Teacher Redistribution in public schools *

Debasis Mishra[†], Soumendu Sarkar[‡], Arunava Sen[§], Jay Sethuraman[¶], Sonal Yadav[‖]

September 15, 2024

Abstract

The Right to Free and Compulsory Education Act (2009) (RTE) of the Government of India prescribes teacher-student ratios for state-run schools. One method advocated by the Act to achieve its goals is the redeployment of teachers from surplus to deficit schools. We consider a model where teachers can either remain in their initially assigned schools or be transferred to a deficit school in their acceptable set. Transfers cannot turn a surplus school into a deficit school and a deficit school cannot be made a surplus school. The planner's objective is specified in terms of the post-transfer deficit vector that can be achieved. We show that there exists a transfer policy that generates a post-transfer deficit vector that Lorenz dominates all achievable post-transfer deficit vectors. We also show that the Lorenz-dominant post-transfer deficit vector can be achieved as the outcome of a strategy-proof mechanism.

JEL classification: C78; D47; D71

Keywords: Matching, teacher tranfers, Lorenz dominance, network flows

*Debasis Mishra acknowledges financial support from the Science and Engineering Research Board (SERB Grant No. SERB/CRG/2021/003099) of India.

[†]Indian Statistical Institute, Delhi.

[‡]Delhi School of Economics, Delhi.

[§]Indian Statistical Institute,Delhi.

[¶]Columbia University, New York.

[‖]University of Liverpool, Liverpool.

1 Introduction

The Right to Education Act in India, 2009 was enacted to improve access to schooling for all children in India. To address the regional disparities in pupil-teacher rations (PTR's) in public schools in a state, the act called for direct transfers of teachers where possible to ensure that all schools meet the minimum teacher requirement. Here the government's objective is to achieve regional balance in the quality of education in public schools in a state. One constraint on redistribution policies is that transfers must be voluntary, so that the transfer policy must be individually rational. The salaries of the public school teachers are fixed and independent of their posting. In this paper, we investigate the possibility of designing a transfer policy that satisfies individual rationality constraints while achieving distributional/fairness objectives and is incentive-compatible.

There is a set of teachers and a set of schools. Each teacher belongs to a single school. We assume for simplicity that each school has a fixed number of students. Given a mandate on minimum teacher-student ratio, an initial distribution may leave some schools with a surplus number of teachers while some may run deficits. Using this minimum teacher-student ratio, we can partition the set of schools into surplus and deficit schools. We assume that there are teachers in the surplus schools who are willing to transfer to deficit schools. However these teachers have a preference over the deficit schools they would like to be transferred to: in particular, we assume that each teacher partitions the set of deficit schools into acceptable and non-acceptable sets. A teacher with a non-empty acceptable set of deficit schools has a "trichotomous " preference where the top indifference class consists of the deficit schools she finds acceptable, the second indifference class is the surplus school that she is currently posted at and the third class is the set of non-acceptable surplus schools.

The objective is to transfer teachers from surplus schools to deficit schools such that no surplus school becomes a deficit school after the transfer and no deficit school becomes a surplus school after the transfer. In addition to this feasibility constraint, the teacher policy scheme must satisfy several criteria. First, the teacher transfer policy must satisfy individual rationality, i.e no teacher can be transferred to a non-acceptable school.

One can conceive of the planner having one of several objectives. A *utilitarian* planner may wish to minimise the aggregate sum of deficits or transfer as many teachers as possible. An *egalitarian* planner may wish to "minimise the worst deficits". Since the set of all possible post-transfer deficit vectors is finite, it follows that the utilitarian and egalitarian solutions exist. In general, there is no reason to believe that these solutions will coincide. The existence of a Lorenz dominant policy would simultaneously reconcile the objectives of utilitarian, egalitarian and welfare maximising planners. The main goal of the paper is to show that a Lorenz dominant policy always exists. Finally we show that a Lorenz dominant transfer policy can be attained by a strategy-proof mechanism.

2 The Model

We let $D = \{d_1, \ldots, d_L\}$, $S = \{s_1, \ldots, s_M\}$ and $T = \{t_1, \ldots, t_N\}$ denote the sets of deficit schools, surplus schools and teachers respectively. We denote a generic deficit school by d_k, a surplus school by s_j and a teacher by t_i. Each teacher t_i belongs to a unique surplus school $O(t_i) \in S$. This is the initial assignment of t_i. For each $s_j \in S$, let $W(s_j) = \{t_i \in T : O(t_i) = s\}$ be the set of teachers currently assigned to school s_j.

Each teacher t_i has a set of *acceptable* schools $A(t_i)$. This set consists of all deficit schools that she would like to be transferred to, i.e. $A(t_i) \subseteq D$. In order to avoid trivialities, we assume $A(t_i)$ contains at least one deficit school. For each deficit school d_k, $\beta(d_k)$ is a strictly positive integer which is the *deficit* of d_k. We will refer to the vector $\beta = (\beta(d_1), \ldots, \beta(d_L))$ as the *deficit vector*. For each surplus school s_j, $\alpha(s_j)$ is a strictly positive integer which is the *surplus* of s_j. We will refer to the vector $\alpha = (\alpha(s_1), \ldots, \alpha(s_M))$ as the *surplus vector*. A Teacher Transfer problem (TT) is a tuple $\Gamma = \langle D, S, T, \{A(t_i)_{t_i \in T}\}, \beta, \alpha\rangle$.

A *transfer policy* in a TT is a map $\sigma : T \to D \cup S$ satisfying the following properties:

1. $\sigma(t_i) \in A(t_i) \cup \{O(t_i)\}$ for all $t_i \in T$,
2. $|\{t_i \in W(s_j) : \sigma(t_i) \neq s_j\}| \leq \alpha(s_j)$ for all $s_j \in S$ and
3. $|\{t_i \in T : \sigma(t_i) = d_k\}| \leq \beta(d_k)$ for all $d_k \in D$.

Various restrictions are imposed on a transfer policy. According to the first, a teacher can only be transferred to an acceptable deficit school or remain in her original assignment. The second imposes the requirement that no surplus school can become a deficit school post-transfers. The third ensures that no deficit school can become a surplus school post-transfers. We regard these restrictions to be reasonable. In Section 6, we show that that the model can be reformulated and several of the restrictions relaxed without any qualititative change in the main result.

We let Σ denote the set of all transfer policies. Every transfer policy $\sigma \in \Sigma$ generates a post-transfer deficit vector β^σ where $\beta^\sigma(d_k) = \beta(d_k) - |\{t_i \in T : \sigma(t_i) = d_k\}|$ for all $d_k \in D$. The goal of the planner is to choose a transfer policy from Σ by evaluating the post-transfer deficit vector by the policy. We discuss various objectives of the planner below.

2.1 Objectives of the planner

One can conceive of the planner having one of several objectives. A *utilitarian* planner may wish to minimise the aggregate sum of deficits or transfer as many teachers as possible. An *egalitarian* planner may wish to "minimise the worst deficits". We define this more formally below.

Let γ be an arbitrary vector in $\Re^L$. Let $[\gamma]$ denote the vector where the components of γ are ordered from highest to lowest, i.e. $[\gamma] = (\gamma_{[1]}, \ldots, \gamma_{[L]})$ and $\gamma_{[1]} \geq \gamma_{[2]} \ldots \geq \gamma_{[L]}$. Consider $\gamma, \gamma' \in \Re^L$. We say γ lexicographically dominates γ' if there exists an integer $r \in \{1, \ldots, L\}$ such that $\gamma_{[1]} = \gamma'_{[1]}, \ldots, \gamma_{[r]} = \gamma'_{[r]}$ and $\gamma_{[r+1]} < \gamma'_{[r+1]}$ or $\gamma_{[r]} = \gamma'_{[r]}$ for all $r \in \{1, \ldots, L\}$. An egalitarian planner would like to choose a transfer policy σ such that β^{σ} lexicographically dominates $\beta^{\sigma'}$ for all other $\sigma' \in \Sigma$.

The set of all possible post-transfer deficit vectors is finite. Therefore it follows that the utilitarian and egalitarian solutions exist. In general, there is no reason to believe that these solutions will coincide.[1] A welfare maximizing planner may wish to......

Consider $\gamma, \gamma' \in \Re^L$. We say γ Lorenz dominates γ' (denoted by $\gamma \succ_{LO} \gamma'$) if

$$\sum_{i=1}^{k} \gamma_{[i]} \leq \sum_{i=1}^{k} \gamma'_{[i]} \text{ for all } k \in \{1, \ldots, L\}.$$

Definition 1 *A transfer policy σ^* is Lorenz dominant if $\beta^{\sigma^*} \succ_{LO} \beta^{\sigma'}$ for all $\sigma' \in \Sigma$.*

The order $\succ_{LO}$ is a partial order, which implies that a Lorenz dominant transfer policy need not exist. For instance, it does not exist in the example in Footnote 1. However if it does, it would simultaneously reconcile the objectives of utilitarian, egalitarian and welfare maximising planners. The main goal of the paper is to show that a Lorenz dominant policy always exists.

3 A network flow formulation and preliminary analysis

A TT problem can be formulated as a single source multiple sink network flow problem.[2] We consider two versions of the problem, one where flows are integers and the other where

[1] Consider the following example. There are two deficit schools and the set of post-transfer deficit vectors is the set $\{(\beta_1, \beta_2) \in \mathbb{N}^2 : 2\beta_1 + \beta_2 \leq 30\}$. The utilitarian and egalitarian solutions would pick $(0, 30)$ and $(10, 10)$ respectively.

[2] For a comprehensive survey of network flows see Ahuja et al. (1988).

flows are allowed to be fractional. Both versions use the same network structure which we describe below.

The set of nodes N in the graph consist of a source N_0, the set of surplus schools S, the set of teachers T and the set of deficit schools D. Formally $N = \{N_0\} \cup S \cup T \cup D$. The set D is the set of sinks. An oriented capacity graph G with edges E is constructed as follows. There is an edge between the source node N_0 and each surplus school $s_j \in S$. There is an edge between each surplus school s_j and each teacher $t_i \in W(s_j)$. There is an edge between each teacher t_i and deficit school d_k if $d_k \in A(t_i)$. All edges in G are oriented from source to sink. The capacity of an edge (N_0, s_j) is $\alpha(s_j)$. The capacity of an edge (s_j, t_i) where $t_i \in W(s_j)$, is 1. Similarly the capacity of an edge (t_i, d_k) where $d_k \in A(t_i)$, is 1.[3] Each deficit school d_k has a node capacity $\beta(d_k)$. We let G denote this network.

An integer flow is a function f from E to the non-negative set of integers such that

1. $f(N_0, s_j) \leq \alpha(s_j)$ for all $s_j \in S$.
2. $\sum_{t_i \in T} f(t_i, d_k) \leq \beta(d_k)$ for all $d_k \in D$.
3. $f(s_j, t_i) \leq 1$ for all edges (s_j, t_i) such that $s_j = O(t_i)$.
4. $f(N_0, s_j) = \sum_{t_i \in W(s_j)} f(s_j, t_i)$ for each surplus school $s_j \in S$.
5. $f(O(t_i), t_i) = \sum_{d_k \in A(t_i)} f(t_i, d_k)$ for each teacher $t_i \in T$.

Let $\mathcal{F}$ denote the set of integer flows in G. We refer to $\Gamma^I = \langle G, \mathcal{F} \rangle$ as the integer network flow problem. We also define a relaxed network flow problem $\Gamma^R = \langle G, \hat{\mathcal{F}} \rangle$ where $\hat{\mathcal{F}}$ is the set of functions (flows) from E to the non-negative set of reals satisfying Conditions 1 to 5 above. Observe that Γ^I and Γ^R use the same network G and $\mathcal{F} \subset \hat{\mathcal{F}}$.

Every transfer policy $\sigma \in \Sigma$ can be uniquely identified with a flow $f \in \mathcal{F}$. For every $f \in \mathcal{F}$, we define a corresponding transfer policy σ as follows.

$$\sigma(t_i) = \begin{cases} d_k, & \text{if } \ f(t_i, d_k) = 1 \\ O(t_i), & \text{if } \ f(t_i, d_k) = 0 \text{ for all } d_k \in A(t_i) \end{cases}$$

[3]Note that it is essential for our analysis, that the capacity of the edges (s_j, t_i) must be 1. However, it is not essential to assume capacity 1 for an edge between a teacher and a deficit school that she finds acceptable.

It is easy to verify that σ satisfies Properties 2 and 3 in the definition of a transfer policy. Similarly given a transfer policy σ, we define a corresponding flow $f \in \mathcal{F}$ as follows. For the edge (t_i, d_k), let

$$f(t_i, d_k) = \begin{cases} 1, & \text{if } \ \sigma(t_i) = d_k \\ 0, & \text{if } \ \sigma(t_i) = O(t_i) \end{cases}$$

For the edge $(O(t_i), t_i)$, let

$$f(O(t_i), t_i) = \begin{cases} 1, & \text{if } \ \sigma(t_i) = d_k \\ 0, & \text{if } \ \sigma(t_i) = O(t_i) \end{cases}$$

For the edge (N_0, s_j), let

$$f(N_0, s_j) = |\{t_i \in W(s_j) : \sigma(t_i) \neq s_j\}|$$

Once again it is easy to verify that f is integer-valued and satisfies conditions 1 to 5 in the definition of a flow.

Example 1 is an instance of a TT problem. The associated network flow formulation of the problem is illustrated in Figure 1.

Example **1** The sets of deficit schools, surplus schools and teachers are $S = \{s_1, s_2, s_3\}$, $D = \{d_1, d_2, d_3, d_4, d_5, d_5, d_7\}$ and $T = \{t_1, t_2, t_3, t_4, t_5\}$. The initial assignment of teachers is given by: $O(t_1) = s_1$, $O(t_2) = O(t_3) = s_2$ and $O(t_4) = O(t_5) = s_3$. The acceptable sets are as follows: $A(t_1) = \{d_1, d_2\}$, $A(t_2) = A(t_3) = \{d_1.d_2, d_3, d_4, d_5\}$, $A(t_6) = \{d_6\}$ and $A(t_5) = \{d_6, d_7\}$. The deficit of every school $d_k \in D$ is $\beta(d_k) = 5$. The surplus of schools s_1, s_2 and s_3 are $\alpha(s_1) = 1$ and $\alpha(s_2) = \alpha(s_3) = 2$.

For the remainder of this section, we restrict attention to the problem Γ^R. Let $X, Y \subset N$ and $f \in \hat{\mathcal{F}}$. The outflow from set X to Y under f is,

$$f(X, Y) = \sum_{(x,y) \in (X,Y) \cap E} f(x, y).$$

For any $B \subseteq D$, the inflow into B is $f(B) = f(N, B)$.

Following Megiddo (1974), the *characteristic function* of Γ^R is a real valued function $v : 2^D \to R$ such that for every $B \subseteq D$,

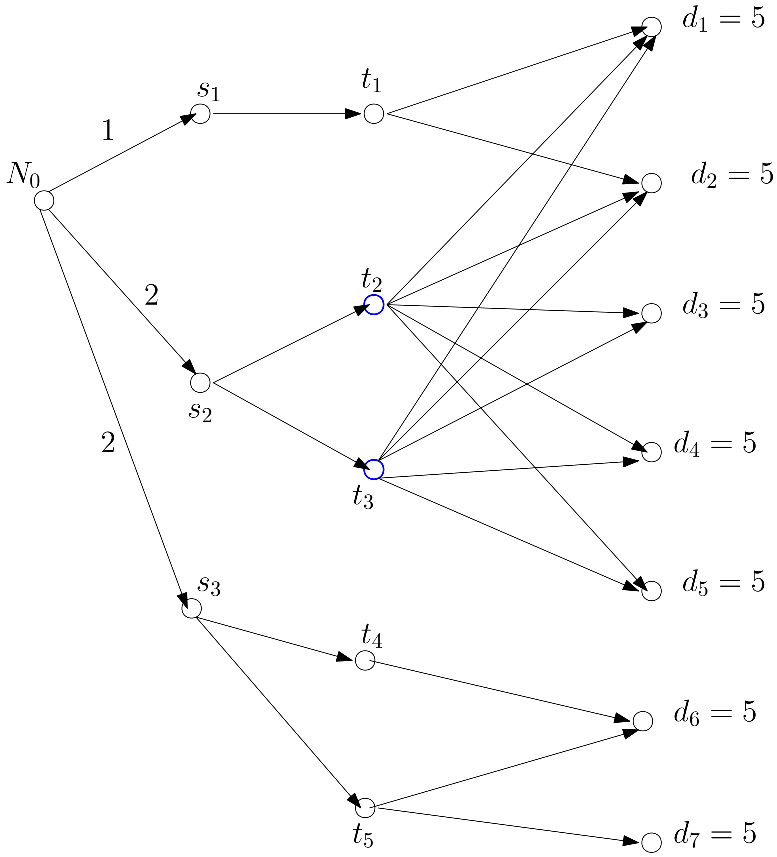

Figure 1: Example 1

$$v(B) = \max\{f(N,B) : f \in \hat{\mathcal{F}}\}.$$

For all $B \subset D$, let $\beta(B) = \sum_{d_k \in B} \beta(d_k)$. Define $w : 2^D \to R$ as follows: for all $B \subseteq D$,

$$w(B) = \beta(B) - v(B).$$

The pairs $\langle D, v\rangle$ and $\langle D, w\rangle$ are cooperative games where D is the set of players and v and w are the respective characteristic functions. Since D will be held fixed throughtout the analysis, we will refer to these games by v and w respectively. Proposition 1 shows that any post-transfer deficit vector that can be achieved by a flow $f \in \hat{\mathcal{F}}$ can be expressed in terms of a condition on w.

Let $h = (h_1, \ldots, h_L)$ be an L-dimensional vector of real numbers. For any $B \subseteq D$, let $h(B) = \sum_{d_k \in B} h_k$. The vector h is *achievable* if there exists $f \in \hat{\mathcal{F}}$ such that $h_k = \beta(d_k) - f(d_k)$ for all $d_k \in D$. In other words, h is the post-transfer deficit vector generated by a flow in the relaxed problem Γ^R.

We state and prove two propositions which we will use in the proof of our main result. Both rely heavily on results in Megiddo (1974).

Proposition 1 *The vector h is achievable if and only if $h(B) \geq w(B)$ for all $B \subseteq D$.*

Proof: Suppose h is achievable. There exists $f \in \hat{\mathcal{F}}$ such that $h_k = \beta(d_k) - f(d_k)$. Pick an arbitrary $B \subseteq D$. By definition $\beta(B) - h(B) = f(B)$. The construction of v implies $f(B) \leq v(B)$. Therefore $h(B) \geq \beta(B) - v(B) = w(B)$.

Let h be an L-dimensional vector satisfying $h(B) \geq w(B)$ for all $B \subseteq D$. For any $d_k \in D$, $f(d_k) = \beta(d_k) - h_k$. Since $h(B) \geq w(B)$, we have $f(B) \leq v(B)$. Applying Lemma 4.1 in Megiddo (1974), we conclude $f \in \hat{\mathcal{F}}$. It follows immediately that h is achievable. ■

Proposition 2 *The game w satifies the following properties.*

1. *$w(B)$ is an integer for all $B \subseteq D$.*

2. *w is convex* [4].

Proof: Pick an arbitrary $B \subseteq D$. By the max-flow min-cut theorem, $v(B)$ is the capacity of the minimum cut separating the source N_0 from B. Since all capacities in Γ^R are integers, it follows that $v(B)$ is an integer. Since $\beta(B)$ is an integer, so is $w(B)$. This completes Part 1.

Lemma 3.2 (Part (ii)) of Megiddo (1974)) proves that v is concave. In order to prove w is convex, pick $B_1 \subseteq B_2 \subset D \setminus \{d_k\}$. Then,

$$\begin{aligned}
w(B_2 \cup \{d_k\}) - w(B_2) &= \beta(B_2 \cup \{d_k\}) - v(B_2 \cup \{d_k\}) - [(\beta(B_2) - v(B_2)] \\
&= \beta(d_k) - [(v(B_2 \cup \{d_k\})) - v(B_2)] \\
&\geq \beta(d_k) - [v(B_1 \cup \{d_k\}) - v(B_1)] \\
&= [\beta(B_1 \cup \{d_k\}) - \beta(B_1)] - [v(B_1 \cup \{d_k\}) - v(B_1)] \\
&= \beta(B_1 \cup \{d_k\}) - v(B_1 \cup \{d_k\}) - [\beta(B_1) - v(B_1)] \\
&= w(B_1 \cup \{d_k\}) - w(B_1).
\end{aligned}$$

The inequality follows from the concavity of v. This completes Part 2. ■

[4] Recall that w is convex if $w(B_2 \cup \{d_k\}) - w(B_2) \geq w(B_1 \cup \{d_k\}) - w(B_1)$ for all $B_1 \subset B_2 \subseteq D \setminus \{d_k\}$. It is concave if $w(B_2 \cup \{d_k\}) - w(B_2) \leq w(B_1 \cup \{d_k\}) - w(B_1)$ for all $B_1 \subset B_2 \subseteq D \setminus \{d_k\}$.

We can summarize the results in this section as follows. Every TT problem can be reformulated as a network flow problem Γ^I with integer flows. A convex cooperative game w is induced by the relaxed network flow problem Γ^R where the set of deficit schools can be interpreted as players. For every coalition $B \subseteq D$, $w(B)$ is an integer. In addition, a post-transfer deficit vector is achievable in the relaxed problem (where fractional teacher flows are allowed) if and only if it belongs to the core of w. We return to Example 1 and illustrate the construction of the game w.

EXAMPLE 1 (continued): We divide the deficit schools into three groups:

$$D_1 = \{d_1, d_2\}; \quad D_2 = \{d_3, d_4, d_5\}; \quad D_3 = \{d_6, d_7\}$$

In Figure 1, every deficit school in D_1 has exactly three incoming edges and every school in D_2 has exactly two incoming edges. School d_6 has two incoming edges and school d_7 has one incoming edge. Using these, we conclude the value of singleton coalitions as:

$$w(d_i) = \begin{cases} 2 & \text{if } d_i \in D_1 \\ 3 & \text{if } d_i \in D_2 \cup \{d_6\} \\ 4 & \text{if } d_i = d_7 \end{cases}$$

For coalitions with two deficit schools, note that any such coalition (i) in D_2 or D_3 can receive a maximum of two teachers; (ii) in D_1 can receive all three teachers. Using these, we conclude the value of doubleton coalitions as:

$$w(d_i, d_j) = \begin{cases} 7 & \text{if } \{d_i, d_j\} = D_1 \\ 8 & \text{if } \{d_i, d_j\} \subseteq D_2 \text{ or } \{d_i, d_j\} \subseteq D_3 \\ 7 & \text{if } d_i \in D_1, d_j \in D_2 \\ 5 & \text{if } d_i \in D_1, d_j = d_6 \\ 6 & \text{if } d_i \in D_1, d_j = d_7 \\ 6 & \text{if } d_i \in D_2, d_j = d_6 \\ 7 & \text{if } d_i \in D_2, d_j = d_7 \end{cases}$$

For coalitions of size three, the calculations can be done by looking at maximum flows in the graph in Figure 1.

$$w(d_i, d_j, d_k) = \begin{cases} 12 & \text{if } \{d_i, d_j\} = D_1 \text{ and } d_k \in D_2 \\ 12 & \text{if } \{d_i, d_j\} \subseteq D_2 \text{ and } d_k \in D_1 \\ 10 & \text{if } \{d_i, d_j\} = D_1 \text{ and } d_k = d_6 \\ 11 & \text{if } \{d_i, d_j\} = D_1 \text{ and } d_k = d_7 \\ 11 & \text{if } \{d_i, d_j\} \subseteq D_2 \text{ and } d_k = d_6 \\ 12 & \text{if } \{d_i, d_j\} \subseteq D_2 \text{ and } d_k = d_7 \\ 13 & \text{if } \{d_i, d_j, d_k\} = D_2 \\ 10 & \text{if } \{d_i, d_j\} = D_3 \text{ and } d_k \in D_1 \\ 11 & \text{if } \{d_i, d_j\} = D_3 \text{ and } d_k \in D_2 \\ 10 & \text{if } d_i \in D_1, d_j \in D_2, \text{ and } d_k = d_6 \\ 11 & \text{if } d_i \in D_1, d_j \in D_2, \text{ and } d_k = d_7 \end{cases}$$

For coalitions of size four, the calculations can be done by looking at maximum flows in the graph in Figure 1. One observation is if all four deficit schools belong to D_1 or D_2, the maximum flow they can receive is three. If three or two of the deficit schools belong to D_1 or D_2, the calculations are more subtle.

$$w(d_i, d_j, d_k, d_\ell) = \begin{cases} 17 & \text{if } \{d_i, d_j, d_k, d_\ell\} \subseteq D_1 \cup D_2 \\ 17 & \text{if } \{d_i, d_j, d_k\} = D_2, d_\ell \in D_1 \text{ or } d_\ell = d_7 \\ 16 & \text{if } \{d_i, d_j, d_k\} = D_2, d_\ell = d_6 \\ 16 & \text{if } \{d_i, d_j, d_k\} \neq D_2, \{d_i, d_j, d_k\} \subseteq D_1 \cup D_2, d_\ell = d_7 \\ 15 & \text{if } \{d_i, d_j, d_k\} \neq D_2, \{d_i, d_j, d_k\} \subseteq D_1 \cup D_2, d_\ell = d_6 \\ 16 & \text{if } \{d_i, d_j\} = D_3, \{d_k, d_\ell\} \subseteq D_2 \\ 15 & \text{if } \{d_i, d_j\} = D_3, \{d_k, d_\ell\} \cap D_1 \neq \emptyset \end{cases}$$

For larger coalitions, the value is computed as follows. First, coalition of size five is computed as follows.

$$w(D_1 \cup D_2) = 22$$

For any $S \subseteq D_1 \cup D_2$ with $|S| = 4$, we have

$$w(S \cup \{d_i\}) = \begin{cases} 20 & \text{if } d_i = d_6 \\ 21 & \text{if } d_i = d_7 \end{cases}$$

For any $S \subseteq D_1 \cup D_2$ with $|S| = 3$, we have

$$w(S \cup D_3) = \begin{cases} 20 & \text{if } S \neq D_2 \\ 21 & \text{if } S = D_2 \end{cases}$$

For coalitions of size six, we can compute as follows.

$$w(D_1 \cup D_2 \cup \{d_i\}) = \begin{cases} 25 & \text{if } d_i = d_6 \\ 26 & \text{if } d_i = d_7 \end{cases}$$

For coalition $S \subseteq D_1 \cup D_2$ with $|S| = 4$,

$$w(S \cup D_3) = 25$$

Finally,

$$w(D) = 30$$

NOTE: Observe that $w(S)$ is an integer for all $S \subseteq D$. It can also be verified that w is convex. For instance, ...

4 The Main Result

We can now state our main result.

Theorem 1 *There exists a Lorenz dominant transfer policy.*

In Section 3, we have shown that the set of post-transfer deficit vectors in Γ^R coincides with the core of a convex game. Dutta and Ray (1989) show that the core of a convex game always contains a Lorenz dominant allocation and provide an algorithm to identify it.[5] We can therefore find an Lorenz dominant post-transfer deficit vector in the relaxed problem Γ^R using the algorithm. We construct an appropriate integer approximation from the Lorenz dominant post-transfer deficit vector in the relaxed problem. This "approximated" vector will typically not be Lorenz dominant among all achievable vectors in the relaxed problem. However, we show that it is Lorenz dominant among all achievable vectors in the integer network flow problem Γ^I. The flow corresponding to this approximated vector is the desired transfer policy. We now proceed to details.

[5] Megiddo (1974) also proposed the same algorithm to construct a lex optimal flow.

Proof: We first construct a Lorenz dominant post-transfer deficit vector h^* in Γ^R using the algorithm in Dutta and Ray (1989) and Megiddo (1974). The algorithm inductively constructs a sequence of sets $\{B_0, \ldots, B_P\}$ and $\{Z_1, \ldots, Z_P\}$ which are subsets of deficit schools.

Initially $B_0 = \emptyset$. For $i = 1, 2, \ldots$, let

$$Z_i = \arg\max_{Y \subseteq D \setminus B_{i-1}} \left\{ \frac{w(Y \cup B_{i-1}) - w(B_{i-1})}{|Y|} \right\}.$$

Break ties arbitrarily in case of multiple solutions. At the end of Step i, set $B_i = B_{i-1} \cup Z_i$. Observe that Z_i is non-empty at every Step i. This implies that the algorithm terminates at some Step P where $B_P = D$.

In any Step i, the sets $Z_1, \ldots, Z_{i-1}$ have been determined. The algorithm picks the subset Z_i from the remaining deficit schools $D \setminus B_{i-1}$, that has the highest average marginal contribution to the set $B_{i-1} = Z_1 \cup \ldots \cup Z_{i-1}$. The collection $\{Z_1, \ldots, Z_P\}$ forms a partition of D. The average marginal contribution computed in Step i ($i \in \{1, \ldots, P\}$) is denoted by g_i, i.e.

$$g_i = \frac{w(Z_i \cup B_{i-1}) - w(B_{i-1})}{|Z_i|}$$

Since $\{Z_1, \ldots, Z_P\}$ is a partition of D, the collection $\{g_i\}_{i \in \{1, \ldots, P\}}$ determines a post-transfer vector h^* in Γ^R as follows: $h^*_k = g_i$ if $d_k \in Z_i$ for all $k = \{1, \ldots L\}$.

Lemma 1 *The vector h^* satisfies the following properties:*

1. *Suppose $d_k \in Z_i$ and $d_{k'} \in Z_{i'}$ where $i < i'$. Then $h^*_k \geq h^*_{k'}$.*
2. *$h^*(\cup_{p=1}^{i} Z_p) = w(\cup_{p=1}^{i} Z_p)$ for all $i \in \{1, \ldots, P\}$.*
3. *h^* is achievable.*

The construction of the algorithm immediately implies Parts 1 and 2 of Lemma 1. The proof of Part 3 of Lemma 1 uses Proposition 1. Details can be found in Megiddo (1974) and Dutta and Ray (1989). In fact, we can use the arguments in Dutta and Ray (1989) to show that h^* is the Lorenz dominant vector in Γ^R. For our purpose, we only use the fact that h^* is achievable.

The post-transfer deficit of school d_k in h^* where $d_k \in Z_i$ is,

$$h_k^* = g_i = \frac{w(\cup_{p=1}^{i} Z_p) - w(\cup_{p=1}^{i-1} Z_p)}{|Z_i|}. \quad (1)$$

According to Part 1 in Proposition 2, $w(B)$ is an integer for all $B \subseteq D$. As a result, the numerator in Equation 1 is an integer, but h^* may not be one. However $g_i|Z_i|$ which is the total post-transfer deficit of all schools in Z_i, is an integer. We use this fact to construct a Lorenz dominant flow in Γ^I.

Lemma **2** *There exists $\hat{f} \in \mathcal{F}$ and a corresponding achievable vector $\hat{h}$ in Γ^I such that*

1. $\hat{h}(\cup_{p=1}^{i} Z_p) = w(\cup_{p=1}^{i} Z_p)$ *for all* $i \in \{1, \ldots, P\}$.
2. $\hat{h}_k \in \{\lfloor g_i \rfloor, \lceil g_i \rceil\}$ *for all* $d_k \in D$ *where* $d_k \in Z_i$.

Proof: We construct an auxillary network Γ^{AU} using $\{Z_1, \ldots, Z_P\}$ and $\{g_1, \ldots, g_P\}$. The set of nodes N' in the graph G' consists of a source N_0, the set of surplus schools S, the set of teachers T, the set of deficit schools D and the set $\{Z_1, \ldots, Z_P\}$. The set $\{Z_1, \ldots, Z_P\}$ is the set of sinks. An oriented capacity graph G' with edges E' is constructed as follows. There is an edge between the source node N_0 and each surplus school $s_j \in S$. There is an edge between each surplus school s_j and each teacher $t_i \in W(s_j)$. There is an edge between each teacher t_i and deficit school d_k if $d_k \in A(t_i)$. For each deficit school d_k, there is an edge between d_k and Z_i if and only if $d_k \in Z_i$. All edges in G' are oriented from source to sink. The capacity of an edge (N_0, s_j) is $\alpha(s_j)$. The capacity of an edge (s_j, t_i) where $t_i \in W(s_j)$, is 1. Similarly the capacity of an edge (t_i, d_k) where $d_k \in A(t_i)$, is 1. An edge (d_k, Z_i) where $d_k \in Z_i$ has upper (capacity) and lower bounds of $\lceil \beta(d_k) - g_i \rceil$ and $\lfloor \beta(d_k) - g_i \rfloor$ respectively. Each deficit school d_k has node capacity $\beta(d_k)$[6], while each node Z_i has capacity $\beta(Z_i) - g_i|Z_i|$. Note that all capacities are integers. All flows are admissible in Γ^{AU} (including fractional flows). Let $\mathcal{F}^{AU}$ denote the set of all flows in Γ^{AU}. The construction of Γ^{AU} is illustrated in Figure ....

Since h^* is achievable (Part 3 in Lemma 1), there exists $f^* \in \hat{\mathcal{F}}$ such that $f^*(d_k) = \beta(d_k) - h_k^*$ for all $d_k \in D$. Construct the flow f^{**} by augmenting f^* as follows: for every d_k,

[6]Do we need to put node capacity on deficit schools? Think it is not required. However keeping it, makes it consistent with the original network G - then G' just adds an extra layer to G.

the flow on the edge (d_k, Z_i) where $d_k \in Z_i$ is $\beta(d_k) - h_k^* = \beta(d_k) - g_i$. Clearly $f^{**} \in \mathcal{F}^{AU}$. The flow received by a sink Z_i in f^{**} is,

$$\begin{aligned} f^{**}(Z_i) &= \sum_{d_k \in Z_i} f^{**}(d_k, Z_i) \\ &= \sum_{d_k \in Z_i} [\beta(d_k) - g_i] \\ &= \sum_{d_k \in Z_i} \beta(d_k) - g_i|Z_i| \\ &= \beta(Z_i) - g_i|Z_i| \end{aligned} \tag{2}$$

The RHS of Equation 2 is the capacity of node Z_i. It follows that f^{**} is a maximal flow in Γ^{AU}.

Since all capacities in Γ^{AU} are integers, the integrality theorem[7] implies that there exists an maximum integer flow $\hat{f} \in \mathcal{F}^{AU}$. Therefore $\sum_{i=1}^{P} f^{**}(Z_i) = \sum_{i=1}^{P} \hat{f}(Z_i)$. Indeed, for node capacities of sinks not to be exceeded in $\hat{f}$, we must have

$$f^{**}(Z_i) = \hat{f}(Z_i) \text{ for all } i \in \{1, \ldots, P\}. \tag{3}$$

Define $\hat{h}_k = \beta(d_k) - \hat{f}(d_k)$ for all $d_k \in D$. Note that $\hat{h}$ is the post-transfer deficit vector corresponding to $\hat{f}$. Observe

$$f^{**}(Z_i) = \beta(Z_i) - g_i|Z_i| = \beta(Z_i) - h^*(Z_i). \tag{4}$$

$$\begin{aligned} \hat{f}(Z_i) &= \sum_{d_k \in Z_i} \hat{f}(d_k) \\ &= \sum_{d_k \in Z_i} [\beta(d_k) - \hat{h}_k] \\ &= \sum_{d_k \in Z_i} \beta(d_k) - \sum_{d_k \in Z_i} \hat{h}_k \\ &= \beta(Z_i) - \hat{h}(Z_i) \end{aligned} \tag{5}$$

Using Equations 3, 4 and 5, we conclude that $\hat{h}(Z_i) = h^*(Z_i)$ for all $i \in \{1, \ldots, P\}$. Since the Z_i's are all disjoint, we can use Part 2 of Lemma 1 to conclude that

[7] Add reference

$$\hat{h}(\cup_{p=1}^{i} Z_p) = h^*(\cup_{p=1}^{i} Z_p) = w(\cup_{p=1}^{i} Z_p) \text{ for all } i \in \{1, \ldots, P\}. \tag{6}$$

By the construction of Γ^{AU}, $\hat{f}$ induces an integer flow in Γ^I. It follows that $\hat{h}$ is an achievable post-transfer deficit vector in Γ^I. Equation 6 establishes Part 1 of Lemma 2.

For any school $d_k \in D$, we know $\hat{f}(d_k) = \beta(d_k) - \hat{h}_k$. By flow conservation for node d_k where $d_k \in Z_i$, we have $\hat{f}(d_k) = \hat{f}(d_k, Z_i)$. Thus,

$$\begin{aligned}
& \lfloor \beta(d_k) - g_i \rfloor \leq \hat{f}(d_k) \leq \lceil \beta(d_k) - g_i \rceil \\
\implies\ & \lfloor \beta(d_k) - g_i \rfloor \leq \beta(d_k) - \hat{h}_k \leq \lceil \beta(d_k) - g_i \rceil \\
\implies\ & \beta(d_k) - \lceil g_i \rceil \leq \beta(d_k) - \hat{h}_k \leq \beta(d_k) - \lfloor g_i \rfloor \\
\implies\ & -\lceil g_i \rceil \leq -\hat{h}_k \leq -\lfloor g_i \rfloor \\
\implies\ & \lfloor g_i \rfloor \leq \hat{h}_k \leq \lceil g_i \rceil
\end{aligned}$$

This establishes Part 2 of Lemma 2 below. ■

Recall that the algorithm constructed a partition $\{Z_1, \ldots, Z_P\}$ and post-transfer deficits $\{g_1, \ldots, g_P\}$ (which may not be integers) where $g_1 \geq g_2 \ldots \geq g_P$ (Proposition 1), i.e. $\lceil g_1 \rceil \geq \lfloor g_1 \rfloor \geq \lceil g_2 \rceil \geq \lfloor g_2 \rfloor \ldots \geq \lceil g_P \rceil \geq \lfloor g_P \rfloor$. Construct a coarsening of $\{Z_1, \ldots, Z_P\}$ which we refer to as $\{\bar{Z}_1, \ldots, \bar{Z}_R\}$ by lumping together different elements of the original partition if their ceilings coincide with each other as do their floors. Formally, $\bar{Z}_i = Z_p \cup Z_{p+1} \ldots \cup Z_{p+s}$ if $\lceil g_p \rceil = \lceil g_{p+1} \rceil \ldots = \lceil g_{p+s} \rceil = x + 1$ and $\lfloor g_p \rfloor = \lfloor g_{p+1} \rfloor \ldots = \lfloor g_{p+s} \rfloor = x$ for some integer x.

Suppose the original partition is $\{Z_1, Z_2, Z_3, Z_4, Z_5\}$ with $g_1 = 6.4$, $g_2 = 6.2$, $g_3 = 6.1$, $g_4 = 5.6$ and $g_5 = 4.8$. Then the coarsened partition is $\{\bar{Z}_1, \bar{Z}_2, \bar{Z}_3\}$ where $\bar{Z}_1 = Z_1 \cup Z_2 \cup Z_3$, $\bar{Z}_2 = Z_4$ and $\bar{Z}_3 = Z_5$. The ceilings and floors associated with $\bar{Z}_1$, $\bar{Z}_2$ and $\bar{Z}_3$ are $\{7, 6\}$, $\{6, 5\}$ and $\{5, 4\}$ respectively.

Observation **1** The following facts hold for the post-transfer deficit vector $\hat{h}$ and the partition $\{\bar{Z}_1, \ldots, \bar{Z}_R\}$.

1. For all $d_k \in \bar{Z}_i$, $\hat{h}_k \in \{x, x+1\}$ where x and $x + 1$ are the floor and ceiling associated with $\bar{Z}_i$.
2. $\hat{h}(\cup_{p=1}^{i} \bar{Z}_p) = w(\cup_{p=1}^{i} \bar{Z}_p)$ for all $i \in \{1, \ldots, R\}$.

Part 1 follows immediately from the construction of the new partition. Part 2 follows from Part 1 of Lemma 2 and the fact that the new partition is constructed by combining consecutive elements in the original partition.

We claim that $\hat{h}$ Lorenz dominates all other achievable vectors h in the problem Γ^I. We need to show that the following inequalites hold for the vectors $[\hat{h}]$ and an arbitrary achievable $[h]$ (in Γ^I).

$$\sum_{i=1}^{m} \hat{h}_{[i]} \leq \sum_{i=1}^{m} h_{[i]} \text{ for all } m = 1, 2, \ldots, |D|. \tag{7}$$

Assume w.l.o.g that components in $\hat{h}$ are arranged according to the partition $\{\bar{Z}_1, \ldots, \bar{Z}_R\}$, i.e. the first $|\bar{Z}_1|$ terms in $\hat{h}$ correspond to deficit schools belonging to Z_1, the next $|\bar{Z}_2|$ terms correspond to deficit schools belonging to Z_2 and so on. Also within each $\bar{Z}_i$, the terms in $\hat{h}$ are arranged in decreasing order. By construction, we know for any $\bar{Z}_p$ and $\bar{Z}_q$ where $p < q$, it must be $\lceil g_p \rceil > \lfloor g_p \rfloor \geq \lceil g_q \rceil > \lfloor g_p \rfloor$, i.e. $[\hat{h}]$ and $\hat{h}$ are identical. Thus,

$$\sum_{i=1}^{m} \hat{h}_{[i]} = \sum_{i=1}^{m} \hat{h}_i \text{ for all } m = 1, 2, \ldots, |D|. \tag{8}$$

The vector $\bar{h}$ is constructed vector from $[h]$ using the partition $\{\bar{Z}_1, \ldots, \bar{Z}_R\}$. The first $|\bar{Z}_1|$ terms in $\bar{h}$ are the deficit schools belonging to Z_1, the next $|\bar{Z}_2|$ terms in $\bar{h}$ are the deficit schools belonging to $\bar{Z}_2$ and so on. Within each $\bar{Z}_i$, the terms are arranged in decreasing order. We are abusing notation here. According to our earlier notation, the ith component $\bar{h}_i$ in a post-transfer deficit vector $\bar{h}$ refers to the deficit of school d_i. However in this case, $\bar{h}_i$ refers to the post-transfer deficit of a school in $\bar{Z}_p$ where $\sum_{i=1}^{p-1} |\bar{Z}_i| < i \leq \sum_{i=1}^{p} |\bar{Z}_i|$.

The following inequalities must hold for the vectors $\bar{h}$ and $[h]$.

$$\sum_{i=1}^{m} \bar{h}_i \leq \sum_{i=1}^{m} h_{[i]} \text{ for all } m = 1, 2, \ldots, |D|. \tag{9}$$

In view of Inequalities 8 and 9, Inequality 10 below suffices to establish Lorenz dominance of $\hat{h}$.

$$\sum_{i=1}^{m} \hat{h}_i \leq \sum_{i=1}^{m} \bar{h}_i \text{ for all } m = 1, 2, \ldots, |D|. \tag{10}$$

Pick $m \in \{1, \ldots, |D|\}$. Since $\hat{h}$ and $\bar{h}$ are arranged according to the partition $\{\bar{Z}_1, \ldots, \bar{Z}_R\}$, the mth term in the sums $\sum_{i=1}^{m} \hat{h}_i$ and $\sum_{i=1}^{m} \bar{h}_i$ belong to the same element of the partition, say $\bar{Z}_p$. Recall $\hat{h}_k \in \{q, q+1\}$ for some integer q, for all $d_k \in \bar{Z}_p$ (Part 2 of Lemma 2). Let r be the highest index such that $\sum_{i=1}^{p-1} |\bar{Z}_i| < r \leq \sum_{i=1}^{p} |\bar{Z}_i|$ and $\hat{h}_r = q + 1$. Then all terms between $(r+1)$ and $\sum_{i=1}^{p} |\bar{Z}_i|$ in $\hat{h}$ belong to $\bar{Z}_p$ and have value q. For convenience, denote $\sum_{i=1}^{p} |\bar{Z}_i|$ by a_p.

There are two possibilities: (A) $m = a_p$ and (B) $a_{p-1} < m \leq a_p$. We will deal with these two cases separately.

Case A: In order to show Inequality 10, we have:

$$
\begin{aligned}
\sum_{i=1}^{m} \bar{h}_i &= \bar{h}(\cup_{k=1}^{p} \bar{Z}_k) \\
&\geq w(\cup_{k=1}^{p} \bar{Z}_k) \\
&= \hat{h}(\cup_{k=1}^{p} \bar{Z}_k) \\
&= \sum_{i=1}^{m} \hat{h}_i
\end{aligned}
$$

The first inequality follows from achievability of $\bar{h}$ and Proposition 1. The second equality follows from Part 1 of Lemma 2.

Case B: The sum $\sum_{i=1}^{m} \hat{h}_i$ can be split into two parts: the sum of all deficit schools that belong to $\cup_{k=1}^{p-1} \bar{Z}_k$ and the remaining terms that belong to $\bar{Z}_p$. We can similarly split the sum $\sum_{i=1}^{m} \bar{h}_i$. The achievability of vector $\bar{h}$ for the set $\cup_{k=1}^{p-1} \bar{Z}_k$ (Proposition 1) and Part 1 of Lemma 2 imply,

$$
\bar{h}(\cup_{k=1}^{p-1} \bar{Z}_k) \geq \hat{h}(\cup_{k=1}^{p-1} \bar{Z}_k). \tag{11}
$$

Inequality 11 can be rewritten as,

$$
\sum_{i=1}^{a_{p-1}} \bar{h}_i \geq \sum_{i=1}^{a_{p-1}} \hat{h}_i. \tag{12}
$$

Now consider the last term $\hat{h}_m$ in the sum $\sum_{i=1}^{m} \hat{h}_m$. There are two subcases to consider depending on where "m cuts $\bar{Z}_p$".

- Case $B.1$: $\hat{h}_m = q$. This means $m > r$ or $m - a_{p-1} > r - a_{p-1}$.
- Case $B.2$: $\hat{h}_m = q + 1$. This means $m \leq r$ or $m - a_{p-1} \leq r - a_{p-1}$.

Case $B.1$: $\hat{h}_m = q$ (or $m > r$). This case will be further split up into two subcases depending upon whether $\bar{h}_m \geq q+1$ or $\bar{h}_m \leq q$.

Case $B.1.1$: $\bar{h}_m \geq q+1$.

Consider the sums in $\bar{h}$ and $\hat{h}$ restriced to $\bar{Z}_p$, i.e. sum of terms in $\bar{h}$ from $a_{p-1}+1$ to m and sum of terms in $\hat{h}$ from $a_{p-1}+1$ to m. Since $\bar{h}_m \geq q+1$, we have

$$\sum_{i=a_{p-1}+1}^{m} \bar{h}_i \geq (m-a_{p-1})(q+1) = (m-a_{p-1})q + (m-a_{p-1}).$$

Also,

$$\begin{aligned}\sum_{i=a_{p-1}+1}^{m} \hat{h}_i &= (r-a_{p-1})(q+1) + (m-r)q \\ &= (r-a_{p-1}+m-r)q + (r-a_{p-1}) \\ &= (m-a_{p-1})q + (r-a_{p-1})\end{aligned}$$

Since $m > r$, we have $(m-a_{p-1})q + (m-a_{p-1}) > (m-a_{p-1})q + (r-a_{p-1})$. This implies

$$\sum_{i=a_{p-1}+1}^{m} \bar{h}_i > \sum_{i=a_{p-1}+1}^{m} \hat{h}_i. \tag{13}$$

We can obtain Inequality 10 by summing Inequalities 12 and 13.

Case $B.1.2$: $\bar{h}_m \leq q$.

Consider the sums of $\bar{h}$ and $\hat{h}$ in the remaining part of $\bar{Z}_p$, i.e. sum of terms in $\bar{h}$ from $m+1$ to a_p and sum of terms in $\hat{h}$ from $m+1$ to a_p. Since $m > r$, we know $\sum_{i=m+1}^{a_p} \hat{h}_i = (a_p - m)q$. By asssumption, $\bar{h}_m \leq q$. So $\sum_{i=m+1}^{a_p} \bar{h}_i \leq (a_p - m)q$. These facts together imply,

$$\sum_{i=m+1}^{a_p} \bar{h}_i \leq \sum_{i=m+1}^{a_p} \hat{h}_i. \tag{14}$$

The achievability of vector $\bar{h}$ for the set $\cup_{k=1}^{p} \bar{Z}_k$ (Proposition 1) and Part 1 of Lemma 2 implies,

$$\sum_{i=1}^{a_p} \bar{h}_i \geq \sum_{i=1}^{a_p} \hat{h}_i. \tag{15}$$

Since $\sum_{i=m+1}^{a_p} \bar{h}_i \leq \sum_{i=m+1}^{a_p} \hat{h}_i$ (Inequality 14), we can subtract $\sum_{i=m+1}^{a_p} \bar{h}_i$ from the LHS and $\sum_{i=1}^{a_p} \hat{h}_i$ from the RHS of Inequality 15 and the inequality will be preserved. We obtain Inequality 10 as follows.

$$\begin{aligned}\sum_{i=1}^{m} \bar{h}_i &= \sum_{i=1}^{a_p} \bar{h}_i - \sum_{i=m+1}^{a_p} \bar{h}_i \\ &\geq \sum_{i=1}^{a_p} \hat{h}_i - \sum_{i=m+1}^{a_p} \hat{h}_i \\ &= \sum_{i=1}^{m} \hat{h}_i\end{aligned}$$

Case $B.2$: $\hat{h}_m = q+1$ (or $m \leq r$). This case will be further split up into two subcases depending upon whether $\bar{h}_m \geq q+1$ or $\bar{h}_m \leq q$.

Case $B.2.1$: $\bar{h}_m \geq q+1$.

Consider the sums in $\bar{h}$ and $\hat{h}$ restriced to $\bar{Z}_p$, i.e sum of terms in $\bar{h}$ from $a_{p-1}+1$ to m and sum of terms in $\hat{h}$ from $a_{p-1}+1$ to m. Since $\bar{h}_m \geq q+1$, we have

$$\sum_{i=a_{p-1}+1}^{m} \bar{h}_i \geq (m - a_{p-1})(q+1).$$

Since $m \leq r$, we have

$$\sum_{i=a_{p-1}+1}^{m} \hat{h}_i = (m - a_{p-1})(q+1).$$

Thus,

$$\sum_{i=a_{p-1}+1}^{m} \bar{h}_i \geq \sum_{i=a_{p-1}+1}^{m} \hat{h}_i. \tag{16}$$

We can obtain Inequality 10 by summing Inequalities 12 and 16.

Case $B.2.2$: $\bar{h}_m \leq q$.

Consider the sums of $\bar{h}$ and $\hat{h}$ in the remaining part of $\bar{Z}_p$, i.e. sum of terms in $\bar{h}$ from $m+1$ to a_p and sum of terms in $\hat{h}$ from $m+1$ to a_p. Since $\bar{h}_m \leq q$, we have $\sum_{i=m+1}^{a_p} \bar{h}_i \leq (a_p - m)q$. Since $m \leq r$, we have

$$\begin{aligned}\sum_{i=m+1}^{a_p} \hat{h}_i &= (r-m)(q+1)+(a_p-r)q \\ &= (r-m+a_p-r)q+(r-m) \\ &= (a_p-m)q+(r-m)\end{aligned}$$

Since $r-m \geq 0$, we have

$$\sum_{i=m+1}^{a_p} \bar{h}_i \leq \sum_{i=m+1}^{a_p} \hat{h}_i. \tag{17}$$

The achievability of $\bar{h}$ for the set $\cup_{k=1}^{p} \bar{Z}_k$ (Proposition 1) and Part 1 of Lemma 2 imply,

$$\sum_{i=1}^{a_p} \bar{h}_i \geq \sum_{i=1}^{a_p} \hat{h}_i. \tag{18}$$

Since $\sum_{i=m+1}^{a_p} \bar{h}_i \leq \sum_{i=m+1}^{a_p} \hat{h}_i$, we can subtract $\sum_{i=m+1}^{a_p} \bar{h}_i$ from the LHS and $\sum_{i=1}^{a_p} \hat{h}_i$ from the RHS of Inequality 18 and the inequality will be preserved. Inequality 10 can be obtained as follows.

$$\begin{aligned}\sum_{i=1}^{m} \bar{h}_i &= \sum_{i=1}^{a_p} \bar{h}_i - \sum_{i=m+1}^{a_p} \bar{h}_i \\ &\geq \sum_{i=1}^{a_p} \hat{h}_i - \sum_{i=m+1}^{a_p} \hat{h}_i \\ &= \sum_{i=1}^{m} \hat{h}_i\end{aligned}$$

We have established that Inequality 10 holds for the vector $\hat{h}$. This completes the proof of Lorenz dominance of $\hat{h}$. The integer flow $\hat{f}$ that results in the post-transfer deficit vector $\hat{h}$ is a Lorenz dominant transfer policy. ■

We illustrate the ideas behind the proof of Theorem 1 with reference to Example 1.

EXAMPLE 1 (continued): We begin by computing the maximum value of $\frac{w(S)}{|S|}$ for all coalition sizes.

$$\max_{S:|S|=1} \frac{w(S)}{|S|} = 4$$
$$\max_{S:|S|=2} \frac{w(S)}{|S|} = 4$$
$$\max_{S:|S|=3} \frac{w(S)}{|S|} = \frac{13}{3} = 4\frac{1}{3}$$
$$\max_{S:|S|=4} \frac{w(S)}{|S|} = \frac{17}{4} = 4.25$$
$$\max_{S:|S|=5} \frac{w(S)}{|S|} = \frac{22}{5} = 4.4$$
$$\max_{S:|S|=6} \frac{w(S)}{|S|} = \frac{26}{6} = 4\frac{1}{3}$$
$$\max_{S:|S|=7} \frac{w(S)}{|S|} = \frac{30}{7} = 4\frac{2}{7}$$

The maximum ratio in Step 1 of the algorithm corresponds to $Z_1 = D_1 \cup D_2$. In Step 2 of the algorithm, we look at the marginal ratios below.

$$w(S_1 \cup \{d_6\}) - w(S_1) = 3$$
$$w(S_1 \cup \{d_7\}) - w(S_1) = 4$$
$$\frac{w(S_1 \cup D_3) - w(S_1)}{2} = 4$$

Hence, (one of) the maxima occurs by adding D_3. So, $Z_2 = D_3$. The partition chosen by the algorithm is $\{D_1 \cup D_2, D_3\}$ with associated values $g_1 = 4.4$ and $g_2 = 4$. The post-transfer deficit vector h^* is defined as follows: $h^*_k = 4.4$ for all $d_k \in D_1 \cup D_2$ and $h^*_k = 4$ for all $d_k \in D_3$. It is clear that the vector h^* cannot be obtained from an integer flow (or a teacher transfer).

An important observation here is that h^* cannot be approximated by an arbitrary integer vector $\hat{h}$. For example, suppose $\hat{h}_1 = \hat{h}_2 = 5$, $\hat{h}_3 = \hat{h}_4 = \hat{h}_5 = 4$, $\hat{h}_6 = \hat{h}_7 = 4$. Although $\sum_{k \in D_1 \cup D_2} \hat{h}_k = w(D_1 \cup D_2) = 22$, $\sum_{k \in D_2} \hat{h}_k = 12 < w(D_2) = 13$. According to Proposition 1, $\hat{h}$ is not achievable. This can also be directly verified by noting that only two teachers t_2 and t_3 can be transferred to D_2. Consequently, the aggregate deficit of D_2 cannot be reduced to 12 from 15.

In order to construct an achievable integer approximation of h^*, we construct the auxilliary network Γ^{AU} shown in Figure below. The vector $\hat{h}$ where $\hat{h}_1 = 5, \hat{h}_2 = \hat{h}_3 = \hat{h}_4 =$

$4, \hat{h}_5 = 5$, $\hat{h}_6 = \hat{h}_7 = 4$ is an achievable post-transfer vector in Γ^I. It can also be shown to Lorenz-dominate all achievable post-transfer vectors in Γ^I.

5 Strategy-Proofness

The objective of this section is to show that the Lorenz dominant post-transfer deficit vector can be achieved by means of a strategy-proof mechanism.

We assume that every teacher t_i has "trichotomous" preferences over schools. The first indifference class consists of deficit schools in $A(t_i)$. The second indifference class consists of her initial assignment $O(t_i)$ and the third indifference class consists of non-acceptable deficit schools $D \setminus A(t_i)$. Each teacher t_i is indifferent between acceptable deficit schools in $A(t_i)$ and strictly prefers being transferred to a school in $A(t_i)$ than remaining in her initial assignment. Her initial assignment is strictly preferred to any school in $D \setminus A(t_i)$.

Recall that a teacher transfer problem (TT) is a tuple $\Gamma = \langle D, S, T, \{A(t_i)_{t_i \in T}\}, \beta, \alpha \rangle$. We assume that the vectors β and α are publicly observable as is the initial assignment of teachers $O(t_i)$, $t_i \in T$. However the set $A(t_i)$, $t_i \in T$, is private information for teacher t_i. A teacher transfer policy must therefore be based on reports of the acceptable sets of teachers. Let $\mathcal{A}$ denote the set of all non-empty subsets of D. A profile A is an N-tuple of acceptable sets, i.e $A = \{(A(t_i), t_i \in T\} \in \mathcal{A}^N$. Let $\mathcal{F}(A)$ denote the set of feasible (integer) flows in the TT problem with profile A. For any flow $f \in \mathcal{F}(A)$, let β^f denote the post-transfer deficit vector generated by f. Let $\bar{\mathcal{F}}(A)$ denote the set of flows in $\mathcal{F}(A)$ that generate the Lorenz dominant post-transfer deficit vector. It is clear that $\beta^f = \beta^{f'}$ for all $f, f' \in \bar{\mathcal{F}}(A)$.

A *teacher transfer policy function* (TTPF) is a map σ that associates a flow $\sigma(A) \in \bar{\mathcal{F}}(A)$ for every $A \in \mathcal{A}^N$. Let $\sigma(A, t_i)$ denote the school assigned to teacher t_i in $\sigma(B)$. For any teacher $t_i \in T$, there are two possibilities: (i) $\sigma(A, t_i) \in A(t_i)$ or (ii) $\sigma(A, t_i) = O(t_i)$. Teacher t_i is not transferred if Case (ii) occurs.

Teacher t_i *manipulates* σ at profile $A = (A(t_i), A(t_{-i})$[8] if there exists $A'(t_i) \subseteq D$ such that (i) teacher t_i is not transferred in $\sigma(A)$ and (ii) $\sigma((A'(t_i), A(t_{-i})), t_i) \in A(t_i)$. The TTPF σ is *strategy-proof* if it cannot be manipulated.

A *teacher assignment* assigns each teacher either to a deficit school or to her initial assignment. Let $\bar{\Sigma}$ denote the set of all teacher assignments. Note that a teacher assignment is defined without any reference to acceptable sets of teachers. A *choice function* is a map $C : 2^{\bar{\Sigma}} \to \bar{\Sigma}$ such that $C(X) \in X$ for all $X \subseteq \bar{\Sigma}$. Pick an arbitrary $A \in \mathcal{A}^N$. Then $\bar{\mathcal{F}}(A) \subseteq \bar{\Sigma}$ and $C(\bar{\mathcal{F}}(A)) \in \bar{\mathcal{F}}(A)$. In other words, a choice function picks a flow that generates a Lorenz

[8] $A(t_{-i})$ denotes the reports of all teachers except t_i in the profile A.

dominant post-transfer deficit vector.

The choice function C satisfies contraction consistency (CC) if for all $X, X' \subseteq \bar{\Sigma}$,

$$[X' \subset X \text{ and } C(X) \in X'] \implies [C(X) = C(X')].$$

This is a standard property of choice functions sometimes referred to Sen's Property α. It is a necessary condition for the rationalizability of choice functions. It is also sufficient with additional richness assumptions on the choice function. It is always satisfied if it is obtained by maximising a fixed anti-symmetric ordering on $\bar{\Sigma}$ (See Rubenstein book, chapter....).

PROPOSITION **3** *Let C be a choice function satisfying CC. The TTPF σ defined by $\sigma(A) \equiv C(\bar{\mathcal{F}}(A))$ for all $A \in \mathcal{A}^N$ is strategy-proof.*

Proof: The proof of the proposition relies on two lemmas.

LEMMA **3** *Pick $t_i \in T$ and $A \in \mathcal{A}^N$. Suppose $A'(t_i) = A(t_i) \cup \{d_k\}$. Then either $\sigma((A'(t_i), A(t_{-i})), t_i) = d_k$ or $\sigma((A'(t_i), A(t_{-i})), t_i) = \sigma(A, t_i)$ holds.*

Proof: Assume for contradiction that there exists a teacher t_i, a profile A and a deficit school d_k such that $A'(t_i) = A(t_i) \cup \{d_k\}$ and $\sigma((A'(t_i), A(t_{-i})), t_i) \notin \{d_k, \sigma(A, t_i)\}$. Thus $\sigma(A)$ and $\sigma(A'(t_i), A(t_{-i}))$ are distinct.

For notational convenience, we will suppress $A(t_{-i})$ in the following expressions. Henceforth $\sigma(A)$ and $\sigma(A'(t_i), A(t_{-i}))$ will be written as $\sigma(A(t_i))$ and $\sigma(A'(t_i))$ respectively. Similarly $\sigma(A, t_i)$ and $\sigma((A'(t_i), A(t_{-i})), t_i)$ will be written as $\sigma(A(t_i), t_i)$ and $\sigma(A'(t_i), t_i)$ respectively. Finally, $\mathcal{F}(A)$, $\bar{\mathcal{F}}(A)$, $\mathcal{F}(A'(t_i), A(t_{-i}))$ and $\bar{\mathcal{F}}(A'(t_i), A(t_{-i}))$ will be written simply as $\mathcal{F}(A(t_i))$, $\bar{\mathcal{F}}(A(t_i))$, $\mathcal{F}(A'(t_i))$ and $\bar{\mathcal{F}}(A'(t_i))$ respectively.

Let f and f' denote the flows $\sigma(A(t_i))$ and $\sigma(A'(t_i))$ respectively. Since f' does not use the edge (t_i, d_k), it follows that $f' \in \mathcal{F}(A(t_i))$. Also $\mathcal{F}(A(t_i)) \subset \mathcal{F}(A'(t_i))$ since $A'(t_i) = A(t_i) \cup \{d_k\}$.

We claim that $\beta^f = \beta^{f'}$. Since $\mathcal{F}(A(t_i)) \subset \mathcal{F}(A'(t_i))$, we know $f \in \mathcal{F}(A'(t_i))$. This implies $\beta^{f'} \succ_{LO} \beta^f$. Since $f' \in \mathcal{F}(A(t_i))$, we have $\beta^f \succ_{LO} \beta^{f'}$. This immediately implies $\beta^f = \beta^{f'}$.

Since $f' \in \mathcal{F}(A(t_i))$ and $\beta^f = \beta^{f'}$, we have $f' \in \bar{\mathcal{F}}(A(t_i))$. Also $\beta^f = \beta^{f'}$ and $\mathcal{F}(A(t_i)) \subset \mathcal{F}(A'(t_i))$ together imply $\bar{\mathcal{F}}(A(t_i)) \subseteq \bar{\mathcal{F}}(A'(t_i))$. The flow f' is chosen in $\bar{\mathcal{F}}(A'(t_i))$ and it belongs to $\bar{\mathcal{F}}(A(t_i))$. By CC, f' must be chosen in $\bar{\mathcal{F}}(A(t_i))$. This leads to a contradiction since $f \neq f'$ by assumption. ■

LEMMA 4 *Pick* $t_i \in T$ *and* $A \in \mathcal{A}^N$. *Suppose* $A'(t_i) = A(t_i) \setminus \{d_k\}$ *where* $\sigma(A, t_i) \neq d_k$. *Then* $\sigma((A'(t_i), A(t_{-i})), t_i) = \sigma(A, t_i)$ *holds.*

Proof: We use the same notation as in the proof of Lemma 3. Assume for contradiction that there exists a teacher t_i, a profile A and a deficit school d_k such that $A'(t_i) = A(t_i) \setminus \{d_k\}$ and $f' = \sigma(A'(t_i), t_i) \neq \sigma(A(t_i), t_i) = f$.

Since $A'(t_i) = A(t_i) \setminus \{d_k\}$, we have $\mathcal{F}(A'(t_i)) \subset \mathcal{F}(A(t_i))$. Also $f \in \mathcal{F}(A'(t_i))$ since f does not use the edge (t_i, d_k).

We claim that $\beta^f = \beta^{f'}$. Since $f \in \mathcal{F}(A'(t_i))$, we have $\beta^{f'} \succ_{LO} \beta^f$. Also $f' \in \mathcal{F}(A(t_i))$ because $\mathcal{F}(A'(t_i)) \subset \mathcal{F}(A(t_i))$. Thus $f \succ_{LO} f'$. This immediately implies $\beta^f = \beta^{f'}$.

Since $\mathcal{F}(A'(t_i)) \subset \mathcal{F}(A(t_i))$ and $\beta^f = \beta^{f'}$ we have $\bar{\mathcal{F}}(A'(t_i)) \subseteq \bar{\mathcal{F}}(A(t_i))$. Also $f \in \mathcal{F}(A'(t_i))$ and $\beta^f = \beta^{f'}$ implies that $f \in \bar{\mathcal{F}}(A'(t_i))$.

We have established $f \in \bar{\mathcal{F}}(A'(t_i))$ and $\bar{\mathcal{F}}(A'(t_i)) \subseteq \bar{\mathcal{F}}(A(t_i))$. This is contradicts CC since $f' \neq f$ by assumption. ■

We now complete the proof of the proposition. Let A be an arbitrary preference profile and t_i be a teacher who is not transferred in $\sigma(A(t_i))$, i.e. $\sigma(A(t_i), t_i) = O(t_i)$. Let $A'(t_i) \in \mathcal{A}$ be an arbitrary set of deficit schools. We claim that $\sigma(A'(t_i), t_i) \notin A(t_i)$ which implies that t_i cannot manipulate.

We proceed in two steps. In the first step, we consider the assignment for t_i in the profile $(A''(t_i), A(t_{-i}))$ where $A''(t_i) = A'(t_i) \cap A(t_i)$. The set $A''(t_i)$ can be obtained from $A(t_i)$ by removing some deficit schools, say $d_1, d_2, \ldots, d_K \in A(t_i)$. Since $\sigma(A(t_i), t_i) \neq d_1$, Lemma 4 can be applied to infer that $\sigma(A(t_i) \setminus \{d_1\}, t_i) = O(t_i)$. Progressively removing schools $d_2, \ldots d_K$ from $A(t_i) \setminus \{d_1\}$ and applying Lemma 4 repeatedly, it follows that $\sigma(A''(t_i), t_i) = O(t_i)$.

Let $A'(t_i) \setminus A''(t_i) = \{d'_1, \ldots d'_S\}$ so that $A'(t_i)$ can be obtained from $A''(t_i)$ by adding schools $d'_1, \ldots d'_S$. In the second step, we progressively add schools $d'_1, \ldots d'_S$ to $A''(t_i)$. Applying Lemma 3,we can infer that $\sigma(A''(t_i) \cup \{d'_1\}, t_i) \in \{d'_1, O(t_i)\}$. Adding $d'_2, \ldots d'_S$ in turn to $A''(t_i) \cup \{d'_1\}$ and applying Lemma 3 repeatedly, it follows that $\sigma(A'(t_i), t_i) \in \{d'_1, \ldots d'_S, O(t_i)\}$. Since $d'_1, \ldots d'_S, O(t_i) \notin A(t_i)$, it follows that $\sigma(A'(t_i), t_i) \notin A(t_i)$.

■

6 Extensions

References

Ahuja, R. K., T. L. Magnanti, and J. B. Orlin (1988): "Network flows," .

Dutta, B. and D. Ray (1989): "A concept of egalitarianism under participation constraints," Econometrica: Journal of the Econometric Society, 615–635.

Megiddo, N. (1974): "Optimal flows in networks with multiple sources and sinks," Mathematical Programming, 7, 97–107.