Effects of Uninsurable Idiosyncratic Income Risks in a Heterogeneous-Agent Incomplete-Insurance Economy with Human Capital

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Abstract

In this paper, I study the effects of uninsurable idiosyncratic income risks on households' consumption and on income inequality in a heterogeneous-agent incomplete-insurance framework with human capital. Households accumulate human capital by withdrawing time from work. To begin with, this is the only asset available to households to hedge against risks to consumption. I study the optimal decision of households to invest time in human capital for income risks of varying persistence and variance and find that time invested in human capital increases steadily as income shocks become more positive, the exact pattern depending on the inherent characteristics of the shock process and households' appetite for risk. Highly persistent and highly variable income shocks cause the highest variation in the dynamics of time invested in human capital, human capital levels, and consumption. Income inequalities are also highest when the risks are more persistent and dispersed. Subsequently, when a financial asset is introduced, time invested in human capital increases with human capital levels for households with low levels of bond holding, and more so when they have bad realizations of the income shock, than when they realize a good income shock. Availability of a financial asset to transfer consumption across time allows households to leverage both assets - human capital and bonds - in complementary ways to insure against income risks which changes the pattern of households' responses to good and bad income shocks in terms of time invested in human capital across levels of human capital and realizations of the income shock.

1 Introduction

How do risks affect households' consumption and income inequality? This is the question I am interested in exploring in this paper. In trying to uncover the effect of uninsurable idiosyncratic income risks on households' consumption, I use human capital as the key component through which these effects are channeled to eventually play out on households' consumption in this economy.

To investigate this question, I set up a heterogeneous-agent incomplete-insurance economy where households face uninsurable idiosyncratic income risks every period. To begin with, the only asset that households have is their human capital. They supply labour to a representative firm which produces goods, and purchase consumption with their labour earnings. Their human capital level together with the time they decide to devote to production is their effective labour supply which determines their total labour earnings. Faced with uncertainty from exogenous risks to income, households face a trade-off each period: whether to spend more time supplying labour today which enhances consumption today or instead invest more time in human capital today which improves consumption tomorrow. It is this critical decision - how much time to invest in human capital - they must make each period to insure, to some extent, against the idiosyncratic income risk and achieve a smoother consumption profile over their lifetime.

I utilize this model to study how uninsurable idiosyncratic risks to households' earnings affect their consumption through their decision to invest in human capital, particularly when there are no financial assets through which they can borrow or save to transfer consumption across time. One way of addressing this question is to subject households to different shock processes that vary in their persistence and variability, making the risk to their consumption more or less persistent and dispersed. The results from the analysis show that as the idiosyncratic income shocks become more positive, the time invested in human capital increases, for any given level of human capital today across different types of shocks. The exact pattern of the response, however, depends on the particular characteristics of the risk process, namely its persistence and variance, and the households' degree of risk aversion.

Having households' incomes be determined endogenously by their labour supply decisions and their human capital investment decisions enables a closer look at the mechanism at work behind the scenes in the way these risks affect households' consumption. It also adds a tinge of realism as human capital is, indeed, a key determinant of households' earnings in the real world. The resulting distributions of income across households are in part the outcome of more fundamental decisions made by households themselves every period. The distribution of income that emerges from the model is rationalized by the decision of households to invest in human capital. This not only makes the dynamics behind income and skill distributions clearer but also makes comparison of income and skill distributions resulting from different risk processes more meaningful. My results show that highly persistent and highly variable risks result in households being widely dispersed over human capital levels. As shocks become less variable, households are more tightly clustered over small ranges of human capital levels.

Comparing income distributions, different patterns in income inequality emerge as an outcome of the differences in persistence and variability of the risks facing households. The endogenous determination of income through human capital decisions of households offers deeper insights into the process behind these patterns. My results show that income inequality is highest when the shocks to earnings are more dispersed, and also more persistent.

Unraveling how households with different levels of human capital respond to different types of risks, with their decision to invest in acquiring human capital reveals interesting patterns in the dynamics of the human capital investment decision itself, the resulting human capital levels, and consumption. Simulations for a household show that the evolution of these variables - their mean value and dispersion around the mean - depends critically on how persistent and variable the risk process is.

To make things more interesting, I subsequently introduce a financial asset - a risk-free discount bond - into the economy, to see how decisions to invest in human capital change. With the availability of bonds, households now enjoy the luxury to save their current earnings for future consumption or to borrow against future earnings for current consumption. Households in this augmented economy now have two mechanisms via which they hedge against risks to consumption for a smoother consumption profile. How far do households leverage each of these assets - human capital and bonds - to cushion against exogenous idiosyncratic income shocks, is what I set forth to analyse using this model. The results from this model show that at lower levels of bond holdings, time invested in human capital actually increases with the level of human capital, and the increase is much starker when the realization of the income shock is lower than when it is higher. This is an interesting departure from the results obtained in the first model, and is the direct effect of the availability of a financial asset that households now leverage to invest more aggressively in human capital while they borrow now to consume when a bad shock hits, securing current and future consumption.

Households' savings and human capital investment decisions have important bearings on the distribution of income across households, as also income inequality. It makes for an interesting comparison to the scenario when households had only their human capital to invest in: income inequality in both models increases with the persistence and variance of the income risk process. How households respond to risks with their decisions to invest time in human capital and to invest in the financial asset translates into interesting patterns in the evolution of human capital, savings, income, and consumption over time.

The rest of the paper is structured as follows. Section 2 reviews the related literature. Section 3 presents the model with human capital and no financial assets: section 3.1 lays out the model; section 3.2 describes a plausible set of parameters for the model; section 3.3 describes the computational method used for solving the model; and section 3.4 presents the results. Section 4 augments this model to incorporate a financial asset along with human capital: section 4.1 presents the augmented model; section 4.2 describes the additional parameters; section 4.3 details the computation of this augmented model; and section 4.4 presents the results. Finally, Section 5 concludes.

2 Related Literature

This paper belongs to the larger literature that develops the Imrohoroglu-Huggett-Aiyagari framework of heterogeneous-agent models to study how heterogeneity can be modeled to explain aggregate outcomes, which the representative-agent framework fails to explain. The most important elements in these models are the presence of uninsurable idiosyncratic income risks and borrowing constraints.

Huggett (1993) provided a possible resolution of the equity-premium puzzle by using a heterogeneous-agent-incomplete-insurance set up where incomes are exogenous idiosyncratic endowments that households receive. Heterogeneity among households and the absence of full-insurance cause risk-free real interest rates to be low in his model, as also observed in data. Aiyagari (1994) includes precautionary savings motive and liquidity constraints in the presence of uninsured idiosyncratic risks to address the impact on aggregate savings rate, and he does this in a production economy set up. My paper departs from Huggett (1993) in that I include human capital (first without financial savings, and subsequently with a financial asset) and that households' incomes are endogenously determined through labour earnings rather than exogenous endowments. While mine is a production economy, unlike Aiyagari (1994), I abstract from physical capital and instead have human capital and time supplied for labour by households as inputs in the goods production process.

There is a vast and growing literature that looks at possible linkages between earnings inequalities and a theory of human capital. Lucas (1988) first presented human capital as a technology that could serve as an engine of growth in the Neoclassical growth model framework. Ben-Porath's (1967) work on production of human capital and the life-cycle of earnings provided a functional form for the accumulation of human capital, some variation of which is used in most studies that incorporate human capital to explain inequalities in earnings.

Heckman, Lochner, and Taber (1998) were the first to study the evolution of wage inequality through the human capital lens. They develop and estimate an overlapping-generations-general-equilibrium model of labor earnings, skill formation and physical capital accumulation with heterogeneous human capital.

Huggett, Ventura, and Yaron (2011) combine idiosyncratic shocks and human capital to analyse whether lifetime inequality is the outcome of human capital developed early in life or just people's luck over their working lifetimes. However, the idiosyncratic shocks in their model arise in the human capital accumulation process making it stochastic. The risky human capital accumulation process is the source of heterogeneity among households in their model.

Guvenen and Kuruscu (2010) construct a life cycle model in which individuals decide each period whether to go to school, work, or stay non-employed, and individuals accumulate skills either in school or while working. They find wage inequality arises from differences across individuals in their ability to learn new skills as well as from idiosyncratic shocks. They argue that the standard Ben-Porath model alone cannot account for changes observed in the data.

Guvenen and Kuruscu (2012) propose an analytically tractable overlapping-generations model of human capital accumulation and study its implications for the evolution of the US wage distribution from 1970 to 2000. Here, the only source of heterogeneity, and resulting wage inequality, is differences in individuals' ability to accumulate human capital. They find their model generates behavior consistent with some prominent trends observed in the US data, including rising income inequality both in the short and long run.

Hendricks (2013), in a general equlibrium overlapping generations model, develops a stochastic Ben-Porath model for human capital accumulation, which is produced in school and on the job. Differences across individuals' learning abilities is an additional source of heterogeneity in his model, apart from wage shocks. In contrast to Guvenen and Kuruscu (2010), he finds that the Ben-Porath model of human capital accumulation can account for almost the entire increase in several measures of wage inequality in US.

Debortoli and Galí (2022) use a basic model of individual and aggregate consumption to delineate the mechanisms through which household heterogeneity may have an effect on aggregate fluctuations. Their model features idiosyncratic income shocks as the only exogenous source of heterogeneity and a risk-free one-period bond is the only asset available. They find that the role of idiosyncratic uncertainty on aggregate fluctuations depends on how changes in uncertainty are distributed across households.

As this discussion makes clear, most of the work in this literature focuses on heterogeneity in the human capital accumulation process, and sometimes idiosyncratic earnings shocks as well. These studies focus on developing a stochastic human capital accumulation process to account for evolving wage inequality measures in US. My model focuses on exogenous idiosyncratic income shocks as the only source of heterogeneity, and uses a deterministic form of the standard Ben-Porath human capital accumulation model to study the propagation of the exogenous idiosyncratic income risks to households' consumption via the human capital decisions made by households. I use this to study the effect of different risks resulting in different distributions of income, and how differences in the nature of risk (varying persistence and variance) affect income inequality among households. Subsequently, I introduce a risk-free discount bond into the existing environment and compare how decisions to invest in human capital change with the availability of a financial asset through which households may save or borrow. It is interesting to observe in this model how households leverage the two assets they have at hand - human capital and bonds - to insure against uncertain income risks. This is the main contribution of this paper: to my best knowledge, there is no work that includes human capital in a Huggett-Aiyagari-style economy to study in depth the role of human capital and the effect of idiosyncratic income risks on households' consumption and income inequality when human capital is the only asset available, and the interplay between human capital and a financial asset when households have access to both.

3 A heterogeneous-agent-incomplete-insurance economy with human capital and no financial assets

In this section, I study a simple economy where households face uninsurable idiosyncratic shocks to their incomes and, in response, invest some portion of their available time in human capital accumulation to hedge against risks to future consumption. However, there are no financial assets so households cannot save their current income for future consumption.

3.1 The model

3.1.1 Model environment

The economy is populated by a unit mass of a continuum of heterogeneous households indexed by $i \in [0, 1]$. Each household has identical preferences over consumption and there is no value for leisure. Households are endowed with one unit of time each period which they can allocate between production of goods and production of human capital. Each household realizes an exogenous idiosyncratic income shock e each period, and, in response, decides to devote a fraction $s \in (0, 1)$ of its unit time endowment to human capital accumulation to safeguard against future risks to consumption. A household's human capital h can be interpreted as their general skill level comparable across individuals in absolute terms (Lucas 1988). Time invested by a household in human capital accumulation takes away from their total time available for supplying labor to a representative firm that engages in production of consumption goods. A household's position at a point in time is described by an individual state vector $x \in X$ and $x = (h, e)$ indicates a household's human capital level h and the realized income shock e. The individual state space is $X = H \times E$, where $H = [0, \infty)$, and $E = \{e_1, e_2, ..., e_{N(e)}\}$, such that $e_1 < e_2 < ... < e_{N(e)}$ and $N(e)$ is the total number of exogenous states that a household can be in.

Preferences: Households are infinitely lived and have preferences defined over stochastic processes for consumption given by a utility function:

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

where $\beta \in (0, 1)$. Period utility is defined by the standard constant relative risk aversion (CRRA) utility function: $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ where $\sigma > 1$. The consumption good produced in this economy is perishable. In the absence of any financial asset to save in, each household must consume all their income each period.

Human capital accumulation: Human capital accumulation for a household takes the standard Ben-Porath (1967) functional form and evolves deterministically according to the following equation:

$$h' = (1 - \delta) \cdot h \; + \; g(s, h) \tag{1}$$

where h' is the human capital level next period and δ is the rate of human capital depreciation. New human capital is created via a decreasing returns to scale production function $g(s, h)$ given as:

$$g(s, h) = \theta \cdot (s \cdot h)^{\alpha}$$

The newly created human capital is a function of the time devoted to human capital accumulation this period s and the existing level of human capital h. Here, $\theta > 0$ is the average learning ability or the average human capital productivity parameter interpreted as the innate ability of individuals to accumulate human capital. The curvature or the elasticity of the human capital production function is given by $\alpha \in (0, 1)$ which determines the degree of diminishing marginal returns to investment in the human capital production function.

Goods production: A representative competitive firm undertakes goods production in this economy using a constant returns production technology:

$$F(N^e) = A \cdot N^e \tag{2}$$

where $F(N^e)$ is the aggregate output in the economy, A is total factor productivity, and N^e is the effective labour engaged in production in the economy, given as the sum of the skill-weighted man-hours devoted to current production. Mathematically, this can be stated as:

$$N^e = \int_0^{\infty} (1 - s(h)) \cdot h \cdot N(h) \; dh \tag{3}$$

where $N(h)$ is the number of workers with skill level h engaged in production and devoting $(1 - s(h))$ time in production. I assume that there is a unit mass of a continuum of workers following from the assumption of a unit mass of a continuum of households who supply labor to the firm. Firms pay wages to households to compensate them for their labour input in the goods production process. Wages adjust so that the firm earns zero profits in equilibrium. The representative firm solves the following problem to maximize profits every period:

$$\Pi = \max_{N^e} F(N^e) - w \cdot N^e \tag{4}$$

or

$$\Pi = \max_{N^e} A \cdot N^e - w \cdot N^e \tag{5}$$

where w is the "wage rate" defined as the labour earnings per efficiency unit of labour, that is, output produced per unit time spent in goods production per unit of human capital. Solving the firm's profit maximization problem gives the equilibrium wage rate $w_{eq} = A$.

An equivalent alternate exposition of the goods production process in this economy is to assume that households engage in home production. That is, each household produces goods for the sole purpose of consumption using the same constant returns production function $F_i(N^e) = A \cdot N_i^e$ where A is again total factor productivity and N_i^e is the effective labour engaged in production at the household where i denotes a given household. In this scenario, there is no trade or exchange of goods among households or between a household and any external agent. In this representation, each household produces for its own consumption, that is, $c_i = y_i$ for all i, and there would be no goods market. For the current model, I use the former exposition where the firm engages in production of goods.

Labour supply: Households trade a fraction $(1-s)$ of their unit time endowment with the representative firm in the labour market in exchange for wages. For an individual i with skill level h_i who devotes $(1-s_i(h_i))$ time in production, labour earnings per worker is given by W_i as:

$$W_i = w \cdot (1 - s_i(h_i)) \cdot h_i \tag{6}$$

where $w = A$.

Households trade their labour earnings in the goods market for consumption goods. The goods market has a trivial price such that the price of one unit of consumption is equal to the price of one unit of output produced by the firm - this is akin to the firm paying wages to households in units of the consumption good so that households no longer need to go to the goods market and purchase goods for consumption.

Exogenous shocks to labour earnings: There exist $N(e)$ idiosyncratic states, and in any period t, households transition between any two such states e and e' with exogenous transition probability $\pi(e' \mid e) = Prob\ (e_{t+1} = e' \mid e_t = e) > 0$ for $e, e' \in E$. The exogenous shock follows a Markov process with stationary transition probabilities that are independent of all other households' current and past labour earnings.

After a household realizes its income shock, its income is given by y as:

$$y_i = e_i \cdot W_i$$

which, substituting the labour earnings and dropping the household index, can be rewritten as:

$$y = e \cdot w \cdot (1 - s) \cdot h$$

or

$$y = e \cdot A \cdot (1 - s) \cdot h \tag{7}$$

3.1.2 Households' decision problem

Having defined the various components of the model, I now define the households' problem. The functional equation that describes the decision problem of a household entering the period in human capital state h and exogenous idiosyncratic state e is, therefore:

$$V(h, e) = \max_{c,s} \left(u(c) + \beta \sum_{e'} \pi(e'|e) V(h', e') \right) \tag{8}$$

subject to:

the budget constraint:

$$c \leq y \text{ where } y = e \cdot w \cdot (1 - s) \cdot h \tag{9}$$

the human capital accumulation constraint:

$$h' = (1 - \delta) \cdot h + \theta \cdot (s \cdot h)^{\alpha} \tag{10}$$

non-negativity constraint for consumption:

$$c \geq 0 \tag{11}$$

time endowment constraint:

$$s \in (0, 1). \tag{12}$$

Optimal path for consumption and human capital: We can characterize the optimal paths for consumption and human capital accumulation for a household in this economy by deriving the Euler equation for h' which is given by the following equation (see Appendix A for derivation):

$$\beta \cdot \mathbb{E}\left[(c')^{-\sigma} \cdot e' \cdot \left[1 + (1 - \delta) \cdot \frac{1}{\alpha \cdot \theta} \cdot (s' \cdot h')^{1-\alpha} \right] \right] = c^{-\sigma} \cdot e \cdot \frac{1}{\alpha \cdot \theta} \cdot (s \cdot h)^{1-\alpha} \tag{13}$$

For any given set of parameters, a higher input $(s \cdot h)$ into human capital production today is likely to slow the growth of consumption over time. This can be seen by imposing certainty equivalence to get rid of the expectation and setting $\delta = 1$. This seems logical - more time invested in human capital leaves less time for production today, lowering consumption today. But it also improves production capability in the next period. Simultaneously, a higher level of human capital increases production and consumption today. The higher investment in human capital today results in a higher level of human capital next period, which increases consumption next period. Therefore, growth in consumption is likely to slow down.

3.1.3 Equilibrium

Heterogeneous households differ in their state vectors. Therefore, for defining an equilibrium concept, I use a probability measure μ defined on subsets of the individual state space to capture heterogeneity among individuals at any given point in time, as in Huggett (1993). Therefore, μ is a probability measure defined on (S, β_S) where $S = H \times E$ and β_S is the Borel $\sigma-$algebra. It follows that $\mu(\mathcal{B})$ represents the mass of individuals whose individual state vectors lie in $\mathcal{B}$ for all $\mathcal{B} \in \beta_S$. Focusing only on stationary equilibria here, I assume that the probability measure μ remains unchanged over time. The definition of a stationary recursive equilibrium for this economy is as follows.

Definition 1. *Stationary recursive equilibrium for this economy is defined as the set of functions* $\{V(h,e), gs(h,e), gc(h,e), \mu(h,e)\}$ *and an equilibrium wage rate* w_{eq} *such that:*

(i) $V(h,e)$ *solves the households' decision problem (equation 8 subject to the constraints 9, 10, 11, and 12);*

(ii) $gs(h,e)$ *and* $gc(h,e)$ *are the optimal decision rules, given* w*;*

(iii) Individual decisions are consistent with the aggregates, that is, $\int_S gc(h,e)\, d\mu(h,e) = \int_S y\, d\mu(h,e)$, *and* $\int_S (1 - gs(h,e)) \cdot h\, d\mu(h,e) = N^e$*;*

(iv) μ *is a stationary probability measure such that it satisfies* $\mu(\mathcal{B}) = \int_S P(x, \mathcal{B})\, d\mu(h,e)$ *for all* $\mathcal{B} \subset \beta_S$ *where* $P(x, \mathcal{B})$ *is a transition function representing the probability that a household with state* $x = (h,e)$ *will have an individual state vector lying in* $\mathcal{B}$ *next period.*

The first and second conditions imply that households optimize and $V(h,e)$ is the value function that results from their optimization problem, and the decision rules for time spent in human capital accumulation $gs(h,e)$ and consumption $gc(h,e)$ are the optimal functions that solve the households' optimization problem, for the equilibrium wage rate w_{eq}. As prices are functions only of the aggregate state, and there is no aggregate state here, the wage rate is a value rather than a function. The third condition specifies consistency between individual decisions and aggregate outcomes. In the labour market, in equilibrium, individual decisions of households to supply labour is consistent with the aggregate demand for effective labour by the representative firm. The equilibrium wage rate w_{eq} clears the labour market, and by Walras' law, the goods market clears as well. In the goods market, in equilibrium, consumption averaged over population is equal to the income averaged over population. The last condition for a stationary equilibrium requires that the distribution of households over the state space remains unchanged over time.

3.2 Setting model parameters

3.2.1 A conventional parameterization

This is an infinite horizon model and the model period is equal to a year. The coefficient of relative risk aversion (CRRA) σ, which is also the parameter governing the elasticity of intertemporal substitution in constant relative risk aversion utility functions, is set to 2 in the baseline specification as in Huggett, Ventura, and Yaron (2011). However, to see how the degree of risk-aversion of households alters their human capital accumulation and consumption responses to shocks, I also look at the cases when $\sigma = 1.5$ and/or $\sigma = 3$ as in Huggett (1993) based on the microeconomic studies reviewed by Mehra and Prescott (1985).

The subjective discount factor β is set to equal $\frac{1}{1+r}$ where r is the risk-free real rate of interest so that in the absence of risk, individuals would perfectly smooth consumption. I set the net annual risk-free real rate of interest $r = 0.042$ as in Huggett (1993) and Hugget, Ventura, and Yaron (2011). Therefore, the subjective discount factor (in annual terms) is $\beta = 0.96$.

The total factor productivity A, which is also equal to the equilibrium wage rate w_{eq}, is normalized to 1.

A brief discussion on the calibration of the parameters of the Ben-Porath model of human capital accumulation is in order for the wide variations in the values used for these parameters across studies. The basic parameters of the Ben-Porath model of human capital accumulation have been estimated in various studies in varying forms by fitting the Ben-Porath model to observed age-wage profiles. Browning, Hansen, and Heckman (1999) from their survey of microeconomic studies show that the estimates for the curvature parameter α have varied widely in the early literature from 0.5 to 0.99 across studies (Rosen 1976; Brown 1976; Heckman 1976) which imposed strong restrictions due to limited data availability. So have the estimates of the depreciation rate of human capital, which vary from 0.0016 (Heckman 1976), to 0.037 (US Bureau of the Census 1960 males), to 0.089 (Heckman 1976). More recent literature has estimated these parameters with more expansive data, but they assume human capital depreciation to be 0.

Smaller values of α imply a higher degree of diminishing returns in the production of human capital making it optimal to spread out investment in human capital over time. On the other hand, with higher values of α, individuals would find it optimal to concentrate investment into human capital such that in the extreme case of $\alpha = 1$, households would put in either all the time they are endowed with or none at all into human capital production in a given period (Guvenen and Kuruscu 2006). It is, therefore, a crucial parameter to be chosen in a model with human capital accumulation. Guvenen and Kuruscu (2006) estimate α to be 0.8; Guvenen and Kuruscu (2010) find the curvature parameter α to typically vary between 0.80 and 0.95; Manuelli and Seshadri (2010) report more recent estimates to be around 0.93. Hendricks (2013) refers to this view of a near-linear human capital production function and no depreciation as the "emerging consensus" and finds that parameter values consistent with the 'emerging consensus' ($\alpha \geq 0.8$ and $\delta < 0.01$) fit the data quite poorly. The calibrated model in Hendricks (2013) implies a curvature parameter near 0.6 and he shows that the preferred values of α are indeed near 0.6 and the rate of human capital depreciation δ takes on values near 0.05. Huggett, Ventura, and Yaron (2011) use measures of earnings dispersion to calibrate a stochastic Ben-Porath model which yields an estimated curvature parameter of 0.70. I set $\alpha = 0.70$ and the depreciation rate of human capital $\delta = 0.05$.

The mean learning ability or the average human capital productivity parameter θ is set to 0.26. The value chosen for θ here may seem arbitrary, but setting θ to higher values (0.29 or higher) results in a degenerate stationary distribution where households are stacked up at very high levels of human capital (as marginal returns to investing in human capital become very high). On the other hand, setting θ to lower values (0.22 or lower) leads to a stationary distribution where households are concentrated at very low levels of human capital (as marginal returns to investing in human capital become very low). In combination with the value chosen for δ based on the evidence presented above, I accordingly set θ to an interior value to get a non-degenerate distribution of households over human capital levels. The chosen values of parameters are collected in Table 1.

3.2.2 Parameterization of the risk process

The idiosyncratic income shock $e_t(i) \in \{e_1, e_2, ..., e_{N(e)}\}$ that households realize every period follows a stationary $N(e)$-state Markov process, independent across households.

Parameter	Description	Value
σ	Coefficient of relative risk aversion	2.0
β	Subjective discount factor	0.96
A	Total factor productivity (wage rate)	1.0
α	Elasticity (or curvature) of human capital production function	0.70
δ	Depreciation rate of human capital	0.05
θ	Mean learning ability or human capital productivity parameter	0.26

Table 1: Chosen parameters

Drawing from the work of Debortoli and Galí (2022), wherein idiosyncratic income shocks are the only exogenous source of heterogeneity, I use a log AR(1) process to model the idiosyncratic income shocks in this model. Their work follows Auclert et. al. (2021) for characterizing and calibrating the income shock. Auclert et. al. (2021) assume that $Prob\ (e' \,|\, e)$ discretizes a log AR(1) process given as:

$$log\, e_t(i) \;=\; \rho\, log\, e_{t-1}(i) \;+\; \varepsilon_t(i) \tag{14}$$

and they do so using the Rouwenhorst method for discretization. For ease of notation, let $z_t = log\, e_t$. So we have:

$$z_t(i) \;=\; \rho\, z_{t-1}(i) \;+\; \varepsilon_t(i) \tag{15}$$

where $\rho \in (0,1)$ is the persistence in the shock process, $z \sim \mathbb{N}(0, \sigma_z^2)$ and innovations to log income shock are $\varepsilon_t \sim \mathbb{N}(0, \sigma_\varepsilon^2)$. The standard deviation of innovations in the log income process is given by $\sigma_\varepsilon = \sigma_z \cdot \sqrt{1-\rho^2}$.

Debortoli and Galí (2022) and Auclert et. al. (2021) set the persistence parameter ρ to 0.966, and the standard deviation of the log income shock σ_z to 0.5. However, since their model is quarterly, I adjust the persistence parameter by raising it to the fourth power to get an approximate measure of persistence for an annual shock since a period in my model is a year. This follows from the logic that if there is a one unit innovation and the persistence for the quarterly shock is ρ_{qrt}, ρ_{qrt} of it remains after the first quarter, then $\rho_{qrt} \cdot \rho_{qrt}$ of it remains after the second quarter and so on, so that after four quarters (a year), what remains of the unit innovation is ρ_{qrt}^4. Since the volatility of innovations is independently and identically distributed, I use the standard deviation of the shock without making any adjustments. The parameterization of the shock process is summarized in Table 2.

Since the goal of this model is to study how different types of uninsurable idiosyncratic risks impact the decisions of households to invest in human capital and, therefore, their consumption, I vary, in turn, the persistence of the shock and the variance of the shock to compare how outcomes change across different levels of human capital and shock realizations, and also over time for a simulated household. I experiment with different combinations of persistence and variance to see how households' responses change when risks are more or less persistent, and have higher or lower variance. I use four independent shocks for this exercise which are characterized as: high persistence, high variance; high persistence, low variance; low persistence, high variance; and low persistence, low variance. The baseline specification for ρ and σ_z characterizes a high persistence-high variance shock. For a shock with low persistence, I set ρ to a lower value $\rho_{low} < \rho$, and for a shock with low variance, I set σ_z to a lower value $\sigma_{z_{low}} < \sigma_z$. The parameter values for the baseline specification of the shock process and for the experiments with persistence and variance is summarized in Table 2.

In experimenting with different risks, I also change the households' risk aversion coefficient to see how robust these changes are, to changes in households' appetite for risk.

Parameter	Description	Value
$N(e)$	Points in Markov chain for e	7
ρ	Persistence of log income shock	0.871
σ_z	Standard deviation of log income shock	0.5
ρ_{low}	Low persistence of log income shock	0.250
$\sigma_{z_{low}}$	Low standard deviation of log income shock	0.125

Table 2: Chosen parameter values for the shock process

3.3 Computation of the model with human capital and no financial assets

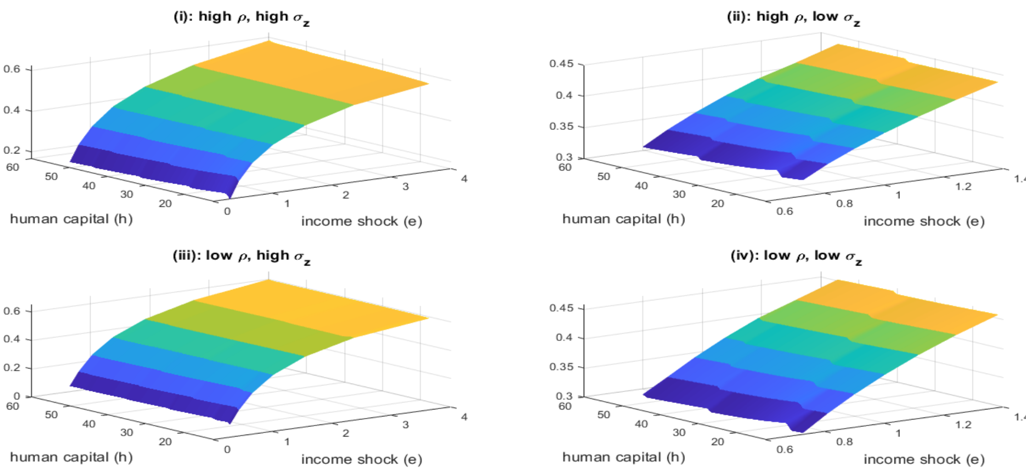
Using the Tauchen algorithm, I develop a Markov chain to approximate the grid on the idiosyncratic income shocks $E = \{e_1, e_2, ..., e_{N(e)}\}$ and the exogenous transition probability matrix. I set up a grid on human capital levels to approximate the value function. I solve the model using standard value function iteration and use the golden section search method and linear interpolation to approximate the optimal decision rule for time invested in human capital in each iteration. The decision to invest time in human capital today has two contrasting effects: first, it reduces the time available for production today, so current income and consumption are impacted negatively, which is accounted for in the utility from consumption today; second, it determines the level of human capital next period which improves production capability in future periods, and therefore, has a positive effect on future income and consumption and future utility. I interpolate the expected value function onto the value of h' corresponding to the s chosen today, to account for this second effect in determining the optimal decision to invest time in human capital accumulation. Since the equilibrium wage is a constant, equal to the total factor productivity, and the price in the goods market is trivial, I do not have any outer loop for determining an equilibrium price in either the labour market or the goods market, and equilibrium is determined by solving the households' decision problem. The detailed procedure used to compute stationary equilibrium in this model is explained in the algorithm provided in Appendix C.

Simulation: To simulate households, I first simulate a Markov chain for the length of the simulation period. I compute the cumulative density function (CDF) using the stationary distribution of households. Using the cdf, I draw random numbers from the uniform distribution and place them on the closest point on the cdf. To initialize the households with starting values of the individual state vector, I locate the corresponding level of human capital h and the shock e using the cdf and assign these as the starting values. Using the simulated Markov chain for the evolution of the income shock e, I trace the paths of their time invested in human capital s, their level of human capital h, and their consumption c over time.

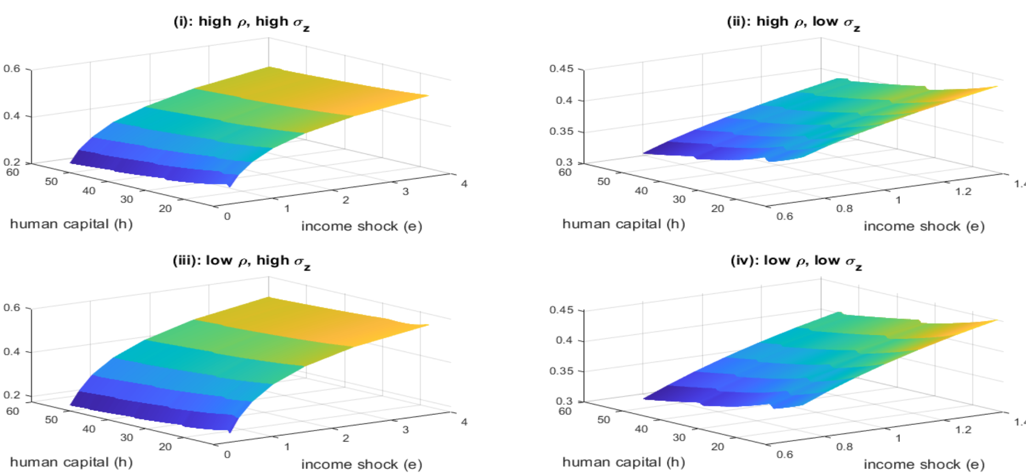
3.4 Results

3.4.1 Time invested in human capital

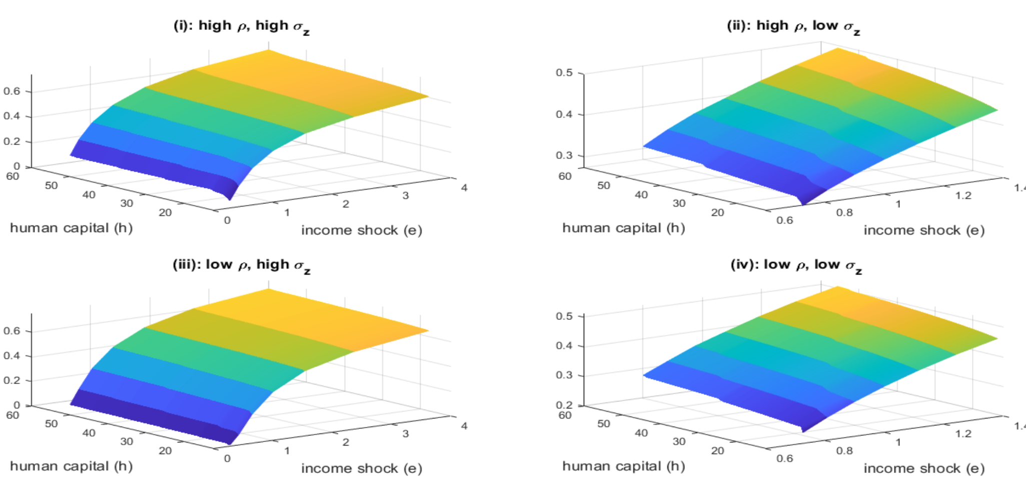
Households spend a larger fraction of their time endowment in human capital production if they are hit by a high-income shock than if they are hit by a low-income shock. The households that realize the lowest income shocks invest the least in human capital. Time invested in human capital production is seen to increase steadily as the shock becomes more positive. This is true for a shock of any persistence and variance, and for any degree of risk aversion of households. This seems logical: a high income shock increases household income for any given level of human capital h and any given fraction of time spent in production $1 - s$, thereby allowing the household the opportunity to invest more time in acquiring human capital at the cost of spending less time in production without taking a fall in consumption. Therefore, with increasingly positive shocks, the incentive to invest in human capital is higher and the fraction of time invested in human capital increases for any given human capital level. Figure 1 shows the optimal decision rules for time invested in human capital for four different shocks of varying persistence and variance when households' risk aversion coefficient takes on three different values.



(a) $\sigma = 2.0$



(b) $\sigma = 1.5$



(c) $\sigma = 3.0$

Figure 1: Households' optimal decision rule for time invested in human capital

Note: ρ is the persistence of the income shock process and σ_z is the standard deviation of innovations to the log income shock process.

The increase in time investment in human capital increases at a decreasing rate as the shocks become more positive, when the shock is of higher variance irrespective of persistence, as can be seen from panels (i) and (iii) in Figure 1a. But when the shock is less dispersed, increase in time invested in human capital declines at a much lower rate. This seems intuitive - when shocks have higher variance, they are more widely dispersed making both good and bad shocks more extreme. As a result, incrementally more positive shocks do not see commensurately large increases in investments in human capital for any given human capital level. This result is robust to the degree of risk aversion of households as seen in Figures 1b and 1c.

However, time invested in human capital, for any given level of shock, behaves somewhat differently across levels of human capital depending on the risk aversion of households. In the benchmark case of $\sigma = 2$ (Figure 1a), for shocks of any persistence and variance, changes in time invested in human capital across levels of human capital are negligible. However, when the risk aversion of households is lower (Figure 1b), human capital investment falls slightly as the human capital level increases. On the other hand, when households are more risk averse (Figure 1c), human capital investment increases with higher levels of human capital. This is likely the case because higher risk aversion of households makes them want to insure against risk to a greater extent. There is greater incentive to invest more in human capital even when they start off with higher levels of human capital in order to insure against risks to future consumption. Contrarily, less risk averse households have greater appetite for risk and feel the need to hedge against future risks to a lesser extent. They take advantage of higher levels of human capital and invest less compared to if they had lower levels of human capital, for a given realization of the income shock.

The risk aversion coefficient of households also ties to their elasticity of intertemporal substitution: when coefficient of risk aversion (σ) is high, elasticity of intertemporal substitution ($\frac{1}{\sigma}$) is lower. When σ is high, households are less willing to substitute between consumption today and consumption tomorrow, so a higher income shock causes a smaller increase in time invested in human capital, compared to when households have a lower coefficient of risk aversion, that is, higher elasticity of intertemporal substitution. This can be seen from the lower time invested in human capital when σ is low (Figure 1b) than when σ is higher (Figure 1c), for any given h and e, and across all types of shocks.

Households invest in human capital to insure against risk to their consumption from the idiosyncratic income shock they face every period. In the absence of a financial asset, human capital is the only asset that households can invest in to hedge against risks from exogenous shocks to their consumption stream in order to smooth their consumption profile. It is interesting to note that accumulation of human capital as a mechanism to insure against unpredictable income risk is equally important for households that start off with relatively high levels of human capital as it is for individuals who are relatively less skilled initially. The importance of human capital as an asset to provide incomplete insurance against income risk is underscored by the absence of any other asset in this model.

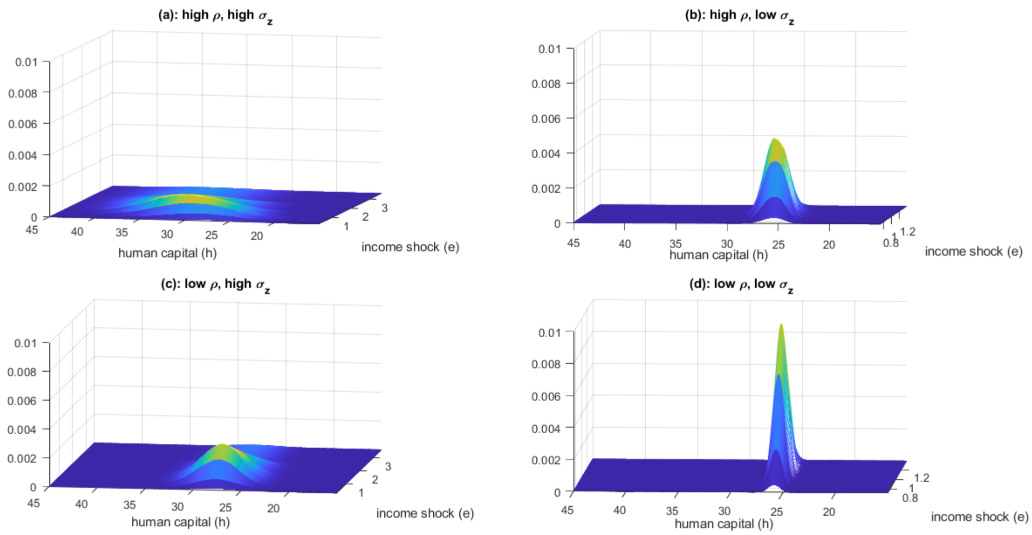
3.4.2 Income distribution: stationary distribution of households

The stationary distribution of households for the benchmark case of $\sigma = 2$ is depicted in Figure 2a. The distribution of households in equilibrium is much more concentrated in the human capital dimension for shocks that have lower variance as seen in panels (b) and (d) of Figure 2a. The households are all concentrated below the median level of human capital, and are smoothly and evenly distributed around this value when the income shock has a lower variance. In contrast, when the shock has high variance, the stationary distribution of households is more loosely and widely spread out as seen in panels (a) and (c) of Figure 2a.

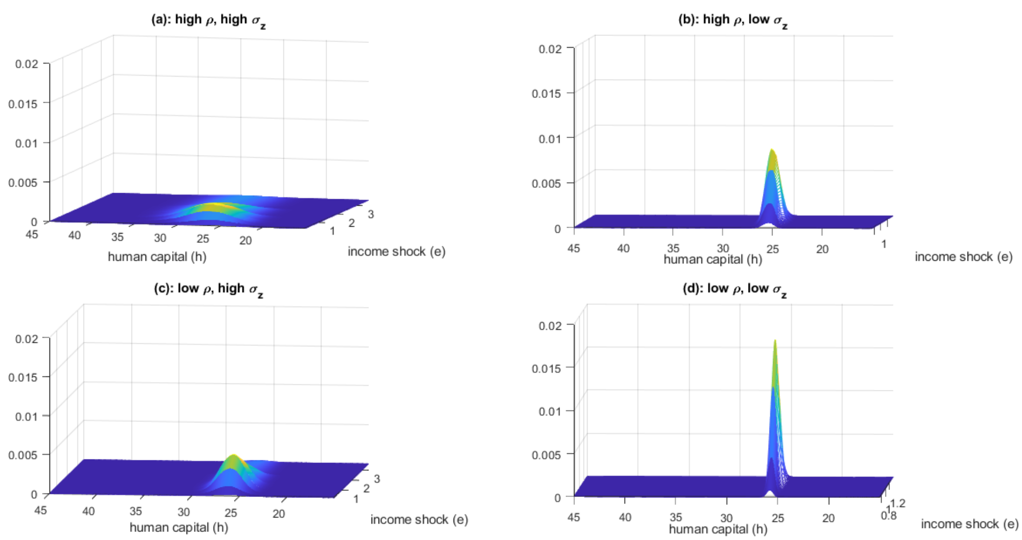
Households prefer to smooth consumption, and the only way they can do it in this economy is through investing time in acquiring human capital. They hedge against risks to consumption by investing in human capital which improves their future production capability. Therefore, when shocks are more variant and widely dispersed so that the good and bad shocks are more extreme, households for a broader range of human capital levels, including those with higher levels of human capital, tend to invest in accumulating more human capital. Therefore, households are more loosely spread out along the human capital dimension in the stationary distribution. In contrast, when the shocks are less variant, households end up more tightly clustered around human capital levels below the median value of human capital, since the shocks are less drastic.

As discussed in the parameterization of θ, a mean learning ability higher than the chosen value results in the stationary distribution of households being concentrated toward higher values of human capital, eventually becoming degenerate for $\theta = 0.29$ or above, and a value of θ less than the chosen value results in households being clustered toward lower values of human capital in the stationary distribution, eventually becoming degenerate at $\theta = 0.22$ or below. To study how differences in the persistence and variance of risk affect the stationary distribution of households, θ was chosen as an interior value within that range.

When households are less risk averse ($\sigma = 1.5$), the distributions becomes tighter along the human capital dimension for the same type of shock, compared to when the risk aversion is higher. This can be seen in panels (a) and (c) of Figure 2b compared to the same panels in Figure 2a. This is for the simple reason that



(a) $\sigma = 2.0$



(b) $\sigma = 1.5$

Figure 2: The stationary distribution of households

Note: ρ is the persistence of the income shock process and σ_z is the standard deviation of innovations to the log income shock process.

a lower risk aversion means there is less motivation to hedge against risks and, therefore, households end up more tightly distributed along the human capital dimension. On the other hand, a higher risk aversion of households ($\sigma = 2$) implies greater need to hedge against risks to consumption which exacerbates the incentive to invest in human capital even at relatively higher levels of human capital, causing the stationary distribution to be more loosely dispersed spanning a wider range of human capital levels.

3.4.3 Simulating households facing shocks with varying persistence and variance

I simulated a household hit by four independent shock processes with different persistence and variance and plotted the evolution of these shock processes for 1000 periods. In this section, I focus on the 100 periods between dates 550 and 650 as one sample of that long simulation to analyze how changes in risks from the four independent idiosyncratic shocks affect the evolution of time invested in human capital s, the level of human capital h, and the consumption c for a household facing these shocks. Figure 3 plots these series along with the evolution of the four shock processes shown in Figure 3a. Since income is equal to consumption in this model, the two terms have been used interchangeably in the discussion that follows, focusing largely on consumption. The results presented here are for the benchmark case when households have a coefficient of relative risk aversion $\sigma = 2$. For robustness, I repeat this experiment for each of these variables over the same period and the same evolution of shocks when $\sigma = 1.5$. Based on these series, I calculate their means and coefficients of variation which are reported in Tables 11, 12, and 13 in Appendix D for reference.

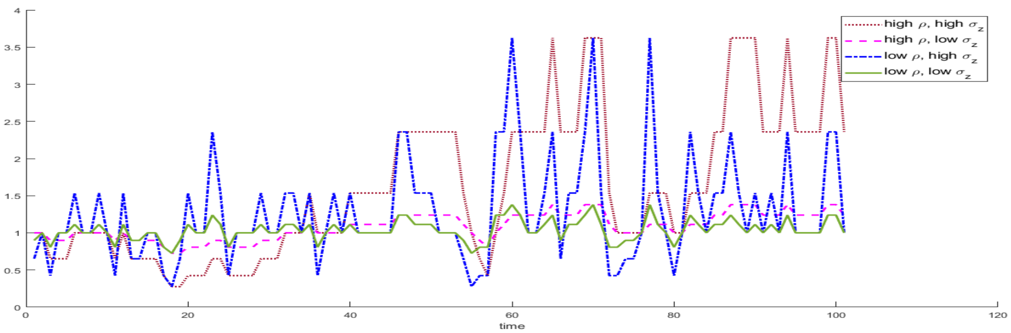
Effect of risk on evolution of time invested in human capital accumulation Figure 3b shows the effect of the four independent shocks with different persistence and variance on a household's decision to invest time in human capital accumulation over time. Table 3 shows the means and coefficients of variation (CV) for time invested in human capital by the household when hit by the different shocks.

Coefficient of Relative Risk Aversion $\sigma = 2.0$			
Mean		Coefficient of Variation (CV)	
high ρ, high σ_z	high ρ, low σ_z	high ρ, high σ_z	high ρ, low σ_z
0.45	0.39	25.45	7.15
low ρ, high σ_z	low ρ, low σ_z	low ρ, high σ_z	low ρ, low σ_z
0.42	0.39	28.79	7.93

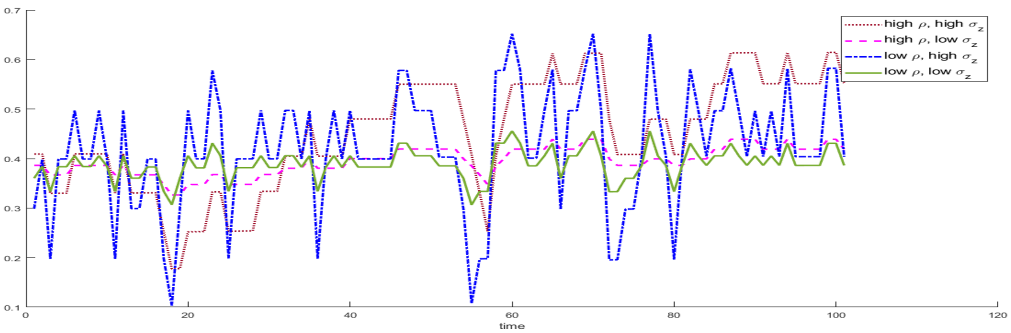
Table 3: Time invested in human capital accumulation

The mean time invested in human capital accumulation over this period is fairly similar under the four different shocks. This can also be seen in Table 3. The mean time invested over this period is marginally higher when shocks have higher variance than when shocks have lower variability, for a given persistence. Higher persistence compounds this effect.

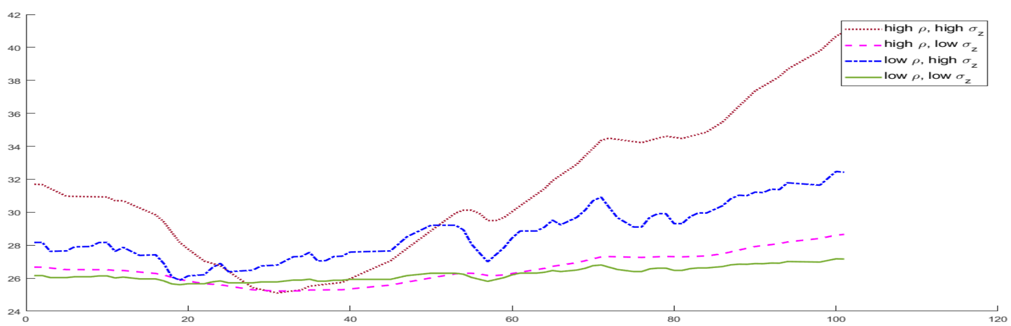
However, the real striking difference can be seen from the wide variations in the coefficient of variation for



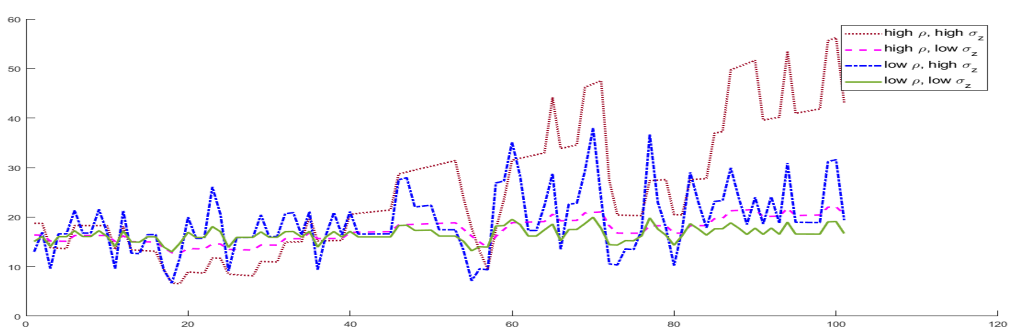
(a) Evolution of income shocks (e)



(b) Evolution of time invested in human capital accumulation (s)



(c) Evolution of human capital (h)



(d) Evolution of consumption (c)

Figure 3: Evolution of s, h, and c for a simulated household facing different shocks ($\sigma = 2.0$)

Note: ρ is the persistence of the income shock process and σ_z is the standard deviation of innovations to the log income shock process.

the four different series of investment in human capital corresponding to the four different shocks that hit the household. The coefficient of variation is a measure of how dispersed the data points in a series are relative to its mean.

A shock with higher variance increases the coefficient of variation in time invested in human capital by over 3.5 times compared to a shock with lower variance, given persistence. For any given persistence of the shock, a higher variance causes more dispersed realizations of the income shock: good times are much better and bad times are much worse than for a shock with low variance, soliciting more extreme responses from the household in the time invested in human capital production and accumulation. This is intuitive: when shocks are more widely dispersed, households react in more drastic ways to the shock by increasing (or decreasing) time invested in human capital when a high (or low) income shock strikes by much more than if the realizations of shock were concentrated closer together, that is, if the shock had lower variance. This effect is stronger when the shock is more persistent. If the shock is also more persistent while being more variable, the effects of the shock linger on for some periods into the future, making the responses more long-drawn while also being more extreme.

When a less persistent shock (with given variance) strikes, the dispersion in human capital investment increases compared to when a highly persistent shock with the same variance strikes. However, the increase in the CV is milder than in the case of higher versus lower variance for given persistence. A higher variability in the shock process exacerbates this effect. A more persistent shock implies that the effect of a shock will linger on: a good shock will allow the household to invest more time in human capital and a bad shock will allow it to invest less time in human capital for as many periods as the effect of the good or bad shock lasts. This lowers periodic variations in the investment in human capital. Accordingly, shocks that are less persistent cause more frequent changes in the human capital investment decision, and therefore, greater variation in the time invested in human capital.

These effects are more pronounced when the household is more risk averse corroborated by the results here compared to the results when $\sigma = 1.5$ presented in Table 11 in Appendix D.

Effect of risk on evolution of human capital: The evolution of human capital resulting from the time invested in its accumulation is shown in Figure 3c. The means and coefficients of variation for the respective series are presented in Table 4.

With more persistent shocks, as a direct outcome of less volatile decisions to invest in human capital every

Coefficient of Relative Risk Aversion $\sigma = 2.0$			
Mean		Coefficient of Variation (CV)	
high ρ, high σ_z	high ρ, low σ_z	high ρ, high σ_z	high ρ, low σ_z
31.18	26.55	13.76	3.57
low ρ, high σ_z	low ρ, low σ_z	low ρ, high σ_z	low ρ, low σ_z
28.68	26.25	5.86	1.59

Table 4: Human capital

period, the human capital level evolves relatively smoothly over time as effects of the shock linger on and periodic variations in investment in human capital are milder. In contrast, less persistent shocks result in more variations in human capital in the short run. This is seen in Figure 3c. The coefficient of variation in human capital for the simulated period is highest for the shock process that is highly persistent and highly variable whereas it is lowest when the shock process has both low persistence as well as low variance.

Figure 3c shows that human capital accumulation is most responsive to the evolution of the shock when the shock process has higher variance as households change their decisions to invest in human capital by greater magnitudes in response to more extreme realizations of the shock. The coefficients of variation are over 3.5 times higher when shocks are more variable, than when shocks are less variable, for any given persistence. This is the outcome of the large variations in time invested in human capital for more variable shocks as discussed above.

Effect of risk on evolution of consumption: The evolution of income and consumption for a household facing shocks that differ in their variance and persistence is shown in Figure 3d. The means and coefficients of variation for the corresponding four series of consumption are presented in Table 5.

Over the simulated period, the household enjoys the highest average consumption when it is faced with a highly persistent and highly variable risk. For the chosen simulation period, the household's average consumption increases with an increase in volatility of the shock process, for any given persistence, as seen from Table 5. Average consumption level also increases when shocks are more persistent, for given variance, for the given evolution of the income shocks.

The effects of the income shock on mean consumption are governed by the nature (good or bad) of the realization of such shocks and their relative evolution over time. Since the realized shocks from the four

Coefficient of Relative Risk Aversion $\sigma = 2.0$			
Mean		Coefficient of Variation (CV)	
high ρ, high σ_z	high ρ, low σ_z	high ρ, high σ_z	high ρ, low σ_z
25.01	17.31	51.99	14.15
low ρ, high σ_z	low ρ, low σ_z	low ρ, high σ_z	low ρ, low σ_z
19.18	16.52	33.52	8.89

Table 5: Consumption

different shock processes differ widely in their nature and also evolve very differently over the time span under study, as depicted in Figure 3a, the above patterns in the mean consumption emerge. These may be masked when the realized shocks are more similar in nature and also evolve more closely.

However, the patterns in variation in consumption across shocks of different persistence and different variance is robust to the particular evolution of the shock during the simulated period, though the degree of such variation may differ. This ties closely to our discussion of the evolution of time invested in human capital under the four circumstances where the different shocks strike. Those decisions bear on time in production, the level of human capital, and consequently, consumption and seek an extended explanation here.

The variation in consumption around the mean increases significantly when the shock is more variable. Increase in variance of shocks increases the coefficient of variation in consumption by no less than 3.5 times, given persistence. Shocks that are widely dispersed (given persistence) cause the household's decision about investing in human capital to diverge more: good (bad) shocks increase (decrease) time invested in human capital by much more. As a result, consumption varies widely when shocks are more widely dispersed. This result holds robustly even when households are less risk averse as shown for $\sigma = 1.5$ in Table 13 in Appendix D.

An increase in persistence also increases the coefficient of variation in consumption, for any given variance of the shock process. The increase is much milder than we see when the shock has higher variance, given persistence. This is the outcome of the contrasting effects of time invested in human capital: it increases future human capital levels, future production capability, and hence, future consumption, but it takes time away from production today and lowers consumption today. This results in a higher coefficient of variation in the income series when a more persistent shock strikes compared to when a less persistent shock strikes.

3.4.4 How risks affect income inequality: the Gini coefficient

In this model without any financial assets, households invest in human capital to insure against risks to consumption posed by unpredictable idiosyncratic shocks to income. I calculate the Gini coefficient to measure income inequality in income distributions resulting from each of the four different shocks to study how different types of risks affect income inequality. Table 6 presents these coefficients for each type of shock when households have a risk aversion coefficient $\sigma = 2$ and also for a lower risk aversion coefficient $\sigma = 1.5$ for robustness.

$\sigma = 2.0$		$\sigma = 1.5$	
high ρ, high σ_z	high ρ, low σ_z	high ρ, high σ_z	high ρ, low σ_z
0.1497	0.0402	0.1357	0.0336
low ρ, high σ_z	low ρ, low σ_z	low ρ, high σ_z	low ρ, low σ_z
0.0516	0.0143	0.0397	0.0112

Table 6: Gini coefficients for income distributions with varying risks

Note: ρ is the persistence of the income shock process and σ_z is the standard deviation of innovations to the log income shock process.

Since a household's income in this model is dependent on the shock realized as well as the current level of human capital of the household, a brief discussion of the method used to calculate the Gini coefficient is in order here. To have the complete distribution of income, I calculate income levels for each possible realization of the shock. From the stationary distribution of households, I locate the income levels with non-zero measures of population. From the non-zero distribution of households over the entire possible distribution of income, I normalize income levels to fall in the $[0, 1]$ range. I then calculate income percentiles and the cumulative sum of households for whom income falls at or below the income level corresponding to each income percentile.

An alternate way of calculating the Gini coefficient could be to create a sample of income distribution using random draws from the possible values that a given type of shock could take to create a vector of shock realizations. We could then calculate the income distributions using these realisations of shocks for each level of human capital with a non-zero measure of population. This sample of income distribution could then be normalized between 0 and 1 and the corresponding income percentiles and cumulative distribution of households could be obtained for calculating the Gini. However, the former approach yields an exhaustive income distribution so I follow that approach for obtaining the estimates below.

As seen in Table 6, the Gini coefficients are higher for income distributions when the income shock process is either more persistent, or has higher variance, or both.

Risks that are more persistent increase income inequality. As discussed earlier, more persistent shocks lead to less volatile time investment in human capital. Therefore, a household that realizes a good shock today invests more time in human capital today, and because the shock is persistent, for a few periods ahead until the effect of the good shock lasts. A household that draws a bad shock today, on the other hand, cannot afford to invest as much time in human capital by withdrawing from production today, and since the shock is persistent, continues to invest less in human capital till the effects of the bad shock wane. This makes the household with a good shock today have higher current income and consumption, and also higher production capability tomorrow due to larger investments in human capital. But the one with a bad shock not only has low income and consumption today, but is stuck for a few periods with low investments in human capital and, therefore, low production capability in the coming periods. As a result, the divergence in income is greater when the risk to income is more persistent.

If the risk is also highly variable, while being highly persistent, then income inequality is highest among all four cases because of the more widely dispersed income shocks that the household could potentially realize, which accentuates the effect of high persistence of risk.

These results are robust to a lower risk aversion of households as shown in Table 6 for $\sigma = 1.5$.

The Gini coefficient often masks the underlying shape of the distribution of income. To highlight this, Table 7 presents the shares of income possessed by top and bottom percentiles of the distribution of households resulting from the four different types of income shocks, and Figure 4 shows the corresponding Lorenz curves for each of these distributions.

The top 10 percent of earners in the distribution possess over 15 percent of the total income when idiosyncratic income risks are highly persistent and have high variability (column 1). The income shares of the top 10 percent of earners falls when either the risk is less persistent, or less variable, or both, the differences not being much among these latter cases. Correspondingly, the income possessed by the bottom 10 percent of earners is lowest (only about 6 percent) when the risk is more persistent and has higher dispersion (variance). This is below the share of income possessed by the bottom 10 percent earners in the distributions resulting from the other three types of income risks, the differences being minimal among them. The top 25 percent possess nearly 34 percent of the total income when the shock is more persistent and variable; the top 25 percent earn much lower shares (26 to 27 percent) of the total income when the shocks are less persistent

	Fraction of income (percent of total income)			
Fraction of households	high ρ, high σ_z	high ρ, low σ_z	low ρ, high σ_z	low ρ, low σ_z
Top 10%	15.23	11.21	11.66	10.40
Bottom 10%	5.95	8.82	8.45	9.78
Top 25%	33.95	27.31	27.88	25.79
Bottom 25%	17.17	23.09	22.18	24.25

Table 7: Shares of income possessed by the top and bottom percentiles of the distribution of households ($\sigma = 2.0$)

Note: ρ is the persistence of the income shock process and σ_z is the standard deviation of innovations to the log income shock process.

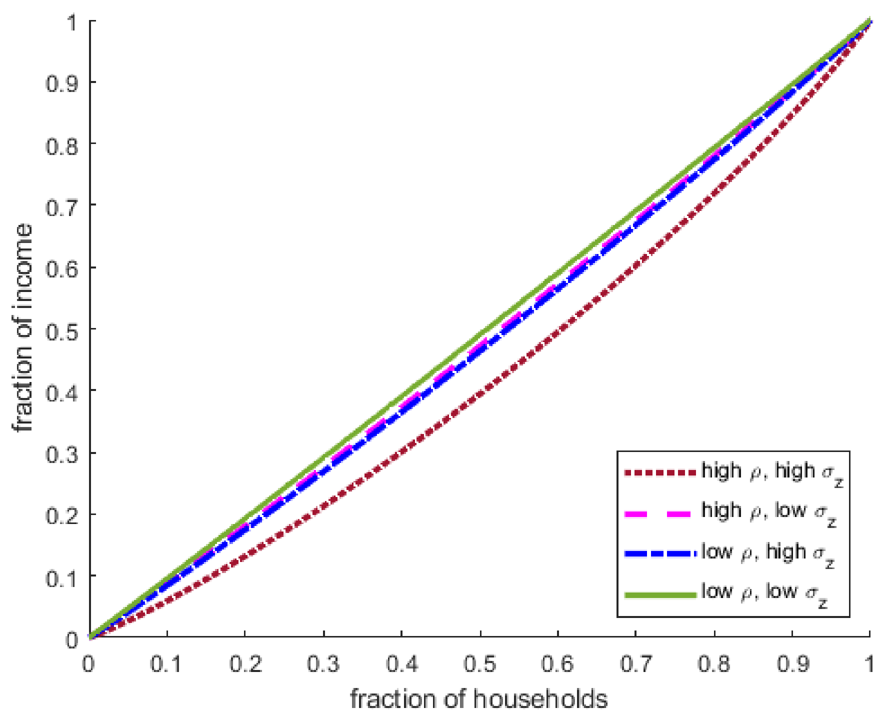


Figure 4: The Lorenz curves for income distributions resulting from different types of shocks ($\sigma = 2.0$)

Note: ρ is the persistence of the income shock process and σ_z is the standard deviation of innovations to the log income shock process.

and/or have lower variance. Accordingly, the bottom 25 percent of earners wind up with a much lower share (17 percent) of the income when persistence and variance of the income risk process are both high, compared to 23 to 24 percent when the risks are less persistent or less variable or both. This shows that the underlying shape of the income distribution is significantly different when the shocks are both more persistent and more variable compared to when either persistence is lower, or the shocks are less variable, or both.

4 A heterogeneous-agent-incomplete-insurance economy with human capital and a financial asset

I now augment the model presented above to have a financial asset in the economy. In this environment, households have access to a single financial instrument - a one-period-ahead sure claim on consumption goods, that is, a risk-free discount bond.

4.1 The Model

4.1.1 Model environment with savings

Possession of b units of a discount bond this period entitles a household to b goods next period. To obtain b' units of consumption goods next period, a household must pay a discounted price of $b' \cdot q$ worth of consumption goods this period, where q is the discount price of next period's claim to a unit of the consumption good. Savings via discount bonds must always remain above a borrowing limit $\underline{b}$, where $\underline{b} < 0$. This is the borrowing constraint that households face: $b' \geq \underline{b}$. A household's position at a point in time is now described by an individual state vector $x \in X$ and $x = (h, b, e)$ indicates a household's human capital level h, its bond holding level or wealth b, and the income shock e. The individual state space is now defined as $X = H \times B \times E$, where $H = [0, \infty)$, $B = (\underline{b}, \infty)$, and $E = \{e_1, e_2, ..., e_{N(e)}\}$, such that $e_1 < e_2 < ... < e_{N(e)}$. All other facets of the model with human capital and without a financial asset, presented in Section 3, are inherited and retained by this model as before.

4.1.2 Households' decision problem with human capital and savings

A period budget constraint for a household that chooses consumption c and next-period bond holding b', given bond holding this period b, and income from production y, is now:

$$c + q \cdot b' \leq b + y$$

where y, as before, is given as $y = e \cdot w \cdot (1 - s) \cdot h$ where $w = A$.

The functional equation describing the decision problem of a household entering the period in human capital state h, bond holding state b, and exogenous idiosyncratic state e is, therefore:

$$V(h, b, e) = \max_{c,\, s,\, b'} \left(u(c) + \beta \sum_{e'} \pi(e' \,|\, e) V(h', b', e') \right) \tag{16}$$

subject to:

the budget constraint:

$$c + q \cdot b' \le b + e \cdot w \cdot (1 - s) \cdot h \tag{17}$$

the human capital accumulation constraint:

$$h' = (1 - \delta) \cdot h + \theta \cdot (s \cdot h)^{\alpha} \tag{18}$$

the borrowing constraint:

$$b' \ge \underline{b} \tag{19}$$

the non-negativity constraint for consumption:

$$c \ge 0 \tag{20}$$

the time endowment constraint:

$$s \in (0, 1). \tag{21}$$

Optimal paths for consumption, human capital, and savings: The Euler equations for h' and b' characterize the optimal paths for consumption, human capital, and savings in this economy. (See Appendix B for derivations.) The Euler equation for h' is given by equation 13 as before and has the same interpretation. The Euler equation for b' is given by the following equation:

$$\frac{\mathbb{E}\ [(c')^{-\sigma}]}{c^{-\sigma}} = \frac{q}{\beta} \tag{22}$$

As the price of the discount bond q increases, the rate of consumption growth in the economy slows down.

4.1.3 Equilibrium

Let μ be a probability measure on (S, β_S), where $S = H \times B \times E$ and β_S is the Borel $\sigma-$algebra. As in the model without savings, μ defined on subsets of the individual state space S represents the distribution of households over such subsets of the individual state space at any given point in time. Therefore, for $\mathcal{B} \in \beta_S$, $\mu(\mathcal{B})$ represents the mass of households whose individual state vectors lie in $\mathcal{B}$. For a stationary equilibrium, I assume that the probability measure μ and the price of discount bonds q remain unchanged over time. In equilibrium, aggregate demand for bonds must equal aggregate supply of bonds for the bond market to clear.

I assume that the net aggregate supply of bonds is zero. Given the heterogeneity among households, this can be achieved without individual demand for and supply of bonds being equal to 0 in equilibrium. A stationary recursive equilibrium in this economy is defined as follows.

Definition 2. *Stationary recursive equilibrium for this economy is defined as the set of functions* $\{V(h,b,e),\ gs\,(h,b,e),\ gb\,(h,b,e),\ gc\,(h,b,e),\ \mu\,(h,b,e)\}$ *and a set of prices* $\{w,\ q^*\}$ *such that:*

(i) $V(h,b,e)$ *solves the households' decision problem (equation 16 subject to the constraints 17, 18, 19, 20, 21);*

(ii) $gs\,(h,b,e)$*,* $gb\,(h,b,e)$*, and* $gc\,(h,b,e)$ *are the optimal decision rules, given* w *and* q^**;*

(iii) Individual decisions are consistent with the aggregates, that is, $\int_S\ gb\,(h,b,e)\ d\mu\,(h,b,e) = 0$*,* $\int_S\ (1 - gs\,(h,b,e))\cdot h\ d\mu\,(h,b,e) = N^e$*, and* $\int_S\ gc\,(h,b,e)\ d\mu\,(h,b,e) = \int_S\ y\ d\mu\,(h,b,e)$*;*

(iv) μ *is a stationary probability measure such that it satisfies* $\mu(\mathcal{B}) = \int_S\ P(x,\mathcal{B})\ d\mu\,(h,b,e)$ *for all* $\mathcal{B} \subset \beta_S$ *where* $P(x,\mathcal{B})$ *is a transition function representing the probability that a household with state* $x = (h,b,e)$ *will have an individual state vector lying in* $\mathcal{B}$ *next period.*

The first and second conditions imply that households optimize and $V(h,b,e)$ is the value function resulting from their optimization problem, and the decision rules for time invested in human capital accumulation $gs\,(h,b,e)$, savings $gb\,(h,b,e)$, and consumption $gc\,(h,b,e)$ are the optimal functions that solve the households' optimization problem, for the equilibrium wage rate w and the equilibrium price of the discount bond q^*. In equilibrium, individual decisions are consistent with the aggregate outcomes. At the equilibrium price of the discount bond q^*, net aggregate demand for and supply of bonds is equal to 0 so the bond market clears. The labour supply decisions of households is consistent with the aggregate effective labour demanded by the representative firm, so the labour market clears. And by Walras' law, the goods market clears as well - in equilibrium, consumption averaged over the population equals income averaged over the population. Lastly, in equilibrium, the distribution of households over states remains constant over time.

4.2 Parameterization

The set of chosen parameter values for the model with human capital and savings is the same as before given in Table 1. One additional parameter to be chosen here is the borrowing limit $\underline{b}$. I computed the model for many different choices of the borrowing limit over a wide range of values from -20 to -4; the results presented in the body of the paper are for a borrowing limit of $\underline{b} = -6$. A note in section 4.3 has a brief discussion on computation issues related to solving this model, and also briefly discusses the nature of results obtained from other choices of the borrowing limit. The shock process also remains the same as before and Table 2 summarizes the parameters for the shock process. For this model, I compare three different shocks: a high persistence, high variance shock, a high persistence, low variance shock, and a low persistence, low variance shock. I keep the households' coefficient of relative risk aversion fixed at the baseline value of $\sigma = 2$ throughout the analysis for this model.

4.3 Computation of the model with human capital and bonds

As before, I use the Tauchen algorithm to develop a Markov chain to approximate the grid on the idiosyncratic income shocks E and the transition probability matrix. I set up grids on human capital levels and bond holding levels for approximating the value function. I solve for the optimal decision rules $gs\,(h, b, e)$ and $gb\,(h, b, e)$ for the households' problem using golden section search and 2-dimensional (2D) linear interpolation, and I use bisection to solve for the equilibrium bond price q^*. The equilibrium optimal decision rules and the equilibrium stationary distribution of households is obtained by using golden section search and 2D linear interpolation for the equilibrium bond price $q = q^*$.

4.4 Results

I solve the model for three different types of shocks which differ in their persistence and variance. These can be characterized as: high persistence, high variance; high persistence, low variance; and low persistence, low variance. The equilibrium bond price for the model solved for the three different shocks are shown in Table 8.

	high ρ, high σ_z	high ρ, low σ_z	low ρ, low σ_z
Equilibrium bond price q^*	1.0395	0.9669	0.9727

Table 8: Equilibrium bond price under three different shock processes

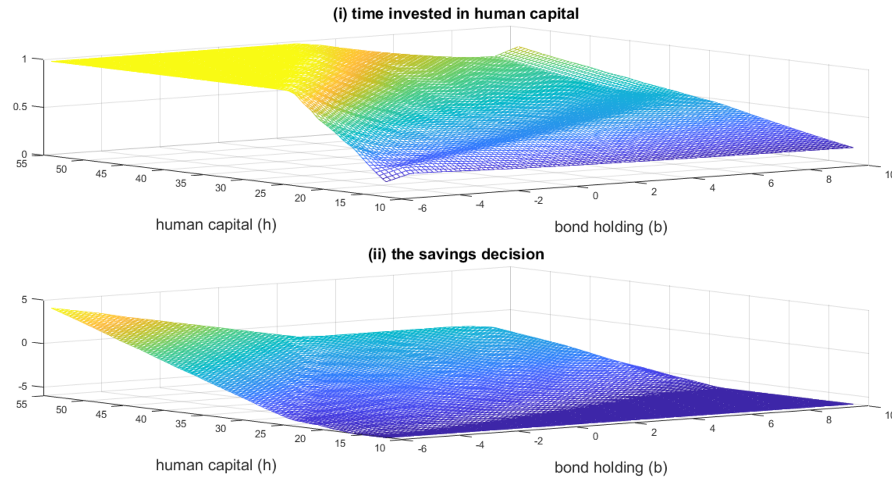
4.4.1 Households' optimal decision rules for investment in human capital and savings

Figure 5 shows the households' equilibrium optimal decision rules for investing in human capital and bonds when they face uninsurable idiosyncratic income risks that have high persistence and high variance. The decision rules are depicted for three different realizations of the income shock e: a low e realization (Figure 5a), realization of the median e value (Figure 5b), and a high e realization (Figure 5c), where the top panel in each figure shows the optimal time invested in human capital and the bottom panel in each figure shows the optimal savings decision, along human capital levels and bond holding levels.

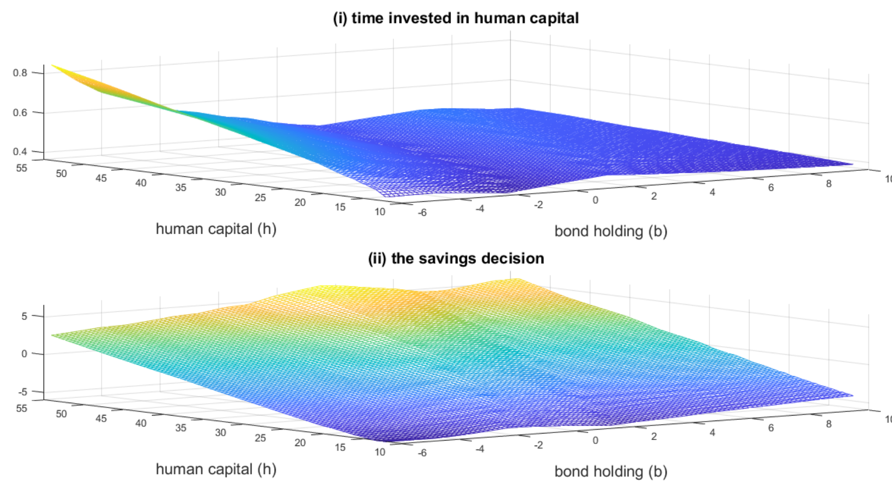
For any realization of the income shock, time invested in human capital increases with the level of human capital for low levels of bond holding. The increase is more pronounced for lower realizations of the income shock than when a higher e is drawn. This is an interesting result, which highlights the importance of introducing a financial asset into the model when households' human capital was the only asset they had available to utilize for safeguarding consumption against the idiosyncrasies of the income shock. This result departs from the pattern observed in time spent in human capital accumulation when households had only human capital as an asset to invest in to insure consumption against risk: time invested in human capital increased as the shock realizations became more favourable. Here, since households now have the ability to borrow or save income to transfer consumption across time, when they have a negative income shock, and their bond holding is very low (they are borrowing to consume today), they spend much more time in human capital accumulation than if they realized a higher income shock. At the same time, they also save more with increasing levels of human capital when saving today is low (or negative), as seen from the rising decision rule along the human capital levels at the low levels of bond holding, for any income shock realized.

The savings decision rule also shows an increase in savings along human capital levels for any level of bond holding. The increase is most even across bond holding levels for moderate realizations of the income shock (Figure 5b). For low realizations of the shock, saving increases more sharply for households with low levels of current bond holding than for those who are already high on savings (Figure 5a). This is sensible as a low income shock coupled with low levels of current bond holding calls for higher savings. For very high income shock realizations, households with high levels of human capital and current bond holding decide to have very high savings as seen in Figure 5c toward the higher levels of human capital and bond holding. This seems intuitive as households which have high levels of human capital as well as large savings, when realize a very positive income shock, decide to save more and invest relatively less time in more human capital accumulation. The two decisions together enable households to smooth their consumption profile.

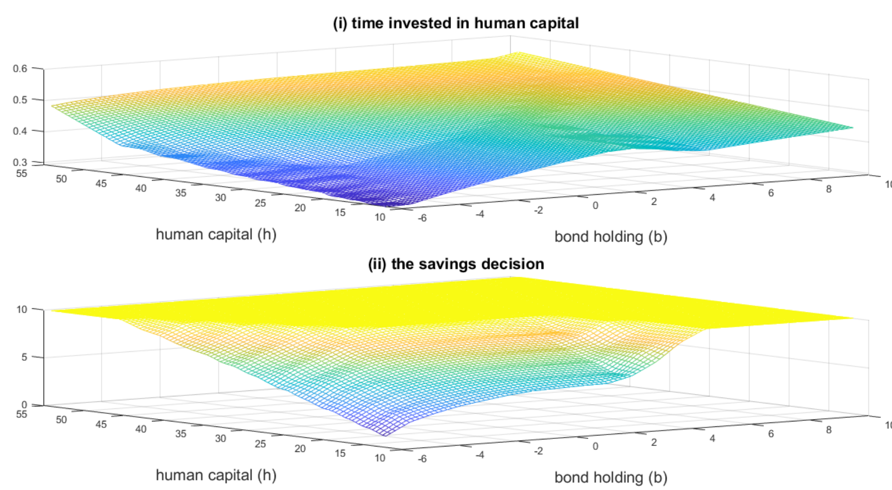
The important thing to note here is that even as households have access to a financial asset via which they can save and borrow to transfer consumption across time, households leverage both assets - their human capital as well as bonds - to insure against the vagaries of the income shock, irrespective of their current human capital level and their current savings. The decision rules depict the extent to which households leverage each of these assets based on their current state of human capital, bond holding, and income shock.



(a) low e

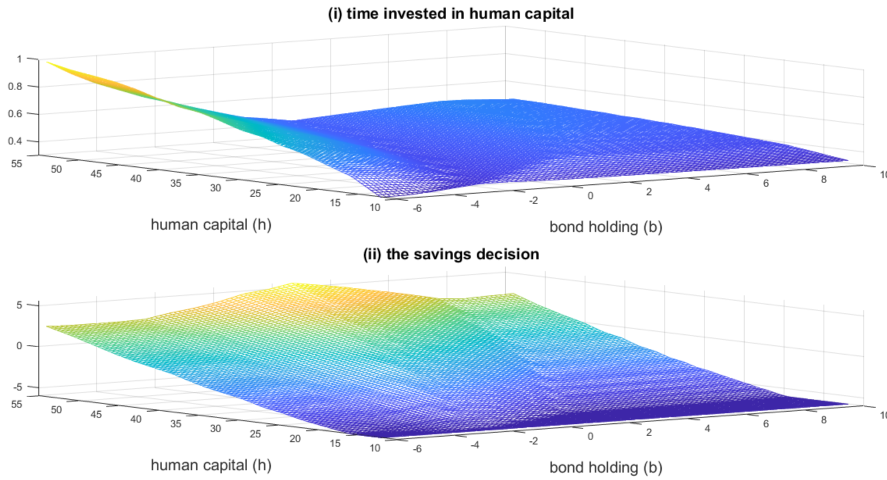


(b) median e

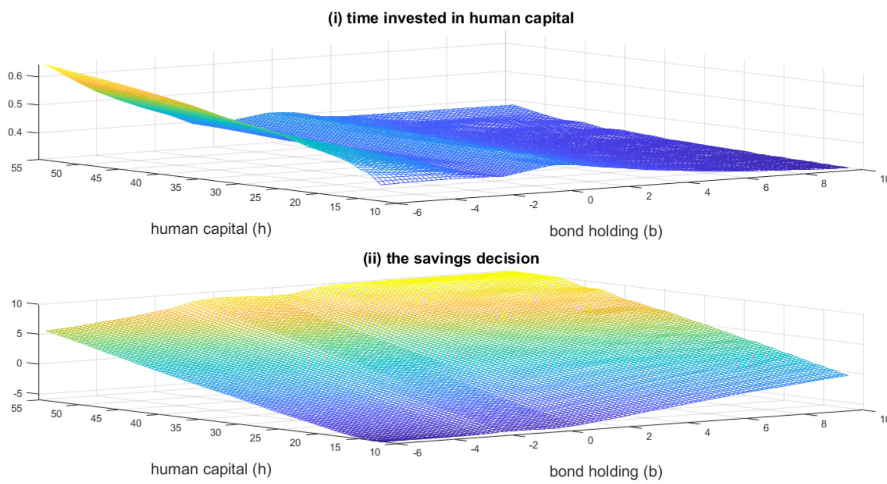


(c) high e

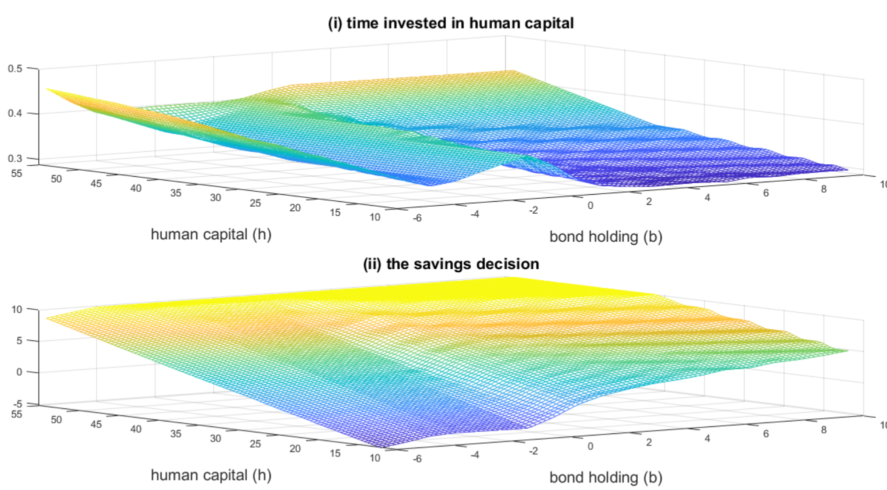
Figure 5: Households' equilibrium decision rules for investment in human capital and bonds when idiosyncratic income shock has high persistence and high variance



(a) low e

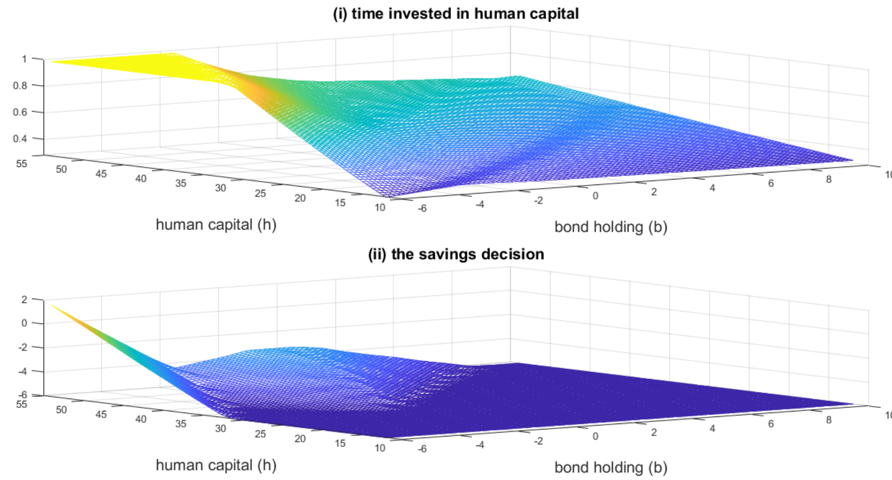


(b) median e

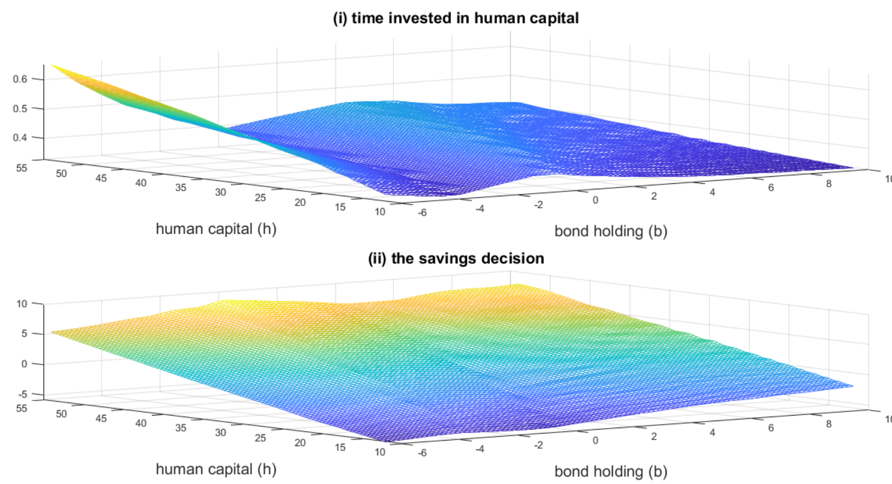


(c) high e

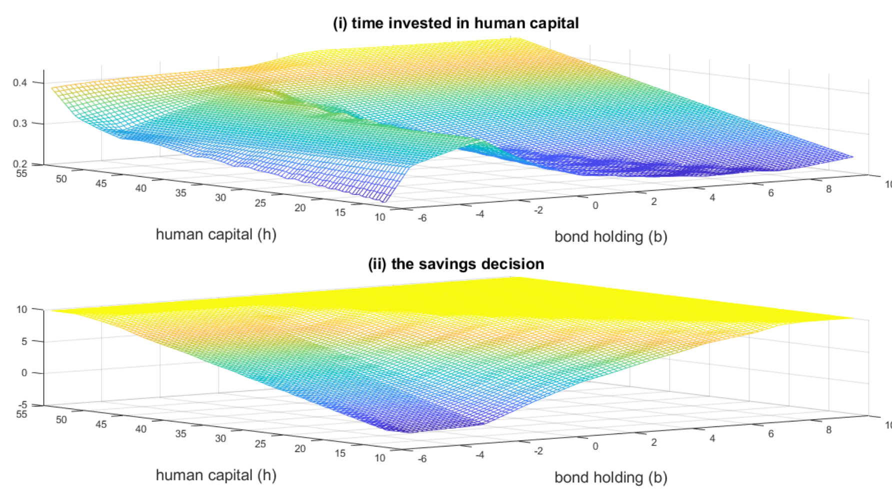
Figure 6: Households' equilibrium decision rules for investment in human capital and bonds when idiosyncratic income shock has high persistence and low variance



(a) low e



(b) median e



(c) high e

Figure 7: Households' equilibrium decision rules for investment in human capital and bonds when idiosyncratic income shock has low persistence and low variance

Slices of the decision rules along human capital levels and idiosyncratic income shocks, conditioned on three different levels of bond holding are shown in Appendix E for a different view of the equilibrium optimal decision rules for households.

Figure 6 shows the households' equilibrium optimal decision rules for time invested in human capital and savings when the shock is highly persistent, but has low variance, and Figure 7 shows households' equilibrium optimal decision rules when the shock is less persistent and also has lower variance. The results appear qualitatively similar. However, as can be noted from a comparison of Figure 5 and Figure 6, a fall in variability of the shock process causes the decisions to be more moderate for the extreme realizations of the income shock, compared to when the shock is more dispersed (Figures 6a and 6c compared to Figures 5a and 5c, respectively).

Analysing carefully the shapes of the decision rule for time invested in human capital versus the decision rule for savings, for higher realizations of the income shock e, we can see the complementarity between the use of human capital and bonds - the two assets available to the households - in trying to insure against unpredictable income risks.

4.4.2 Income distribution: stationary distribution of households

Figure 8 shows the stationary distribution of households when the income shock is highly persistent and has high variance. When the realized shock is very negative, the distribution of households is concentrated around very low levels of human capital, as shown in Figure 8a. When the shock is very positive, the households tend to get clustered around very high levels of human capital, as shown in Figure 8c. The distribution is relatively more dispersed along levels of human capital for a realized income shock which is neither too bad nor too good (Figure 8b). Households are, however, much more evenly distributed along levels of bond holding.

We see from the stationary distribution of households that despite the decision to have very high savings when households' individual human capital state, their bond holding state, and their income shock state are all very high, the distributions are fairly evenly distributed along levels of bond holding as shown in Figure 8, as also can be seen in the distribution over levels of human capital and income shocks conditioned on given values of bond holding as in Figure 13.

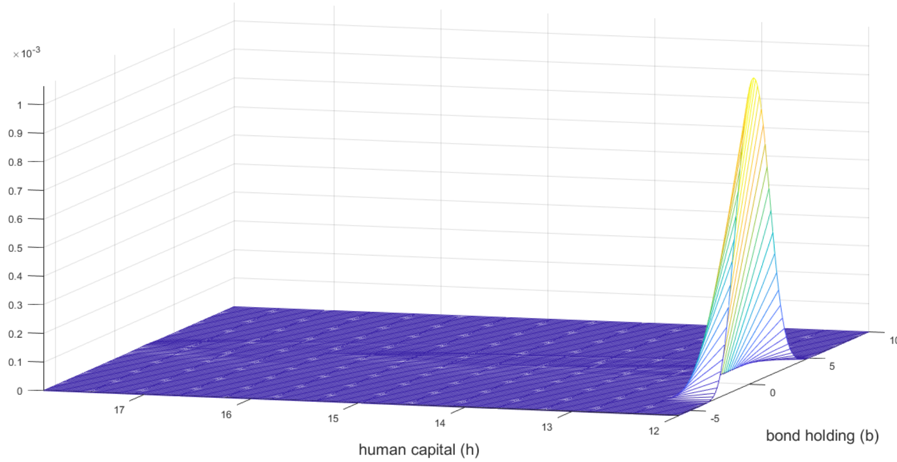
A different view of the equilibrium stationary distribution of households is depicted in Figure 13 in Appendix E where the distribution is shown along levels of human capital h and income shocks e, conditioned on the level of bond holding b.

Figure 9 and Figure 10 show the stationary distribution of households along levels of human capital and bond holding, for given realizations of the income shock e. As the variance of the shock decreases (Figure 9 compared to Figure 8), the distributions spread out more loosely along human capital levels. This is most evident for moderate realizations of the income shock, but it also shows in the distributions captured at more extreme realizations of e. Lower variance of the income shock process decreases the variability of the shocks realized and moderates households' responses to positive and negative shocks. As a result, the distribution of households obtained is more dispersed over human capital levels when the variance of the shock is lower. Further, lowering the persistence makes the shocks more volatile and, as a result, households adjust their human capital investment and savings decisions more abruptly causing households to be less smoothly distributed over human capital and bond holding levels as shown in Figure 10.

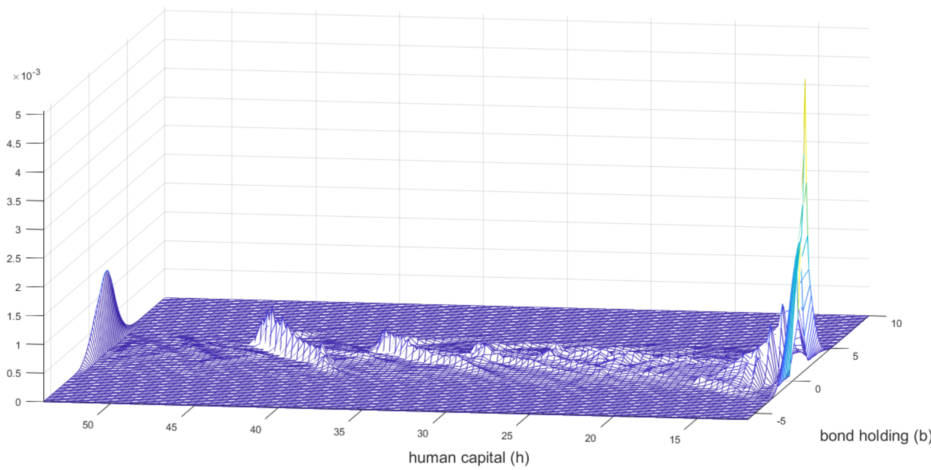
4.4.3 Simulating households facing shocks with varying persistence and variance

I simulated a household hit by three independent shock processes with different persistence and variance and plotted the evolution of these shock processes for 1000 periods. As before, I focus on the same 100 periods between dates 550 and 650 as one sample of that long simulation to see how changes in risks affect the evolution of time invested in human capital s , the level of human capital h, savings b', income y, cash-on-hand, and consumption c for a household facing these shocks. Figure 11 plots these series along with the evolution of the shock processes shown in Figure 11a. The means and variances of the simulated series are shown in Table 9. The shock with high persistence and high variance causes the most variation in all the series except the savings series, which shows most variation for the shock with low persistence and low variance.

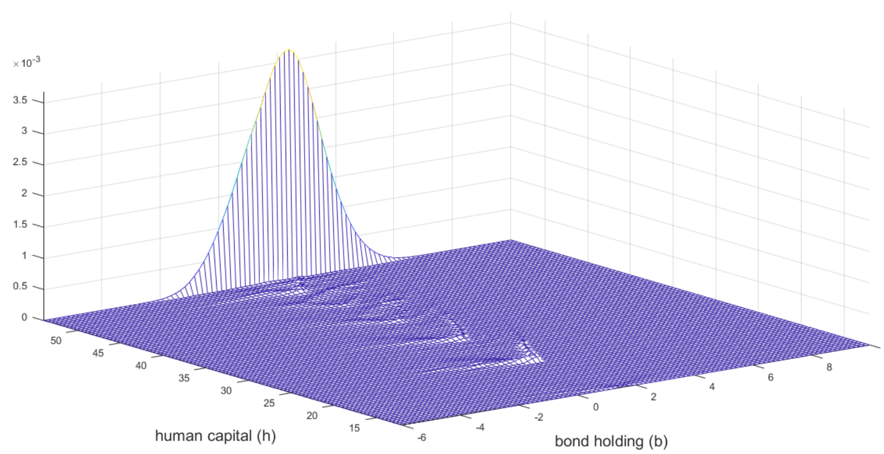
When the shock is highly persistent as well as widely dispersed, the decisions to invest time in human capital do not fluctuate as frequently with the realization of the shock (Figure 11b) and trend broadly in the direction of the shock rather than being adjusted periodically because the effect of the shock, in terms of its nature, lingers on. This translates into a lagged but steady response in the human capital level (Figure 11c). Due to higher variance, the shocks are widely spread out and both positive and negative shocks are more extreme. As positive shocks get realized as shown in Figure 11a, households are also more inclined to save more (Figure 11d). As a result, higher human capital causes income to rise (Figure 11e), and with higher savings, cash-on-hand increases (Figure 11f), and consumption trends steadily in the direction of the shocks (Figure 11g). When shocks are less persistent, or less variable, or both, the higher volatility and narrower dispersion of



(a) low e

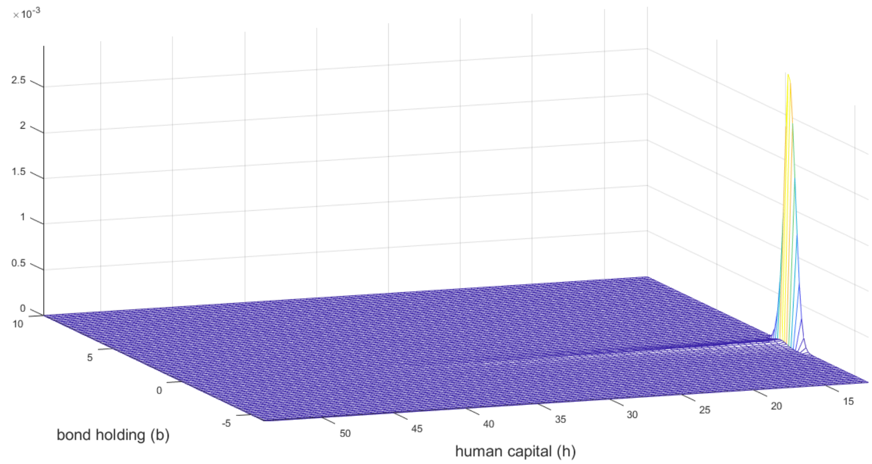


(b) median e

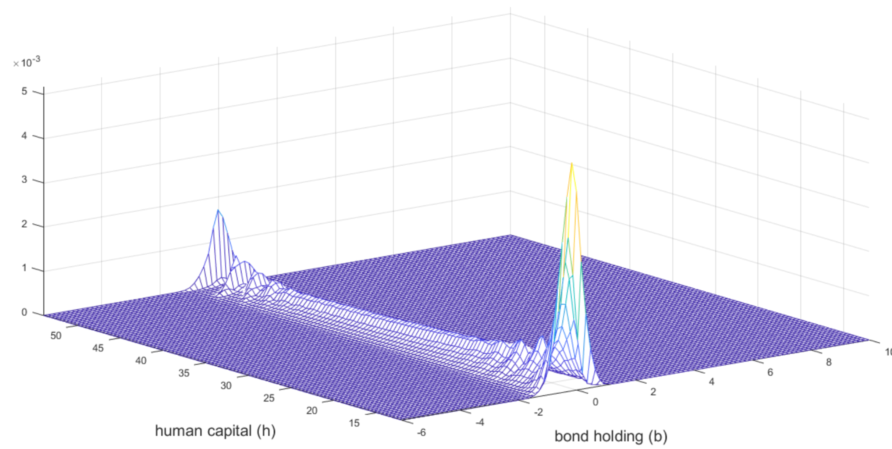


(c) high e

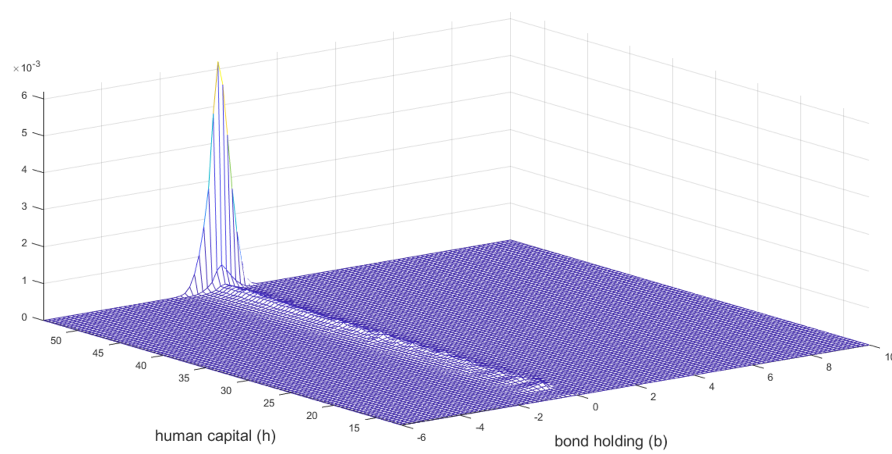
Figure 8: The stationary distribution of households when idiosyncratic income shock has high persistence and high variance



(a) low e

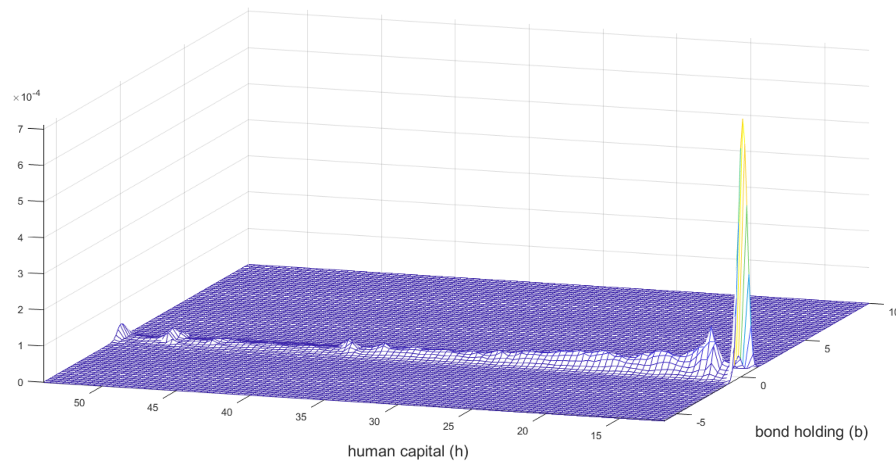


(b) median e

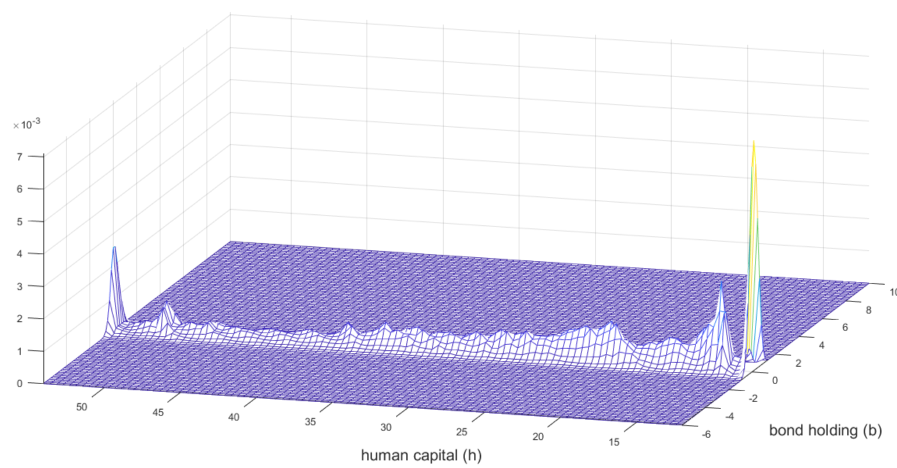


(c) high e

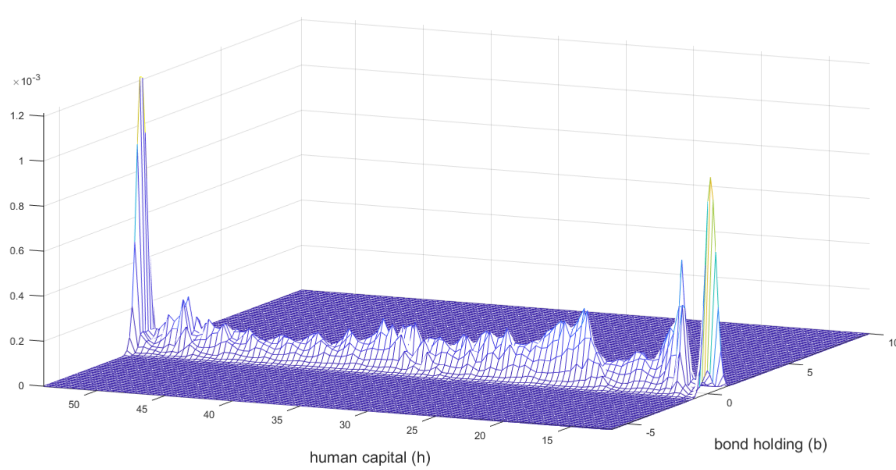
Figure 9: The stationary distribution of households when idiosyncratic income shock has high persistence and low variance



(a) low e

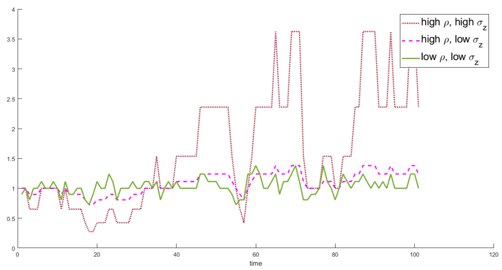


(b) median e

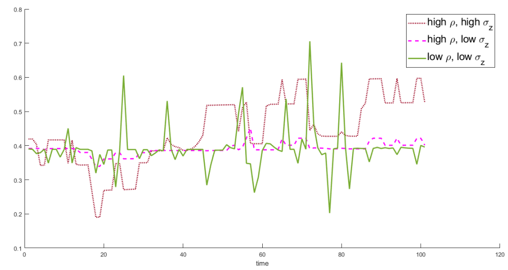


(c) high e

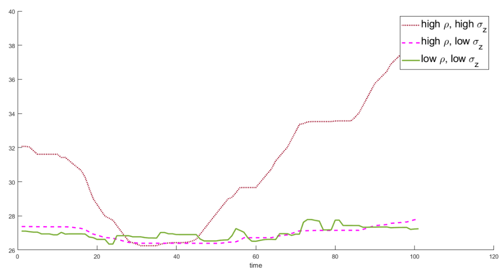
Figure 10: The stationary distribution of households when idiosyncratic income shock has low persistence and low variance



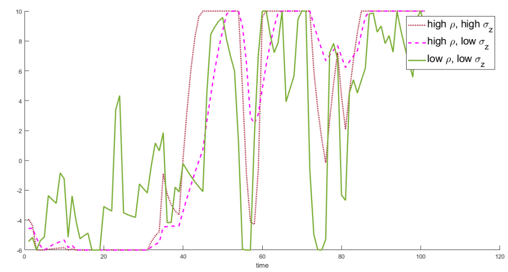
(a) Income shock e



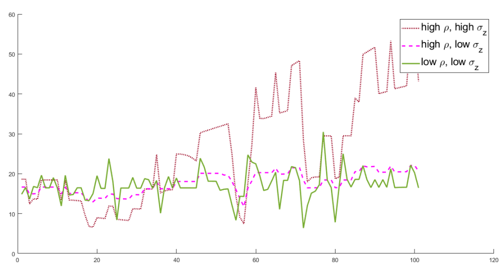
(b) Time invested in human capital s



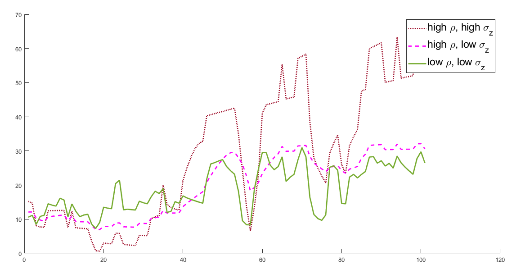
(c) Human capital h



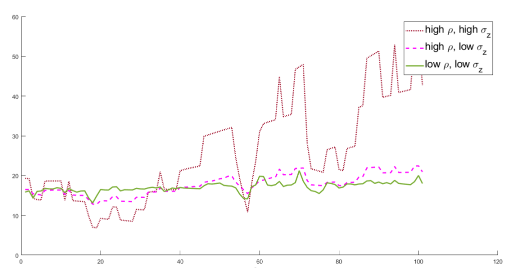
(d) Savings b'



(e) Income y



(f) Cash-on-hand



(g) Consumption c

Figure 11: Evolution of e, s, h, b', y, cash-on-hand, and c for a simulated household facing different shocks

Note: ρ is the persistence of the income shock process and σ_z is the standard deviation of innovations to the log income shock process.

	Mean			Coefficient of Variation (CV)		
	high ρ, high σ_z	high ρ, low σ_z	low ρ, low σ_z	high ρ, high σ_z	high ρ, low σ_z	low ρ, low σ_z
time invested in human capital (s)	0.44	0.39	0.39	22.18	4.42	16.67
human capital (h)	30.92	26.94	26.99	11.27	1.59	1.25
savings (b')	2.61	2.38	1.76	272.03	291.75	326.58
income (y)	25.69	17.73	17.18	51.98	14.97	20.90
cash-on-hand (coh)	28.18	19.97	18.81	69.21	45.60	35.70
consumption (c)	25.46	17.67	17.09	50.23	14.67	7.26

Table 9: Mean and coefficient of variation of series of interest obtained for a simulated household

Note: ρ is the persistence of the income shock process and σ_z is the standard deviation of innovations to the log income shock process.

shocks makes the household respond more frequently and less drastically to shock realizations. The changes in time invested in human capital, the level of human capital, income, or consumption is, therefore, much subdued.

4.4.4 Comparing income inequality: the Gini coefficient

Table 10 shows the Gini coefficient for income inequality which does not include wealth in the form of bonds held by the household. Income inequality is highest when the shock is more persistent and has higher variance as was the case earlier. Higher variability of the shock process also increases inequality in incomes. The shares of income possessed by the top and bottom percentiles under the three different shocks is also shown in Table 10. It highlights the different characteristics of the underlying distributions which may not be brought out by differences in the Gini coefficient itself. The distribution of income resulting from the shock with high persistence and high variance is, indeed, the most unequal compared to distributions resulting from the other two types of shocks. The top 10 percent earners have over 15 percent of total income, and the top 25 percent of earners have over 33 percent of total income, when the shock is more persistent and more dispersed. In contrast, the bottom 10 percent have a little over 6 percent of income and the bottom 25 percent have over 17 percent of income when such a shock hits the households. The shares of income possessed by the top and bottom percentiles of earners is very similar in the distribution of income when the shock is more persistent but has lower variance, or when it has both lower persistence and lower variance.

As further work in this direction, I would look at how the Gini coefficients for wealth inequality compare across different types of shocks in this model. It would also be interesting to observe the evolution of the Gini coefficient for income inequality over time in each of the models discussed in this paper to see how income

	high ρ, high σ_z	high ρ, low σ_z	low ρ, low σ_z
Gini coefficient for income inequality	0.1477	0.0397	0.0309
Top 10%	15.40	10.61	10.54
Bottom 10%	6.32	9.25	9.56
Top 25%	33.67	27.81	24.82
Bottom 25%	17.76	23.50	23.23

Table 10: Gini coefficients for income inequality in distributions resulting from different types of shocks and shares of income possessed by the top and bottom percentiles of earners in such distributions

inequality is affected over time by the evolution of the type of shock in question.

5 Conclusion

This paper develops a heterogeneous-agent incomplete-insurance model with uninsurable idiosyncratic income risks and human capital to study how exogenous income risks affect households' consumption and inequality in income distributions. I study these effects using different types of risks by changing the persistence and variance of the income shock process to see how decisions to invest in human capital change with the level of human capital and the realization of the shock. Households invest more in human capital as the shocks become more positive, for any persistence and variance of the shock process. Highly persistent and more variable shocks cause the highest variations in evolution of human capital investment, the level of human capital, and consumption around their respective means. Households are distributed more widely across human capital levels, in equilibrium, and income inequality is highest in the distribution of income resulting from such income shocks.

Households, when faced with the uncertainty from idiosyncratic shocks to their income, invest in their human capital as a means to insure against the vagaries of income risks. As shocks improve, investment in human capital increases as households take advantage of the good state today to secure future consumption. The nature of the shock process in terms of its persistence and variance, together with the households' appetite for risk, determines the exact nature of the patterns seen in time invested in human capital across levels of human capital and across the possible realizations of the shock.

I augment this model by introducing a financial asset - a risk-free discount bond - into the model to analyse how households' decisions to invest in human capital changes when they have access to an additional asset through which they can save or borrow to transfer consumption across time. With access to two assets - their human capital, and discount bonds - households leverage both these assets to insure against the idiosyncratic income risk and secure current and future consumption. The time invested in human capital now increases unambiguously with the level of human capital, irrespective of realizing a good shock or a bad shock, at lower levels of bond holding. Additionally, there is an interesting departure in the pattern of time invested in human capital here from what we observe in the model without bonds - with a low income shock realization, households now invest more in human capital compared to when they have a high income shock realization because they now have access to an additional asset - bonds - to cover for consumption in a bad state, so they invest more in building human capital for improving their production capacity to enhance consumption in future periods.

Households end up being fairly evenly distributed in equilibrium along levels of bond holding, but dispersion along human capital levels depends on the realized shock and the nature of the shock process generating those shocks. As before, income inequality is highest when the shock is more persistent and has higher variance. It is also these shocks for which the evolution of the time invested in human capital, the level of human capital, income, and consumption have the highest variation around their respective means.

These results point to interesting dynamics at work behind households' decisions to invest in human capital versus their motivation to invest in a financial asset to hedge against the risks from idiosyncratic income shocks.

My outlook for furthering this work is to see how aggregate shocks in this environment play out in households' decisions to invest in human capital and bonds in a general equilibrium setting, and to see how these decisions bear on the stationary distribution of income and the resulting inequalities in income, wealth, and human capital.

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A Deriving the Euler equation for human capital accumulation when there is no financial asset

Setting up the Lagrangian for an individual i:

$$\mathcal{L} = u(c_i) + \beta \cdot \mathbb{E}\ V(h_i', e_i') + \lambda_i \cdot [y_i - c_i] = u(c_i) + \beta \cdot \mathbb{E}\ V(h_i', e_i') + \lambda_i \cdot [e_i \cdot A \cdot (1 - s_i) \cdot h_i - c_i]$$

For ease of exposition, I suppress the individual subscript i and substitute for s from the human capital accumulation constraint:

$$h' = (1 - \delta) \cdot h + \theta \cdot (s \cdot h)^{\alpha}$$

Therefore, substituting $s = \left[\frac{h' - (1-\delta) \cdot h}{\theta \cdot h^{\alpha}}\right]^{\frac{1}{\alpha}}$ in the Lagrangian $\mathcal{L}$, we have:

$$\mathcal{L} = \frac{c^{1-\sigma}}{1-\sigma} + \beta \cdot \mathbb{E}\ V(h', e') + \lambda \cdot \left[e \cdot A \cdot \left(1 - \left[\frac{h' - (1-\delta) \cdot h}{\theta \cdot h^{\alpha}}\right]^{\frac{1}{\alpha}}\right) \cdot h - c\right]$$

Taking first-order conditions with respect to c and h':

$$[\mathbf{c}] : c^{-\sigma} = \lambda \tag{23}$$

$$[\mathbf{h'}] : \beta \cdot \mathbb{E}\ V'(h', e') - \lambda \cdot e \cdot A \cdot \frac{1}{\alpha \cdot \theta} \left(\frac{h' - (1-\delta) \cdot h}{\theta}\right)^{\frac{1}{\alpha} - 1} = 0 \tag{24}$$

Using the Benveniste-Scheinkman condition and leading it forward:

$$V'(h', e') = \lambda' \cdot e' \cdot A - \lambda' \cdot e' \cdot A \cdot \frac{1}{\alpha} \cdot \left(\frac{h'' - (1-\delta) \cdot h'}{\theta}\right)^{\frac{1}{\alpha} - 1} \cdot \left(\frac{-(1-\delta)}{\theta}\right)$$

or

$$V'(h', e') = \lambda' \cdot e' \cdot A \cdot \left[1 + \frac{(1-\delta)}{\alpha \cdot \theta} \cdot \left(\frac{h'' - (1-\delta) \cdot h'}{\theta}\right)^{\frac{1}{\alpha} - 1}\right] \tag{25}$$

Substituting equation 23 and equation 25 into the first-order condition with respect to h' (equation 24) gives:

$$\beta \cdot \mathbb{E}\left[\lambda' \cdot e' \cdot A \cdot \left[1 + \frac{(1-\delta)}{\alpha \cdot \theta} \left(\frac{h'' - (1-\delta) \cdot h'}{\theta}\right)^{\frac{1}{\alpha} - 1}\right]\right] = \lambda \cdot e \cdot A \cdot \frac{1}{\alpha \cdot \theta} \left(\frac{h' - (1-\delta) \cdot h}{\theta}\right)^{\frac{1}{\alpha} - 1}$$

or

$$\beta \cdot \mathbb{E}\left[(c')^{-\sigma} \cdot e' \cdot \left[1 + \frac{(1-\delta)}{\alpha \cdot \theta}\left(\frac{h'' - (1-\delta) \cdot h'}{\theta}\right)^{\frac{1}{\alpha}-1}\right]\right] = c^{-\sigma} \cdot e \cdot \frac{1}{\alpha \cdot \theta}\left(\frac{h' - (1-\delta) \cdot h}{\theta}\right)^{\frac{1}{\alpha}-1}$$

Substituting $s = \left[\frac{h'-(1-\delta)\cdot h}{\theta \cdot h^{\alpha}}\right]^{\frac{1}{\alpha}}$ or $s \cdot h = \left[\frac{h'-(1-\delta)\cdot h}{\theta}\right]^{\frac{1}{\alpha}}$ from the human capital accumulation constraint into the above equation, we have:

$$\beta \cdot \mathbb{E}\left[(c')^{-\sigma} \cdot e' \cdot \left[1 + (1-\delta) \cdot \frac{1}{\alpha \cdot \theta} \cdot (s' \cdot h')^{1-\alpha}\right]\right] = c^{-\sigma} \cdot e \cdot \frac{1}{\alpha \cdot \theta} \cdot (s \cdot h)^{1-\alpha}$$

This is the Euler equation for h'.

B Deriving the Euler equation for human capital accumulation and savings in the model with human capital and risk-free discount bonds

$$V(h,b,e) = \max_{c,\,s,\,b'} \left(u(c) + \beta \sum_{e'} \pi(e' \mid e) V(h',b',e') \right)$$

subject to:

the budget constraint:

$$c \,+\, q \cdot b' \leq b \,+\, e \cdot w \cdot (1-s) \cdot h$$

where $w = A$, and

the human capital accumulation constraint:

$$h' = (1-\delta) \cdot h + \theta \cdot (s \cdot h)^{\alpha}$$

Setting up the Lagrangian:

$$\mathcal{L} = \frac{c^{1-\sigma}}{1-\sigma} + \beta \cdot \mathbb{E}\; V(h',b',e') + \lambda \cdot [b + e \cdot A \cdot (1-s) \cdot h - q \cdot b' - c]$$

Substituting $s = \left[\frac{h'-(1-\delta)\cdot h}{\theta \cdot h^{\alpha}}\right]^{\frac{1}{\alpha}}$ in the Lagrangian $\mathcal{L}$, we have:

$$\mathcal{L} = \frac{c^{1-\sigma}}{1-\sigma} + \beta \cdot \mathbb{E}\; V(h',b',e') + \lambda \cdot \left[b + e \cdot A \cdot \left(1 - \left[\frac{h'-(1-\delta)\cdot h}{\theta \cdot h^{\alpha}} \right]^{\frac{1}{\alpha}} \right) \cdot h - q \cdot b' - c \right]$$

Taking first-order conditions with respect to c, b', and h':

$$[\mathbf{c}] : c^{-\sigma} = \lambda \tag{26}$$

$$[\mathbf{b'}] : \beta \cdot \mathbb{E}\; V_2'(h',b',e') = \lambda \cdot q \tag{27}$$

$$[\mathbf{h'}] : \beta \cdot \mathbb{E}\; V_1'(h',b',e') - \lambda \cdot e \cdot A \cdot \frac{1}{\alpha \cdot \theta} \left(\frac{h'-(1-\delta)\cdot h}{\theta} \right)^{\frac{1}{\alpha}-1} = 0 \tag{28}$$

Using the Benveniste-Scheinkman condition and leading it forward:

$$V_2'(h',b',e') = \lambda' \cdot 1 \tag{29}$$

and

$$V_1'(h',b',e') = \lambda' \cdot e' \cdot A - \lambda' \cdot e' \cdot A \cdot \frac{1}{\alpha} \cdot \left(\frac{h'' - (1-\delta) \cdot h'}{\theta}\right)^{\frac{1}{\alpha}-1} \cdot \left(\frac{-(1-\delta)}{\theta}\right)$$

or

$$V_1'(h',b',e') = \lambda' \cdot e' \cdot A \cdot \left[1 + \frac{(1-\delta)}{\alpha \cdot \theta} \cdot \left(\frac{h'' - (1-\delta) \cdot h'}{\theta}\right)^{\frac{1}{\alpha}-1}\right] \tag{30}$$

Substituting equation 26 and equation 29 into the first-order condition with respect to b' (equation 27) gives:

$$\beta \cdot \mathbb{E}\ \lambda' = \lambda \cdot q$$

or

$$\frac{\mathbb{E}\ [(c')^{-\sigma}]}{c^{-\sigma}} = \frac{q}{\beta} \tag{31}$$

This is the Euler equation for b'.

Substituting equation 26 and equation 30 into the first-order condition with respect to h' (28) gives:

$$\beta \cdot \mathbb{E}\left[\lambda' \cdot e' \cdot A \cdot \left[1 + \frac{(1-\delta)}{\alpha \cdot \theta}\left(\frac{h'' - (1-\delta) \cdot h'}{\theta}\right)^{\frac{1}{\alpha}-1}\right]\right] = \lambda \cdot e \cdot A \cdot \frac{1}{\alpha \cdot \theta}\left(\frac{h' - (1-\delta) \cdot h}{\theta}\right)^{\frac{1}{\alpha}-1}$$

or

$$\beta \cdot \mathbb{E}\left[(c')^{-\sigma} \cdot e' \cdot \left[1 + \frac{(1-\delta)}{\alpha \cdot \theta}\left(\frac{h'' - (1-\delta) \cdot h'}{\theta}\right)^{\frac{1}{\alpha}-1}\right]\right] = c^{-\sigma} \cdot e \cdot \frac{1}{\alpha \cdot \theta}\left(\frac{h' - (1-\delta) \cdot h}{\theta}\right)^{\frac{1}{\alpha}-1}$$

Substituting $s = \left[\frac{h' - (1-\delta) \cdot h}{\theta \cdot h^{\alpha}}\right]^{\frac{1}{\alpha}}$ or $s \cdot h = \left[\frac{h' - (1-\delta) \cdot h}{\theta}\right]^{\frac{1}{\alpha}}$ from the human capital accumulation constraint into the above equation, we have:

$$\beta \cdot \mathbb{E}\left[(c')^{-\sigma} \cdot e' \cdot \left[1 + (1-\delta) \cdot \frac{1}{\alpha \cdot \theta} \cdot (s' \cdot h')^{1-\alpha}\right]\right] = c^{-\sigma} \cdot e \cdot \frac{1}{\alpha \cdot \theta} \cdot (s \cdot h)^{1-\alpha} \tag{32}$$

This is the Euler equation for h'.

C Algorithm for computation of the model with human capital and no financial asset

i. Initialize value function $v(h, e)$ defined over the state space $H \times E$

ii. Calculate conditional expectation of the value function using transition probabilities approximated using the Tauchen algorithm

iii. For each $h \in H$ and each $e \in E$, and with s_{low} and s_{high} being the lower and upper bounds, respectively, on the values that time invested in human capital s can take, I search for the optimal decision rule $gs\,(h, e)$ using golden section search

iv. I iterate on the value function until it converges to a tolerance of $1e - 5$, as the golden section search must be more precise than the value function iteration

v. I iterate on the distribution of households which is the probability measure $\mu\,(h, e)$ defined over the finer grid of h values, that is, $\mu_{n+1}(\mathcal{B}) = \int_S^{\infty} \; P\,(x, \mathcal{B})\; d\mu_n$ where $x = (h, e)$, and starting with some arbitrarily initialized $\mu_0 \in M(S)$ where $M(S)$ is the space of probability measures on (S, β_S)

   (i) For each h in the finer H grid and each e in the E grid for the exogenous idiosyncratic shocks, I have the optimal decision rule $mugs\,(h, e)$

   (ii) These decision rules allow us to calculate the distribution next period μ_{t+1} such that each household at a point today can be placed onto two possible points tomorrow

vi. I iterate on the probability measure until convergence to some level of tolerance (here, $1e - 5$) to obtain the stationary distribution of households

D Model with human capital and no financial asset: means and coefficients of variation in s, h, and c for a simulated household when $\sigma = 1.5$

Coefficient of Relative Risk Aversion $\sigma = 1.5$			
Mean		Coefficient of Variation (CV)	
high ρ, high σ_z	high ρ, low σ_z	high ρ, high σ_z	high ρ, low σ_z
0.42	0.39	17.06	4.18
low ρ, high σ_z	low ρ, low σ_z	low ρ, high σ_z	low ρ, low σ_z
0.40	0.39	19.62	4.79

Table 11: Time invested in human capital accumulation

Coefficient of Relative Risk Aversion $\sigma = 1.5$			
Mean		Coefficient of Variation (CV)	
high ρ, high σ_z	high ρ, low σ_z	high ρ, high σ_z	high ρ, low σ_z
27.73	26.11	9.24	2.10
low ρ, high σ_z	low ρ, low σ_z	low ρ, high σ_z	low ρ, low σ_z
26.99	25.98	4.03	0.97

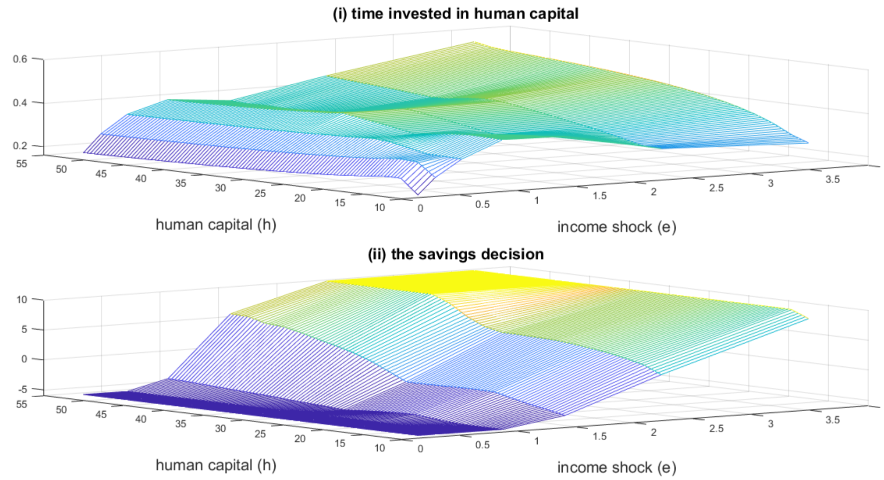
Table 12: Human capital

Coefficient of Relative Risk Aversion $\sigma = 1.5$			
Mean		Coefficient of Variation (CV)	
high ρ, high σ_z	high ρ, low σ_z	high ρ, high σ_z	high ρ, low σ_z
24.74	17.25	56.67	15.16
low ρ, high σ_z	low ρ, low σ_z	low ρ, high σ_z	low ρ, low σ_z
19.42	16.49	41.11	10.74

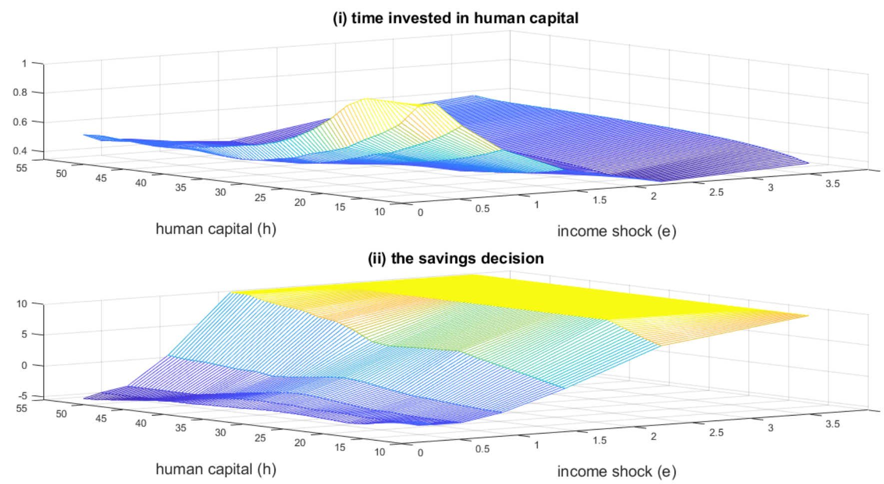
Table 13: Consumption

E Model with human capital and risk-free discount bonds:

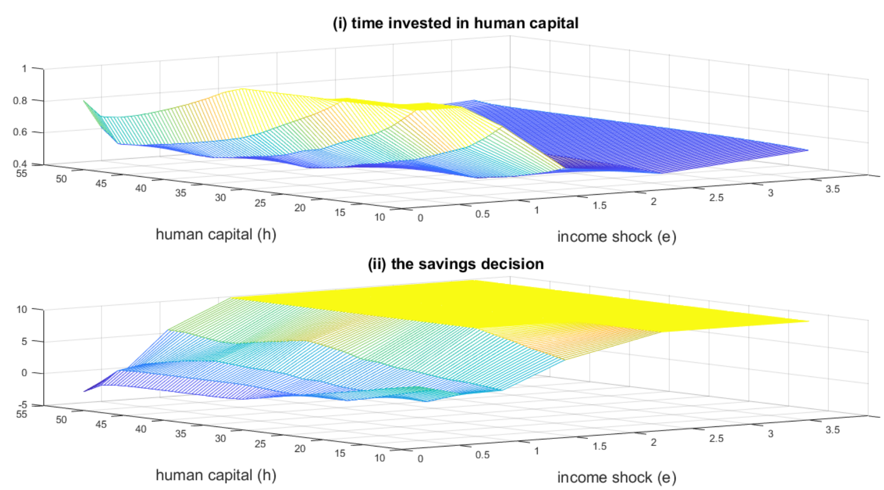
The figures below show the equilibrium optimal decision rules for the households and the equilibrium stationary distribution of households along human capital levels and shock realizations, conditioned on different levels of bond holding, when the risk is highly persistent and highly variable. This is to show a different view of the equilibrium decision rules obtained. These figures are available for each of the other two risk processes as well which have not been included here but they can be made available if desired.



(a) low b

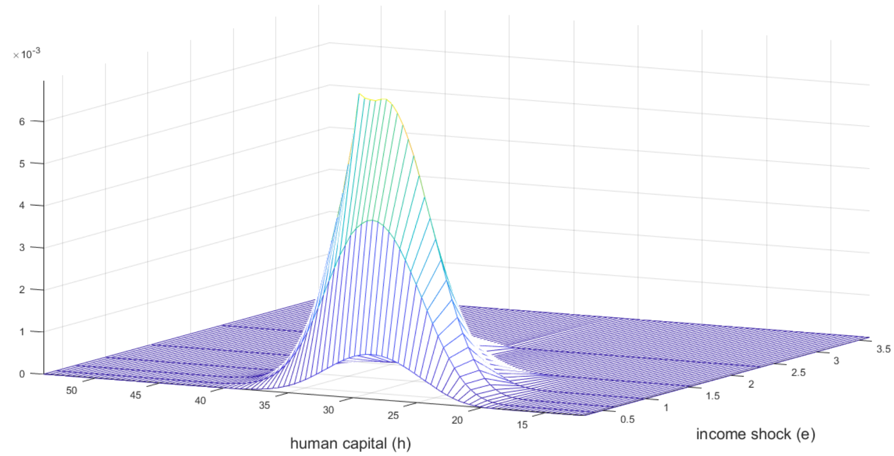


(b) median b

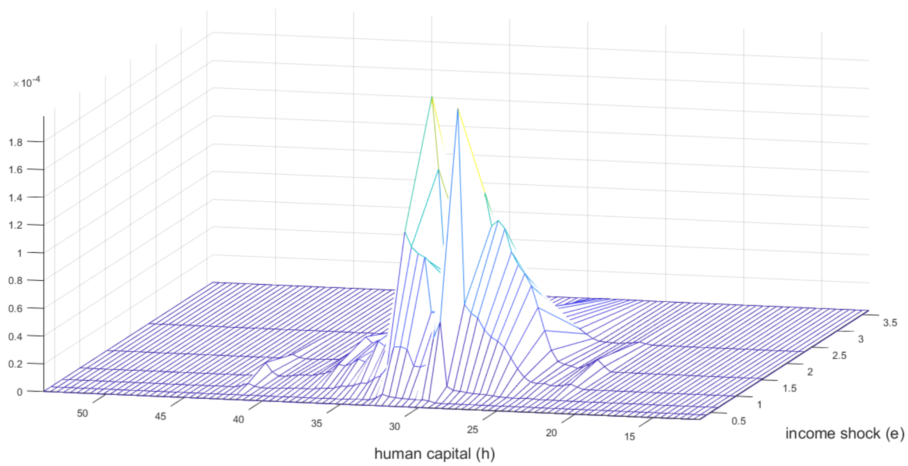


(c) high b

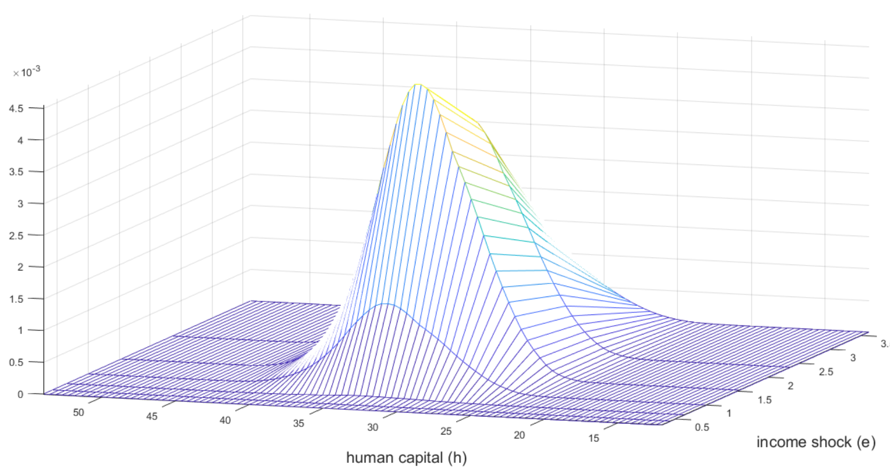
Figure 12: Households' equilibrium decision rules for investment in human capital and bonds when idiosyncratic income shock has high persistence and high variance



(a) low b



(b) median b



(c) high b

Figure 13: Stationary distribution of households when idiosyncratic income shock has high persistence and high variance