha_H-c_H)^2[\gamma^2 n^2(n+1-2\gamma^2 n)+4n^2(n+1)-2\gamma^2 n^2\{2(n+1)-\gamma^2 n] \\
&+ (n+1)^2(\alpha_L-\bar{c})(\alpha_H-c_H)[-2\gamma n(n+1-2\gamma^2 n)+2\gamma n\{2(n+1)-\gamma^2 n\}-4\gamma n^2]\Big] \\
= \frac{1}{2(n+1)^2\{2(n+1)-\gamma^2 n\}^2}\Big[&(n+1)^2(\alpha_L-\bar{c})^2\{1+(1-\gamma^2)n^2+2n(1-\gamma^2)\} \\
&+ (n+1)(\alpha_H-c_H)^2[\gamma^2 n^2\{n+1-2\gamma^2-4(n+1)+2\gamma^2\}+4n^2(n+1)] \\
&+ (n+1)^2(\alpha_L-\bar{c})(\alpha_H-c_H)[2\gamma n\{-(n+1)+2\gamma^2 n+2(n+1)-\gamma^2 n\}-4\gamma n^2]\Big] \\
= \frac{1}{2(n+1)^2\{2(n+1)-\gamma^2 n\}^2}\Big[&(n+1)^2(\alpha_L-\bar{c})^2\{1+(1-\gamma^2)(n^2+2n)\}+n^2(n+1)^2(\alpha_H-c_H)^2(4-3\gamma^2) \\
&+ (n+1)^2(\alpha_L-\bar{c})(\alpha_H-c_H)\{2\gamma n(1+\gamma^2 n-n)\}\Big] \\
= \frac{n^2(4-3\gamma^2)(\alpha_H-c_H)^2+\{1+(1-\gamma^2)(n^2+2n)\}(\alpha_L-\bar{c})^2+2\gamma n(1+\gamma^2 n-n)(\alpha_L-\bar{c})(\alpha_H-c_H)}{2[2(n+1)-\gamma^2 n]^2}&.
\end{aligned}
$$

Hence, the result follows. $\square$

6.3 Proof of $\lim\limits_{\gamma \to 0} \delta(\gamma) > 0$

.

Let $S := \frac{\bar{c}+\underline{c}}{2}$. Now, solving the expected consumer surplus from (15) using the uniform distribution assumption, we get

$$\mathbb{E}(\bar{CS}) = R_1R_2 + nR_3R_4 - \gamma n R_3 R_2 - \frac{R_2^2}{2} - \frac{n^2}{2}R_3^2 - \frac{SR_1}{2} + \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24}.$$

Substituting the values of R1, R2, R3, and R4, we get that $\Delta(\gamma) = \mathbb{E}(\bar{CS}) - \hat{\hat{CS}}(K^*(\gamma))$ is as follows:

$$\begin{aligned}
\Delta = {} & \frac{4(n+1)(n+1-\gamma^2 n)\alpha_L^2 + 4\gamma^3 n^2(\alpha_L - S)(\alpha_L - \alpha_H) + 2\gamma^4 n^2 \alpha_L S - 4\gamma^2 n^2(\alpha_H - c_H)^2 - \gamma^4 n^2 S^2}{4[2(n+1) - \gamma^2 n]^2} + \frac{4n^2(2-\gamma^2)(\alpha_H - c_H)^2}{2[2(n+1)-\gamma^2 n]^2} \\
& + \frac{4\gamma n(\alpha_H - c_H)\alpha_L + 2\gamma n^2(2-\gamma^2)(\alpha_H - c_H)S - 2\gamma^2 n(\alpha_L - S)\alpha_L - 2\gamma n^2(2-\gamma^2)(\alpha_L - S)(\alpha_H - c_H) - \gamma^2 n^2(2-\gamma^2)(\alpha_L - S)S}{2[2(n+1)-\gamma^2 n]^2} \\
& - \frac{4\gamma n(n+1)(\alpha_H - c_H)\alpha_L - 4\gamma^2 n^2(\alpha_H - c_H)^2 - 2\gamma^3 n^2(\alpha_H - c_H)S - 2\gamma^2 n(n+1)(\alpha_L - S)\alpha_L + 2\gamma^3 n^2(\alpha_H - c_H)(\alpha_L - S) + \gamma^4 n^2(\alpha_L - S)S}{2[2(n+1)-\gamma^2 n]^2} \\
& - \frac{4(n+1)^2\alpha_L^2 + 4\gamma^2 n^2(\alpha_H - c_H)^2 + \gamma^4 n^2 S^2 - 8\gamma n(n+1)(\alpha_H - c_H)\alpha_L + 4\gamma^3 n^2(\alpha_H - c_H)S - 4\gamma^2 n(n+1)\alpha_L S}{8[2(n+1)-\gamma^2 n]^2} \\
& - \frac{4n^2(\alpha_H - c_H^2) + \gamma^2 n^2(\alpha_L - S)^2 - 4\gamma n^2(\alpha_L - S)(\alpha_H - c_H)}{2[2(n+1)-\gamma^2 n]^2} - \frac{2(n+1-\gamma^2 n)\alpha_L S + 2\gamma n(\alpha_H - c_H)S + \gamma^2 n S^2}{4[2(n+1)-\gamma^2 n]} + \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} \\
& - \frac{n^2(4-3\gamma^2)(\alpha_H - c_H)^2 + \{1 + (1-\gamma^2)(n^2+2n)\}(\alpha_L - \bar{c})^2 + 2\gamma n(1+\gamma^2 n - n)(\alpha_L - \bar{c})(\alpha_H - c_H)}{2[2(n+1)-\gamma^2 n]^2} \\
= {} & \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{1}{8[2(n+1)-\gamma^2 n]^2}\Big[8(n+1)(n+1-\gamma^2 n)\alpha_L^2 + 8\gamma^3 n^2(\alpha_L - S)(\alpha_L - \alpha_H) + 4\gamma^4 n^2 \alpha_L S - 8\gamma^2 n^2(\alpha_H - c_H)^2 - 2\gamma^4 n^2 S^2 \\
& + 16n^2(2-\gamma^2)(\alpha_H - c_H)^2 + 16\gamma n(\alpha_H - c_H)\alpha_L + 8\gamma n^2(2-\gamma^2)(\alpha_H - c_H)S - 8\gamma^2 n(\alpha_L - S)\alpha_L \\
& - 8\gamma n^2(2-\gamma^2)(\alpha_L - S)(\alpha_H - c_H) - 4\gamma^2 n^2(2-\gamma^2)(\alpha_L - S)S - 4\gamma n(n+1)(\alpha_H - c_H)\alpha_L + 16\gamma^2 n^2(\alpha_H - c_H)^2 \\
& + 8\gamma^3 n^2(\alpha_H - c_H)S + 8\gamma^2 n(n+1)(\alpha_L - S)\alpha_L - 8\gamma^3 n^2(\alpha_H - c_H)(\alpha_L - S) - 4\gamma^4 n^2(\alpha_L - S)S - 4(n+1)^2\alpha_L^2 \\
& - 4\gamma^2 n^2(\alpha_H - c_H)^2 - \gamma^4 n^2 S^2 + 8\gamma n(n+1)(\alpha_H - c_H)\alpha_L - 4\gamma^3 n^2(\alpha_H - c_H)S + 4\gamma^2 n(n+1)\alpha_L S \\
& - 16n^2(\alpha_H - c_H)^2 - 4\gamma^2 n^2(\alpha_L - S)^2 + 16\gamma n^2(\alpha_L - S)(\alpha_H - c_H) - 4(n+1-\gamma^2 n)(2n+2-\gamma^2 n)\alpha_L S \\
& - 4\gamma n(2n+2-\gamma^2 n)(\alpha_H - c_H)S - 2(2n+2-\gamma^2 n)\gamma^2 n S^2 - 4n^2(4-3\gamma^2)(\alpha_H - c_H)^2 \\
& - 8\gamma n(1+\gamma^2 n - n)(\alpha_H - c_H)\alpha_L + 8\gamma n(1+\gamma^2 n - n)(\alpha_H - c_H)\bar{c} - 4\{1 + (1-\gamma^2)(n^2+2n)\}\alpha_L^2 \\
& - 4\{1 + (1-\gamma^2)(n^2+2n)\}\bar{c}^2 + 8\{1 + (1-\gamma^2)(n^2+2n)\}\alpha_L \bar{c}\Big] \\
= {} & \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{1}{8[2(n+1)-\gamma^2 n]^2}\Big[4\alpha_L^2\{2(n+1)(n+1-\gamma^2 n) - (n+1)^2 - (1 + n^2 + 2n - \gamma^2 n^2 - 2\gamma^2 n)\} + 4n^2(\alpha_H - c_H)^2\{-2\gamma^2 + 4(2-\gamma^2) \\
& + 4\gamma^2 - \gamma^2 - 4 + 3\gamma^2\} + 8\gamma n^2(\alpha_L - S)(\alpha_H - c_H)\{\gamma^2 - (2-\gamma^2) - \gamma^2 + 2\} + 8\gamma n(\alpha_H - c_H)\alpha_L \\
& \{2 - 2(n+1) + (n+1) - (1 + \gamma^2 n - n)\} + 4\gamma n(\alpha_H - c_H)S\{2n(2-\gamma^2) + 2\gamma^2 n - \gamma^2 n - (2n+2-\gamma^2 n)\} - \gamma^2 n S^2 \\
& \{2\gamma^2 n + \gamma^2 n + 4n + 4 - 2\gamma^2 n\} + 4\gamma^2 n(\alpha_L - S)\{-2\alpha_L - n(2-\gamma^{)}S + 2(n+1)\alpha_L - \gamma^2 n S - n(\alpha_L - S)\} \\
& + 4\alpha_L S\{\gamma^4 n^2 + \gamma^2(n+1) - (2n+2-\gamma^n)(n+1-\gamma^2 n)\} + 8\gamma n(1+\gamma^2 n - n)(\alpha_H - c_H)\bar{c} \\
& - 4\{1 + (1-\gamma^2)(n^2+2n)\}\bar{c}^2 + 8\{1 + (1-\gamma^2)(n^2+2n)\}\alpha_L \bar{c}\Big] \\
= {} & \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{1}{8[2(n+1)-\gamma^2 n]^2}\Big[-4\gamma^2 n^2 \alpha_L^2 + 8\gamma^3 n^2(\alpha_L - S)(\alpha_H - c_H) - 8\gamma^3 n^2(\alpha_H - c_H)\alpha_L + 8\gamma n(n-1)(\alpha_H - c_H)S \\
& + 4\gamma^2 n^2(\alpha_L^2 + S^2 - 2\alpha_L S) - 8\alpha_L S\{1 + n^2 + 2n - 2\gamma^2 n^2 - 2\gamma^2 n\} - \gamma^2 n\{4(n+1) + \gamma^2 n\}S^2 \\
& + 8\gamma n(1+\gamma^2 n - n)(\alpha_H - c_H)\bar{c} - 4\{1 + (1-\gamma^2)(n^2+2n)\}\bar{c}^2 + 8\{1 + (1-\gamma^2)(n^2+2n)\}\alpha_L \bar{c}\Big] \\
= {} & \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{1}{8[2(n+1)-\gamma^2 n]^2}\Big[-8\gamma^3 n^2(\alpha_H - c_H)S + 8\gamma n(n-1)(\alpha_H - c_H)S + 4\gamma^2 n^2 S^2 - 8\gamma^2 n^2 \alpha_L S - 8\alpha_L S\{1 + n^2 + 2n - 2\gamma^2 n^2 - 2\gamma^2 n\} \\
& - \gamma^2 n\{4(n+1) + \gamma^2 n\}S^2 + 8\gamma n(1+\gamma^2 n - n)(\alpha_H - c_H)\bar{c} - 4\{1 + (1-\gamma^2)(n^2+2n)\}\bar{c}^2 \\
& 8\{1 + (1-\gamma^2)(n^2+2n)\}\alpha_L \bar{c}\Big]
\end{aligned}$$

$$= \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{1}{8[2(n+1) - \gamma^2 n]^2}\Big[- 8\gamma n S(\alpha_H - c_H)(1 + \gamma^2 n - n) + \gamma^2 n S^2(4n - 4n - 4 - \gamma^2 n) - 8\alpha_L S(1 + n^2 + 2n - 2\gamma^2 n^2 - \gamma^2 n + \gamma^2 n^2)$$
$$8\gamma n(1 + \gamma^2 n - n)(\alpha_H - c_H)\bar{c} - 4\{1 + (1-\gamma^2)(n^2 + 2n)\}\bar{c}^2 + 8\{1 + (1-\gamma^2)(n^2+2n)\}\alpha_L \bar{c}\Big]$$
$$= \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{1}{8[2(n+1) - \gamma^2 n]^2}\Big[8\gamma n(1 + \gamma^2 n - n)(\alpha_H - c_H)(\bar{c} - S) - \gamma^2 n S^2(4 + \gamma^2 n) + 8\{1 + (1-\gamma^2)(n^2+2n)\}(\bar{c} - S)\alpha_L$$
$$- 4\{1 + (1-\gamma^2)(n^2+2n)\}\bar{c}^2\Big]$$
$$= \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{1}{8[2(n+1) - \gamma^2 n]^2}\Big[4\{1 + (1-\gamma^2)(n^2+2n)\}\{2(\bar{c} - S)\alpha_L - \bar{c}^2\} - 4\gamma n\{n(1-\gamma^2) - 1\}\{2(\bar{c} - S)(\alpha_H - c_H)\} - \gamma^2 n S^2(4 + \gamma^2 n)\Big]$$
$$= \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{1}{8[2(n+1) - \gamma^2 n]^2}\Big[4n\{(1-\gamma^2)(n+2)\}\{2(\bar{c} - S)\alpha_L - \bar{c}^2\} - 4\gamma n\{n(1-\gamma^2) - 1\}\{2(\bar{c} - S)(\alpha_H - c_H)\}$$
$$+ 4\{2(\bar{c} - S)\alpha_L - \bar{c}^2\} - \gamma^2 n S^2(4 + \gamma^2 n)\Big]$$

Therefore, we get that:

$$\begin{aligned}
\lim_{\gamma \to 0} \Delta(\gamma) &= \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{4\{1 + n(n+2)\}\{2(\bar{c} - S)\alpha_L - \bar{c}^2\}}{8[2(n+1)]^2} \\
&= \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{\{1 + n(n+2)\}\{(\bar{c} - \underline{c})\alpha_L - \bar{c}^2\}}{8(n+1)^2} \\
&> \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} + \frac{(n+1)^2\{(\bar{c} - \underline{c})\bar{c} - \bar{c}^2\}}{8(n+1)^2} \qquad \because \alpha_L > \bar{c} \text{ (by regularity condition)} \\
&= \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c}}{24} - \frac{\bar{c}\underline{c}}{8} \\
&= \frac{\bar{c}^2 + \underline{c}^2 + \bar{c}\underline{c} - 3\bar{c}\underline{c}}{24} \\
&= \frac{(\bar{c} - \underline{c})^2}{24} \\
&> 0.
\end{aligned}$$

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