# State Dependent Sharing and Efficiency in Teams

Shasikanta Nandeibam[1]
and
Lucy O'Shea

Department of Economics
University of Bath
Bath, BA2 7AY
United Kingdom

ABSTRACT

Most models in the existing literature on moral hazard in teams do not capture the kind of team production settings that we often observe where the random element affecting the final team output can also be observed *ex-post*. In such situations it is natural to allow the team's sharing rule to depend on the observed realisations of both the final output and the random element. So we examine a team problem in which the share of each team member is a function of the observed realisations of the final output and the random element. We provide a necessary and sufficient condition for implementing an outcome in Nash equilibrium. This condition imposes restrictions on those deviations from the outcome that could have been caused unilaterally by each and every member of the team. Using this characterization we also derive a necessary and sufficient condition for implementing an efficient outcome. When the production function has a separable structure, namely, the output can be written as a function of the random element and a composite action: (i) we show that efficient outcomes cannot be implemented if everyone has quasi-linear utility functions; (ii) when the quasi-linearity assumption is dropped, we present some examples in which there are implementable efficient outcomes and also derive some conditions for implementing efficient outcomes. When the production function does not have a separable structure, we present an example which shows that efficient outcomes may be implementable even when all individuals have quasi-linear utility functions and also provide a necessary and sufficient condition for locally implementing efficient outcomes. Finally, when validity of the first-order approach and nondecreasing share functions are required for implementing any given outcome, we show that it is without loss of generality to consider only the class of sharing rules that are linear in the final output and efficient outcomes cannot be implemented in this case.

Key words: team, outcome, sharing rule, implementation, efficient outcome.

*JEL* Classification: D82, D2, C72, J54.

[1]Corresponding author. Email: ecssn@bath.ac.uk

## 1. INTRODUCTION

Moral hazard in teams can be observed within a variety of situations such as partnerships, labour-managed firms, share cropping arrangements, non-point source pollution, etc. In all of these situations an aggregate measure, which can be output, profits or ambient pollution is the only observable and verifiable indicator of inputs or emission levels. Moral hazard in teams arises because while the total welfare of the team would be higher if all team members exerted high levels of effort, there is an incentive for each member to exert less effort because such effort is costly. Since individual effort levels cannot be observed (at reasonable cost) but only the final output, identification and subsequent punishment of the shirkers is not possible. The scenario just described can apply equally well to the non-point source pollution context in which polluters as a group must share among themselves the cost of environmental damage due to the ambient level of pollution, where individual emissions are unobservable but aggregate or ambient pollution level can be observed through monitoring of the receptor. The presence of uncertainty in the relationship between inputs/emissions and output/pollution could further compound the problem of moral hazard as individuals or firms can hide behind the veil of uncertainty concerning who was at fault (Holmstrom (1982)).

The literature on moral hazard in teams focuses on ways to design appropriate incentive mechanisms to mitigate the incentive to free-ride (e.g. Alchian and Demsetz (1972), Eswaran and Kotwal (1984), Holmstrom (1982), Legros and Matsushima (1991), Legros and Matthews (1993), Radner and Williams (1992), Rasmusen (1987)). The literature on non-point source pollution has focused on how to apply the theory of moral hazard in teams to the environmental context (e.g. Herriges *et al.* (1994), Segerson (1988), Strand (1999)).[2] Perfect monitoring of individual actions is usually impossible or prohibitively costly and instead, imperfect indicators of individual actions such as final output or ambient pollution are used as a basis for contracting. The contract may be improved upon if other factors which yield information about actions can be included within the information base (e.g. Nandeibam (2003) widened the information base of the sharing rule to include intermediate as well as final outputs).

In the traditional setting where uncertainty enters into the relationship between individual actions and output, it is implicitly assumed that it is not possible to disentangle the random component affecting how actions are converted into output even after the uncertainty is resolved, so a low output cannot be taken to imply inadequate actions. An example of this is Eswaran and Kotwal (1985) who suggest weather as a possible stochastic variable which can alter the relationship between inputs and agricultural output. In that case, a bad harvest could be due to adverse weather conditions and not lack of effort. Their analysis does not allow for the fact that adverse weather conditions are often observable *ex-post*, information which could be taken

[2]Both Segerson's and Strand's contributions allow for budget-breaking, thereby enabling the design of incentive mechanisms to induce polluters to act efficiently.

into account when designing contracts. Thus, we maintain that in certain situations it may be possible to disentangle the realisation of the random element *ex-post*, thereby allowing it to be observed. In the non-point source pollution context, ambient pollution is a function of individual emission levels and stochastic factors such as rainfall. Prior to emitting pollutants, future rainfall in unknown but it becomes known at the time of monitoring the ambient level of pollution. We can think of other contexts which share a similar characteristic as that just described. Consider a team whose final revenue/profit depends on the market conditions of its observable inputs and outputs (essentially their prices) which are uncertain at the time when the individual actions are chosen. Once the final outputs are generated and sold, the revenue/profit as well as the realised market conditions are observable. Eswaran and Kotwal (1985) provide examples of this within the agrarian context, where the choice of crops is dependent on a range of factors such as expected prices, water availability, etc. For example, in the context of share-cropping arrangements or labour-managed cooperatives in the agrarian sector, there are anecdotal evidence of the shares depending not only on the output but also on other factors such as monsoon conditions, occurence or otherwise of draught, favourable or not so favourable agricultural market conditions, etc. So, by acknowledging that there is a wide range of situations in which the realisation of the random element is observable, we depart from the standard analysis in which the sharing rule is based solely on the final joint output.

The model we consider closely resembles the non-deterministic team production model in Holmstrom (1982). However, unlike Holmstrom (1982), we assume that, after the resolution of the uncertainty, the realisation of the random element can be observed, and hence, each member's share of the final output could be made to depend on the observed final output itself and the realised value of the random element. In our setting, although the realisation of the random element is observable, because the actions are taken before the uncertainty is resolved, team members will choose their actions noncooperatively to maximize their respective expected utilities. In order to derive the expected utilities, team members have to consider outcomes from an *ex-ante* perspective. So an outcome is defined as a combination of the actions chosen by the team members and a distribution of the final output corresponding to each realisation of the random element among the team members. Thus, we are interested in studying outcomes that are implementable noncooperatively in the sense that it could be generated by a pure strategy Nash equilibrium of the production game conditional on some sharing rule.

Our first result provides a necessary and sufficient condition for implementing a given outcome. Roughly speaking, this condition imposes restrictions on final output levels that do not correspond to the combination of actions in the given outcome but could be generated by each and every member of the team deviating unilaterally from his/her given action. These restrictions allow sufficient punishments to be incorporated into the sharing rule to deter such unilateral deviations whilst at the same time ensuring that the team's budget is balanced at every realisation of the random element.

The second set of results are on the issue of achieving efficiency. In the general case, we show that restricting the necessary and sufficient condition for implementation to only the individual efforts below the efficient levels is necessary and sufficient for implementing an efficient outcome. In order to gain more insight, we then distinguish between two types of production functions, namely, those that have a weakly separable structure between the random factor and a composite action[3] and those that do not have the weakly separable structure. For weakly separable production functions, we first prove that efficiency cannot be achieved if everyone has quasi-linear utility function. However, when there are individuals with utility functions that are not quasi-linear, we present examples in which some efficient outcome can be implemented and derive a necessary and sufficient condition for implementing an efficient outcome. The intuition for these efficiency results seem to be similar to those in the existing literature, such as Legros and Matsushima (1991), Legros and Matthews (1993) and Rasmusen (1987), which suggests that the results in our framework could be seen as a generalization of the existing literature when the production functions are weakly separable. For production functions that are not weakly separable, we first present an example in which everyone has quasi-linear utility function and efficient outcomes are implementable. So the kind of intuitive argument used in Rasmusen (1987), which relies on risk aversion, is not applicable here. We also use the example to show that, as in the traditional literature, if the shares cannot be made contingent on the random factor, then efficient outcomes are not implementable. We also derive a necessary and sufficient condition for implementing an efficient outcome locally when everyone has quasi-linear utility function. Thus, in the case of production functions that are not weakly separable, our framework may add something new to what we already know from the existing literature, such as Legros and Matsushima (1991), Legros and Matthews (1993) and Rasmusen (1987).

For our final set of results, we consider the restricted case where the first-order approach is valid and the share functions are nondecreasing in a neighbourhood of the outcome considered for implementation. Under these restriction, we show that any implementable outcome could be achieved by adopting a sharing rule which is linear in the final output. This result could be contrasted with those of Kim and Wang (1998)[4] and Nandeibam (2002). In the former, Kim and Wang show that, when there is uncertainty which is unobservable *ex-post*, the linear sharing rule result holds only when the production function has the weakly separable structure and team members have quasi-linear utility functions. On the otherhand, Nandeibam shows that the linear sharing rule result is valid with fairly general utility functions and production technology when there is no uncertainty. As a consequence of our linear sharing rule result, we also show that efficient outcomes are not achievable if validity of the first-order approach is required and the

[3]This is the case often considered in the literature, e.g. Bhattacharyya and Lafontaine (1995), Eswaran and Kotwal (1985), Kim and Wang (1998), Romano (1994), etc.

[4]See also Bhattacharyya and Lafontaine (1995) and Romano (1994).

share functions have to be nondecreasing in a neighbourhood of the efficient outcome. The contrast between this result and some of the earlier positive efficiency results mentioned above hints at a possible trade off between the desire for some degree of simplicity in the sharing rule to be adopted on the one hand and the desire for achieving efficiency on the otherhand.

In the next section we describe our model and define the notion of implementability of an outcome. In the third section we present a necessary and sufficient condition for implementing a given outcome. In section 4 we consider the issue of implementability of efficient outcomes. In the fifth section we examine what happens if validity of the first-order approach is imposed and the share functions are required to be nondecreasing in a neighbourhood of the given outcome. We conclude in the final section.

## 2. THE MODEL

We consider a team comprising of $N \geq 2$ individuals in which each individual $i$'s unobservable and unverifiable action is denoted by $a_i \in [0, \xi]$, where we allow for the possibility of $\xi = \infty$ with a slight abuse of notation. The vector of actions $\mathbf{a} = (a_1, ..., a_N)$ of the $N$ individuals together with a random variable $\theta \in \Theta$ determine a joint monetary output according to the production function $f : [0, \xi]^N \times \Theta \to \Re_+$. We endow the state space $\Theta$ with a Borel field $\mathcal{F}$ and represent the distribution of the random variable $\theta$ by a probability measure $\mu$ on $\mathcal{F}$. Each individual $i$'s utility over money and action pairs is given by

$$U_i(s_i, a_i) = u_i(s_i) - c_i(a_i),$$

where $s_i \in [m, \infty)$ is $i$'s income.[5] Since an individual may be permitted to receive negative shares in certain situations, which in the extreme case could take the form of unlimited liability, by abusing notation, we allow for the possibility of $m = -\infty$. Throughout this paper we maintain the following assumptions:

**A1.** $f$ is continuously differentiable in the actions and $f_i$ denotes the derivative of $f$ with respect to the action of individual $i$; $f$ is strictly increasing in the actions whenever everyone's action is positive; $f$ is concave in the actions.

**A2.** $f$ is measurable with respect to the probability measure $\mu$.

**A3.** For each $i$, $u_i$ and $c_i$ are continuously differentiable; $u_i'$ and $c_i'$ respectively denote the derivatives of $u_i$ and $c_i$; $u_i$ and $c_i$ are strictly increasing; $u_i$ is concave and $c_i$ is convex.

Assumptions **A1** and **A3** are quite standard and need no further explanation. Assumption **A2** is a technical assumption.[6]

[5]The additively separable form we have adopted for the individual utility functions is purely for simplicity.

[6]Our framework could be adapted in a straightforward manner to the context of non-point source pollution

The realisation of the random variable takes place after the actions have been chosen. However, at the end of the production process, both the monetary output and the realised value of the random variable are observable.

In our framework, because the actions are chosen before the realisation of the random factor, at the time of taking their actions, each team member would take into consideration the prospective possible outcomes, i.e. outcomes from an *ex-ante* perspective. So an outcome would comprise of a single action vector and one share/payment vector for each realisation of the random variable which allocates amongst the team members the final output corresponding to the action vector and this random variable. However, because of the problem of enforcing sharing rules that do not distribute the entire final output within the team (see Eswaran and Kotwal (1984)), we need the shares to add up to the final output in each state of the world. Thus, an outcome is $(\mathbf{a}, (\mathbf{p}^\theta)_{\theta\in\Theta})$, where $\mathbf{p}^\theta = (p_1^\theta, ..., p_N^\theta) \in [m, \infty)^N$ for each $\theta$ is such that $\sum_{i=1}^N p_i^\theta = f(\mathbf{a}, \theta)$.[7]

Since actions are unobservable, each individual's share of the final output can only be a function of the observable final output and the observed realisation of the random variable. So a *share function* of individual $i$ is a function $s_i : \Re_+ \times \Theta \to [m, \infty)$, where $s_i(q, \theta)$ is the amount/share individual $i$ receives when $q$ is the final output and $\theta$ is the realisation of the random variable. An individual's share function can be seen as a menu of contracts in which each contract corresponds to a realisation of the random variable and specifies the relationship between the final output and the individual's share at that realisation of the random variable. Because of the reasons mentioned above, we will require the team to balance its budget at each realisation of the random variable. Thus, a *sharing rule* is a collection of $N$ share functions $\mathbf{s} = (s_1, ..., s_N)$ such that the following budget balancing condition holds:

$$\sum_{i=1}^N s_i(q, \theta) = q \quad \text{for all } (q, \theta).$$

Each sharing rule $\mathbf{s}$ in the class of admissible sharing rules we consider is such that, for each individual $i$ and for each action vector $\mathbf{a}$, $s_i(f(\mathbf{a}, .), .)$ is measurable with respect to the probability measure $\mu$.

Once the team adopts a sharing rule $\mathbf{s}$, the members of the team play a noncooperative game in choosing their actions in the production process. In this game conditional on $\mathbf{s}$, the payoff of each individual $i$ is given by

$$\int_\Theta u_i(s_i(f(\mathbf{a}, \theta), \theta))d\mu - c_i(a_i), \tag{1}$$

problem where the unobservable action chosen by each polluter is its individual emission level. By invoking the polluter pays principle, the monetary transfers/shares would refer to the fee that polluters are charged to cover the cost of environmental damages. The utility function for polluter $i$ could be written as $U^i(t_i, e_i) = B_i(e_i) - D_i(t_i)$, where $t_i$ and $e_i$ respectively refer to $i$'s transfer payment/share and emission level. In this case $f$ becomes the cost function of environmental damage caused by pollution. Since the generation of emissions in the production process produces benefits whilst the transfer payment is a cost for the individual polluter, we have $B_i' > 0$ and $D_i' > 0$.

[7] In the non-point source pollution case an outcome would consist of an emissions vector and a charge on each polluter for each realisation of the random variable such that the sum of the charges paid by all polluters covers the total cost of environmental damages arising from pollution in each state.

when the actions in **a** are chosen. Clearly, given a sharing rule **s**, **a** is a Nash equilibrium conditional on **s** if, for each $i$ and for all $a'_i$,

$$\int_\Theta u_i(s_i(f(\mathbf{a},\theta),\theta))d\mu - c_i(a_i) \geq \int_\Theta u_i(s_i(f(\mathbf{a}_{-i},a'_i,\theta),\theta))d\mu - c_i(a'_i)$$

We implicitly assume that the noncooperative production game mentioned above is the second stage of a two stage process where in the first stage, before production takes place, the team uses some procedure to select a sharing rule. For example, this could be a bargaining process or a welfare maximization problem for the team. This will ultimately generate an outcome which is realized in a Nash equilibrium of the second stage game conditional on the sharing rule adopted in the first stage. Thus, we are interested in the class of outcomes that could be implemented in the following sense:

**Implementation**: An outcome $(\mathbf{a}, (\mathbf{p}^\theta)_{\theta\in\Theta})$ is implementable if there exists a sharing rule **s** such that:

(i) **a** is a Nash equilibrium conditional on **s**;

(ii) $p_i^\theta = s_i(f(\mathbf{a},\theta),\theta)$ for each $i$ and each $\theta$.

For obvious reasons, we are only interested in implementing outcomes that involve positive actions from every team member. Thus, in the rest of this paper we assume that every team member takes positive action in any outcome we consider for implementation.

In our framework an efficient outcome is simply an outcome which cannot be dominated in the Pareto sense *ex-ante*.

**Efficiency**: An outcome $(\mathbf{a}, (\mathbf{p}^\theta)_{\theta\in\Theta})$ is an efficient outcome if there does not exists another outcome $(\mathbf{a}', (\mathbf{p}'^\theta)_{\theta\in\Theta})$ such that $\int_\Theta u_i(p_i'^\theta)d\mu - c_i(a'_i) \geq \int_\Theta u_i(p_i^\theta)d\mu - c_i(a_i)$ for all $i$, with strict inequality holding for some $i$.

## 3. IMPLEMENTABLE OUTCOMES

In this section, for completeness, we will examine the requirements that have to be satisfied for implementability in the general case with no further restrictions on the production and utility functions than those specified in the previous section. Without loss of generality, let us fix an outcome $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta\in\Theta})$ to be implemented and let $\hat{U}_i$ be the expected utility of individual $i$ at this outcome, i.e. $\hat{U}_i = \int_\Theta u_i(\hat{p}_i^\theta)d\mu - c_i(\hat{a}_i)$ for each $i$. Discouraging each team member from unilaterally deviating from his/her action in the outcome is crucial for implementing it. The most problematic deviations in the final outputs from the levels corresponding to the given outcome are those for which no team member could be excluded from causing it with a unilaterally deviation, and such deviations would have to satisfy some restrictions for implementation.

For each $\theta$, let

$$\hat{Q}(\theta) = \{q \in \Re_+ : \text{there exists } a_i \text{ with } f(\hat{\mathbf{a}}_{-i}, a_i, \theta) = q \text{ for all } i\}.$$

When a state $\theta$ is realised and an output level from $\hat{Q}(\theta)$ is observed, we cannot rule out any team member $i$ from generating it by unilaterally deviating from $\hat{a}_i$. So $\hat{Q}(\theta)$ is the most problematic set of deviations where we would need to specify some restrictions.

For each $i$ and each $a_i$, let

$$W_i(a_i, \hat{U}_i) = \left\{(w^\theta)_{\theta\in\Theta} : w^\theta \in [m, \infty) \text{ for each } \theta; \int_\Theta u_i(w^\theta) d\mu - c_i(a_i) = \hat{U}_i\right\}$$

Whenever an individual $i$ deviates unilaterally and an output and random variable combination $(q, \theta)$ is observed, using assumption **A1**, it can be verified that there is a unique action to which $i$ could have deviated unilaterally to generate the output $q$ in state $\theta$. We denote this unique action by $a_i^{q\theta}$, i.e. $f(\hat{\mathbf{a}}_{-i}, a_i^{q\theta}, \theta) = q$.

It is clear from the definition of $W_i(a_i, \hat{U}_i)$ that, if we could make one of the menu of payments in $W_i(a_i, \hat{U}_i)$ act as an upper bound (state by state) on individual $i$'s shares when he/she deviates unilaterally to $a_i$, then individual $i$ could be discouraged from deviating to $a_i$. We have to be able to do this simultaneously for the entire team at every possible deviation in $\hat{Q}(\theta)$ for all $\theta$. However, satisfying these restrictions whilst ensuring that the team's budget is always balanced may not be possible. This leads us to the following condition for implementability of the given outcome, which requires both the restrictions we have just discussed and the budget-balancing condition.

**(I)** For each $i$ and each $a_i$, there exists $(w_i^\theta(a_i))_{\theta\in\Theta} \in W_i(a_i, \hat{U}_i)$ such that, for every $\theta$: $\sum_{i=1}^N w_i^\theta(a_i^{q\theta}) \geq q$ for all $q \in \hat{Q}(\theta)$.

Using the above intuition, we will construct a sharing rule which will be used to show that condition **(I)** is sufficient for implementing the given outcome. In fact, our first result shows that condition **(I)** is also necessary for implementation.

Suppose condition **(I)** is satisfied. Given any $(q, \theta)$ with $q \in \hat{Q}(\theta)$, let $w_i(q, \theta)$ for each $i$ be such that:

(i) $w_i(q, \theta) \leq w_i^\theta(a_i^{q\theta})$;

(ii) $\sum_{i=1}^N w_i(q, \theta) = q$.

Also, for each $(q, \theta)$, let $\eta^{q\theta} = \{i : f(\hat{\mathbf{a}}_{-i}, \xi, \theta) \geq q \geq f(\hat{\mathbf{a}}_{-i}, 0, \theta)\}$. So $|\eta^{q\theta}| = N$ (where $|\eta^{q\theta}|$ is the number of individuals in $\eta^{q\theta}$) if and only if $q \in \hat{Q}(\theta)$.

Let us define $\hat{\mathbf{s}} = (\hat{s}_1, ..., \hat{s}_N)$ to be the sharing rule which satisfies the following for each $i$ and each $(q, \theta)$:

(A) if $q = f(\hat{\mathbf{a}}, \theta)$, then $\hat{s}_i(q, \theta) = \hat{p}_i^\theta$;

(B) if $q \in \hat{Q}(\theta)$ but $q \neq f(\hat{\mathbf{a}}, \theta)$, then $\hat{s}_i(q, \theta) = w_i(q, \theta)$;

(C) if $q \notin \hat{Q}(\theta)$ but $i \in \eta^{q\theta}$, then $\hat{s}_i(q, \theta) = w_i^\theta(a_i^{q\theta})$;

(D) if $i \notin \eta^{q\theta}$, then $\hat{s}_i(q, \theta) = [q - \sum_{j \in \eta^{q\theta}} w_j^\theta(a_j^{q\theta})]/[N - |\eta^{q\theta}|]$.

It can be checked that $\hat{\mathbf{s}}$ is a sharing rule, because it ensures that the team's budget is always balanced. In the above construction, condition (A) is self-explanatory, whilst conditions (B) and (C) are sufficient to deter anyone from deviating unilaterally.

**Proposition 1**: *The outcome* $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta \in \Theta})$ *can be implemented if and only if it satisfies condition* **(I)**.

*Proof*: See the appendix.

## 4. EFFICIENCY

In order to address the issue of implementability of efficient outcomes, as in the previous section, without loss of generality, pick an arbitrary efficient outcome $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta \in \Theta})$ and let $\hat{U}_i = \int_\Theta u_i(\hat{p}_i^\theta) d\mu - c_i(\hat{a}_i)$ for each $i$. As mentioned in section 3, we assume that $\hat{a}_i > 0$ for all $i$. It can be verified that $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta \in \Theta})$ is efficient if and only if it is a solution of the following problem:

$$(\mathbf{EP}) \qquad \max_{(\mathbf{a}, (\mathbf{p}_{-1}^\theta)_{\theta \in \Theta})} \int_\Theta u_1(f(\mathbf{a}, \theta) - \sum_{i=2}^N p_i^\theta) d\mu - c_1(a_1)$$

subject to:

$$\int_\Theta u_i(p_i^\theta) d\mu - c_i(a_i) \geq \hat{U}_i \quad \text{for all } i \geq 2.$$

Given the concavity assumptions for the production function and the utility functions in **A1** and **A3** respectively, the following first-order conditions are necessary and sufficient for an interior solution of this problem:

$$\int_\Theta u_1'(f(\mathbf{a}, \theta) - \sum_{i=2}^N p_i^\theta) f_1(\mathbf{a}, \theta) d\mu - c_1'(a_1) = 0 \tag{2}$$

$$\int_\Theta u_1'(f(\mathbf{a}, \theta) - \sum_{i=2}^N p_i^\theta) f_i(\mathbf{a}, \theta) d\mu - \lambda_i c_i'(a_i) = 0 \quad \forall i \geq 2 \tag{3}$$

$$-u_1'(f(\mathbf{a}, \theta) - \sum_{i=2}^N p_i^\theta) + \lambda_i u_i'(p_i^\theta) = 0 \quad \forall\, i \geq 2 \text{ and } \forall\, \theta \tag{4}$$

$$\int_\Theta u^i(p_i^\theta) d\mu - c_i(a_i) = \hat{U}_i \quad \forall i \geq 2 \tag{5}$$

where the $\lambda_i$s are the Lagrange multipliers.

Since actions are costly to individuals, our intuition would suggest that controlling unilateral deviations by anyone above his/her efficient action should not be problematic. This suggests that the reasoning we used to derive the implementability condition **(I)** might not be needed for unilateral deviations above the efficient levels. For the more problematic unilateral deviations below the efficient levels, our reasoning for condition **(I)** seems to be tight.

**Proposition 2**: *The efficient outcome* $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^{\theta})_{\theta\in\Theta})$ *can be implemented if and only if it satisfies the restriction of condition* **(I)** *to* $a_i < \hat{a}_i$ *for all* $i$.

*Proof*: See the appendix.

The result in the literature that is closest to Propositions 1 and 2 is that of Legros and Matthews (1993) even though they only consider the case of non-stochastic production. In fact it can be seen from the following reasoning that our result is a generalization of theirs. Suppose $\Theta$ contains a single state, say $\hat{\theta}$, i.e. production is non-stochastic. Given any $q \in \hat{Q}(\hat{\theta})$, for each $i$, let $\hat{w}_i(q, \hat{\theta})$ be the unique number such that $u_i(\hat{w}_i(q, \hat{\theta})) - c_i(a_i^{q\hat{\theta}}) = u_i(\hat{p}_i^{\hat{\theta}}) - c_i(\hat{a}_i)$. In this case, the condition in Legros and Matthew (1993) is equivalent to requiring $\sum_{i=1}^{N} \hat{w}_i(q, \hat{\theta}) \geq q$ for all $q \in \hat{Q}(\hat{\theta})$. When we generalize to the stochastic production case, for the problematic unilateral deviations, equality of utility state by state is replaced by equality of utility in expected terms. This allows us to translate the above condition in a fairly natural way to condition **(I)** or its restriction as in Proposition 2. So, at the incomes that appear in the efficiency condition for each of the problematic deviations, in contrast to the non-stochastic case where the actual/*ex-post* utilities have to be equal to those at the efficient outcome, in our stochastic set up the actual/*ex-post* utilities may be lower or higher than those at the efficient outcome. Thus, it is intuitively possible to see that our set up might provide more freedom to incorporate sufficient punishments into the sharing rule to deter the problematic deviations. We will examine whether this is the case or not under two broad classes of production functions that differ from each other according to a particular form of separability.

### 4.1: Separable Production Function

In this sub-section we will consider the class of production functions with a weakly separable structure between the actions and the random factor, i.e. the production function is of the form

$$f(\mathbf{a}, \theta) = F(A(\mathbf{a}), \theta) \quad \text{for all } (\mathbf{a}, \theta) \in [0, \xi]^N \times \Theta,$$

where the functions $A : [0, \xi]^N \to \Re_+$ and $F : \Re_+ \times \Theta \to \Re_+$ are consistent with assumptions **(A1)** and **(A2)**.

Given the weakly separable production function, let us first look at the case where all team members have quasi-linear utility functions, i.e. every team member is risk neutral in income. So each individual $i$'s utility function is given by:

$$U_i(s_i, a_i) = s_i - c_i(a_i)$$

Consider the following expected total surplus maximization problem:

$$\max_{\mathbf{a}} \left[ \int_{\Theta} F(A(\mathbf{a}), \theta) d\mu - \sum_{i=1}^{N} c_i(a_i) \right]$$

For expositional convenience, we will assume that $\hat{\mathbf{a}}$ is the unique interior solution of the above problem. So it is clear from the transferrable utility setup implied by the quasi-linear utility functions that an outcome is efficient if and only if the actions in it are $\hat{\mathbf{a}}$.

When the production function is weakly separable and the utility functions are quasi-linear, the smoothness of the production and utility functions make it impossible to incorporate sufficient incentives in a sharing rule to deter everyone from unilaterally deviating marginally below their respective efficient actions, whilst at the same time ensuring that the team's budget is balanced.

**Proposition 3**: *Efficient outcomes are not implementable if the production function is weakly separable and every team member has a quasi-linear utility function.*

*Proof*: See the appendix.

Using some examples, we will now demonstrate that Proposition 3 may no longer be valid if the assumption of quasi-linear utility functions is dropped.

*Example 1*

Consider a team of 2 individuals in which the utility function of each individual $i$ is given by $U^i(s_i, a_i) = (1 - e^{-s_i}) - (a_i^2/8)$ for all $(s_i, a_i) \in \Re \times \Re_+$. The production function is given by $f(a_1, a_2, \theta) = \theta(a_1 + a_2)$, where the productivity parameter $\theta \in \Re_+$ is a random variable whose realization can only be observed after the individual actions are taken and is distributed according to the cumulative distribution function $G(\theta) = (1 - e^{-\theta})$. Using equations (2) - (5), it can be verified that $(\mathbf{a}^*, (\mathbf{p}^{\theta *})_{\theta \in \Re_+})$, where $a_1^* = a_2^* = 1$ and $p_1^{\theta *} = p_2^{\theta *} = \theta$ for all $\theta$, is an efficient outcome. The expected utilities at this efficient outcome are $u_1^* = u_2^* = 3/8$.

Let $\Theta_1 = [0, \ln(8/3))$ and $\Theta_2 = [\ln(8/3), \infty)$. Also, let $\mathbf{s}^* = (s_1^*, s_2^*)$ be the sharing rule such that, for each $i$ and every $(q, \theta) \in \Re_+^2$,

$$s_i^*(q, \theta) = \begin{cases} q/2 & \text{if } q \geq 2\theta \\ q & \text{if } q < 2\theta \text{ and } \theta \in \Theta_i \\ 0 & \text{if } q < 2\theta \text{ and } \theta \notin \Theta_i \end{cases}$$

In the appendix we show that the sharing rule $\mathbf{s}^*$ implements the efficient outcome $(\mathbf{a}^*, (\mathbf{p}^{\theta *})_{\theta \in \Re_+})$.

*Example 2*

Consider a team of 2 individuals in which the utility functions are given by $U^1(s_1, a_1) = s_1 - (3/4)a_1^2$ for all $(s_1, a_1) \in \Re \times \Re_+$ and $U^2(s_2, a_2) = (1 - e^{-s_2}) - (1/4)(e^{a_2} - a_2 - 1)$ for all $(s_2, a_2) \in \Re \times \Re_+$. The production function is given by $f(a_1, a_2, \theta) = \theta(a_1 + a_2)$, where the productivity parameter $\theta \in \{1, 2\}$ is a random variable whose realization can only be observed after the individual actions are taken and is such that $\theta = 1$ and $\theta = 2$ are equally likely. Using equations (2) - (5), it can be verified that $(\mathbf{a}^*, (\mathbf{p}^{\theta*})_{\theta\in\{1,2\}})$, where $a_1^* = a_2^* = 1$, $p_1^{1*} = 2 - \ln(6/(e-1))$, $p_1^{2*} = 4 - \ln(6/(e-1))$, $p_2^{1*} = \ln(6/(e-1))$ and $p_2^{2*} = \ln(6/(e-1))$, is an efficient outcome. The expected utilities at this efficient outcome are $u_1^* = (9/4) - \ln(6/(e-1))$ and $u_2^* = (5/3) - (5e/12)$.

Let $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2)$ be the sharing rule such that, for each $q \in \Re_+$,

$$\begin{aligned}
\tilde{s}_1(q,1) &= \begin{cases} 4 - \ln(3/(e-1)) - q & \text{if } q < 2 \\ 1 - \ln(6/(e-1)) + (q/2) & \text{if } q \geq 2 \end{cases} \\
\tilde{s}_1(q,2) &= \begin{cases} 2q - 4 - \ln(12/(e-1)) & \text{if } q < 4 \\ 2 - \ln(6/(e-1)) + (q/2) & \text{if } q \geq 4 \end{cases} \\
\tilde{s}_2(q,\theta) &= q - \tilde{s}_1(q,\theta) \quad \text{for each } \theta.
\end{aligned}$$

In the appendix we show that $\tilde{\mathbf{s}}$ implements the efficient outcome $(\mathbf{a}^*, (\mathbf{p}^{\theta*})_{\theta\in\{1,2\}})$.

The non-linearity or risk aversion in income of the utility functions of some individuals is crucial in the above examples for showing that the sharing rules we have constructed can achieve the given efficient outcomes. In these examples, we were able to assign punishment states and reward states to each individual (when the output is below the efficient level, for each individual $i$, his/her punishment and reward states are respectively $\Theta_j$ and $\Theta_i$ in example 1, and $\theta = i$ and $\theta = j$ in example 2, where $i, j = 1, 2$ and $j \neq i$) and incorporate sufficient punishments and rewards appropriately into the sharing rules to deter everyone from unilateral deviations from their respective efficient actions.

Thus, the contrast between the examples and the negative efficiency result in Proposition 3 suggest the following observation regarding our earlier intuition that our framework might provide more freedom for incorporating punishments into the sharing rules to deter the problematic unilateral deviations compared to the non-stochastic set up. In the case of weakly separable production functions, the additional freedom could be enough in some situations to incorporate sufficient punishments into the sharing rules to deter all unilateral deviations from the efficient actions if there are individuals with utility functions that are not quasi-linear, but is not enough if everyone has quasi-linear utility functions.

It may be noted that our framework reduces to that of Rasmusen (1987) if the output is invariant to the state, i.e. for each $\mathbf{a}$, $f(\mathbf{a}, \theta') = f(\mathbf{a}, \theta'')$ for all $\theta', \theta'' \in \Theta$, except $\Theta$ and $\mu$ are part of the design of the randomized sharing rule in his set up but are exogeneously fixed here. So, depending on the exogeneous $\Theta$ and $\mu$, the intuitive reasoning used by Rasmusen could work in our framework when there are risk averse team members. This is confirmed by the intuitive similarity of the reasoning behind our examples and that of Rasmusen (1987). As in Rasmusen, our examples rely on assigning punishment states and reward states appropriately to each individual for undesirable unilateral deviations in such a way that his/her degree of risk aversion may make the undesirable risk of facing the punishments sufficiently stronger than the attraction of the rewards when deviating unilaterally. In fact, the examples show that the intuition is applicable more generally because, unlike Rasmusen (1987): (i) the output varies with the state in all examples; (ii) as shown by examples 2, we do not have to require every individual to be risk averse. Inspite of the intuitive similarity, there are crucial differences in the formal structure and motivation of the two frameworks. Firstly, Rasmusen considers a deterministic production process, which means that he is looking at *ex-post* efficient outcomes. However, in order to implement the *ex-post* efficient outcome, he allows randomization of the sharing rule, because of which the individuals have to maximize *ex-ante* expected utility in choosing their respective actions. In contrast, we can only consider *ex-ante* efficient outcomes for implementation, because our production process is nondeterministic and the individuals have to choose their respective actions before the uncertainty is resolved. Furthermore, because the uncertainty is an integral part of our model albiet observable *ex-post*, our sharing rule should be viewed as a menu of deterministic contracts rather than as allowing randomization by choosing an appropriate randomization device. Secondly, Rasmusen's motivation for allowing randomization in a risk-free (deterministic) environment is to introduce risk and exploit the risk aversion of the individuals to achieve the efficient outcome. This may raise serious issues about the acceptability of such randomized sharing rules to the individual participants. On the otherhand, our motivation for allowing state dependent sharing rules when the state is observable *ex-post* is just the opposite, namely, to reduce the risk faced by the individuals from the uncertainties that are beyond their control. Therefore, our sharing rules are more likely to be acceptable to the individual participants.

Given the possibility of achieving some efficient outcomes as demonstrated by examples 1 and 2, it seems both interesting and natural to ask whether we could derive efficiency conditions that are more transparent than those in Proposition 2 for the case of weakly separable production functions. We will make a modest attempt to provide some answers to this question by first considering the weaker notion of *local implementation* and then the more general notion of implementation as we have defined. The reason we think it is worthwhile to pursue the issue of local implementation follows from our earlier observation in the context of Proposition 3 that one of the main difficulty in achieving an efficient outcome was the impossibility of deterring everyone from unilaterally deviating marginally below their respective efficient actions.

When considering local implementation, we will only allow sharing rules in which the share function of each individual is piecewise continuous and piecewise continuously differentiable in the output and denote this class of sharing rules by $\mathcal{S}$. Although this is a restrictive class, we think that the restriction is fairly natural and almost all sharing rules one could realistically adopt would satisfy it. The notion of local implementation we use is the following straightforward modification of our definition of implementation.

**Local Implementation**: The efficient outcome $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta \in \Theta})$ is locally implementable if there exist a sharing rule $\mathbf{s} \in \mathcal{S}$ and $\epsilon > 0$ such that, for each $i$:

(i) $\hat{p}_i^\theta = s_i(f(\hat{\mathbf{a}}, \theta), \theta)$ for all $\theta$;

(ii) $\int_\Theta u_i(s_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta))d\mu - c_i(a_i) \leq \hat{U}_i$ for all $a_i \in (\hat{a}_i - \epsilon, \xi)$.

**Proposition 4**: *Suppose the production function is weakly separable. Then the efficient outcome $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta\in\Theta})$ is locally implementable if and only if there exists another outcome $(\hat{\mathbf{a}}, (\bar{\mathbf{p}}^\theta)_{\theta\in\Theta})$ such that $\int_\Theta u_i(\bar{p}_i^\theta)d\mu - c_i(\hat{a}_i) < \hat{U}_i$ for every $i$.*

*Proof*: See the appendix.

Given Proposition 3, it is obvious that the condition given in Proposition 4 would not be satisfied if all team members are risk neutral. When there are risk averse team members, the following observation is key to proving the proposition. Whenever a sharing rule $\mathbf{s} \in \mathcal{S}$ implements the given efficient outcome, there would be an individual $i$ whose expected utility from unilateral deviations based on $\mathbf{s}$ is discontinuous at his/her efficient action $\hat{a}_i$. This discontinuity provides enough freedom to construct the payment schedule $(\bar{\mathbf{p}}^\theta)_{\theta\in\Theta}$ which satisfies the condition given in the proposition. Conversely, it seems intuitively straightforward that the condition in the proposition would allow us to construct a sharing rule in $\mathcal{S}$ which will deter any unilateral deviations in a neighbourhood of the efficient actions.

In order to derive a condition for full implementation, we will introduce two auxiliary functions. For each $x \leq A(\hat{\mathbf{a}})$ and each $i$, let $\hat{\rho}_i(x)$ be the unique action of individual $i$ such that $A(\hat{\mathbf{a}}_{-i}, \hat{\rho}_i(x)) = x$, where the uniqueness follows from the requirement that the function $A$ is consistent with the monotonicity assumption in **A1**. For each $x < A(\hat{\mathbf{a}})$, define $\hat{V}(x)$ as follows:

$$\begin{aligned} \hat{V}(x) \quad &\equiv \quad \min_{(\mathbf{p}_{-1}^\theta)_{\theta\in\Theta}} \int_\Theta u_1(F(x,\theta) - \sum_{i=2}^N p_i^\theta)d\mu \\ &\qquad \text{subject to:} \quad \int_\Theta u_i(p_i^\theta)d\mu \leq \hat{U}_i + c_i(\hat{\rho}_i(x)) \quad \text{for all } i \geq 2 \end{aligned}$$

Using the function $\hat{V}$, we can provide the following characterization for implementability of the efficient outcome.

**Proposition 5**: *Suppose the production function is weakly separable. Then the efficient outcome $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta\in\Theta})$ is implementable if and only if $\hat{V}(x) \leq \hat{U}_1 + c_1(\hat{\rho}_1(x))$ for every $x < A(\hat{\mathbf{a}})$.*

*Proof*: See the appendix.

Because of Proposition 3, it is again obvious that the condition given in Proposition 5 would not be satisfied if all team members are risk neutral. When there are risk averse team members, the intuition for the condition in Proposition 5 is quite simple. The structure imposed by the weakly separable production function allows us to concentrate only on the deviations that are roughly at an aggregate level, namely, the composite action $A(\mathbf{a})$ rather than dealing with deviations in each individual action. This in turn makes it possible to derive an efficiency condition based on the single variable $x < A(\hat{\mathbf{a}})$.

### 4.2: Non-separable Production Function

In this sub-section we will consider the class of production functions that do not have the weakly separable structure between the actions and the random factor. To highlight the main contrast with the case of weakly separable production function, we will first present an example which shows that, when the production function is not weakly separable, we might be able to escape the negative efficiency result in Proposition 3.

*Example 3*

Consider a team of 2 individuals in which the utility function of each individual $i$ is given by $U^i(s_i, a_i) = s_i - (3/16)a_i^2$ for all $(s_i, a_i) \in \Re \times [0, 2]$. The production function is such that, for each pair of actions $(a_1, a_2)$ and each $\theta = (\theta_1, \theta_2) \in \{1/4, 1/2\}^2$, $f(a_1, a_2, \theta) = a_1^{\theta_1} a_2^{\theta_2}$. For each individual $i$, the productivity parameter $\theta_i$ is a random variable whose realization is only observable after the actions have been taken. It is also assumed that the productivity parameters of the two individuals are independently distributed with $\theta_i = 1/4$ and $\theta_i = 1/2$ equally likely for each $i$. Clearly, the production function in this example is not weakly separable. Let $\theta^1 = (\theta_1^1, \theta_2^1) = (1/4, 1/4)$, $\theta^2 = (\theta_1^2, \theta_2^2) = (1/4, 1/2)$, $\theta^3 = (\theta_1^3, \theta_2^3) = (1/2, 1/4)$ and $\theta^4 = (\theta_1^4, \theta_2^4) = (1/2, 1/2)$, which implies that the probability of $\theta = \theta^j$ is 1/4 for each $j = 1, 2, 3, 4$. Consider the following expected total surplus maximization problem:

$$\max_{(a_1,a_2)} \left[ (1/4) \sum_{j=1}^{4} a_1^{\theta_1^j} a_2^{\theta_2^j} - (3/16)a_1^2 - (3/16)a_2^2 \right]$$

It can be checked that $(a_1^*, a_2^*) = (1, 1)$ is the unique solution of this problem. So it follows from the quasi-linear utility functions that any $(\mathbf{a}, (\mathbf{p}^\theta)_{\theta\in\Theta})$ is an efficient outcome if and only if $\mathbf{a} = (1, 1)$ and $p_1^{\theta^j} + p_2^{\theta^j} = 1$ for each $j = 1, 2, 3, 4$. Pick any efficient outcome $(\mathbf{a}^*, (\hat{\mathbf{p}}^\theta)_{\theta\in\Theta})$ and let $\hat{\alpha}^1 = \hat{p}_1^{\theta^1} - 1/2$, $\hat{\alpha}^2 = \hat{p}_1^{\theta^2} + 5/2$, $\hat{\alpha}^3 = \hat{p}_1^{\theta^3} - 7/2$ and $\hat{\alpha}^4 = \hat{p}_1^{\theta^4} - 1/2$. Define the sharing rule $\hat{\mathbf{s}} = (\hat{s}_1, \hat{s}_2)$ as follows:

$$\begin{aligned} \hat{s}_1(q, \theta^j) &= \begin{cases} (1/2)q + \hat{\alpha}^j & \text{if } j = 1 \text{ or } 4 \\ -(5/2)q + \hat{\alpha}^2 & \text{if } j = 2 \\ (7/2)q + \hat{\alpha}^3 & \text{if } j = 3 \end{cases} \\ \hat{s}_2(q, \theta^j) &= q - \hat{s}_1(q, \theta^j) \end{aligned}$$

It can be easily verified that $\hat{s}_i(f(\mathbf{a}^*, \theta^j), \theta^j) = \hat{p}_i^{\theta^j}$ for each $i$ and each $j$. In the appendix we show that the sharing rule $\hat{\mathbf{s}}$ implements the efficient outcome $(\mathbf{a}^*, (\hat{\mathbf{p}}^\theta)_{\theta\in\Theta})$. Thus, in this example every efficient outcome is implementable. However, some might argue that the sharing rule $\hat{\mathbf{s}}$ has an undesirable property, namely, $\hat{s}_1$ is decreasing in the output for $\theta = \theta^2$ and $\hat{s}_2$ is decreasing in the output for $\theta = \theta^3$.

In this example, because both individuals are risk neutral, a similar intuitive reasoning as in Rasmusen (1987) is not likely to be behind the positive result. When the random variable is observable *ex-post* and can be contracted upon, it enriches the information structure that could be used in constructing sharing rules. It is most likely that this enhanced degree of informativeness would depend crucially on the structure of the production function. We will provide a line of reasoning which hints at the possibility that, although the enhanced information structure is not rich enough to implement efficient outcomes in the set up of Proposition 3, when the production function is not weakly separable, such as in example 3, the enhanced information structure could be rich enough to implement efficient outcomes even if all team members are risk neutral.

In example 3, as in the existing literature, suppose the state $\theta$ is not observable/verifiable *ex-post*. So the enhanced information structure discussed above is no longer available. In this case the sharing rule can only depend on the final output, i.e. a sharing rule $\mathbf{s}$ is such that $s_i : [0, 2] \to \Re$ for each $i$ and $s_1(q) + s_2(q) = q$ for all $q$. In this case, we show in the appendix that there is no sharing rule which can support the efficient actions $\mathbf{a}^* = (1, 1)$ as a Nash equilibrium. Thus, there seems to be some support for our intuition.

Suppose in example 3 the action space is discretised to

$$E_i = \{x/(64)^2 : \; x \text{ is any non-negative integer not exceeding } 2(64)^2\} \quad \text{for each } i.$$

Then it is straightforward to reformulate the utility functions in a way to make example 3, with the discrete action space and unobservable/nonverifiable state, an example of the framework used by Legros and Matsushima (1991). Clearly, when the state can be contracted upon, the sharing rule $\hat{\mathbf{s}}$ we constructed above, with the obvious modification of the output space implied by the discretisation, would still implement the efficient outcome. However, using the same argument as in the continuous case, we show in the appendix that the efficient actions $\mathbf{a}^* = (1, 1)$ cannot be supported as a Nash equilibrium when the state can no longer be contracted upon. Thus, although Proposition 3 also seems to suggest that our efficiency condition could be seen as a generalisation of Legros and Matsushima (1991), we could argue that this suggestion may no longer be true if the production function is not weakly separable.

When there are individuals with risk averse utility functions, the line of intuitive reasoning we applied in the case of weakly separable production funtions would still be valid here. As in examples 1 and 2, it would be relatively straightforward to construct examples that exploit the risk aversion to assign punishment states and reward states to each individual and incorporate sufficient punishments and rewards appropriately into the sharing rules to deter every one from unilateral deviations from their respective efficient actions. Thus, in the rest of this section we will only consider the case in which all team members have quasi-linear utility functions.

In what follows we will only look at the case with a finite number of possible states. Let $\Theta = \{\theta^1, ..., \theta^M\}$, where $M$ is finite, and $\pi^h$ is the probability of state $\theta^h$ being realised for $h = 1, ..., M$. Furthermore, we assume that all team members have quasi-linear utility functions, i.e. $U^i(s_i, a_i) = s_i - c_i(a_i)$ for all $i$. Consider the following expected total surplus maximization problem:

$$\max_{\mathbf{a}} \left[ \sum_{h=1}^{M} \pi^h f(\mathbf{a}, \theta^h) - \sum_{i=1}^{N} c_i(a_i) \right]$$

For expositional convenience, we will assume that the above problem has an interior solution $\hat{\mathbf{a}}$ and is unique. So it is clear from the transferrable utility setup implied by the quasi-linear utility functions that an outcome is efficient if and only if the actions in it are $\hat{\mathbf{a}}$.

Without loss of generality, pick an arbitrary efficient outcome $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta \in \Theta})$ and let $\hat{U}_i = \sum_{h=1}^{M} \pi^h \hat{p}_i^{\theta^h} - c_i(\hat{a}_i)$ for each $i$. When considering implementation of $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta \in \Theta})$, we will restrict ourselves to $\mathcal{S}$, which is the class of sharing rules with piecewise continuous and piecewise continuously differentiable share functions for each state. As noted earlier, this seems a fairly natural and reasonable restriction.

The class of production functions that are not weakly separable between the actions and the random factor is far more general than the class that is weakly separable. This means that looking at implementation of efficient outcomes would require us to consider restrictions on the structure of the production function away from a neighbourhood of the efficient actions $\hat{\mathbf{a}}$. However, it is likely that such restrictions could turn out to be quite ad hoc. Furthermore, we noted earlier in the context of Proposition 3 that, when all individuals are risk neutral, one of the main dificulty in implementing efficient outcomes has to do with the inability to mitigate the incentive to deviate unilaterally in a neighbourhood of the efficient action. Therefore, we will only examine the issue of local implementation of the efficient outcome $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta \in \Theta})$, where local implementation is as defined earlier with the obvious understanding that the production function is no longer weakly separable and the utility functions are all quasi-linear.

In order to motivate the intuition for the efficiency result presented below, let us first consider the general case in which the utility functions may or may not be quasi-linear. According to equation (4), for all individuals $i, j$ and for all states $\theta^h, \theta^l$, we have

$$\frac{u_i'(\hat{p}_i^{\theta^h})}{u_j'(\hat{p}_j^{\theta^h})} = \frac{u_i'(\hat{p}_i^{\theta^l})}{u_j'(\hat{p}_j^{\theta^l})}$$

Notice that the above equation is still valid for any redistribution of the total outputs at $\hat{\mathbf{a}}$ if everyone has quasi-linear utility functions. However, when there are risk averse individuals, it may be possible to redistribute the total outputs at $\hat{\mathbf{a}}$ in such a way that the equation does not hold. Furthermore, in the case of weakly separable production functions we know that

$$\frac{f_i(\hat{\mathbf{a}}, \theta^h)}{f_j(\hat{\mathbf{a}}, \theta^h)} = \frac{f_i(\hat{\mathbf{a}}, \theta^l)}{f_j(\hat{\mathbf{a}}, \theta^l)} \quad \text{for all } i, j \text{ and for all } h, l.$$

From these observations we can conclude the following distinction in the case of weakly separable productions:

(i) If all individuals have quasi-linear utility function, then there is no other outcome $(\hat{\mathbf{a}}, (\mathbf{p}^\theta)_{\theta\in\Theta})$ such that

$$\frac{u_i'(p_i^{\theta^h})f_i(\hat{\mathbf{a}},\theta^h)}{u_j'(p_j^{\theta^h})f_j(\hat{\mathbf{a}},\theta^h)} \neq \frac{u_i'(p_i^{\theta^l})f_i(\hat{\mathbf{a}},\theta^l)}{u_j'(p_j^{\theta^l})f_j(\hat{\mathbf{a}},\theta^l)} \quad \text{for some } i,j \text{ and some } h,l.$$

(ii) If there are risk averse individuals, then it may be possible to find other outcomes $(\hat{\mathbf{a}}, (\mathbf{p}^\theta)_{\theta\in\Theta})$ such that

$$\frac{u_i'(p_i^{\theta^h})f_i(\hat{\mathbf{a}},\theta^h)}{u_j'(p_j^{\theta^h})f_j(\hat{\mathbf{a}},\theta^h)} \neq \frac{u_i'(p_i^{\theta^l})f_i(\hat{\mathbf{a}},\theta^l)}{u_j'(p_j^{\theta^l})f_j(\hat{\mathbf{a}},\theta^l)} \quad \text{for some } i,j \text{ and some } h,l.$$

This distinction is the key to the difference in the efficiency results in Proposition 3 on the one hand and examples 1 and 2 and Propositions 4 and 5 on the other hand.

Let us now apply a similar intuitive line of reasoning to the present case in which all team members have quasi-linear utility functions and the production function is not weakly separable. For any outcome $(\hat{\mathbf{a}}, (\mathbf{p}^\theta)_{\theta\in\Theta})$, we have $u_i'(p_i^{\theta^h})/u_j'(p_j^{\theta^h}) = 1$ for all $i,j$ and for all $h$. However, because the production function is not weakly separable, it is possible to have

$$\frac{f_i(\hat{\mathbf{a}},\theta^h)}{f_j(\hat{\mathbf{a}},\theta^h)} \neq \frac{f_i(\hat{\mathbf{a}},\theta^l)}{f_j(\hat{\mathbf{a}},\theta^l)} \quad \text{for some } i,j \text{ and some } h,l.$$

Thus, even if all team members have quasi-linear utility functions, unlike in the case of weakly separable production function, it might be possible that

$$\frac{u_i'(\hat{p}_i^{\theta^h})f_i(\hat{\mathbf{a}},\theta^h)}{u_j'(\hat{p}_j^{\theta^h})f_j(\hat{\mathbf{a}},\theta^h)} \neq \frac{u_i'(\hat{p}_i^{\theta^l})f_i(\hat{\mathbf{a}},\theta^l)}{u_j'(\hat{p}_j^{\theta^l})f_j(\hat{\mathbf{a}},\theta^l)} \quad \text{for some } i,j \text{ and some } h,l.$$

This observation is the key to example 3 and also the characterization of local implementation of efficient outcomes.

**Proposition 6**: *If the production fuction is not weakly separable, every team member has a quasi-linear utility function and $\Theta = \{\theta^1, ..., \theta^M\}$, then the efficient outcome $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta\in\Theta})$ is locally implementable if and only if there exist $i,j$ and $h,l$ such that $f_i(\hat{\mathbf{a}},\theta^h)/f_j(\hat{\mathbf{a}},\theta^h) \neq f_i(\hat{\mathbf{a}},\theta^l)/f_j(\hat{\mathbf{a}},\theta^l)$.*

*Proof*: See the appendix.

The examples and propositions presented in this section have shown that, although the results in our framework could be seen as a generalization of the existing literature (such as Legros and Matsushima (1991), Legros and Matthews (1993) and Rasmusen (1987)) if the production function is weakly separable, our framework may add something new if the production function is not weakly separable.

## 5. LINEAR SHARING RULES

In this section we will concentrate on a restricted class of implementable outcomes. In particular, we will only consider those outcomes that are implementable by using sharing rules for which the first-order approach is valid.

In the literature, the random variable $\theta$ is often suppressed by viewing the output $q$ as a random variable with a distribution function $G(q|\mathbf{a})$ parametrised by the actions (e.g. see Holmstrom (1982), Kim and Wang (1998)) and conditions are often imposed on $G$ to generate contracts that are nondecreasing in output. We do not want to suppress the random variable $\theta$ in our context as it has an essential role to play in the analysis, but we could explore conditions on the production function $f$ and the probability measure $\mu$ to avoid a negative relationship between payments and output. However, instead of this approarch, we will rely on the fairly usual casual empiricism that individual payments are often related non-negatively to the final output level. Thus, throughout this section we will also consider only the class of outcomes that are implementable by using sharing rules in which the share functions are nondecreasing in the output in a neighbourhood of the Nash equilibrium outputs.

Consider an implementable outcome $(\bar{\mathbf{a}}, (\bar{\mathbf{p}}^\theta)_{\theta\in\Theta})$ and let the sharing rule $\bar{\mathbf{s}}$ implement this outcome. So we have

$$\bar{s}_i(f(\bar{\mathbf{a}},\theta),\theta) = \bar{p}_i^\theta \quad \text{for each } i \text{ and for all } \theta.$$

In the appendix we derive the following equation for each $i$:

$$\begin{aligned}
\int_\Theta [u_i'(\bar{p}_i^\theta)\bar{s}_i'^-(f(\bar{\mathbf{a}},\theta),\theta)f_i(\bar{\mathbf{a}},\theta)]d\mu - c_i'(\bar{a}_i) &= \\
\int_\Theta [u_i'(\bar{p}_i^\theta)\bar{s}_i'^+(f(\bar{\mathbf{a}},\theta),\theta)f_i(\bar{\mathbf{a}},\theta)]d\mu - c_i'(\bar{a}_i) &= \\
\frac{d}{da_i}\int_\Theta u_i(\bar{s}_i(f(\bar{\mathbf{a}},\theta),\theta))d\mu - c_i'(\bar{a}_i) &= 0 \qquad (6)
\end{aligned}$$

where $\bar{s}_i'^+(f(\bar{\mathbf{a}},\theta),\theta)$ and $\bar{s}_i'^-(f(\bar{\mathbf{a}},\theta),\theta)$ are respectively the upper and lower derivatives of $\bar{s}_i(q,\theta)$ at $q = f(\bar{\mathbf{a}},\theta)$. As $\bar{s}_i$ is nondecreasing in a neighbourhood of $f(\bar{\mathbf{a}},\theta)$ for each $i$ and each $\theta$, we have $\bar{s}_i'^+(f(\bar{\mathbf{a}},\theta),\theta) \geq 0$. Furthermore, the budget balancing condition implies that $\sum_{i=1}^N \bar{s}_i'^+(f(\bar{\mathbf{a}},\theta),\theta) = 1$ for all $\theta$.

For each $i$ and each $\theta$, let $\tilde{\gamma}_i^\theta$ and $\tilde{\alpha}_i^\theta$ be defined as follows:

$$\tilde{\gamma}_i^\theta = \bar{s}_i'^+(f(\bar{\mathbf{a}},\theta),\theta) \qquad (7)$$

$$\tilde{\alpha}_i^\theta = \bar{p}_i^\theta - \tilde{\gamma}_i^\theta f(\bar{\mathbf{a}},\theta) \qquad (8)$$

So we know that $\tilde{\gamma}_i^\theta \geq 0$ for all $i$ and for all $\theta$ and

$$\sum_{i=1}^N \tilde{\gamma}_i^\theta = 1 \quad \text{for each } \theta. \qquad (9)$$

It can also be verified that

$$\sum_{i=1}^N \tilde{\alpha}_i^\theta = 0 \quad \text{for each } \theta. \qquad (10)$$

Using the above derivations, we will now construct a sharing rule which is linear in the output for each realisation of the random variable and show that this sharing rule also implements the given outcome $(\bar{\mathbf{a}}, (\bar{\mathbf{p}}^\theta)_{\theta\in\Theta})$. Let $\tilde{\mathbf{s}} = (\tilde{s}_1, ..., \tilde{s}_N)$ be the sharing rule such that, for each $i$,

$$\tilde{s}_i(q,\theta) = \tilde{\alpha}_i^\theta + \tilde{\gamma}_i^\theta q \quad \text{for all } (q,\theta). \tag{11}$$

It can be verified by using equations (9) and (10) that the above definition does satisfy the budget balancing condition.

**Proposition 7**: *The outcome* $(\bar{\mathbf{a}}, (\bar{\mathbf{p}}^\theta)_{\theta\in\Theta})$ *can be implemented by using the sharing rule* $\tilde{\mathbf{s}}$.

*Proof*: See the appendix.

In order to look at the efficiency issue for the restricted class of implementable outcomes considered here, let us now pick an efficient outcome $(\mathbf{a}^E, (\mathbf{p}^{E\theta})_{\theta\in\Theta})$ and suppose that it could be implemented. Using equations (2) - (4), it is straightforward to see that the following is satisfied for each $i$:

$$\int_\Theta [u_i'(p_i^{E\theta}) f_i(\mathbf{a}^E, \theta)] d\mu - c_i'(a_i^E) = 0. \tag{12}$$

According to Proposition 7, $(\mathbf{a}^E, (\mathbf{p}^{E\theta})_{\theta\in\Theta})$ must be implementable by a linear sharing rule $\mathbf{s}^E$ such that, for each $i$, $s_i^E(q,\theta) = \alpha_i^{E\theta} + \gamma_i^{E\theta} q$ for all $(q,\theta)$, with $\gamma_i^{E\theta} \geq 0$, $\sum_{j=1}^N \gamma_j^{E\theta} = 1$ and $\sum_{j=1}^N \alpha_j^{E\theta} = 0$. Then from the first-order condition for expected utility maximization at a Nash equilibrium, we can derive the following for each $i$:

$$\int_\Theta [u_i'(p_i^{E\theta}) \gamma_i^{E\theta} f_i(\mathbf{a}^E, \theta)] d\mu - c_i'(a_i^E) = 0. \tag{13}$$

However, it can be seen clearly that equations (12) and (13) cannot hold simultaneously for some $i$. Therefore, we have established the following negative efficiency result.

**Proposition 8**: *There are no efficient outcomes in the restricted class of implementable outcomes.*

Note that, although in example 3 the efficient outcomes were implementable by using linear sharing rules, this does not contradict Proposition 8. The linear sharing rules in the example do not satisfy one of the condition required in Proposition 8, because some of the individual share functions were negatively sloped. This is not so surprising once we realise that, if the individual share functions are no longer required to be nondecreasing in a neighbourhood of the efficient output levels, i.e. the $\gamma_i^{E\theta}$s are not restricted to be nonnegative, equations (12) and (13) could hold simultaneously for all $i$.

Imposing validity of the first-order approach requires the share functions to be smooth in a neighbourhood of the Nash equilibrium output levels. So, although this is a technical regularity condition, it could in some sense be viewed as a requirement on the sharing rule to be simple to a certain extent. One could also argue that sharing rules with share functions that are decreasing in output are not reasonable and could face difficulty in making the team accept such sharing rules. These interpretations provide an important motivation for imposing validity of the first-order approach and requiring the share functions to be locally nondecreasing around the desired output levels in some situations, namely, they could be seen as requiring the team to only adopt sharing rules with a certain degree of simplicity and reasonableness/acceptability. In the context of such situations, Proposition 7 is important as it implies that the team does not have to look for anything more complicated than linear sharing rules.

Our Proposition 7 could be contrasted to the linear sharing rule result of Kim and Wang (1998) that was derived under the setting in which the production technology exhibited weak separability between the actions and the random variable and the random variable was not observable *ex-post* or could not be contracted upon.[8] Kim and Wang were able to derive their linear sharing rule result only under the assumption of quasi-linear utility functions. However, even under the assumption of validity of the first-order approach, they showed that the optimal contract is not linear when there is risk aversion. They argued that this was because risk sharing must be taken into account in addition to incentive provision in designing a contract. When state contingent contracts are not permissible, the risk element is still present *ex-post* but this dissipates once state contingent contracts are allowed. This suggests that, as argued by Kim and Wang, risk sharing will not be taken into account in our framework even if there is risk aversion. However, following this intuition for our linear sharing rule result is not so straightforward, because individuals still face risk at the time of choosing their actions as these are done *ex-ante*. This suggests the possibility of taking risk sharing into consideration in designing the sharing rule if individuals are risk averse. Thus, it follows from examples 1 and 2 with efficient implementable outcomes and the negative efficiency result of Proposition 8 that, apart from the absence of the risk element *ex-post*, the assumption of validity of the first-order approach itself and the requirement of the share functions to be nondecreasing locally around the desired output levels have some role to play in our linear sharing rule result in Proposition 7.

Our linear sharing rule result could also be compared to that of Nandeibam (2002) derived in a framework with no uncertainty. Although there are similarities in the line of reasoning used in both results, there are two crucial differences. Firstly, Nandeibam (2002) did not impose validity of the first-order approach, instead the validity of the first-order approach itself was derived. Secondly, the share functions were not required to be nondecreasing in a neighbourhood of the Nash equilibrium output, instead this was also derived. These differences are significant because, if we relax the restrictions imposed in this section, then Proposition 8 and examples 1 and 2 imply that the team may have to adopt a sharing rule that is more complex than one that is linear in the output with nonnegative slopes. This suggests that Proposition 7 is not a straightforward generalisation of the linear sharing rule result of Nandeibam (2002).

[8]See also Bhattacharyya and Lafontaine (1995) and Romano (1994).

If we accept the interpretation that a possible motivation for the two key restrictions imposed in this section could be the desire for some degree of simplicity and reasonableness in the sharing rule, then the contrast between Proposition 8 and the efficiency results of the previous section suggest the possibility of a trade off between this desire and efficiency in the sense that, in some situations where efficiency is achievable, the desire for simplicity and reasonableness may prevent it. In fact, using a similar line of reasoning, it would be interesting to explore whether every outcome that is implementable by adopting a sharing rule for which the two key restrictions are valid might be *ex-ante* Pareto dominated by another outcome that is also implementable (but not necessarily efficient), possibly by adopting a sharing rule for which the restrictions do not hold. It may be noted that such a trade off is not implied by the linear sharing rule results of Kim and Wang (1998) in the non-deterministic case and Nandeibam (2002) in the deterministic case, because neither of them required validity of the first-order approach and share functions that are nondecreasing in a neighbourhood of the Nash equilibrium output.

A straightforward corollary of Proposition 7 is the following characterization of the restricted class of implementable outcomes considered in this section.

**Proposition 9**: *An outcome* $(\mathbf{a}, (\mathbf{p}^\theta)_{\theta\in\Theta})$ *belongs to the restricted class of implementable outcomes if and only if there exist* $\gamma_i^\theta \geq 0$ *for each* $i$ *and each* $\theta$ *such that*

(i) $\sum_{i=1}^{N} \gamma_i^\theta = 1$;

(ii) $\int_\Theta u_i'(p_i^\theta)\gamma_i^\theta f_i(\mathbf{a}, \theta) d\mu = c_i'(a_i)$.

*Proof*: See the appendix.

## 6. SUMMARY AND CONCLUSION

In a moral hazard in team setting, apart from the final output, any additional information that allows better inference of the actions may bring us closer to the first-best outcome. We argue that there are many situations where, although uncertainty enters into the relationship between the actions and the final output, this uncertainty is resolved and more crucially, the realised value of the randon factor is observable. For example, we can think of the case of non-point source pollution problem where a random element like rainfall combines with individual emissions to produce an ambient level of pollution. Prior to pollution discharge, future rainfall is not known with certainty. However, at the time when the ambient level of pollution is observed, the rainfall that preceded this point in time is also known and observable. There are other situations, some of which we have already mentioned, that are similar to the non-point source pollution problem. Thus, the potential to observe the realised value of the random factor *ex-post* seems quite prevalent in problems characterised by moral hazard in teams. This observation suggests that it makes sense to include this knowledge in the construction of a sharing rule aimed at reducing free-riding incentives and may improve on current schemes suggested in the literature.[9] Accordingly, we considered a set up in which the team's sharing rule took the form of a menu of contracts by incorporating the random factor as an additional variable that could be contracted upon.

We first provided a general characterization of outcomes that could be implemented in our framework. Intuitively, this condition imposed restrictions on the set of unilateral deviations for which no one could be ruled out and this allowed sufficient punishments to be incorporated into the sharing rule to deter such deviations, whilst ensuring that the team's budget is balanced at the same time. We also showed that, for implementing an efficient outcome without imposing any additional restrictions on the utility functions and the production function, it is necessary and sufficient to require the implementability condition only on actions that are below the efficient levels. To explore further the issue of implementing efficient outcomes, we considered separately the case of production functions that are weakly separable between the actions and the random factor and those that are not. In the former case, we first proved that efficient outcomes could not be implemented when everyone has quasi-linear utility function and then presented two examples to show that we might be able to escape the negative result if there are team members who do not have quasi-linear utility funcyions. For weakly separable production functions with some individual utility functions that are not quasi-linear, in addition to providing a more transparent necessary and sufficient condition for implementing an efficient outcome, we also derived a condition that is necessary and sufficient for local implementation of an efficient outcome. For the latter case with production functions which are not weakly separable between the actions and the random factor, using an example we first showed that efficient outcomes might be implementable even if everyone has a quasi-linear utility function. We also presented a necessary and sufficient condition for local implementation of an efficient outcome. It is worth pointing out that, although the results for weakly separable production functions could be seen as generalizations of analogous results in the existing literature, there does not seem to be any existing result that is analogous to ours in the case of production functions which are not weakly separable.

When we narrowed attention to the restricted class of implementable outcomes where the first-order approach is valid and the share functions are locally nondecreasing around the desired output levels, we found that it was possible to restrict the team's search for a suitable sharing rule to simple rules that are linear and nondecreasing in output. This might also provide some explanation as to why, although contrary to predictions by theory, simple linear sharing rules are prevalent.[10] Using the linear sharing rule result, we showed that there are no efficient out-

[9] For example, in the non-point source pollution context, by not accounting for the random factor, polluters may find that they have to pay a tax because the ambient level of pollution exceeded a specified target due to the random factor and not because they over emitted pollution (Herriges *et al.* (1994), Horan *et al.* (1998), Segerson (1988), Xepapadeas (1995)).

[10] Romano (1994) and Bhattacharyya and Lafontaine (1995) provide another explanation of why linear sharing rules tend to be the norm in practice. They find that, in the presence of double moral hazard and risk neutrality of the agent, a simple linear sharing rule implements the desired outcome.

comes in the restricted class of implementable outcomes and provided a characterization of this restricted class. Although we have not explored it here, the line of reasoning used in our general characterization of implementable efficient outcomes suggests that, in some situations it might be possible to show that the outcomes in the restricted class could be *ex-ante* Pareto dominated by other implementable outcomes that are not in the restricted class. Thus, imposing validity of the first-order approach and share functions that are nondecreasing around the desired output levels might be seen as restrictive requirements.

It is worth pointing out that a possible weakness of our approach is the supposition that the realisation of the random element is exactly observable. Arguably, this supposition could be considered to be somewhat extreme. There are various situations one can think of where, although some amount of information about the random element will be available *ex-post*, it is not as precise as we have assumed in this paper. To that extent, our extreme assumption could be seen as a first attempt at trying to understand the problem by looking at the simplest case. An interesting possible extension would be to consider coarser information partitions to capture situations with less precise degree of observability of the random element *ex-post* and see to what extent our results could be generalized.

## REFERENCES

ALCHIAN, A. and H. DEMSETZ (1972), "Production, information costs, and economic organization", *Amer. Econ. Rev.* **62**, 777–795.

BHATTACHARYYA, S. and F. LAFONTAINE (1995), "Double-sided moral hazard and the nature of share contracts", *RAND J. Econ.* **26**, 761–781.

ESWARAN, M. and A. KOTWAL (1984), "The Moral Hazard of Budget-breaking", *RAND J. Econ.* **15**, 578–581.

ESWARAN, M. and A. KOTWAL (1985), "A theory of contractual structure in agriculture", *Amer. Econ. Rev.* **75**, 352–367.

HERRIGES, M.A., R. GOVINDASAMY and J.F. SHOGREN (1994), "Budget-balancing incentive mechanism", *J. Environ. Econ. Mgt.* **27**, 275–285.

HOLMSTROM, B. (1979), "Moral hazard and observability", *Bell J. Econ.* **10**, 74–91.

HOLMSTROM, B. (1982), "Moral hazard in teams", *Bell J. Econ.* **13**, 324–340.

HORAN, R., J.S. SHORTLE and D.G. ABLER (1998), "Ambient taxes when polluters have multiple choices", *J. Environ. Econ. Mgt.* **36**, 186–199.

KIM, S.K. and S. WANG (1998), "Linear contracts and the double moral hazard", *J. Econ. Theory*, **82**, 342–378.

LEGROS, P. and H. MATSUSHIMA (1991), "Efficiency in partnerships", *J. Econ. Theory*, **55**, 296–322.

LEGROS, P. and S. MATTHEWS (1993), "Efficient and nearly efficient partnerships", *Rev. Econ. Stud.* **68**, 599–611.

NANDEIBAM, S. (2002), "Sharing rules in teams", *J. Econ. Theory*, **107**, 407–420.

NANDEIBAM, S. (2003), "Implementation in teams", *Econ. Theory*, **22**, 569–581.

RADNER, R. and S. WILLIAMS (1992), "Efficiency in partnerships when the joint output is uncertain", in: J. Ledyard, ed., *The Economics of Informational Decentralization*, Kluwer, Boston.

RASMUSEN, E. (1987), "Moral hazard in risk-averse teams", *RAND J. Econ.* **18**, 428–435.

ROCKAFELLAR, T. R. (1970), *Convex Analysis*, Princeton University Press, Princeton.

ROMANO, R. (1994), "Double moral hazard and resale price maintenance", *RAND J. Econ.* **25**, 455–466.

SEGERSON, K. (1988), "Uncertainty and incentives for nonpoint pollution control", *J. Environ. Econ. Mgt.* **15**, 87–98.

STRAND, J. (1999), "Efficient environmental taxation under moral hazard", *Eur. J. Polt. Econ.* **15**, 73–88.

XEPAPADEAS, A.P. (1995), "Observability and choice of instrument mix in the control of externalities", *J. Pub. Econ.* **56**, 485–498.

APPENDIX

*Proof of Proposition* 1: Suppose $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta\in\Theta})$ can be implemented. Let $\check{\mathbf{s}}$ be a sharing rule such that $\hat{\mathbf{a}}$ is a Nash equilibrium conditional on $\check{\mathbf{s}}$ and $\check{s}_i(f(\hat{\mathbf{a}}, \theta), \theta) = \hat{p}_i^\theta$ for all $i$ and for all $\theta$. Consider $(\check{q}, \check{\theta})$ such that $\check{q} \in \hat{Q}(\check{\theta})$. For each $i$, let $\check{a}_i$ be such that $f(\hat{\mathbf{a}}_{-i}, \check{a}_i, \check{\theta}) = \check{q}$. We need to find $(w_i^\theta(\check{a}_i))_{\theta\in\Theta} \in W_i(\check{a}_i, \hat{U}_i)$ for each $i$ such that $\sum_{i=1}^N w_i^{\check{\theta}}(\check{a}_i) \geq \check{q}$. Clearly, $\sum_{i=1}^N \check{s}_i(\check{q}, \check{\theta}) = \check{q}$ and for each $i$,

$$\int_\Theta u_i(\check{s}_i(f(\hat{\mathbf{a}}_{-i}, \check{a}_i, \theta), \theta))d\mu - c_i(\check{a}_i) \leq \int_\Theta u_i(\check{s}_i(f(\hat{\mathbf{a}}, \theta), \theta))d\mu - c_i(\hat{a}_i)$$

So there exists $\epsilon_i \geq 0$ for each $i$ such that

$$\int_\Theta u_i(\check{s}_i(f(\hat{\mathbf{a}}_{-i}, \check{a}_i, \theta), \theta) + \epsilon_i)d\mu - c_i(\check{a}_i) = \int_\Theta u_i(\hat{p}_i^\theta)d\mu - c_i(\hat{a}_i)$$

For each $i$, let $\check{w}_i^\theta = \check{s}_i(f(\hat{\mathbf{a}}_{-i}, \check{a}_i, \theta), \theta) + \epsilon_i$ for all $\theta$. Then $(\check{w}_i^\theta)_{\theta\in\Theta} \in W_i(\check{a}_i, \hat{U}_i)$ for each $i$ and

$$\sum_{i=1}^N \check{w}_i^{\check{\theta}} = \sum_{i=1}^N [\check{s}_i(\check{q}_i, \check{\theta}) + \epsilon_i] \geq \check{q}.$$

We will next prove the converse by using the sharing rule $\hat{\mathbf{s}}$ constructed in section 3. Consider any $i$ and any $a_i$. Because of (A) in the definition of $\hat{\mathbf{s}}$, it is sufficient to show that

$$\int_\Theta u_i(\hat{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta))d\mu - c_i(a_i) \leq \int_\Theta u_i(\hat{s}_i(f(\hat{\mathbf{a}}, \theta), \theta))d\mu - c_i(\hat{a}_i)$$

From (B) and (C) in the definition of $\hat{\mathbf{s}}$, it can be verified that $\hat{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta) \leq w_i^\theta(a_i)$ for every $\theta$. We also know that $(w_i^\theta(a_i))_{\theta\in\Theta} \in W_i(a_i, \hat{U}_i)$. Thus, the monotonicity condition in assumption **A3** implies

$$\int_\Theta u_i(\hat{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta))d\mu - c_i(a_i) \leq \int_\Theta u_i(\hat{s}_i(f(\hat{\mathbf{a}}, \theta), \theta))d\mu - c_i(\hat{a}_i). \quad \|$$

*Proof of Proposition* 2: From Proposition 1 it is clear that the restriction of condition **(I)** to $a_i < \hat{a}_i$ for all $i$ is necessary for implementing the efficient outcome $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta\in\Theta})$.

To prove sufficiency, let us suppose that $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta\in\Theta})$ satisfies the restriction of condition **(I)** to $a_i < \hat{a}_i$ for all $i$. We will modify the sharing rule used in the proof of Proposition 1 appropriately to complete the proof. For each $i$ and each $\theta$, let $\hat{\alpha}_i^\theta = \hat{p}_i^\theta - f(\hat{\mathbf{a}}, \theta)/N$.

For each $i$, let $\hat{s}_i$ be the share function which satisfies the following for each $(q, \theta)$:

(A) if $q \geq f(\hat{\mathbf{a}}, \theta)$, then $\hat{s}_i(q, \theta) = \hat{\alpha}_i^\theta + q/N$;

(B) if $q < f(\hat{\mathbf{a}}, \theta)$ and $q \in \hat{Q}(\theta)$, then $\hat{s}_i(q, \theta) = w_i(q, \theta)$;

(C) if $q < f(\hat{\mathbf{a}}, \theta)$, $q \notin \hat{Q}(\theta)$ and $i \in \eta^{q\theta}$, then $\hat{s}_i(q, \theta) = w_i^\theta(a_i^{q\theta})$;

(D) if $q < f(\hat{\mathbf{a}}, \theta)$ and $i \notin \eta^{q\theta}$, then $\hat{s}_i(q, \theta) = [q - \sum_{j\in\eta^{q\theta}} w_j^\theta(a_j^{q\theta})]/[N - |\eta^{q\theta}|]$.

It can be checked that $\hat{\mathbf{s}} = (\hat{s}_1, ..., \hat{s}_N)$ is a sharing rule, because it ensures that the team's budget is always balanced.

It can be easily checked that, for each $i$ and each $\theta$, $\hat{s}_i(f(\hat{\mathbf{a}}, \theta), \theta) = \hat{p}_i^\theta$. Using (A) in the definition of $\hat{\mathbf{s}}$, for each $i$ and each $a_i \geq \hat{a}_i$, let

$$\hat{u}_i(a_i) = \int_\Theta u_i(\hat{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta))d\mu - c_i(a_i) = \int_\Theta u_i\left(\hat{\alpha}_i^\theta + \frac{f(\hat{\mathbf{a}}_{-i}, a_i, \theta)}{N}\right)d\mu - c_i(a_i)$$

Given any $i$, consider $a_i', a_i'' \geq \hat{a}_i$, and let $a_i^\lambda = \lambda a_i' + (1-\lambda)a_i''$ for each $\lambda \in (0,1)$. For each $\theta$, the concavity of $u_i$ and the convexity of $c_i$ impy

$$\begin{aligned} u_i(\lambda(\hat{\alpha}_i^\theta + f(\hat{\mathbf{a}}_{-i}, a_i', \theta)/N) + (1-\lambda)(\hat{\alpha}_i^\theta + f(\hat{\mathbf{a}}_{-i}, a_i'', \theta)/N)) - c_i(a_i^\lambda) &\geq \\ \lambda[u_i(\hat{\alpha}_i^\theta + f(\hat{\mathbf{a}}_{-i}, a_i', \theta)/N) - c_i(a_i')] + (1-\lambda)[u_i(\hat{\alpha}_i^\theta + f(\hat{\mathbf{a}}_{-i}, a_i'', \theta)/N) - c_i(a_i'')]. \end{aligned} \quad (14)$$

For each $\theta$, the concavity of $f$ in the actions also implies

$$\hat{\alpha}_i^\theta + f(\hat{\mathbf{a}}_{-i}, a_i^\lambda, \theta)/N \geq \lambda(\hat{\alpha}_i^\theta + f(\hat{\mathbf{a}}_{-i}, a_i', \theta)/N) + (1-\lambda)(\hat{\alpha}_i^\theta + f(\hat{\mathbf{a}}_{-i}, a_i'', \theta)/N). \quad (15)$$

Using (14), (15) and the monotonicity of $u_i$, we get

$$\begin{aligned} u_i(\hat{\alpha}_i^\theta + f(\hat{\mathbf{a}}_{-i}, a_i^\lambda, \theta)/N) - c_i(a_i^\lambda) \geq \\ \lambda[u_i(\hat{\alpha}_i^\theta + f(\hat{\mathbf{a}}_{-i}, a_i', \theta)/N) - c_i(a_i')] + (1-\lambda)[u_i(\hat{\alpha}_i^\theta + f(\hat{\mathbf{a}}_{-i}, a_i'', \theta)/N) - c_i(a_i'')]. \end{aligned}$$

So $\hat{u}_i$ is concave. Using the first-order conditions (2) - (4) of the efficiency problem **(EP)**, it can also be verified that the first-order derivative of $\hat{u}_i$ is negative at $\hat{a}_i$. Hence, $\hat{u}_i$ is maximized at $\hat{a}_i$.

Using a similar reasoning as in the proof of Proposition 1, we can also conclude from (B) and (C) in the definition of $\hat{\mathbf{s}}$ that, for each $i$ and every $a_i < \hat{a}_i$,

$$\int_\Theta u_i(\hat{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta))d\mu - c_i(a_i) \leq \int_\Theta u_i(\hat{s}_i(f(\hat{\mathbf{a}}, \theta), \theta))d\mu - c_i(\hat{a}_i). \quad \|$$

*Proof of Proposition* 3: Let $\hat{x} = \int_\Theta f(\hat{\mathbf{a}}, \theta)d\mu$ and $\underline{x} = \max_i \int_\Theta f(\hat{\mathbf{a}}_{-i}, 0, \theta)d\mu$. Then for any $x \geq \underline{x}$, let $a_i(x)$ be the unique action of individual $i$ such that $\int_\Theta f(\hat{\mathbf{a}}_{-i}, a_i(x), \theta)d\mu = x$, where the uniqueness follows from the monotonicity condition in assumption **A1**. Our assumptions also imply that $a_i(x)$ is continuously differentiable. It can be verified by using the weakly separable production function that, for any two individuals $i$ and $j$ and for any $x \geq \underline{x}$, $f(\hat{\mathbf{a}}_{-i}, a_i(x), \theta) = f(\hat{\mathbf{a}}_{-j}, a_j(x), \theta)$ for all $\theta \in \Theta$, and we let $q^{x\theta} = f(\hat{\mathbf{a}}_{-i}, a_i(x), \theta)$.

Suppose the proposition is false, so that, condition **(I)** holds. For each $i$ and each $a_i$, let $(w_i^\theta(a_i))_{\theta\in\Theta} \in W_i(a_i, \hat{U}_i)$ be such that $\sum_{i=1}^N w_i^\theta(a_i^{q\theta}) \geq q$ for all $(q, \theta)$ with $q \in \hat{Q}(\theta)$. So we have the following for each $x \geq \underline{x}$:

$$\int_\Theta w_i^\theta(a_i(x))d\mu = \int_\Theta \hat{p}_i^\theta d\mu - c_i(\hat{a}_i) + c_i(a_i(x)) \quad \forall i$$

which implies that

$$\sum_{i=1}^{N}\left[\int_{\Theta}\hat{p}_i^{\theta}d\mu - c_i(\hat{a}_i) + c_i(a_i(x))\right] = \sum_{i=1}^{N}\int_{\Theta} w_i^{\theta}(a_i(x))d\mu \geq \int_{\Theta} q^{x\theta}d\mu = x$$

For each $x \geq \underline{x}$, define $K(x)$ as follows:

$$K(x) = \sum_{i=1}^{N}\left[\int_{\Theta}\hat{p}_i^{\theta}d\mu - c_i(\hat{a}_i) + c_i(a_i(x))\right] - x$$

Then we know that $K(x) \geq 0$ for all $x \geq \underline{x}$ and $K(\hat{x}) = 0$. It can also be verified that

$$K'(x) = \sum_{i=1}^{N} c_i'(a_i(x))\left[\int_{\Theta} f_i(\hat{\mathbf{a}}_i, a_i(x), \theta)d\mu\right]^{-1} - 1$$

where $K'$ is the derivative of $K$. By using equations (2) - (5) in the above expression, it can be easily verified that $K'(\hat{x}) = N - 1 > 0$. Hence, there exists $x \in [\underline{x}, \hat{x})$ such that $K(x) < K(\hat{x}) = 0$, which is a contradiction. $\quad \|$

*Example* 1: From the definition of $\mathbf{s}^*$, we know that $s_i^*(f(\mathbf{a}^*, \theta), \theta) = \theta = p_i^{\theta*}$ for each $i$ and each $\theta$. By a similar reasoning as in the proof of Proposition 2, we can also conclude that, for each $i$ and for all $a_i \geq a_i^*$,

$$\int_{\Re_+} u_i(s_i^*(f(\mathbf{a}^*, \theta), \theta))e^{-\theta}d\theta - c_i(a_i^*) \geq \int_{\Re_+} u_i(s_i^*(f(a_{-i}^*, a_i, \theta), \theta))e^{-\theta}d\theta - c_i(a_i).$$

For $a_1 < 1$, it can be verified that

$$\begin{aligned}\int_{\Re_+} u_1(s_1^*(f(a_1, a_2^*, \theta), \theta))e^{-\theta}d\theta - c_1(a_1) &= \int_{\Theta_1}[1 - e^{-\theta(a_1+1)}]e^{-\theta}d\theta - (a_1^2/8) \\ &= \left(\frac{5 - a_1^2}{8}\right) + (2 + a_1)^{-1}e^{-(2+a_1)\ln(8/3)} - (2 + a_1)^{-1}\end{aligned}$$

In the second line of the above equality, it can be checked that the first two terms are decreasing and the last term is increasing in $a_1$. So, for $a_1 < 1$, we have

$$\int_{\Re_+} u_1(s_1^*(f(a_1, a_2^*, \theta), \theta))e^{-\theta}d\theta - c_1(a_1) < \frac{3}{8} = \int_{\Re_+} u_1(s_1^*(f(\mathbf{a}^*, \theta), \theta))e^{-\theta}d\theta - c_1(a_1^*).$$

Similarly, for $a_2 < 1$, it can be verified that

$$\begin{aligned}\int_{\Re_+} u_2(s_2^*(f(a_1^*, a_2, \theta), \theta))e^{-\theta}d\theta - c_2(a_2) &= \int_{\Theta_2}[1 - e^{-\theta(a_2+1)}]e^{-\theta}d\theta - (a_2^2/8) \\ &= \frac{3}{8} - (2 + a_2)^{-1}e^{-(2+a_2)\ln(8/3)} - \frac{a_2^2}{8} \\ &< \frac{3}{8} = \int_{\Re_+} u_2(s_2^*(f(\mathbf{a}^*, \theta), \theta))e^{-\theta}d\theta - c_2(a_2^*).\end{aligned}$$

*Example* 2: Suppose the team uses the sharing rule $\tilde{\mathbf{s}}$ as defined in the example. It can be checked that, when both individuals take their efficient actions, $\tilde{\mathbf{s}}$ generates the shares in the given efficient outcome. When both individuals take their efficient actions or one of them unilaterally deviates above it, it can be verified that $\tilde{\mathbf{s}}$ is the same as the sharing rule used in the proof of Proposition 2. Hence, we can conclude that neither individual has the incentive to deviate unilaterally above his/her efficient action.

For each $i$ and each $a_i < 1$, let $Y_i(a_i)$ be the expected utility of individual $i$ when he/she unilaterally deviates to $a_i$. Using $\tilde{\mathbf{s}}$ in the utility functions, we can derive the following for each $i$ and each $a_i < 1$:

$$\begin{aligned} Y_1(a_1) &= \frac{1}{2}\left[3 + 3a_1 - \ln\left(\frac{36}{(e-1)^2}\right)\right] - \frac{3}{4}(a_1)^2 \\ Y_2(a_2) &= \frac{1}{2}\left[2 - e^{-2(1+a_2)-\ln(3/[e-1])+4} - e^{2(1+a_2)-\ln(12/[e-1])-4}\right] - \frac{1}{4}(e^{a_2} - a_2 - 1) \end{aligned}$$

It can be checked that

$$\begin{aligned} \lim_{a_1\to 1} Y_1(a_1) &= (9/4) - \ln(6/[e-1]) \;=\; u_1^* \\ \lim_{a_2\to 1} Y_2(a_2) &< (3/2) - (5/12)e \;<\; u_2^* \end{aligned}$$

One can also easily verify that, for each $i$, $\lim_{a_i\to 1} Y_i'(a_i) = 0$ and $Y_i''(a_i) < 0$ for all $a_i < 1$, where $Y_i'$ and $Y_i''$ are respectively the first and second order derivatives of $Y_i$. Thus, for each $i$, $Y_i(a_i) \leq u_i^*$ for all $a_i < 1$. Therefore, the efficient outcome given in the example is implementable.

*Proof of Proposition 4*: Let $\tilde{\mathbf{s}} \in \mathcal{S}$ locally implement $(\hat{\mathbf{a}}, (\hat{\mathbf{p}}^\theta)_{\theta\in\Theta})$. To complete the proof of necessity, we will first show that $\int_\Theta u_i(\tilde{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta))d\mu$ is discontinuous at $\hat{a}_i$ for some $i$.

Suppose $\int_\Theta u_i(\tilde{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta))d\mu$ is continuous at $\hat{a}_i$ for all $i$. Then

$$\lim_{a_i\to\hat{a}_i^-} \int_\Theta u_i(\tilde{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta))d\mu = \hat{U}_i + c_i(\hat{a}_i) \quad \text{for all } i$$

which implies that $\int_\Theta u_i(p_i^{\theta-})d\mu = \hat{U}_i + c_i(\hat{a}_i)$ for all $i$, where $p_i^{\theta-} = \lim_{a_i\to\hat{a}_i^-} \tilde{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta)$ for each $i$ and each $\theta$. From the continuity and monotonicity of $f$ and the budget balancing condition, it can be verified that $\sum_{i=1}^N p_i^{\theta-} = f(\hat{\mathbf{a}}, \theta)$ for all $\theta$, which implies that $(\hat{\mathbf{a}}, (\mathbf{p}^{\theta-})_{\theta\in\Theta})$ is also an efficient outcome and satisfies (2) - (5). So, for each $i \geq 2$ and each $\theta$, we have $u_i'(p_i^{\theta-}) = u_1'(p_1^{\theta-})/\lambda_i$. Also, from the weak separability of the production function, for each $i \geq 2$ and each $\theta$, we have $f_i(\hat{\mathbf{a}}, \theta) = f_1(\hat{\mathbf{a}}, \theta)/\tau_i$, where $\tau_i = A_1(\hat{\mathbf{a}})/A_i(\hat{\mathbf{a}})$. From the Nash equilibrium condition we also have

$$\int_\Theta [u_i'(p_i^{\theta-})\tilde{s}_i'^-(f(\hat{\mathbf{a}}, \theta), \theta) f_i(\hat{\mathbf{a}}, \theta)]d\mu - c_i'(\hat{a}_i) \geq 0 \quad \text{for all } i$$

where $\tilde{s}_i'^-(f(\hat{\mathbf{a}}, \theta), \theta)$ is the lower derivative of $\tilde{s}_i$ at $f(\hat{\mathbf{a}}, \theta)$. So we get

$$\int_\Theta [u_1'(p_1^{\theta-})\tilde{s}_i'^-(f(\hat{\mathbf{a}}, \theta), \theta) f_1(\hat{\mathbf{a}}, \theta)]d\mu - \tau_i\lambda_i c_i'(\hat{a}_i) \geq 0 \quad \text{for all } i \geq 2$$

From (2) -(4), we also have

$$-\int_\Theta [u_1'(p_1^{\theta-}) f_1(\hat{\mathbf{a}}, \theta)]d\mu = \tau_i\lambda_i c_i'(\hat{a}_i) \quad \text{for all } i$$

where we let $\tau_1 = \lambda_1 = 1$. So we get

$$\int_\Theta [u_1'(p_1^{\theta-}) f_1(\hat{\mathbf{a}}, \theta)[\tilde{s}_i'^{-}(f(\hat{\mathbf{a}}, \theta), \theta) - 1]] d\mu \geq 0 \quad \text{for all } i$$

which implies that

$$\int_\Theta \left[ u_1'(p_1^{\theta-}) f_1(\hat{\mathbf{a}}, \theta) \left( \sum_{i=1}^{N} \tilde{s}_i'^{-}(f(\hat{\mathbf{a}}, \theta), \theta) - N \right) \right] d\mu \geq 0$$

Because of the budget balancing condition, it is straightforward to verify that $\sum_{i=1}^{N} \tilde{s}_i'^{-}(f(\hat{\mathbf{a}}, \theta), \theta) = 1$ for all $\theta$. Hence, we get $(1-N) \int_\Theta [u_1'(p_1^{\theta-}) f_1(\hat{\mathbf{a}}, \theta)] d\mu \geq 0$, which is a contradiction as $(1-N) < 0$, $u_1' > 0$ and $f_1 > 0$.

From the proof of Proposition 2 it can be verified that, without loss of generality, we can assume

$$\int_\Theta u_i(\tilde{s}_i^{+}(f(\hat{\mathbf{a}}, \theta), \theta)) d\mu = \hat{U}_i + c_i(\hat{a}_i) \quad \text{for all } i$$

where $\tilde{s}_i^{+}(f(\hat{\mathbf{a}}, \theta), \theta) = \lim_{a_i \to \hat{a}_i^{+}} \tilde{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta)$. Then the Nash equilibrium condition and the discontinuity of $\int_\Theta u_i(\tilde{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta)) d\mu$ at $\hat{a}_i$ for some $i$ imply that

$$\int_\Theta u_i(p_i^{\theta-}) d\mu \leq \hat{U}_i + c_i(\hat{a}_i) \quad \text{for all } i$$

with strict inequality holding for some $i$. Furthermore, we know from the budget balancing condition that $\sum_{i=1}^{N} p_i^{\theta-} = f(\hat{\mathbf{a}}, \theta)$ for all $\theta$. Now, because of the monotonicity of $u_i$, there is no loss of generality in assuming that $\int_\Theta u_i(p_i^{\theta-}) d\mu < \hat{U}_i + c_i(\hat{a}_i)$ for all $i$.

For the sufficiency, let $\tilde{\mathbf{s}} \in \mathcal{S}$ be a sharing rule such that it is the same as the sharing rule used in the proof of Proposition 2 for all $(q, \theta)$ with $q \geq f(\hat{\mathbf{a}}, \theta)$ and $\lim_{q \to f(\hat{\mathbf{a}}, \theta)^-} \tilde{s}_i(q, \theta) = \bar{p}_i^{\theta}$ for each $i$ and each $\theta$. So, from the proof of Proposition 2 we know that, for each $i$,

$$\begin{aligned} \int_\Theta u_i(\tilde{s}_i(f(\hat{\mathbf{a}}, \theta), \theta)) d\mu &= \hat{U}_i + c_i(\hat{a}_i) \\ \int_\Theta u_i(\tilde{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta)) d\mu &\leq \hat{U}_i + c_i(a_i) \quad \text{for all } a_i \geq \hat{a}_i \end{aligned}$$

We also have

$$\lim_{a_i \to \hat{a}_i^-} \int_\Theta u_i(\tilde{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta)) d\mu = \int_\Theta u_i(\bar{p}_i^{\theta}) d\mu < \hat{U}_i + c_i(\hat{a}_i) \quad \text{for all } i$$

Thus, there exists $\epsilon > 0$ such that, for each $i$,

$$\int_\Theta u_i(\tilde{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta)) d\mu - c_i(a_i) \leq \hat{U}_i \quad \text{for all } a_i \in (\hat{a}_i - \epsilon, \hat{a}_i). \quad \|$$

*Proof of Proposition* 5: To prove the necessity, suppose the given efficient outcome can be implemented using a sharing rule $\hat{\mathbf{s}}$. For each $i \geq 2$ and each $\theta$, let

$$p_i^{\theta x} = \hat{s}_i(F(x, \theta), \theta) = \hat{s}_i(f(\hat{\mathbf{a}}_{-i}, \hat{\rho}_i(x), \theta), \theta) \quad \text{for all } x < A(\hat{\mathbf{a}})$$

Then, by the budget balancing condition, for each $\theta$,

$$\begin{aligned}\hat{s}_1(F(x,\theta),\theta) &= F(x,\theta) - \sum_{i=2}^{N} \hat{s}_i(F(x,\theta),\theta) \\ &= F(x,\theta) - \sum_{i=2}^{N} p_i^{\theta x} \quad \text{for all } x < A(\hat{\mathbf{a}})\end{aligned}$$

For each $i \geq 2$ and each $x < A(\hat{\mathbf{a}})$, we have

$$\int_\Theta u_i(p_i^{\theta x})d\mu = \int_\Theta u_i(\hat{s}_i(f(\hat{\mathbf{a}}_{-i}, \hat{\rho}_i(x), \theta), \theta))d\mu \leq \hat{U}_i + c_i(\hat{\rho}_i(x))$$

For each $x < A(\hat{\mathbf{a}})$, we also have

$$\int_\Theta u_1(F(x,\theta) - \sum_{i=2}^{N} p_i^{\theta x})d\mu = \int_\Theta u_1(\hat{s}_1(F(x,\theta),\theta))d\mu \leq \hat{U}_1 + c_1(\hat{\rho}_1(x))$$

Hence, $\hat{V}(x) \leq \hat{U}_1 + c_1(\hat{\rho}_1(x))$ for every $x < A(\hat{\mathbf{a}})$.

To prove the sufficiency, suppose $\hat{V}(x) \leq \hat{U}_1 + c_1(\hat{\rho}_1(x))$ for every $x < A(\hat{\mathbf{a}})$. For each $x < A(\hat{\mathbf{a}})$, let $(\bar{\mathbf{p}}_{-1}^{\theta x})_{\theta \in \Theta}$ be a solution of the problem that defines $\hat{V}(x)$. Given any $(q, \theta)$ with $q < f(\hat{\mathbf{a}}, \theta)$, let $x(q,\theta)$ be such that $F(x(q,\theta),\theta) = q$. Define the sharing rule $\bar{\mathbf{s}}$ as follows: for each $(q,\theta)$,

(i) $\bar{\mathbf{s}}$ is the same as the sharing rule in the proof of Proposition 2 when $q \geq f(\hat{\mathbf{a}}, \theta)$;

(ii) if $q < f(\hat{\mathbf{a}}, \theta)$, then

$$\bar{s}_i(q,\theta) = \begin{cases} \bar{p}_i^{\theta x(q,\theta)} & \text{if } i \geq 2 \\ F(x(q,\theta),\theta) - \sum_{i=2}^{N} \bar{p}_i^{\theta x(q,\theta)} & \text{if } i = 1 \end{cases}$$

Then from the proof of Proposition 2 we have the following for each $i$:

$$\int_\Theta u_i(\bar{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta))d\mu \leq \hat{U}_i + c_i(a_i) \text{ for all } a_i \geq \hat{a}_i \text{ and } \bar{s}_i(f(\hat{\mathbf{a}},\theta),\theta) = \hat{p}_i^\theta \text{ for all } \theta.$$

For each $i$ and each $\theta$, it can be verified from the definitions that, for every $a_i < \hat{a}_i$,

$$x(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta) = A(\hat{\mathbf{a}}_{-i}, a_i) \text{ and } \hat{\rho}_i(x(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta)) = a_i$$

So, for each $i \geq 2$ and each $a_i < \hat{a}_i$, we have

$$\int_\Theta u_i(\bar{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta), \theta))d\mu = \int_\Theta u_i(\bar{p}_i^{\theta A(\hat{\mathbf{a}}_{-i}, a_i)})d\mu \leq \hat{U}_i + c_i(a_i).$$

Furthermore, for each $a_1 < \hat{a}_1$, we have

$$\int_\Theta u_1(\bar{s}_1(f(\hat{\mathbf{a}}_{-1}, a_1, \theta), \theta))d\mu = \int_\Theta u_1(F(A(\hat{\mathbf{a}}_{-1}, a_1), \theta) - \sum_{i=2}^{N} \bar{p}_i^{\theta A(\hat{\mathbf{a}}_{-1}, a_1)})d\mu \leq \hat{U}_1 + c_1(a_1). \quad \|$$

*Example* 3: Suppose the team uses the sharing rule $\hat{\mathbf{s}}$ as defined in the example. For each $i$ and each $a_i$, let $Z_i(a_i)$ be $i$'s expected utility when he/she unilaterally deviates to $a_i$. Then it can be verified that, for each $i$ and each $a_i$,

$$Z_i(a_i) = (1/4)[4(a_i)^{1/2} - 2(a_i)^{1/4} + K_i] - (3/16)(a_i)^2, \quad \text{where } -K_2 = K_1 = \textstyle\sum_{j=1}^{4} \hat{\alpha}^j.$$

So we have the following for each $i$ :

$$\begin{aligned} Z_i'(a_i) &= (1/4)[2(a_i)^{-1/2} - (1/2)(a_i)^{-3/4}] - (3/8)a_i \text{ for all } a_i \\ Z_i''(a_i) &= (1/4)[-(a_i)^{-3/2} + (3/8)(a_i)^{-7/4}] - (3/8) \text{ for all } a_i \end{aligned}$$

which implies that $Z_i'(a_i^*) = 0$ and $Z_i''(a_i^*) < 0$. For each $i$, although $Z_i(a_i)$ is not globally concave, it can be verified that there is a $\delta_i \in (0,1)$ such that

$$Z_i'(a_i) \begin{cases} = & 0 \quad \text{if } a_i = \delta_i \text{ or } 1 \\ < & 0 \quad \text{if } a_i \in [0, \delta_i) \cup (1,2] \\ > & 0 \quad \text{if } a_i \in (\delta_i), 1) \end{cases}$$

Also, for each $i$,

$$Z_i(0) = (K_i/4) < (K_i/4) + (5/16) = Z_i(a_i^*).$$

Hence, $\mathbf{a}^* = (1,1)$ is a Nash equilibrium conditional on $\hat{\mathbf{s}}$ and the efficient outcome $(\mathbf{a}^*, (\hat{\mathbf{p}}^\theta)_{\theta \in \Theta})$ is implementable.

Now, suppose the state is not observable/verifiable, but there is a sharing rule $\mathbf{s}$ (which is a function only of the final output and not the state) that can support $\mathbf{a}^*$ as a Nash equilibrium. Then, for each $i$ and for all $a_i$,

$$(1/4)\left[\sum_{j=1}^{4} s_i(f(a_{-1}^*, a_i, \theta^j))\right] - (3/16)(a_i)^2 \leq (1/4)\left[\sum_{j=1}^{4} s_i(f(\mathbf{a}^*, \theta^j))\right] - (3/16) = s_i(1) - (3/16)$$

which implies that

$$(1/2)[s_i(a^{1/2}) + s_i(a^{1/4})] - (3/16)a^2 \leq s_i(1) - (3/16) \text{ for all } a \in [0,2]$$

So we have

$$(1/2)\sum_{i=1}^{2}[s_i(a^{1/2}) + s_i(a^{1/4})] - (3/8)a^2 \leq \sum_{i=1}^{2} s_i(1) - (3/8) \text{ for all } a \in [0,2]$$

which together with the budget balancing condition imply

$$(1/2)[a^{1/2} + a^{1/4}] - (3/8)a^2 \leq 1 - (3/8) = (5/8) \text{ for all } a \in [0,2].$$

For each $a \in [0,2]$, let $D(a) = (1/2)[a^{1/2} + a^{1/4}] - (3/8)a^2 - (5/8)$. So we know that $D(a) \leq 0$ for all $a \leq 1$ and $D(1) = 0$. By differentiating $D(a)$ it can be easily checked that $D'(1) < 0$ and $D''(a) < 0$ for all $a$. Hence, there exists $\tilde{a} < 1$ such that $D(a) > 0$ for all $a \in (\tilde{a}, 1)$, a contradiction. Thus, we have shown that, if the state is not observable/verifiable, then there is no sharing rule which can support $\mathbf{a}^*$ as a Nash equilibrium.

Now, suppose the action spaces are discretised, so that, the action space of each individual $i$ is $E_i$ (as given in the example). Then it immediately follows from what we have shown above that, if the state is observable/verifiable, then all efficient outcomes are still implementable. It can also be readily verified that the interval $(\tilde{a}, 1)$ we derived above has a nonempty intersection with the discrete action space $E_i$ for each $i$. Therefore, when the state is not observable/verifiable, there is no sharing rule that can support $\mathbf{a}^*$ as a Nash equilibrium.

*Proof of Proposition* 6: To prove the necessity, suppose the given efficient outcome can be implemented using a sharing rule $\bar{\mathbf{s}}$. From the Nash equilibrium condition and $\bar{\mathbf{s}} \in \mathcal{S}$, we have

$$\lim_{a_i \to \hat{a}_i^-} \sum_{h=1}^{M} \pi^h \bar{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta^h), \theta^h) - c_i(\hat{a}_i) \leq \hat{U}_i \quad \text{for each } i$$

which together with the assumptions in **A1** imply

$$\sum_{h=1}^{M} \pi^h \left[ \lim_{q \to f(\hat{\mathbf{a}}, \theta^h)^-} \bar{s}_i(q, \theta^h) \right] - c_i(\hat{a}_i) \leq \hat{U}_i \quad \text{for each } i.$$

So by summing over all individuals we get

$$\sum_{h=1}^{M} \pi^h \left[ \lim_{q \to f(\hat{\mathbf{a}}, \theta^h)^-} \sum_{i=1}^{N} \bar{s}_i(q, \theta^h) \right] - \sum_{i=1}^{N} c_i(\hat{a}_i) \leq \sum_{i=1}^{N} \hat{U}_i.$$

From the budget balancing condition we also have

$$\lim_{q \to f(\hat{\mathbf{a}}, \theta^h)^-} \sum_{i=1}^{N} \bar{s}_i(q, \theta^h) = f(\hat{\mathbf{a}}, \theta^h) \quad \text{for each } h.$$

So we get $\sum_{h=1}^{M} \pi^h f(\hat{\mathbf{a}}, \theta^h) - \sum_{i=1}^{N} c_i(\hat{a}_i) \leq \sum_{i=1}^{N} \hat{U}_i$. Hence, it must be the case that

$$\lim_{a_i \to \hat{a}_i^-} \sum_{h=1}^{M} \pi^h \bar{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta^h), \theta^h) - c_i(\hat{a}_i) = \hat{U}_i \quad \text{for each } i$$

Then the Nash equilibrium condition and $\bar{\mathbf{s}} \in \mathcal{S}$ imply that

$$\sum_{h=1}^{M} \pi^h \bar{s}_i'^-(f(\hat{\mathbf{a}}, \theta^h), \theta^h) f_i(\hat{\mathbf{a}}, \theta^h) - c_i'(\hat{a}_i) \geq 0 \quad \text{for each } i$$

where $\bar{s}_i'^-(f(\hat{\mathbf{a}}, \theta^h), \theta^h)$ is the lower derivative of $\bar{s}_i$ at $f(\hat{\mathbf{a}}, \theta^h)$. For each $i \leq N-1$ and each $h$, let $\gamma_i^h = \bar{s}_i'^-(f(\hat{\mathbf{a}}, \theta^h), \theta^h)$. Then the budget balancing condition implies $\bar{s}_N'^-(f(\hat{\mathbf{a}}, \theta^h), \theta^h) = 1 - \sum_{i=1}^{N-1} \gamma_i^h$ for all $h$. So we have the following

$$-\sum_{h=1}^{M} \pi^h f_i(\hat{\mathbf{a}}, \theta^h) \gamma_i^h \leq -c_i'(\hat{a}_i) \quad \text{for all } i \leq N-1; \tag{16}$$

$$\sum_{i=1}^{N-1} \sum_{h=1}^{M} \pi^h f_N(\hat{\mathbf{a}}, \theta^h) \gamma_i^h \leq \sum_{h=1}^{M} \pi^h f_N(\hat{\mathbf{a}}, \theta^h) - c_N'(\hat{a}_N) = 0 \tag{17}$$

where the last equality follows from the first order conditions of the total surplus maximization problem for deriving efficient outcomes. Now suppose $f_i(\hat{\mathbf{a}}, \theta^h)/f_j(\hat{\mathbf{a}}, \theta^h) = f_i(\hat{\mathbf{a}}, \theta^l)/f_j(\hat{\mathbf{a}}, \theta^l)$ for every $i, j$ and every $h, l$. For each $i \leq N-1$, let $\tau_i = f_N(\hat{\mathbf{a}}, \theta^h)/f_i(\hat{\mathbf{a}}, \theta^h) > 0$ for all $h$. Then we have the following

$$\pi^h f_N(\hat{\mathbf{a}}, \theta^h) - \tau_i \pi^h f_i(\hat{\mathbf{a}}, \theta^h) = 0 \quad \text{for all } i \leq N-1 \text{ and for all } h; \tag{18}$$

$$-\sum_{i=1}^{N-1} \tau_i c_i'(\hat{a}_i) < 0. \tag{19}$$

However, according to Theorem 22.1 in Rockafellar (1970), the two systems of inequalities given by (16)-(17) and (18)-(19) cannot hold simultaneously. This completes the proof of necessity.

To prove sufficiency, suppose there exist $i, j$ such that $f_i(\hat{\mathbf{a}}, \theta^h)/f_j(\hat{\mathbf{a}}, \theta^h) \neq f_i(\hat{\mathbf{a}}, \theta^l)/f_j(\hat{\mathbf{a}}, \theta^l)$ for some $h, l$. Then it can be checked that there are no non-negative real numbers $\bar{\tau}_1, ..., \bar{\tau}_{N-1}$ such that $\pi^h f_N(\hat{\mathbf{a}}, \theta^h) - \bar{\tau}_i \pi^h f_i(\hat{\mathbf{a}}, \theta^h) = 0$ for all $i \leq N-1$ and for all $h$. Hence, Theorem 22.1 in Rockafellar (1970) implies that there exist real numbers $\acute{\gamma}_i^h$ for each $i \leq N-1$ and each $h$ such that

$$\begin{aligned} -\sum_{h=1}^{M} \pi^h f_i(\hat{\mathbf{a}}, \theta^h)\acute{\gamma}_i^h &\leq -c_i'(\hat{a}_i) \quad \text{for all } i \leq N-1; \\ \sum_{i=1}^{N-1}\sum_{h=1}^{M} \pi^h f_N(\hat{\mathbf{a}}, \theta^h)\acute{\gamma}_i^h &\leq \sum_{h=1}^{M} \pi^h f_N(\hat{\mathbf{a}}, \theta^h) - c_N'(\hat{a}_N) = 0 \end{aligned}$$

For each $h$, let $\acute{\gamma}_N^h = 1 - \sum_{i=1}^{N-1} \acute{\gamma}_i^h$. So we have

$$\sum_{h=1}^{M} \pi^h f_i(\hat{\mathbf{a}}, \theta^h)\acute{\gamma}_i^h - c_i'(\hat{a}_i) \geq 0 \quad \text{for all } i. \tag{20}$$

Without loss of generality, let $\pi^1 f_N(\hat{\mathbf{a}}, \theta^1)/\pi^2 f_N(\hat{\mathbf{a}}, \theta^2) < \pi^1 f_1(\hat{\mathbf{a}}, \theta^1)/\pi^2 f_1(\hat{\mathbf{a}}, \theta^2)$ and pick any $\delta^1, \delta^2 > 0$ such that $\delta_2/\delta_1 = \pi^1 f_N(\hat{\mathbf{a}}, \theta^1)/\pi^2 f_N(\hat{\mathbf{a}}, \theta^2)$. Also, for each $i$ and each $h$, let $\bar{\gamma}_i^h$ be the same as $\acute{\gamma}_i^h$ except $\bar{\gamma}_1^1 = \acute{\gamma}_1^1 + \delta^1$, $\bar{\gamma}_1^2 = \acute{\gamma}_1^2 - \delta^2$, $\bar{\gamma}_N^1 = \acute{\gamma}_N^1 - \delta^1$ and $\bar{\gamma}_N^2 = \acute{\gamma}_N^2 + \delta^2$. Then it can be verified that $\sum_{i=1}^{N} \bar{\gamma}_i^h = 1$ for each $h$,

$$\begin{aligned} \sum_{h=1}^{M} \pi^h f_1(\hat{\mathbf{a}}, \theta^h)\bar{\gamma}_1^h - c_1'(\hat{a}_1) &= \\ \sum_{h=1}^{M} \pi^h f_1(\hat{\mathbf{a}}, \theta^h)\acute{\gamma}_1^h + \delta^1\pi^1 f_1(\hat{\mathbf{a}}, \theta^1) - \delta^2\pi^2 f_1(\hat{\mathbf{a}}, \theta^2) - c_1'(\hat{a}_1) &> 0, \\ \sum_{h=1}^{M} \pi^h f_N(\hat{\mathbf{a}}, \theta^h)\bar{\gamma}_N^h - c_N'(\hat{a}_N) &= \\ \sum_{h=1}^{M} \pi^h f_N(\hat{\mathbf{a}}, \theta^h)\acute{\gamma}_N^h - \delta^1\pi^1 f_N(\hat{\mathbf{a}}, \theta^1) + \delta^2\pi^2 f_N(\hat{\mathbf{a}}, \theta^2) - c_N'(\hat{a}_N) &\geq 0, \\ \sum_{h=1}^{M} \pi^h f_i(\hat{\mathbf{a}}, \theta^h)\bar{\gamma}_i^h - c_i'(\hat{a}_i) &= \\ \sum_{h=1}^{M} \pi^h f_i(\hat{\mathbf{a}}, \theta^h)\acute{\gamma}_i^h - c_i'(\hat{a}_i) &\geq 0 \quad \text{for all } i \in \{2, ..., N-1\}. \end{aligned}$$

Thus, because $f_i(\hat{\mathbf{a}}, \theta^h) > 0$ for all $i$ and for all $h$, we can rewrite (20) as

$$\sum_{h=1}^{M} \pi^h f_i(\hat{\mathbf{a}}, \theta^h)\acute{\gamma}_i^h - c_i'(\hat{a}_i) > 0 \quad \text{for all } i. \tag{21}$$

For each $i$ and each $h$, define $\acute{\alpha}_i^h = \hat{p}_i^{\theta^h} - \acute{\gamma}_i^h f(\hat{\mathbf{a}}, \theta^h)$. So $\sum_{i=1}^{N} \acute{\gamma}_i^h = 1$ implies $\sum_{i=1}^{N} \acute{\alpha}_i^h = 0$ for each $h$. Let $\acute{\mathbf{s}}$ be the sharing rule which is the same as the one used in the proof of Proposition

2 for all $(q, \theta^h)$ with $q \geq f(\hat{\mathbf{a}}, \theta^h)$, and for each $(q, \theta^h)$ with $q < f(\hat{\mathbf{a}}, \theta^h)$, $\acute{s}_i(q, \theta^h) = \acute{\alpha}_i^h + \acute{\gamma}_i^h q$ for each $i$. From the construction it can be verified that $\acute{s}_i(f(\hat{\mathbf{a}}, \theta^h), \theta^h) = \hat{p}_i^{\theta^h}$ for each $i$ and each $h$, and no team member has the incentive to unilaterally deviate above his/her efficient action when $\acute{\mathbf{s}}$ is adopted. Furthermore, when $\acute{\mathbf{s}}$ is adopted, (21) implies that there exists $\epsilon > 0$ such that, for each $i$,

$$\sum_{h=1}^{M} \acute{s}_i(f(\hat{\mathbf{a}}_{-i}, a_i, \theta^h), \theta^h) - c_i(a_i) \leq \hat{U}_i \quad \text{for all } a_i \in (\hat{a}_i - \epsilon, \hat{a}_i). \quad \|$$

*Proof of Proposition* 7: Since $\bar{s}_i(q, \theta)$ is nondecreasing in a neighbourhood of $f(\bar{\mathbf{a}}, \theta)$ for each $i$ and each $\theta$, the upper and lower limits of $\bar{s}_i(q, \theta)$ exist at $q = f(\bar{\mathbf{a}}, \theta)$. So the budget balancing condition implies that $\bar{s}_i(q, \theta)$ is continuous at $q = f(\bar{\mathbf{a}}, \theta)$. Because of the validity of the first-order approach and assumptions **A1, A2** and **A3**, we also know that, for each $i$ and each $\theta$, the upper and lower derivatives of $\bar{s}_i(q, \theta)$ exist at $q = f(\bar{\mathbf{a}}, \theta)$. Thus, (6) follows from the first-order conditions of Nash equilibrium.

It can be checked from the definitions that

$$\tilde{s}_i(f(\bar{\mathbf{a}}, \theta), \theta) = \bar{p}_i^\theta \quad \text{for each } i \text{ and for all } \theta. \tag{22}$$

Furthermore, from equations (6), (7), (11) and (22), we can derive the following first-order conditions for $\bar{\mathbf{a}}$ to be a Nash equilibrium when $\tilde{\mathbf{s}}$ is used:

$$\int_\Theta [u_i'(\tilde{s}_i(f(\bar{\mathbf{a}}, \theta), \theta))\tilde{s}_i'(f(\bar{\mathbf{a}}, \theta), \theta) f_i(\bar{\mathbf{a}}, \theta)]\, d\mu - c_i'(\bar{a}_i) = 0 \quad \text{for all } i.$$

To complete the proof, it is then sufficient to show that, for each $i$, $u_i(\tilde{s}_i(f(\mathbf{a}, \theta), \theta))$ is concave in $\mathbf{a}$ for every $\theta$.

Consider $\mathbf{a}', \mathbf{a}''$ and $\lambda \in (0, 1)$. Let $\mathbf{a}^\lambda = \lambda \mathbf{a}' + (1-\lambda)\mathbf{a}''$. For each $i$, the concavity of $u_i$ implies

$$\begin{aligned} u_i(\lambda \tilde{s}_i(f(\mathbf{a}', \theta), \theta) + (1-\lambda)\tilde{s}_i(f(\mathbf{a}'', \theta), \theta)) &\geq \\ \lambda u_i(\tilde{s}_i(f(\mathbf{a}', \theta), \theta)) + (1-\lambda) u_i(\tilde{s}_i(f(\mathbf{a}'', \theta), \theta)) &\quad \text{for all } \theta. \end{aligned} \tag{23}$$

Using (11), $\lambda \tilde{s}_i(f(\mathbf{a}', \theta), \theta) + (1-\lambda)\tilde{s}_i(f(\mathbf{a}'', \theta), \theta) = \tilde{\alpha}_i^\theta + \tilde{\gamma}_i^\theta[\lambda f(\mathbf{a}', \theta) + (1-\lambda) f(\mathbf{a}'', \theta)]$ for each $i$ and each $\theta$. For each $i$ and each $\theta$, the concavity of $f$ and $\tilde{\gamma}_i^\theta \geq 0$ also imply $\tilde{\gamma}_i^\theta f(\mathbf{a}^\lambda, \theta) \geq \tilde{\gamma}_i^\theta[\lambda f(\mathbf{a}', \theta) + (1-\lambda) f(\mathbf{a}'', \theta)]$. So, for each $i$ and each $\theta$, we have

$$\tilde{\alpha}_i^\theta + \tilde{\gamma}_i^\theta f(\mathbf{a}^\lambda, \theta) \geq \lambda \tilde{s}_i(f(\mathbf{a}', \theta), \theta) + (1-\lambda)\tilde{s}_i(f(\mathbf{a}'', \theta), \theta) \tag{24}$$

Hence, (11), (23), (24) and the monotonicity of the utility functions imply $u_i(\tilde{s}_i(f(\mathbf{a}^\lambda, \theta), \theta)) \geq \lambda u_i(\tilde{s}_i(f(\mathbf{a}', \theta), \theta)) + (1-\lambda) u_i(\tilde{s}_i(f(\mathbf{a}'', \theta), \theta))$ for each $i$ and each $\theta$. Thus, for each $i$, $u_i(\tilde{s}_i(f(\mathbf{a}, \theta), \theta))$ is concave in $\mathbf{a}$ for every $\theta$. $\|$

*Proof of Proposition* 9: Suppose the given outcome $(\mathbf{a}, (\mathbf{p}^\theta)_{\theta \in \Theta})$ belongs to the restricted class of implementable outcomes. According to Proposition 7, $(\mathbf{a}, (\mathbf{p}^\theta)_{\theta \in \Theta})$ must be implementable by a linear sharing rule **s** such that, for each $i$, $s_i(q, \theta) = \alpha_i^\theta + \gamma_i^\theta q$ for all $(q, \theta)$, with $\gamma_i^\theta \geq 0$ for each $i$ and each $\theta$, $\sum_{j=1}^N \gamma_j^\theta = 1$ and $\sum_{j=1}^N \alpha_j^\theta = 0$. Then from the first-order condition for expected utility maximization at a Nash equilibrium, we have the following for each $i$:

$$\int_\Theta \left[ u_i'(p_i^\theta)\gamma_i^\theta f_i(\mathbf{a}, \theta) \right] d\mu - c_i'(a_i) = 0.$$

Suppose the given outcome $(\mathbf{a}, (\mathbf{p}^\theta)_{\theta \in \Theta})$ is such that there exist $\gamma_i^\theta \geq 0$ for each $i$ and each $\theta$, with $\sum_{j=1}^N \gamma_j^\theta = 1$ and

$$\int_\Theta \left[ u_i'(p_i^\theta)\gamma_i^\theta f_i(\mathbf{a}, \theta) \right] d\mu - c_i'(a_i) = 0.$$

For each $i$ and each $\theta$, let $\alpha_i^\theta = p_i^\theta - \gamma_i^\theta f(\mathbf{a}, \theta)$. Then $\sum_{j=1}^N \alpha_j^\theta = \sum_{j=1}^N p_j^\theta - f(\mathbf{a}, \theta) \sum_{j=1}^N \gamma_j^\theta = 0$ for all $\theta$. So we can define a linear sharing rule **s** by $s_i(q, \theta) = \alpha_i^\theta + \gamma_i^\theta q$ for each $i$ and for all $(q, \theta)$. Therefore, using a similar reasoning as in the proof of Proposition 7, we can conclude that **s** implements the outcome $(\mathbf{a}, (\mathbf{p}^\theta)_{\theta \in \Theta})$. ‖