Vertical Differentiation with Partial Share in Revenue and Profit

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Abstract

We examine duopoly competition between firms with asymmetric quality, wherein firms compete sequentially in quality and price. We find that the partial shareholding of the high-quality firm in revenue (or profit) of the low-quality firm softens the competition. The market share of the high-quality firm decreases as the percentage of the share in revenue (or profit) increases. Further, we find that the improvement in quality by high-quality firm is lesser than by low-quality firms. The price charged by the high-quality firm is higher than that of the low-quality firm as the high-quality firm continues to have the quality advantage. Comparing the two scenarios, revenue sharing is more desirable than profit sharing for firms, giving higher total profits. Consumers and social planner prefer profit sharing between the firms as it leads to a higher surplus.

Keywords - Revenue share, Profit share, Vertical differentiation, Hoteling line

1 Introduction

Various companies diversify their product portfolio by buying shares of other firms in the same industry. One such example is Coca-Cola. It has entered into partnerships and joint ventures with different beverage brands globally

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depending on the local conditions across various countries. Coca-Cola owns approximately non-controlling 17% shares in California-based Monster Beverage Corporation, which produces many energy drinks. Coca-Cola fully owns the coffee brand Costa Cofee; however, it usually has co-ownership in its operations with retail outlets. In 2009, Coca-cola invested in Innocent Drinks, a UK-based company, and acquired 18% shares. Later on, it invested further and owned majority shares of 58%, and eventually, by 2013, Coca-Cola Coal owned a full company. Intially Coca-Cola co-owned Fairlife before fully acquiring it in 2020. Coca-Cola also invested in US-based Honest tea with acquiring 48% share in the company while the founders maintained independent operations and product development. Later in 2011, Coca-Cola took over the remaining 60% of shares. Coca-Cola discontuined the brand in 2022. Coca-Cola co-owns a premium sports drink brand BodyArmor. Coca-Cola initially owned majority shares in Coca-Cola Beverages Africa (CCBA) and later it sold the shares in investors in Africa while maintaining the co-ownership.Coca-Cola co-owns brands like Barbican through its significant stakes in the Aujan industries. There are many brands that Coca-Cola fully acquired in Indian market like Maaza, Gold Spot, ThumbsUp, Limca, Minute Maid and Minute Maid Pulpy.

The discussion above indicated how Coca-Cola expanded its operations globally by acquiring partial or full stakes in other beverage-producing companies. These were co-owned companies of Coca-Cola while also being competitors in the beverage industry. This motivates our study, where we analyze the impact of partial ownership on the competition between firms in the same industry. Many such examples can be found in various industries like food, apparel, automobiles, etc.

Sometimes, the companies do not acquire shares of another company but enter into a contract where only revenue is shared. In such cases, the cost is not shared between the companies entering the contract. For example, Amazon has its own brand with the name “Amazon Basics,” selling many products and also competing with other brands selling the same products on the platform. The other brands selling the products share a predetermined percentage of their revenue with Amazon. Similar features of revenue sharing can be observed in offline stores also, like Walmart, which sells products from its own brand and other brands in the same store. In the airline industry, different companies share technical codes with each other and share revenues. Major telecom operators lease out their infrastructure to rival telecom companies on a revenue-sharing basis; one such example is Verizon and AT&T in the US.

In all the examples above, we see partial or one-sided ownership of profits or revenues between the firms. Even though the firms share profits or revenue, they compete with each other. In this study, we model this for the price competition between the firms. Further, we also look at the incentive for improvement in the product quality given that there is an initial difference in the quality. The different scenarios raise some questions like (a) What is the extent of improvement in quality by both the firms when there is partial cross-ownership in revenue/profit? (b) How do the demand, level of quality improvement, and prices change as the revenue/profit share changes? (c) What is the impact of partial cross-ownership of the consumer surplus and total welfare? (d) Does the competition increase or decrease under partial cross-ownership with quality differences? We also compare the partial ownership (revenue/profit) with no cross-ownership with the initial difference in the quality.

We use a basic framework to answer these questions, whereby the firms maximize their profits. The firms are asymmetric in their initial quality; there is a high-quality firm and a low-quality form. The firms interact with each other in two stages. In the first stage, the firms choose the quality improvement level simultaneously. In stage two, the firms compete in the prices. There are three different scenarios: (1) no revenue/profit sharing, (2) the high-quality firm owns a minority share in the revenue of the low-quality firm, and (3) the high-quality firm owns a minority and non-controlling share in the profit of the low-quality firm.

In the preliminary results, we show that where there is no profit/revenue sharing, the high-quality firm has more incentive to improve quality and charges a higher price, and earns more profits than the low-quality firm. Further, in the revenue sharing case, quality improvement by the high-quality firm may or may not be higher than that by the low-quality firm, and the price charged by the high-quality firm is higher than that by the low-quality firm. In the profit sharing case, the difference in price charged by the high-quality and low-quality firms depends upon the degree of the profit share and the difference in the initial quality level. The total profit earned by the high-quality firm is higher than that of the low-quality firm under revenue and profit sharing.

The study is organized as follows. Section 2 discusses the related literature. Section 3 presents basic and extended models. Section 4 provides the comparative statics between different regimes. Section 6 concludes.

2 Related Literature

There is vast research on the impact of partial cross-ownership (PCO) and overlapping ownership on the competition. One of the studies closely related to this study is by Li et al. (2023a), which examines price competition between firms with both innovation and overlapping ownership. It shows that overlapping ownership reduces the low-quality firm's incentive to invest further in quality and increases vertical differentiation. Our study differs from and extends the model in the direction of partial cross-ownership in profit and partial share in the revenue of the rival firm. Some other studies like Malueg (1992), López and Vives (2019), Li and Zhang (2021), Ederer and Pellegrino (2022), Bayona et al. (2022) also examines competition effects of overlapping ownership between firms.

Amongst the early studies in partial cross-ownership, Reynolds and Snapp (1986) focuses on the competitive effects side of the trade-off between efficiency and competition in case of partial ownership arrangements. The paper finds that in markets where entry is difficult, PCO may lead to lower output and higher prices compared to standard quantity competition models. This study differs and examines the impact of PCO on the price competition between the firms. Gilo et al. (2006) examines the incentive to collude in an infinitely repeated interaction between competing firms in an oligopoly structure with partial cross-ownership. Brito et al. (2010) studies the PCO between the firms where shareholders hold voting and non-voting stocks in both the firms. Shelegia and Spiegel (2012) examines Bertrand competition between n asymmetric firms and potential anti-competitive effect of PCO. Fanti (2015) considers Cournot Duopoly with asymmetric costs and PCO. This study finds that as the partial share of a firm (in profits of the competing firm) increases, its own output decreases. Also, for a sufficiently large market PCO is welfare improving. Nain and Wang (2018) empirically examines the product market impact of partial equity ownership between rival firms. The study finds changes in output prices and price-cost margins followed by minority stake acquisitions in the U.S. manufacturing industry. This motivates us to examine the product market competition in the case of PCO theoretically in our study and fill the gap in the literature. Brito et al. (2019) compares Cournot competition under PCO with Monopoly and finds that PCO can lessen competition more than a monopoly.

There is wide research on vertical differentiation and quality choice in a duopoly with overlapping ownership. However, the research on vertical differentiation in the case of partial cross-ownership is limited. Wauthy (1996) provides the characterization of results in a vertically differentiated duopoly where the firms choose quality and prices sequentially. It finds that one firm chooses the best quality, and the other chooses a fixed proportion of the best quality. The stage game of the model in our study is also similar. The firms have an initial quality level, which is not the same, and then the firms choose the level of quality improvement in stage 1, followed by price competition in stage 2. Brito et al. (2020) investigates the impact of overlapping ownership on quality levels, profits, consumer surplus, and welfare, with vertically differentiated duopoly firms choosing quality endogenously.

Shelegia and Spiegel (2022) looks at a two-stage model where firms decide to innovate and then compete in prices. Both firms own a minor share in each other's profit. The study finds that asymmetry in ownership structure can lead to significant differences in investment levels and innovation incentives. This subsequently affects the pricing decision in stage two. Li et al. (2023b) show that overlapping ownership can raise high-quality firms' incentive to improve product quality.

There are a few studies on revenue sharing among competing firms. Leonardos et al. (2021) studies the technology licensing in a Cournot duopoly where one firm has better technology than the other and holds a partial share in the profits of the other firm. Our study explores a similar structure via revenue sharing. Hervas-Drane and Shelegia, 2022 has examined the impact of revenue-sharing and profit-sharing stakes under the duopoly model by Varian, 1980. They have found that a revenue-sharing stake has a more substantial competition-dampening effect, resulting in higher prices than a profit-sharing stake.

Our paper contributes to the literature of vertical differentiation and partial cross ownership (PCO). The effects on product market competition in the case of one firm having a revenue share is not explored much in the literature. Paying a royalty to a competing firm is common in the case of a franchise, technology transfer, brand royalty, etc. Further, a limited number of studies in the literature explore vertical differentiation between two competing firms in the case of partial cross-ownership. This paper examines two scenarios of partial cross-ownership and revenue share of one firm into another and compares the competition effects of the two-stage duopoly model.

3 The Model

We consider a market with two firms (1 and 2), located on the Hotelling line. The consumers are uniformly distributed on the Hotelling line, and the location of the firm 1 and firm 2 is exogenously fixed at the endpoints of the line; that is, firm 1 is located at 0, and firm 2 is located at 1. The location of the indifferent consumer determines the demand faced by each firm.

The initial quality of each firm is represented by S_i and is given exogenously. We assume that $S_1 > 0$ and $S_2 = 0$ for simplicity. This implies that firm 1 has better quality at the beginning than firm 2.

We investigate a two-stage model where the two firms choose the quality level V_i (where $i = 1, 2$) in Stage 1. The choice of quality is costly and is measured using a quadratic cost function. In the second stage, both the firms compete in prices and choose P_i where $i = 1, 2$.

The utility function faced by the consumer located at point x, is structured below

$$U = \begin{cases} S_1 + V_1 - P_1 - tx, & \text{if buying from firm 1} \\ S_2 + V_2 - P_2 - t(1-x), & \text{if buying from firm 2} \end{cases}$$

where S_i for $i = \{1, 2\}$ is the initial quality level of the firms, and t is the transportation cost, normalized to 1.

We assume the reservation utility is high enough to ensure full market coverage. In this paper, we do not analyze the case of partial market coverage. The location of the indifferent consumer is at $\hat{x} = \frac{1+S_1+V_1-V_2-(P_1-P_2)}{2}$. $\hat{x}$ depends positively on S_1, on the difference between the level of quality improvement and the difference in price charged by the two firms. Demand faced by firm 1 is, therefore, $\hat{x}$ and that by firm 2 is $1 - \hat{x}$.

We assume that the marginal cost of production for both firms is constant and zero. However, the firms have to incur a cost of $\frac{V_i^2}{2}$ for quality improvement. Firms are maximizing their respective profit. The profit made by firm 1 and firm 2 is given by $\pi_1 = \hat{x}P_1 - \frac{V_1^2}{2}$ and $\pi_2 = (1-\hat{x})P_2 - \frac{V_2^2}{2}$.

We will first consider the base model where the firms do not have any share in the revenue or profit of the other firm. Both the firms have the profits accruing to themselves only. Li et al. (2023a) has studied the impact of cross-ownership among the firms using the same model, wherein the firms have the same profits as accruing to themselves only and the comparison model as one where the firms have cross-ownership and hence, a proportion of each other's profit accrues to the opposite firm. Hence, similarly, we compare the base model regime (Regime 1) with Regime 2 when firm 1 has τ_R share in the revenue of firm 2 and Regime 3 when firm 1 has τ_P share in the profit of firm 2. We assume $\tau_R, \tau_P \in [0, \frac{1}{2})$ to ensure minority shareholding by firm 1.

To summarize, we solve the model for the following three regimes:

Regime 1: Firms do not have any share in the revenue or profit of the rival firm

Regime 2: Firm 1 holds τ_R share in revenue of firm 2.

Regime 3: Firm 1 holds τ_P share in profit of firm 2.

In all the three regimes, there are two stages in the game described below:

Stage 1: The two firms simultaneously choose the level of quality improvement for their product, V_1 and V_2.

Stage 2: The two firms simultaneously choose the price P_1 and P_2 for their product.

We will solve the Subgame Perfect Nash Equilibrium using backward induction.

3.1 Regime 1: Firms do not have any share in the revenue or profit of the rival firm

This is the baseline model of this study. We consider the case where no firm has a share in the revenue/profit of the rival firm. The firms choose the level of improvement of quality and prices. The detailed solution is in Appendix 7.1. In equilibrium firms choose: $V_1 = \frac{1}{21}(7 + 3S_1)$ and $V_2 = \frac{1}{21}(7 - 3S_1)$. The high-quality firm chooses the higher level of improvement as compared to the low-quality firm, $V_1 > V_2$ for $S_1 > 0$. In equilibrium,$P_1 = 1 + \frac{3S_1}{7}$ and $P_2 = 1 - \frac{3S_1}{7}$. We see that $P_1 > P_2$ for $S_1 > 0$. The high-quality firm chooses a higher level of improvement and, therefore, higher prices than the low-quality firm. The demand faced by firm 1 is $\hat{x} = \frac{1}{14}(7+3S_1)$ which is greater than $\frac{1}{2}$ for $S_1 > 0$. We see that firm 1 captures the majority of the market share, further incentivizing it to improve the quality more than firm 2, even if it's costly. Also, $\pi_1 > \pi_2$ because of the impact of higher initial quality on demand and price, which outweighs the cost of quality improvement. The higher the initial quality, the higher the improvement in quality in Stage 2, the higher the price charged in Stage 1, and subsequently, the higher the demand faced by firm 1. This leads to higher profit for firm 1. We also see that the better initial quality of the opponent firm discourages improvement in quality by firm 2, therefore resulting in lower prices, lower demand, and thus lower profit. Clearly, firm 1 has an advantage over firm 2 because of the higher initial level of quality. This is the model proposed by Li et al. (2023a); we simplify it further by normalizing the cost coefficient to 1. In this study, we extend the model to understand the impact of one-sided revenue sharing and partial cross-ownership in profit on the competition between the firms.

3.2 Regime 2: Firm 1 holds revenue share τ_R in firm 2

In this case, the high-quality firm has a share in the revenue of the low-quality firm. This implies that firm 1 does not look into the operations, costs, etc, of firm 2. It only receives a certain percentage of the revenue for every unit. The profit accruing to the two firms is as follows:

$$\pi_1^R = \hat{x}_R P_1 - \frac{V_1^2}{2} + \tau_R((1-\hat{x}_R)P_2)$$

$$\pi_2^R = (1-\tau_R)(1-\hat{x}_R)P_2 - \frac{V_2^2}{2}$$

where $\tau_R \in [0, \frac{1}{2})$ is the percentage of revenue share of firm 1 in firm 2.

Proposition 1. *When firm 1 holds τ_R share in firm 2's revenue, there exists a SPNE when $\tau_R \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_1^R$ where $\bar{S}_1^R = \frac{7-2\tau_R}{3-\tau_R}$ wherein,*

$$V_1^R = \frac{7+S_1(3-\tau_R)-\tau_R(\tau_R^2-7\tau_R+15)}{(3-\tau_R)(7-(5-\tau_R)\tau_R)}$$

$$V_2^R = \frac{(7-S_1(3-\tau_R)-2\tau_R)(1-\tau_R)}{(3-\tau_R)(7-(5-\tau_R)\tau_R)}$$

$$P_1^R = \frac{7+S_1(3-\tau_R)(1-\tau_R)-\tau_R}{7-(5-\tau_R)\tau_R}$$

$$P_2^R = \frac{7-S_1(3-\tau_R)-2\tau_R}{7-(5-\tau_R)\tau_R}$$

Proof. The proof is in Appendix 7.2. □

Corollary 1. *For the case when $S_1 > \bar{S}_1^R$, it will not be profitable for firm 2 to invest in quality improvement and $V_2^R = 0$.*

Proof. $V_2^R \leq 0 \Leftrightarrow S_1 > \frac{7-2\tau_R}{3-\tau_R} = \bar{S}_1^R$. □

Lemma 1. *(i) Demand of firm 1 decreases as τ_R increases.*

(ii) The level of quality improvement by firm 1 decreases as τ_R increases.

(iii) The level quality improvement by firm 2 decreases as τ_R increases when $S_1 < S_1^{V_2^R}$ and increases when $S_1 > S_1^{V_2^R}$.

(iv) Average level of quality declines in the market with the increase in τ_R.

(v) The difference in the level of quality between the firms decreases with an increase in τ_R.

(vi) The price level of firm 2 increases with the increase in τ_R, however, price level of firm 1 increases with increase in τ_R at $S_1 \leq S_1^{P_1^R}$ and declines at $S_1 \geq S_1^{P_1^R}$.

(vii) Average price level in the market increases with an increase in τ_R at $S_1 \leq S_1^{P^R}$ and declines at $S_1 \geq S_1^{P^R}$.

Proof. The proof is in Appendix 7.2. □

Lemma 2. *(i) Profits accruing to firm 1 increases with increase in τ_R at $S_1 \leq S_1^{\pi_1^R}$ and decreases at $S_1 \geq S_1^{\pi_1^R}$.*

(iii) For $S_1 \leq \bar{S_1}^R$, total profit increases with increases in τ_R

Proof. The proof is in Appendix 7.2. □

Lemma 3. *(i) Consumer Surplus decreases with increase in τ_R.*

(ii) Total Welfare decreases with increase in τ_R.

Proof. The proof is in Appendix 7.2. □

We observe that if $S_1 \geq \tau_R$ then $V_1^R > V_2^R$. Also, P_1^R is always greater than P_2^R. We also see that demand for firm 1, $\hat{x}_R$, falls as τ_R increases for any S_1^R. This implies that demand for firm 1 can also be below $\frac{1}{2}$. Also, $\pi_1^R > \pi_2^R$, even if firm 1 may not hold the majority market share and may choose to improve the quality less than firm 2. The high-quality firm (Firm 1) earns higher profits when it holds a share in the revenue of the low-quality firm, i.e. $\pi_1^R > \pi_1$.

As τ_R increases, firm 1's share in firm 2's revenue increases and the competition softens. Firm 1 earns revenue through firm 2; therefore, it chooses the lower level of quality improvement. Since a part of the marginal revenue earned by firm 2 due to its quality improvement also accrues to firm 1; Firm 2 should be less incentivized to increase its quality level. The improvement in quality level by the firm 2 would be higher if $\tau_R = 0$. However, we observe that the level of quality improvement by firm 2 increases as τ_R increases. This is because of two reasons: (1) a higher share of firm 2 in demand and (2) lower quality improvement by firm 1. As competition softens, firm 1 chooses the lower level of quality improvement which further reduces the market share of firm 1. Firm 1 initially is a high-quality firm, and even after the quality improvement by both the firms, firm 1 still stays the high-quality firm which leads to $P_1^R > P_2^R$.

The average quality in the market declines as τ_R increases because the quality improvement by both firms declines with increasing revenue share. As $S_1 + V_1 \geq V_2$, the decline in the share of firm 1 in market demand also adds to the decrease in the average quality level of the firm. This can also be seen from the Figure 1, where we have assumed $S_1 = 0.4$.

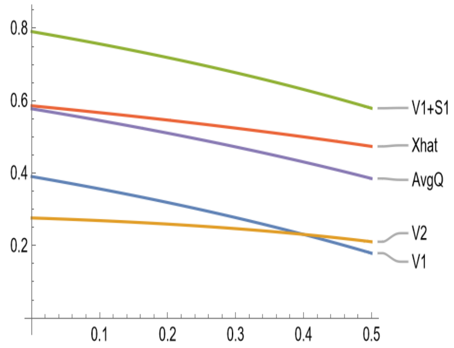

Figure 1: Quality Comparisons assuming $S_1 = 0.4$

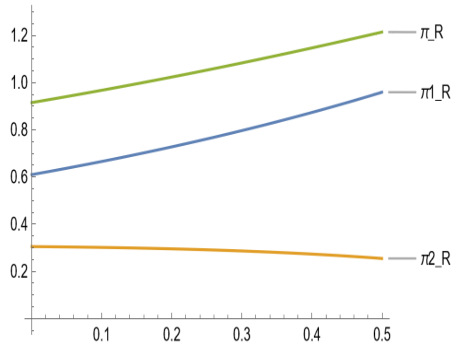

Figure 2: Firm level profit and total profit in case of revenue share assuming $S_1 = 0.4$

The profit of firm 1 increases with an increase in its share of the revenue of firm 2. Firm 1 internalizes the impact on profit of firm 2 while choosing its quality and price. Figure 2 gives the profit of both firms and the total profit when $S_1 = 0.4$.

The total profit accruing to both firms increases with an increase in τ_R. Hence, there is a possibility for the two firms to profitably enter into a contract where one firm has a revenue share in the other firm.

Consumer surplus depends upon the average quality, average price in the market, and $\hat{x}_R$.

$$CS^R = AQ^R - AP^R - \hat{x_R}^2 + \hat{x_R} - \tfrac{1}{2} \text{ where } AP^R = \hat{x_R} P_1^R + (1 - \hat{x_R}) P_2^R$$

$\hat{x_R}$ affects consumer surplus through a change in transportation cost. As τ_R increases, $\hat{x_R}$ declines. If $\hat{x_R} < \frac{1}{2}$, it has a negative impact on consumer surplus. Further, with an increase in τ_R, average quality falls, and average price increases and the consumer surplus declines. The change in the total welfare depends on declining consumer surplus and increasing total profits. The net effect is that it decreases with an increase in τ_R.

3.3 Regime 3: Firm 1 holds profit share τ_P in firm 2

In this sub-section, we have assumed that firm 1 holds τ_P share in the profit of firm 2. We assume that firm 1 holds a non-controlling share in firm 2's profit. The profit accruing to both firms can be computed as below:

$$\pi_1^P = \hat{x}_P P_1 - \frac{V_1^2}{2} + \tau_P((1-\hat{x}_P)P_2 - \frac{V_2^2}{2})$$

$$\pi_2^P = (1-\tau_P)((1-\hat{x}_P)P_2 - \frac{V_2^2}{2})$$

where $\tau_P \in [0, \frac{1}{2})$ is the percentage of profit share of firm 1 in firm 2.

The profit maximization condition for firm 2 in Regime 3 is the same as in Regime 1 as the total profit of firm 2 gets slashed by $1-\tau_P$, and hence, the first order maximization condition remains the same. In the case of Regime 2, the revenue of firm 2 gets slashed by $1-\tau_R$, and the cost remains the same, thus affecting the first-order maximization condition.

Proposition 2. *When firm 1 holds τ_P share in firm 2's profit, there exists a SPNE when $\tau_P \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S_1}^P$ where $\bar{S_1}^P = \frac{7-2\tau_P}{3-\tau_P}$ wherein,*

$$V_1^P = \frac{7+S_1(3-\tau_P)-\tau_P^3+8\tau_P^2-17\tau_P}{(3-\tau_P)(7-(6-\tau_P)\tau_P)}$$

$$V_2^P = \frac{7-S_1(3-\tau_P)-2\tau_P}{(3-\tau_P)(7-(6-\tau_P)\tau_P)}$$

$$P_1^P = \frac{(7+S_1(3-\tau_P)(1-\tau_P)-3\tau_P)}{(7-(6-\tau_P)\tau_P)}$$

$$P_2^P = \frac{(7-S_1(3-\tau_P)-2\tau_P)}{(7-(6-\tau_P)\tau_P)}$$

Proof. Proof is in Appendix 7.3 □

Corollary 2. *For $S_1 > \bar{S_1}^P$, it will not be profitable for firm 2 to invest in quality improvement and $V_2 = 0$.*

Proof. $V_2 \leq 0 \Leftrightarrow S_1 > \frac{7-2\tau_P}{3-\tau_P} = \bar{S_1}^P$. □

Lemma 4. *(i) Demand of firm 1 declines as τ_P increases.*

(ii) The level of quality improvement by firm 1 decreases as τ_P increases.

(iii) The level of quality improvement by firm 2 increases as τ_P increases.

(iv) Change in level of average quality increases as τ_P increases for $S_1 \leq S_1^{AQ^P}$ and decreases for $S_1 \geq S_1^{AQ^P}$.

(v) *The difference in the level of quality between the firms decreases with an increase in τ_P.*

(vi) *The price level of firm 2 increases with the increase in τ_P, however, price level of firm 1 increases with increase in τ_P at $S_1 \leq S_1^{P_1^P}$ and declines at $S_1 \geq S_1^{P_1^P}$.*

(vii) *Average market price increases at $S_1 \leq S_1^{AP^P}$ and declines at $S_1 \geq S_1^{AP^P}$, with increase in τ_P.*

Proof. The proof is in Appendix 7.3. □

Lemma 5. *(i) Profit accruing to firm 1 increase with increase in τ_P for $S_1 \leq S_1^{\pi_1^P}$ and decreases for $S_1 \geq S_1^{\pi_1^P}$.*

(ii) *Total profit increases with increase in τ_P for $S_1 \leq S_1^{\Pi^P}$ and decreases for $S_1 \geq S_1^{\Pi^P}$.*

Proof. The proof is in Appendix 7.3. □

Lemma 6. *(i) Consumer Surplus decreases with increase in τ_P.*

(ii) *Total Welfare decreases with increase in τ_P.*

Proof. The proof is in Appendix 7.3. □

As τ_P increases, the level of quality improvement by the high-quality firm (V_1^P) decreases, and the level of quality improvement by the low-quality firm (V_2^P) increases. The intuition behind this result is that the cost incurred to improve the quality by firm 2 is also shared in a higher proportion by firm 1 as τ_P increases. The net burden of the cost of improvement in the level of quality reduces for firm 2. Further, as V_2^P increases, $\hat{x_P}$ decreases, which implies that the market share of firm 2 increases as the quality becomes better.

We also see that $S_1 \leq \frac{5\tau_P - \tau_P^2}{2} \Rightarrow V_1^P \leq V_2^P$. For $S_1 \geq \frac{\tau_P}{\tau_P^2 - 5\tau_P + 6}$, $P_1 \geq P_2$. So, it depends on the value of S_1 whether firm 1 level of improvement in quality is higher or lower than Firm 2. Similarly, the relative price charged by both firms also depends on S_1. Since $\hat{x}_P$ decreases as τ_P increases, firm 1 may or may not capture the majority market share.

The average quality increases with an increase in τ_P when S_1 is very low. In the Figure 3, we have shown the average quality in the market as the profit

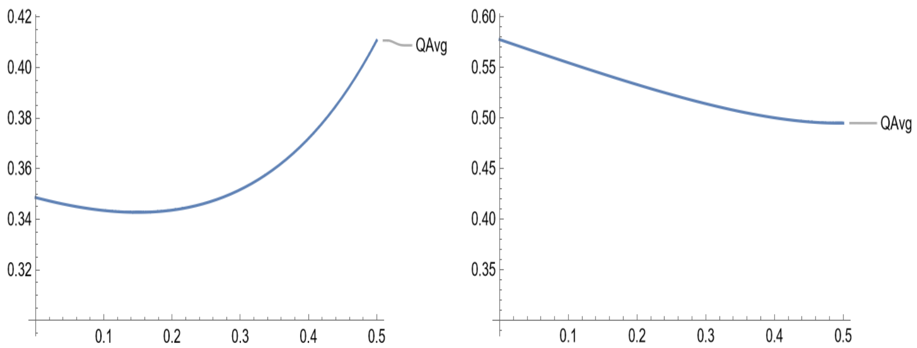

Figure 3: Average Quality in case of profit share when $S_1 = 0.3$ (Left panel) and when $S_1 = 0.4$ (Right panel)

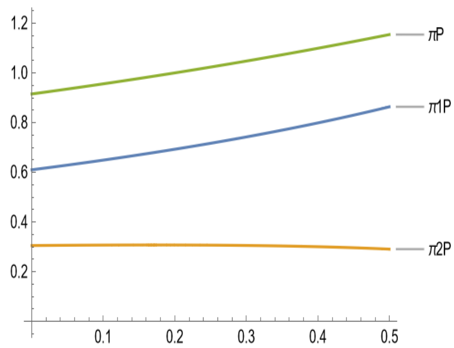

Figure 4: Firm level profit and total profit in case of profit share assuming S1=0.4

share increases. We have assumed two cases, (a) when $S_1 = 0.3$, and average quality is increasing, (b) when $S_1 = 0.4$, and average quality is decreasing.

The profit of a high-quality firm is always higher than that of the low-quality firm (Figure 4). The total profit accruing to both the firms increases with an increase in τ_P when S_1 is low enough. This happens due to the softening of competition as firm 1 takes into account the increase in its total cost with an increase in τ_P while choosing V_1^P and P_1^P. We also observe that both the consumer surplus and the total welfare decrease with an increase in τ_P.

4 Comparative Statics

In this section, we will compare the results across all regimes. For this purpose, we fix $\tau_R = \tau_P = \tau$. Also, when $\tau_R = \tau_P = \tau$, then $\bar{S_1}^R = \bar{S_1}^P = \frac{7-2\tau}{3-\tau} = \bar{S}_{1\tau}$. Note that if $\tau_R = \tau_P = 0$, then Regime 2 and Regime 3 collapse to the base model in Regime 1. We will assume that $S_1 < \bar{S}_{1\tau}$ so that the lower-quality firm is not driven out of the market.

Lemma 7. *(i) $\hat{x}_P \leq \hat{x}_R$ when $S_1 < \bar{S}_{1\tau}$ for all $\tau \in [0, \frac{1}{2})$.*

(ii) The level of quality improvement by firm 1 in Regime 2 is higher as compared to Regime 3. The decline in quality is greater in Regime 3 as compared to Regime 2 as τ increases.

(iii) The level of quality improvement by firm 2 in Regime 2 is lower as compared to Regime 3. The decline in quality is lesser in Regime 3 as compared to Regime 2 as τ increases.

(iv) The level of average quality is low at $S_1 \leq S_1^{AQ}$ and higher at $S_1 \geq S_1^{AQ}$ in Regime 2 as compared to Regime 3.

(v) The difference in the level of quality between the firms is higher in Regime 2 as compared to Regime 3. Moreover, the decline in the difference in quality level between the firms is flatter in Regime 2 than in Regime 3.

(vi) The price charged by firm 1 is higher in Regime 2 in comparison to Regime 3. Price charged by firm 2 is higher in Regime 3 as compared to that in Regime 2.

(vii) The average market price is higher in Regime 2 as compared to Regime 3 when $S_1 \geq S_1^{AP}$ and lesser for $S_1 \leq S_1^{AP}$

Proof. The proof is in Appendix 7.4 □

Lemma 8. *(i) For $S_1 < \bar{S}_{1\tau}$, profit of firm 1 is higher in Regime 2 as compared to Regime 3 for $\tau \in [0, \frac{1}{2})$.*

(ii) For $S_1 \leq \bar{S}_{1\tau}$, total profit is higher in Regime 2 as compared to Regime 3 for $\tau \in [0, \frac{1}{2})$.

Proof. The proof is in Appendix 7.4 □

Lemma 9. *(i) Consumer Surplus is low in Regime 2 as compared to Regime 3.*

(ii) Total Welfare is low in Regime 2 as compared to Regime 3 when $S_1 < S_1^W$ and high in Regime 2 as compared to Regime 3 when $S_1 > S_1^W$.

Proof. The proof is in Appendix 7.4. □

In general, the level of quality can be improved by any firm at a cost. Better quality adds to the utility of the consumer, and hence, the products entail a higher price in the market. $V_1^R > V_1^P$ and $P_1^R > P_1^P$. In Regime 2, firm 2's revenue is slashed by τ_R. However, the cost remains the same. Hence, the marginal revenue with each unit of quality improvement declines by τ_R, while the marginal cost remains the same. Therefore, in equilibrium $V_2^R < V_2^P$ and $P_2^R < P_2^P$.This is also summarised in the Table 1.

The reason for this result is that in Regime 2, firm 1 does not share the cost of the improvement in quality unlike in Regime 3.

Table 1: Regime 2 compared to Regime 3

	Price	Quality
Firm 1	High	High
Firm 2	Low	Low

In Figure 5 and Figure 6, we show the level of V_2 in Regime 2 and Regime 3. In Figure 5, we have shown $S_1 < S_1^{V_2^R}$ and V_2^R declines, and in Figure 6, we have shown that $S_1 > S_1^{V_2^R}$, thus V_2^R increases. The level of quality improvement by firm 2 is lower in Regime 2 as compared to that in Regime 3 for all feasible values of S_1 and τ.

In Figure 7, we have shown how the quality difference between firm 1 and firm 2 changes as τ increases.We have assumed that $S_1 = 1$ and show that the quality difference between both firms declines as the degree of revenue share and profit share increases. The curve with a profit share is steeper than

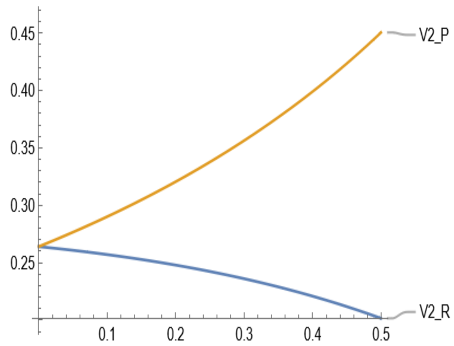

Figure 5: Quality improvement by firm 2 in revenue share and in profit share when V2 is decreasing in τ

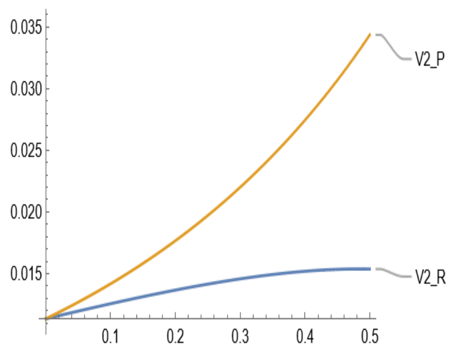

Figure 6: Quality improvement by firm 2 in revenue share and in profit share when V2 is increasing in τ

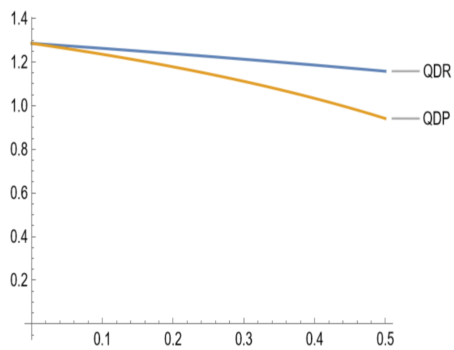

Figure 7: Quality Differential between the two firms in revenue share and in profit share assuming S1=1

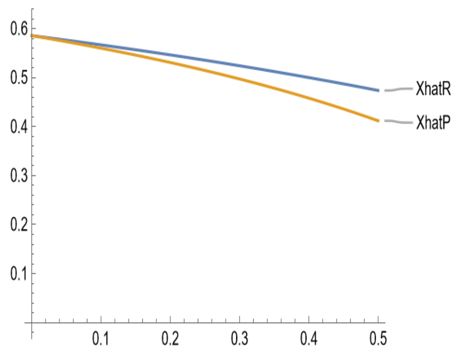

Figure 8: Demand of firm 1 under Regime 2 and Regime 3 assuming S1=0.4

that with a revenue share. Initially, with $\tau = 0$, firm 1 (high-quality firm with $S_1 = 1$) invests more in quality improvement. However, with revenue and profit share, the investment in quality level by firm 1 declines. It declines more in case of partial cross-ownership in profit.

We summarize the key results of the quality improvement under different regimes below:

1. V_1^R and V_1^P declines because of partial cross-ownership in revenue and profit, respectively. This softens the competition between both firms, and as a result, the market share of the firm 1 also declines.

2. V_2^R declines as the marginal cost of investing in quality improvement is higher vis-a-vis marginal benefit than Regime 1 where $\tau = 0$.

3. V_2^P increases as τ increases. Even though the profit-maximizing problem for firm 2 remains unchanged, τ indirectly affects the quality decision of firm 2 through expansion in the market share.

In Regime 2 , firm 1's higher level of quality improvement is accompanied by higher market share and higher price than Regime 3. This is also shown in Figure 8 below. In the figure, we observe the following:

1. Demand of firm 1 declines as τ_R increase. (as shown in Lemma 1)

2. Demand of firm 1 declines as τ_P increase. (as shown in Lemma 4)

3. The decline in demand is steeper under profit sharing (Regime 3) than revenue sharing (Regime 2) when $\tau_R = \tau_P = \tau$.

Higher quality and higher price are followed by larger market share and higher profits for firm 1 in Regime 2. The increase in profit of firm 1 has two

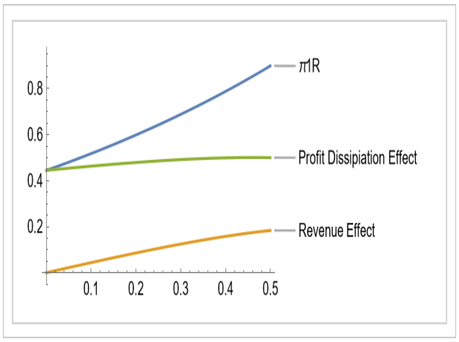

Figure 9: Segregation of π_1 in Regime 2 into Profit Dissipation Effect and Revenue Effect when $S_1 = \frac{1}{2}$

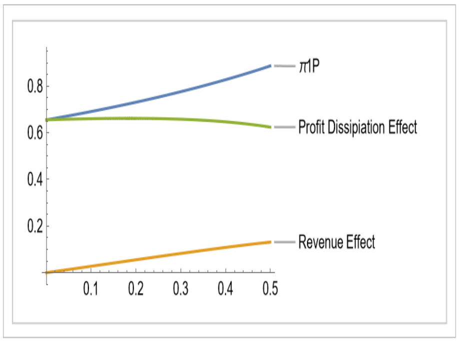

Figure 10: Segregation of π_1 in Regime 3 into Profit Dissipation Effect and Revenue Effect when $S_1 = \frac{1}{2}$

sources- (1) profit dissipation effect, which is the reduction of the potential profits that the high-quality firm could earn, as a result of competition softening and lower market share to high-quality firm, on account of having a share in the low-quality firm and (2) revenue effect, which is the amount of revenue accruing from the low-quality firm to the high-quality firm because $\tau \in (0, \frac{1}{2})$. In Figure 9 and 10 below, we segregate the profit of firm 1 into profit dissipation effect and revenue effect under both Regime 2 and Regime 3. The total profit of both firms is higher in Regime 2 as compared to Regime 3.

The consumer surplus is derived from the average quality and the average price. Its is lower in Regime 2, as compared to Regime 3. The consumer surplus is positively related to the average quality and negatively associated with the average price under both regimes. Although the impact of the average quality and average price on consumer surplus is in the opposite direction, the net effect is that the consumer surplus is lower in Regime 2 compared to Regime 3. The total welfare in Regime 2 and Regime 3 depends upon the total profit of the firms and consumer surplus. Since the total profit is higher in Regime 2 as compared to Regime 3 and consumer surplus is lower in Regime 2 as compared to Regime 3, the net effect on the total welfare is ambiguous.

5 Extensions

In the proposition 1 and 2, we have considered two cases wherein the high-quality firm has a share in the revenue of the low-quality firm and when the high-quality firm has a partial cross holding in the low-quality firm respectively.

We extended the model to alternative regimes:(1) the low-quality firm (firm 2) has a share in the revenue of the high-quality firm (firm 1), and (2) the low-quality firm has a share in the profit of the high-quality firm. For example, when Whole Foods was acquired by Amazon in 2017, the former was well known for its high-quality organic and premium grocery items. After being acquired by Amazon, Whole Foods was integrated into Amazon's delivery mechanisms. The key findings are: Firstly, in revenue sharing, quality improvement by firm 2 is lesser for each level of S_1 and τ compared to the case where firm 1's has a share in the revenue of firm 2. Firm 2's holding share in Firm 1's revenue disincentivizes to invest further in quality for firm 2. Second, in profit sharing, firm 2 has almost no incentive to improve quality for a very low level of the initial quality of firm 1. The gap between the quality of the firms widens further. Third, the average quality is higher when firm 2 holds a share in the revenue or profit of firm 1 as compared to when firm 1 holds a share in the revenue or profit of firm 2. This is also shown in Figure 11.

Fourth, the difference in the quality between the firms is higher when the low-quality firm has a share in a high-quality firm's revenue or profit rather than when the high-quality firm has a share in a low-quality firm's revenue or profit. Figure 12 indicates the same in case of profit sharing.

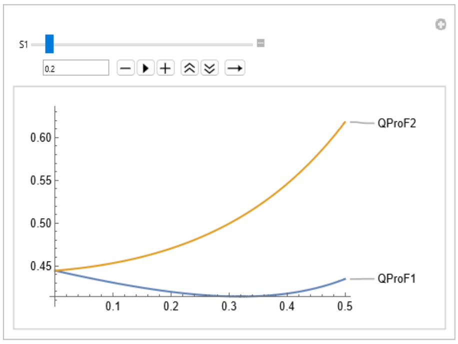

Figure 11: Average quality (Regime 2:Firm 1 has profit share in firm 1. Regime 5: Firm 2 has profit share in firm 1

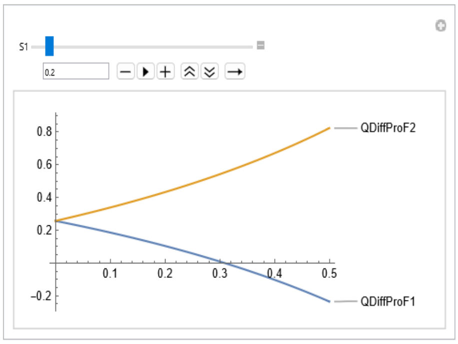

Figure 12: Vertical Differentiation (Regime 2: Firm 1 has profit share in firm 2, Regime 5: Firm 2 has profit share in firm 1)

6 Conclusion

In this paper, we study duopoly competition with partial cross-ownership between firms where firms are asymmetric in initial quality. The firms sequentially choose the level of improvement of quality and then compete in prices. We compare the results between two regimes: (1) the high-quality firm (firm 1) holds a share in the revenue of the low-quality firm (firm 2), and (2) firm 1 holds a share in the profit of firm 2. We discuss the results on the quality improvement, price, market share, profit, consumer surplus, and welfare. Some of the main findings are: First, the market share of firm 1 declines as its revenue or profit share increases; moreover, the market share of firm 1 is lower under profit sharing compared to revenue sharing. Second, firm 1 chooses a lower level of improvement in quality as its revenue or profit share increases. Also, firm 1's level of quality improvement in the revenue sharing regime is higher in comparison to the profit sharing regime. Third, firm 2 chooses a higher level of improvement in quality as firm 1's share in its profit increase. This is an interesting result driven by the sharing of costs between firms and the softening of competition. Fourth, the total level of quality is always higher for firm 1 compared to firm 2, even if it chooses a lower level in first stage under both revenue and profit sharing. This means that the initial difference in quality matters in the determination of the final outcome. Fifth, firm 1 earns more profit in revenue sharing than profit sharing for a low enough level of initial quality. Sixth, the consumer surplus is lower in revenue sharing as compared to profit sharing. Lastly, consumer surplus and total welfare, both decrease with an increase in firm 1's share in revenue or profit of firm 2. This implies that partial ownership is not desirable from a consumer's and social planner's standpoint. We also conclude that from the perspective of firms, revenue sharing is more beneficial and leads to higher profits, whereas for consumers and social planner profit sharing gives more surplus.

7 Appendix

7.1 Regime 1: Firms do not have any share in the revenue or profit of the other firm

We solve for the equilibrium

Stage 2:

$max\ \pi_1 = \hat{x}P_1 - \frac{V_1^2}{2}$ w.r.t P_1

$max\ \pi_2 = (1-\hat{x})P_2 - \frac{V_2^2}{2}$ w.r.t P_2.

We get the best response functions as

$P_1 = \frac{1}{2}(P_2 + S_1 + V_1 - V_2 + 1)$

$P_2 = \frac{1}{2}(P_1 - S_1 - V_1 + V_2 + 1)$

Simultaneously solving the two equations gives the values of P_1 and P_2

$P_1 = \frac{1}{3}(V_1 - V_2 + S_1 + 3)$

$P_2 = \frac{1}{3}(V_2 - V_1 - S_1 + 3)$

Stage 1:

Plugging in the values of P_1 and P_2 in the profit functions, we get,

$$\pi_1 = \frac{1}{18}(S_1 - 2V_1 - V_2 + 3)(S_1 + 4V_1 - V_2 + 3)$$

$$\pi_2 = \frac{1}{18}(3 - S_1 - V_1 + 4V_2)(3 - S_1 - V_1 - 2V_2)$$

Maximising the firms' profit w.r.t. the respective quality improvement levels, V_1 and V_2, gives the best response functions as

$$V_1 = \frac{1}{8}(3 + S_1 - V_2)$$

$$V_2 = \frac{1}{8}(3 - S_1 - V_1).$$

Simultaneously solving both the equations, we get

$$V_1 = \frac{1}{21}(7 + 3S_1)$$

$$V_2 = \frac{1}{21}(7 - 3S_1).$$

The profits of two firms in base model regime is as follows:

$$\pi_1 = \frac{4}{441}(7 + 3S_1)^2$$

$$\pi_2 = \frac{4}{441}(7 - 3S_1)^2$$

Plugging in the values of V_1 and V_2 in P_1, P_2 and $\hat{x}$, we get their optimum values:

$$P_1 = 1 + \frac{3S_1}{7}$$

$$P_2 = 1 - \frac{3S_1}{7}$$

$$\hat{x} = \frac{1}{14}(7 + 3S_1)$$

We restrict $S_1 \leq \bar{S}_1$ where $\bar{S}_1 = \frac{7}{3} \Leftrightarrow P_2 \geq 0$.

7.2 Regime 2

Proof. Proof of Proposition 1 Solving for SPNE using backward induction.

Stage 2:

$max\ \pi_1^P = \hat{x}_R P_1 - \frac{V_1^2}{2} + \tau_P((1-\hat{x}_R)P_2 - \frac{V_2^2}{2})$ w.r.t P_1

$max\ \pi_2^P = (1-\tau_P)((1-\hat{x}_R)P_2 - \frac{V_2^2}{2})$ w.r.t P_2

We get the following first order conditions:

$\frac{\partial \pi_1^P}{\partial P_1} = \frac{1}{2}(1-(P_1-P_2)+(S_1+V_1-V_2)) - \frac{P_1}{2} + \frac{P_2\tau_P}{2} = 0$
$\Rightarrow P_1 = \frac{1}{2}(P_2\tau_P + P_2 + S_1 + V_1 - V_2 + 1)$, best response of firm 1.

$\frac{\partial \pi_2^P}{\partial P_2} = (1-\tau_P)\left(\frac{1}{2}((P_1-P_2)-(S_1+V_1-V_2)-1) - \frac{P_2}{2} + 1\right) = 0$
$\Rightarrow P_2 = \frac{1}{2}(P_1 - (S_1+V_1-V_2)+1)$, best response of Firm 2.

Simultaneously solving the best response function, we get the values of P_1 and P_2:

$P_1 = \frac{3+S_1+V_1-V_2-\tau_P(S_1+V_1-V_2-1)}{3-\tau_P}$

$P_2 = \frac{3-(S_1+V_1)+V_2}{3-\tau_P}$

Plugging in the values of P_1 and P_2 in the profit functions of the firms, we get the profit functions of two firms as below:

$\pi_1^P = \frac{3-(S_1+V_1-V_2)}{3-\tau_P} + \frac{(3-(S_1+V_1-V_2))^2}{2(3-\tau_P)^2} + S_1 - \frac{V_1^2}{2} + V_1 - \frac{\tau_P V_2^2}{2} - V_2 - 1$

$\pi_2^P = \frac{(1-\tau_P)(3-S_1-V_1+(4-\tau_P)V_2)(3-S_1-V_1-(2-\tau_P)V_2)}{2(3-\tau_P)^2}$

Maximising π_i^P w.r.t. V_i for $i \in 1,2$ for $\tau_P \in (0,\frac{1}{2})$, we get the best response functions of choice pf quality level of both the firms.

$V_1 = \frac{S_1+\tau_P^2-5\tau_P-V_2+3}{\tau_P^2-6\tau_P+8}$

$V_2 = \frac{3-S_1-V_1}{\tau_P^2-6\tau_P+8}$

Simultaneously solving the best response function, we get the values of V_1 and V_2,

$V_1^P = \frac{7+S_1(3-\tau_P)-\tau_P^3+8\tau_P^2-17\tau_P}{(3-\tau_P)(7-(6-\tau_P)\tau_P)}$

$V_2^P = \frac{7-S_1(3-\tau_P)-2\tau_P}{(3-\tau_P)(7-(6-\tau_P)\tau_P)}$

Hence,

$P_1^P = \frac{7+S_1(3-\tau_P)(1-\tau_P)-3\tau_P}{7-(6-\tau_P)\tau_P}$

$P_2^P = \frac{7-S_1(3-\tau_P)-2\tau_P}{7-(6-\tau_P)\tau_P}$

□

Proof. Proof of Lemma 1

(i) Plugging in the equilibrium values of V_1^R, V_2^R, P_1^R and P_2^R in $\hat{x}_R$ from Proposition 1, we get:

$$\hat{x}_R = \frac{7 + S_1(3-\tau_R) - 2(4-\tau_R)\tau_R}{2(7-(5-\tau_R)\tau_R)}$$

Differentiating w.r.t τ_R, we get:

$$\frac{\partial \hat{x}_R}{\partial \tau_R} = \frac{(-21 + S_1(4-\tau_R)(2-\tau_R) + 2(7-\tau_R)\tau_R)}{(2(7-(5-\tau_R)\tau_R)^2)}$$

which is negative for $\tau_R \in [0, \frac{1}{2})$ and $S_1 < \bar{S_1}^R$.

(ii) Using Proposition 1 we get V_1^R as:

$$V_1^R = \frac{7 + S_1(3-\tau_R) - \tau_R(\tau_R^2 - 7\tau_R + 15)}{(3-\tau_R)(7-(5-\tau_R)\tau_R)}$$

Differentiating w.r.t. τ_R, we get

$$\frac{\partial V_1^R}{\partial \tau_R} = \frac{S_1(5-2\tau_R) + \tau_R^2 - 7}{(7-(5-\tau_R)\tau_R)^2} - \frac{2}{(3-\tau_R)^2}$$

which is negative for $\tau_R \in [0, \frac{1}{2})$ and $S_1 < \bar{S_1}^R$.

(iii) Using proposition 1 we get V_2^R as:

$$V_2^R = \frac{(7 - S_1(3-\tau_R) - 2\tau_R)(1-\tau_R)}{(3-\tau_R)(7-(5-\tau_R)\tau_R)}$$

Differentiating w.r.t. τ_R, we get:

$$\frac{\partial V_2^R}{\partial \tau_R} = \frac{4 - S_1}{(\tau_R - 5)\tau_R + 7} + \frac{-3S_1(\tau_R - 3) + 6\tau_R - 21}{((\tau_R - 5)\tau_R + 7)^2} - \frac{2}{(\tau_R - 3)^2}$$

which is < 0 for $S_1 \leq S_1^{V_2^R}$ and > 0 for $S_1 > S_1^{V_2^R}$, where

$$S_1^{V_2^R} = \frac{\tau_R(\tau_R(2(\tau_R - 9)\tau_R + 49) - 28) - 35}{(\tau_R - 3)^2((\tau_R - 2)\tau_R - 2)}$$

.

(iv) Using parts (i), (ii) and (iii) above, Average Quality = $(S_1 + V_1^R)\hat{x}_R + V_2^R(1 - \hat{x}_R)$

$$AQ^R = \frac{7 - S_1^2 + 2S_1\tau_R - 5\tau_R}{2(7 - (5 - \tau_R)\tau_R)^2} + \frac{S_1(-S_1(\tau_R - 4) + \tau_R - 7) - 1}{2(7 - (5 - \tau_R)\tau_R)} + S_1 - \frac{2}{3 - \tau_R} + 1$$

Differentiating AQ^R w.r.t τ_R, we get

$$\frac{\partial AQ^R}{\partial \tau_R} = \frac{(S_1(5 - 2\tau_R) + \tau_R^2 - 7)\,(S_1((\tau_R - 5)\tau_R + 5) + 2\tau_R)}{2((\tau_R - 5)\tau_R + 7)^3} - \frac{2}{(\tau_R - 3)^2}$$

$\frac{\partial AQ^R}{\partial \tau_R} < 0$ for $\tau_R \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_1{}^R$

(v) Quality Differential is given by:

$$\begin{aligned} QD^R &= S_1 + V_1^R - V_2^R \\ &= S_1 + \frac{7 + S_1(3 - \tau_R) - \tau_R(\tau_R^2 - 7\tau_R + 15)}{(3 - \tau_R)(7 - (5 - \tau_R)\tau_R)} - \left(\frac{(7 - S_1(3 - \tau_R) - 2\tau_R)(1 - \tau_R)}{(3 - \tau_R)(7 - (5 - \tau_R)\tau_R)}\right) \\ &= \frac{S_1(\tau_R - 3)^2 + (\tau_R - 2)\tau_R}{(\tau_R - 5)\tau_R + 7} \end{aligned}$$

Differentiating QD^R w.r.t τ_R, we get

$$\frac{\partial QD^R}{\partial \tau_R} = S_1(\tau_R - 3)(\tau_R - 1) + \tau_R(14 - 3\tau_R) - 14(7 - (5 - \tau_R)\tau_R)^2$$

which is $\frac{\partial QD^R}{\partial \tau_R} < 0$ for $\tau_R \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_1{}^R$

(vi) From Proposition 1 above

$$P_1^R = \frac{7 + S_1(3-\tau_R)(1-\tau_R) - \tau_R}{7-(5-\tau_R)\tau_R}$$

$$P_2^R = \frac{7 - S_1(3-\tau_R) - 2\tau_R}{7-(5-\tau_R)\tau_R}$$

Taking differential w.r.t τ, we get

$$\frac{\partial P_1^R}{\partial \tau_R} = \frac{-S_1((\tau_R-8)\tau_R+13)+(\tau_R-14)\tau_R+28}{((\tau_R-5)\tau_R+7)^2}$$

which is > 0 for $S_1 < S_1^{P_1^R}$ and < 0 for $S_1 > S_1^{P_1^R}$ where

$$S_1^{P_1^R} = \frac{(\tau_R-14)\tau_R+28}{(\tau_R-8)\tau_R+13}$$

$$\frac{\partial P_2^R}{\partial \tau_R} = \frac{-S_1(\tau_R-4)(\tau_R-2)+2(\tau_R-7)\tau_R+21}{((\tau_R-5)\tau_R+7)^2}$$

$\frac{\partial P_2^R}{\partial \tau_R} > 0$ for $\tau_R \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_1^{\ R}$

(vii) From part (i) and (vi) above, we get

$$AP^R = P_1^R(\hat{x}_R) + P_2^R(1-\hat{x}_R)$$

$$= \frac{-S_1^2(\tau_R-2)(\tau_R-3)^2+2S_1(\tau_R-2)\tau_R(\tau_R-3)^2+\tau_R(-2(\tau_R-13)\tau_R-91)+98}{2((\tau_R-5)\tau_R+7)^2}$$

$$\frac{\partial AP^R}{\partial \tau_R} = \frac{S_1^2(\tau_R-3)(\tau_R-1)((\tau_R-7)\tau_R+11)-4S_1(\tau_R-3)(\tau_R((\tau_R-10)\tau_R+27)-21)}{2((\tau_R-5)\tau_R+7)^3} + \frac{\tau_R(\tau_R(2(\tau_R-21)\tau_R+231)-483)+343}{2((\tau_R-5)\tau_R+7)^3}$$

which is > 0 for $S_1 < S_1^{AP^R}$ and < 0 for $S_1 > S_1^{AP^R}$, where

$$S_1^{AP^R} = \frac{34-14\tau_R}{5(\tau_R-7)\tau_R+55} - \sqrt{\frac{((\tau_R-5)\tau_R+7)^2(\tau_R(2(\tau_R-7)\tau_R+35)-31)}{(\tau_R-3)(\tau_R-1)^2((\tau_R-7)\tau_R+11)^2}} + \frac{6}{5-5\tau_R} + 2$$

.

$\square$

Proof. Proof of Lemma 2

(i) Using proposition 1 and plugging in equilibrium values of V_i and P_i in the profit equation, we get

$$\pi_1^R = \frac{(\tau_R-4)\left(S_1^2(\tau_R-2)(\tau_R-3)^2+2S_1(\tau_R-2)(\tau_R((\tau_R-7)\tau_R+15)-7)(\tau_R-3)+\tau_R(28-(\tau_R-4)\tau_R((\tau_R-8)\tau_R+17))-98\right)}{2(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2}$$

Taking first order derivative w.r.t. τ_R, we find that that $\frac{\partial \pi_1^R}{\partial \tau_R}$ is ≥ 0 for $S_1 < {S_1}^{\Pi_1^R}$ and ≤ 0 for $S_1 > {S_1}^{\Pi_1^R}$

where

$$\begin{aligned} {S_1}^{\Pi_1^R} &= \frac{((\tau_R-5)\tau_R+7)\sqrt{(\tau_R-2)(\tau_R(\tau_R(\tau_R(\tau_R(5\tau_R-69)+379)-1038)+1421)-779)}}{(\tau_R-3)^2(\tau_R((\tau_R-9)\tau_R+24)-19)} \\ &+ \frac{4\tau_R^5-57\tau_R^4+323\tau_R^3-904\tau_R^2+1242\tau_R-665}{(\tau_R-3)^2(\tau_R((\tau_R-9)\tau_R+24)-19)} \end{aligned}$$

(ii) Using proposition 1 and plugging in equilibrium values of V_i and P_i in the profit equation, we get

$$\pi_1^R = \frac{(\tau_R-4)\left(S_1^2(\tau_R-2)(\tau_R-3)^2+2S_1(\tau_R-2)(\tau_R((\tau_R-7)\tau_R+15)-7)(\tau_R-3)+\tau_R(28-(\tau_R-4)\tau_R((\tau_R-8)\tau_R+17))-98\right)}{2(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2}$$

$$\pi_2^R = -\frac{(\tau_R-1)((\tau_R-5)\tau_R+8)(S_1(\tau_R-3)-2\tau_R+7)^2}{2(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2}$$

Now, calculating total profit in case of revenue share and profit share,

$$\begin{aligned} \Pi^R &= \pi_1^R + \pi_2^R \\ &= \frac{2S_1\tau_R((\tau_R-6)\tau_R+11)((\tau_R-5)\tau_R+5)(\tau_R-3)-S_1^2(\tau_R((\tau_R-7)\tau_R+19)-16)(\tau_R-3)^2}{2(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2} \\ &- \frac{\tau_R(\tau_R(\tau_R(\tau_R((\tau_R-12)\tau_R+45)+5)-446)+1071)-784}{2(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial \pi^R}{\partial \tau_R} &= \frac{2(3S_1-7)}{(\tau_R-3)^2} + \frac{(S_1-16)S_1+32}{2((\tau_R-5)\tau_R+7)} + \frac{(S_1-4)S_1\tau_R+2(S_1-7)S_1-6\tau_R+45}{2((\tau_R-5)\tau_R+7)^2} \\ &+ \frac{6(S_1(S_1(\tau_R-3)-4\tau_R+14)+3\tau_R-14)}{((\tau_R-5)\tau_R+7)^3} + \frac{8}{(\tau_R-3)^3} \end{aligned}$$

which is > 0 for $S_1 < {S_1}^{\Pi_R}$ and < 0 for $S_1 > {S_1}^{\Pi_R}$ where

$$\begin{aligned} {S_1}^{\Pi_R} &= \frac{2\tau_R^6-36\tau_R^5+273\tau_R^4-1103\tau_R^3+2448\tau_R^2-2745\tau_R+1155}{(\tau_R-3)^2(\tau_R(\tau_R((\tau_R-9)\tau_R+36)-61)+27)} \\ &- \frac{((\tau_R-5)\tau_R+7)\sqrt{\tau_R(\tau_R(\tau_R(\tau_R(\tau_R((55-2\tau_R)\tau_R-582)+3198)-10050)+18283)-179}}{(\tau_R-3)^2(\tau_R(\tau_R((\tau_R-9)\tau_R+36)-61)+27)} \end{aligned}$$

□

Proof. This is Proof of Lemma 3

(i) $CS^R = \int_0^{\hat{x}_R} S_1 + V_1 - P_1 - x.dx + \int_{\hat{x}_R}^1 V_2 - P_2 - (1-x).dx$

$CS^R = \frac{S_1^2(\tau-3)^3 - 2S_1(\tau(2(\tau-9)\tau+51)-49)(\tau-3)+\tau(2\tau(\tau((\tau-9)\tau+17)+61)-525)+539}{4(\tau-3)((\tau-5)\tau+7)^2}$

$\frac{\partial CS^R}{\partial \tau_R} = \frac{1}{2}\left(\frac{{S_1}^2(\tau-4)-2S_1(\tau-7)-2\tau-7}{((\tau-5)\tau+7)^3} + \frac{2S_1}{(\tau-5)\tau+7} + \frac{S_1(-(S_1+6)\tau+4S_1+3)+24(\tau-2)}{((\tau-5)\tau+7)^2} - \frac{4}{(\tau-3)^2}\right)$

which is < 0 for $\tau_R \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S_1}^R$

(ii)

$$W^R = \Pi^R + CS^R$$

From Lemma 2 and part (i) above, we get

$$W^R = \frac{2S_1(\tau_R(\tau_R(2\tau_R((\tau_R-12)\tau_R+58)-275)+312)-147)(\tau_R-3)}{4(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2} + \frac{\tau_R\left(-2\tau_R^3+10\tau_R^2+\tau_R-28\right)-49-S_1^2(\tau_R(\tau_R(2\tau_R-15)+44)-41)(\tau_R-3)^2}{4(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2}$$

$$\frac{\partial W^R}{\partial \tau_R} = \frac{S_1^2(\tau_R-1)(\tau_R((\tau_R-9)\tau_R+36)-51)(\tau_R-3)^3}{2(\tau_R-3)^3((\tau_R-5)\tau_R+7)^3} - \frac{S_1(\tau_R(\tau_R(\tau_R(\tau_R(2(\tau_R-17)\tau_R+261)-1107)+2570)-2901)+1113)(\tau_R-3)}{2(\tau_R-3)^3((\tau_R-5)\tau_R+7)^3} + \frac{\tau_R(\tau_R(\tau_R(\tau_R(\tau_R(2\tau_R-15)-6)+272)-504)-497)+1372}{2(\tau_R-3)^3((\tau_R-5)\tau_R+7)^3}$$

which is $\frac{\partial W^R}{\partial \tau_R} < 0$ for $\tau_R \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S_1}^R$

□

7.3 Regime 3

Proof. Proof of Proposition 2

Solving for SPNE using backward induction.

Stage 2**:**

$max\ \pi_1^P = \hat{x}_P P_1 - \frac{V_1^2}{2} + \tau_P((1-\hat{x}_P)P_2 - \frac{V_2^2}{2})$ w.r.t P_1

$max\ \pi_2^P = (1-\tau_P)((1-\hat{x}_P)P_2 - \frac{V_2^2}{2})$ w.r.t P_2

We get the following first order conditions:

$$\frac{\partial \pi_1^P}{\partial P_1} = \frac{1}{2}(1-(P_1-P_2)+(S_1+V_1-V_2)) - \frac{P_1}{2} + \frac{P_2\tau_P}{2} = 0$$
$$\Rightarrow P_1 = \frac{1}{2}(P_2\tau_P + P_2 + S_1 + V_1 - V_2 + 1),$$ best response of firm 1.

$$\frac{\partial \pi_2^P}{\partial P_2} = (1-\tau_P)\left(\frac{1}{2}((P_1-P_2)-(S_1+V_1-V_2)-1) - \frac{P_2}{2} + 1\right) = 0$$
$$\Rightarrow P_2 = \frac{1}{2}(P_1 - (S_1+V_1-V_2)+1),$$ best response of firm 2.

Simultaneously solving the best response function, we get the values of P_1 and P_2:

$$P_1 = \frac{3+S_1+V_1-V_2-\tau_P(S_1+V_1-V_2-1)}{3-\tau_P}$$

$$P_2 = \frac{3-(S_1+V_1)+V_2}{3-\tau_P}$$

Plugging in the values of P_1 and P_2 in the profit functions of the firms, we get the profit functions of two firms as below:

$$\pi_1^P = \frac{3-(S_1+V_1-V_2)}{3-\tau_P} + \frac{(3-(S_1+V_1-V_2))^2}{2(3-\tau_P)^2} + S_1 - \frac{V_1^2}{2} + V_1 - \frac{\tau_P V_2^2}{2} - V_2 - 1$$

$$\pi_2^P = \frac{(1-\tau_P)(3-S_1-V_1+(4-\tau_P)V_2)(3-S_1-V_1-(2-\tau_P)V_2)}{2(3-\tau_P)^2}$$

Maximising π_i^P w.r.t. V_i for $i \in 1,2$ for $\tau_P \in (0,\frac{1}{2})$, we get the best response functions of choice pf quality level of both the firms.

$$V_1 = \frac{S_1+\tau_P^2-5\tau_P-V_2+3}{\tau_P^2-6\tau_P+8}$$

$$V_2 = \frac{3-S_1-V_1}{\tau_P^2-6\tau_P+8}$$

Simultaneously solving the best response function, we get the values of V_1 and V_2,

$$V_1^P = \frac{7+S_1(3-\tau_P)-\tau_P^3+8\tau_P^2-17\tau_P}{(3-\tau_P)(7-(6-\tau_P)\tau_P)}$$

$$V_2^P = \frac{7-S_1(3-\tau_P)-2\tau_P}{(3-\tau_P)(7-(6-\tau_P)\tau_P)}$$

Hence,

$$P_1^P = \frac{7+S_1(3-\tau_P)(1-\tau_P)-3\tau_P}{7-(6-\tau_P)\tau_P}$$

$P_2^P = \frac{7-S_1(3-\tau_P)-2\tau_P}{7-(6-\tau_P)\tau_P}$

□

Proof. This is proof of Lemma 4

(i) Plugging in the equilibrium values of V_1^P, V_2^P, P_1^P and P_2^P in $\hat{x}_P$ from Proposition 2, we get

$$\hat{x}_P = \frac{7 + S_1(3 - \tau_P) - 2(5 - \tau_P)\tau_P}{2(7 - (6 - \tau_P)\tau_P)}$$

Differentiating w.r.t τ_P, we get

$$\frac{\partial \hat{x}_P}{\partial \tau_P} = \frac{-28 + 11S_1 - 2(-7 + \tau_P)\tau_P + S1(-6 + \tau_P)\tau_P}{2(7 + (-6 + \tau_P)\tau_P)^2}$$

When $S_1 < \bar{S_1}^P$, $\frac{\partial \hat{x}_P}{\partial \tau_P} < 0$.

(ii) Using Proposition 2, we get V_1^P as:

$$V_1^P = \frac{7 + S_1(3 - \tau_P) - \tau_P^3 + 8\tau_P^2 - 17\tau_P}{(3 - \tau_P)(7 - (6 - \tau_P)\tau_P)}$$

Differentiating w.r.t. τ, we get

$$\frac{\partial V_1^P}{\partial \tau_P} = \frac{1}{2}\left(\frac{1}{7 - (6 - \tau_P)\tau_P} - \frac{2(3 - \tau_P)(7 - 2S_1 - \tau_P)}{(7 - (6 - \tau_P)\tau_P)^2} - \frac{1}{(3 - \tau_P)^2}\right)$$

which is negative for $\tau_P \in [0, \frac{1}{2})$ and $S_1 < \bar{S_1}^P$.

(iii) Using proposition 2, we get V_2^P as:

$$V_2^P = \frac{7 - S_1(3 - \tau_P) - 2\tau_P}{(3 - \tau_P)(7 - (6 - \tau_P)\tau_P)}$$

Differentiating w.r.t. τ_P, we get:

$$\frac{\partial V_2^P}{\partial \tau_P} = \frac{133 + 2S_1(-3 + \tau_P)^3 + \tau_P(-126 + (39 - 4\tau_P)\tau_P)}{(-3 + \tau_P)^2(7 + (-6 + \tau_P)\tau_P)^2}$$

which is positive for $\tau_P \in [0, \frac{1}{2})$ and $S_1 < \bar{S_1}^P$

(iv) Average Quality $= (S_1 + V_1)\hat{x}_P + V_2(1 - \hat{x}_P)$

$AQ^P = \frac{-S_1^2(\tau_P-3)^4+S_1(\tau_P(\tau_P(\tau_P(2\tau_P-23)+91)-135)+49)(\tau_P-3)+\tau_P(\tau_P(\tau_P(2(\tau_P-13)\tau_P+121)-244)+217)-98}{2(\tau_P-3)((\tau_P-6)\tau_P+7)^2}$

Differentiating AQ^P w.r.t τ_P, we get

$\frac{\partial AQ^P}{\partial \tau_P} = \frac{1}{2}\left(\frac{6S_1^2-2S_1(\tau_P+7)+6\tau_P-3}{((\tau_P-6)\tau_P+7)^2} + \frac{S_1-3}{(\tau_P-6)\tau_P+7} + \frac{8(2S_1(S_1+\tau_P-7)-4\tau_P+21)}{((\tau_P-6)\tau_P+7)^3} - \frac{1}{(\tau_P-3)^2}\right)$

$\frac{\partial AQ^P}{\partial \tau_P} \geq 0$ for $S_1 \leq S_1^{AQ^P}$ and ≤ 0 for $S_1 \in (S_1^{AQ^P}, \bar{S}_1{}^P)$

where $S_1^{AQ^P} = \frac{-\tau_P((\tau_P-8)(\tau_P-6)\tau_P+2)+\frac{((\tau_P-6)\tau_P+7)\sqrt{\tau_P(\tau_P(\tau_P(\tau_P((\tau_P-22)\tau_P+251)-1596)+5483)-9462)+6385}}{\tau_P-3}+161}{4(3(\tau_P-6)\tau_P+29)}$

(v) Quality Differential is given by:

$$\begin{aligned} QD^P &= S_1 + V_1^P - V_2^P \\ &= S_1 + \frac{7 + S_1(3-\tau_P) - \tau_P^3 + 8\tau_P^2 - 17\tau_P}{(3-\tau_P)\,(7-(6-\tau_P)\tau_P)} - \left(\frac{7 - S_1(3-\tau_P) - 2\tau_P}{(3-\tau_P)(7-(6-\tau_P)\tau_P)}\right) \\ &= \frac{2S_1 + \tau_P - 7}{(\tau_P - 6)\tau_P + 7} + S_1 + 1 \end{aligned}$$

Differentiating QD^P w.r.t τ_P, we get

$$\frac{\partial QD^P}{\partial \tau_P} = -\frac{4S_1(\tau_P - 3) + (\tau_P - 14)\tau_P + 35}{((\tau_P - 6)\tau_P + 7)^2}$$

which is $\frac{\partial QD^P}{\partial \tau_P} < 0$ for $\tau_P \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_1{}^P$

(vi) From Proposition 2 above

$P_1^P = \frac{(7+S_1(3-\tau_P)(1-\tau_P)-3\tau_P)}{(7-(6-\tau_P)\tau_P)}$

$P_2^P = \frac{(7-S_1(3-\tau_P)-2\tau_P)}{(7-(6-\tau_P)\tau_P)}$

Taking differential w.r.t τ, we get

$$\frac{\partial P_1^P}{\partial \tau_R} = \frac{-2S_1((\tau-4)\tau+5)+\tau(3\tau-14)+21}{((\tau-6)\tau+7)^2}$$

which is $>$ 0 for $S_1 < S_1^{P_1^P}$ and $<$ 0 for $S_1 > S_1^{P_1^P}$ where $S_1^{P_1^P} = \frac{3-\tau}{(\tau-4)\tau+5} + \frac{3}{2}$

$$\frac{\partial P_2^P}{\partial \tau_R} = \frac{-S_1((\tau-6)\tau+11)+2(\tau-7)\tau+28}{((\tau-6)\tau+7)^2}$$

$\frac{\partial P_2^R}{\partial \tau_R} > 0$ for $\tau_R \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S_1}^R$

(vii) From part (i) and (vi) above, we get

$$AP^P = P_1^P(\hat{x}_P) + P_2^P(1-\hat{x}_P)$$

$$= \frac{-S_1^2(\tau-2)(\tau-3)^2 + 2S_1(\tau-4)(\tau-2)\tau(\tau-3) + \tau(-6(\tau-8)\tau-119)+98}{2((\tau-6)\tau+7)^2}$$

$$\frac{\partial AP^P}{\partial \tau_P} = \frac{S_1^2(\tau-3)(\tau((\tau-7)\tau+21)-23)+S_1(2\tau(220-3\tau((\tau-10)\tau+39))-336)+\tau(3\tau(2(\tau-10)\tau+77)-434)+343}{2((\tau-6)\tau+7)^3}$$

Which is > 0 for $S_1 < S_1^{AP^P}$ and < 0 for $S_1 > S_1^{AP^P}$, where

$$S_1^{AP^P} = \frac{3\tau^4-30\tau^3+117\tau^2-220\tau-((\tau-6)\tau+7)\sqrt{\tau(3\tau((\tau-8)\tau+27)-136)+93}+168}{(\tau-3)(\tau((\tau-7)\tau+21)-23)}$$

$\square$

Proof. This is proof of Lemma 5

(i)

$$\pi_1^P = \frac{S_1^2((\tau_P-7)\tau_P+8)(\tau_P-3)^2 - \tau_P(\tau_P(\tau_P(\tau_P((\tau_P-14)\tau_P+70)-126)-72)+483)}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2}$$
$$+ \frac{392 + 2S_1(\tau_P(\tau_P(\tau_P((\tau_P-14)\tau_P+73)-171)+171)-56)(\tau_P-3)}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2}$$

Taking first order derivative w.r.t. τ_P, we find that that $\frac{\partial \pi_1^P}{\partial \tau_P}$ is ≥ 0 for $S_1 < {S_1}^{\pi_1^P}$ and ≤ 0 for $S_1 > {S_1}^{\pi_1^P}$ where

$${S_1}^{\pi_1^P} = -\sqrt{\frac{((\tau_P-6)\tau_P+7)^2(\tau_P(\tau_P(\tau_P(\tau_P(\tau_P(\tau_P((\tau_P-20)\tau_P+172)-834)+2514)-4874)+5987)-4270)+1348)}{(\tau_P-3)^4(\tau_P(\tau_P(2\tau_P-21)+60)-47)^2}} - \frac{51\tau_P^2-522\tau_P+597}{188-4\tau_P(\tau_P(2\tau_P-21)+60)} - \frac{\tau_P}{2} - \frac{12}{\tau_P-3} + \frac{2}{(\tau_P-3)^2} + \frac{7}{4}$$

(ii) Using proposition 2 and plugging in equilibrium values of V_i and P_i in the profit equation, we get

$$\pi_1^P = \frac{S_1^2((\tau_P-7)\tau_P+8)(\tau_P-3)^2 - \tau_P(\tau_P(\tau_P(\tau_P((\tau_P-14)\tau_P+70)-126)-72)+483)}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2}$$
$$+ \frac{392 + 2S_1(\tau_P(\tau_P(\tau_P((\tau_P-14)\tau_P+73)-171)+171)-56)(\tau_P-3)}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2}$$

$\pi_2^P = -\frac{(\tau_P-4)(\tau_P-2)(\tau_P-1)(S_1(\tau_P-3)-2\tau_P+7)^2}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2}$

Now, calculating total profit in case of revenue share and profit share,

$\Pi^P = \pi_1^P + \pi_2^P$

$$\Pi^P = \frac{2S_1\tau_P(\tau_P((\tau_P-9)\tau_P+25)-19)(\tau_P-3)^2 - S_1^2(\tau_P((\tau_P-8)\tau_P+21)-16)(\tau_P-3)^2}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2} - \frac{\tau_P(\tau_P(\tau_P(\tau_P((\tau_P-10)\tau_P+14)+175)-839)+1393)-784}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2}$$

$$\frac{\partial\pi^P}{\partial\tau_P} = \frac{(S_1-6)S_1+7}{2(\tau_P-6)\tau_P+7} + \frac{2(S_1-4)S_1\tau_P+4(S_1-3)S_1+5\tau_P+16}{2((\tau_P-6)\tau_P+7)^2} + \frac{8(2S_1(S_1(\tau_P-2)-3\tau_P+7)+5\tau_P-14)}{2((\tau_P-6)\tau_P+7)^3} + \frac{1}{2(\tau_P-3)^2} + \frac{1}{2(\tau_P-3)^3}$$

which is $\frac{\partial\pi^P}{\partial\tau_P} > 0$ for $S_1 < {S_1}^{\Pi_P}$ and < 0 for $S_1 > {S_1}^{\Pi_P}$ where

${S_1}^{\Pi_P} = -\frac{4(\tau_P+3)}{3(\tau_P-6)\tau_P+45} - \sqrt{\frac{((\tau_P-6)\tau_P+7)^2(\tau_P(\tau_P(\tau_P((\tau_P-15)\tau_P+89)-257)+366)-204)}{(\tau_P-3)^4(\tau_P-1)((\tau_P-6)\tau_P+15)^2}} - \frac{2}{3(\tau_P-3)} + 3$

□

Proof. This is proof of Lemma 6

(i) $CS^P = \int_0^{\hat{x}_P}(S_1+V_1-P_1-x).dx + \int_{\hat{x}_P}^1(V_2-P_2-(1-x)).dx$

$CS^P = \frac{S_1^2(\tau_P-3)^3-2S_1(\tau_P(4(\tau_P-8)\tau_P+75)-49)(\tau_P-3)+\tau_P(2\tau_P((\tau_P-8)\tau_P(\tau_P+1)+160)-777)+539}{4(\tau_P-3)((\tau_P-6)\tau_P+7)^2}$

$\frac{\partial CS^P}{\partial\tau_P} = \frac{1}{2}\left(\frac{4S_1-7}{(\tau_P-6)\tau_P+7} + \frac{-4S_1(S_1(\tau_P-3)-4\tau_P+14)-18\tau_P+70}{((\tau_P-6)\tau_P+7)^3} - \frac{S_1(S_1(\tau_P-3)-8\tau_P+11)+4\tau_P+10}{((\tau_P-6)\tau_P+7)^2} - \frac{1}{(\tau_P-3)^2}\right)$

which is < 0 for $\tau_P \in [0, \frac{1}{2})$ and $S_1 \le \bar{S_1}^P$

(ii) $W^P = \Pi^P + CS^P$ From lemma 6 and part (i), we get

$W^P = \frac{1}{4}\left(\frac{S_1^2(5-2\tau_P)+4S_1(\tau_P-3)-1}{(\tau_P-6)\tau_P+7} + \frac{2S_1(S_1(3-2\tau_P)+5\tau_P-7)-6\tau_P+7}{((\tau_P-6)\tau_P+7)^2} + 4S_1 - \frac{1}{(\tau_P-3)^2}\right)$

$\frac{\partial W^P}{\partial\tau_P} = \frac{1}{2}\left(\frac{(S_1-2)S_1}{(\tau_P-6)\tau_P+7} + \frac{S_1(S_1(\tau_P+7)-23)+\tau_P+6}{((\tau_P-6)\tau_P+7)^2} + \frac{4S_1(S_1(3\tau_P-5)-8\tau_P+14)+22\tau_P-42}{((\tau_P-6)\tau_P+7)^3} + \frac{1}{(\tau_P-3)^3}\right)$

which is < 0 for $\tau_P \in [0, \frac{1}{2})$ and $S_1 \le \bar{S_1}^P$

□

7.4 Comparison between Regime 2 and Regime 3

Proof. This is proof of Lemma 7

(i) Using Lemma 1 and 4, the difference $\hat{x}_P$ and $\hat{x_R}$ is given by:

$$\hat{x}_R - \hat{x}_P = [\frac{7 + S_1(3-\tau_R) - 2(4-\tau_R)\tau_R}{2(7-(5-\tau_R)\tau_R)}] - [\frac{7 + S_1(3-\tau_P) - 2(5-\tau_P)\tau_P}{2(7-(6-\tau_P)\tau_P)}]$$

We put $\tau_R = \tau_P = \tau$, we get:

$$\hat{x}_R - \hat{x}_P = \frac{\tau(7 - S_1(3-\tau) - 2\tau)}{2(7-(6-\tau)\tau)(7-(5-\tau)\tau)}$$

For $S_1 < \bar{S}_{1\tau}$, $\hat{x}_P \leq \hat{x}_R$.

(ii) From Proposition 1 and 2, we get the quality level by firm 1 as below:

$$V_1^R = \frac{7 + S_1(3-\tau_R) - \tau_R(\tau_R^2 - 7\tau_R + 15)}{(3-\tau_R)(7-(5-\tau_R)\tau_R)}$$

$$V_1^P = \frac{7 + S_1(3-\tau_P) - \tau_P^3 + 8\tau_P^2 - 17\tau_P}{(3-\tau_P)\,(7-(6-\tau_P)\tau_P)}$$

For, $\tau_R = \tau_P = \tau$,

$$V_1^R - V_1^P = \frac{\tau(7 - S_1(3-\tau) - 2\tau)}{(3-\tau)(7-(6-\tau)\tau)(7-(5-\tau)\tau)}$$

which is > 0 for $\tau \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_{1\tau}$

It also follows that

$$0 > \frac{\partial V_1^R}{\partial \tau} \geq \frac{\partial V_1^P}{\partial \tau}$$

for $\tau \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_{1\tau}$

V_1^R decreases as τ_R increases for $\tau_R \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S_1}^R$. V_1^P increases as τ_P increases for $\tau_P \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S_1}^P$. Further $V_1^R = V_1^P$ if $\tau = 0$ as both the regimes are same as Regime 1 in that case. Hence, $V_1^R < V_1^P$ for $\tau \in [0.\frac{1}{2})$ and $S_1 \leq \bar{S}_{1\tau}$

(iii) From Proposition 1 and 2, we know that

$$V_2^R = \frac{(7 - S_1(3-\tau_R) - 2\tau_R)(1-\tau_R)}{(3-\tau_R)(7-(5-\tau_R)\tau_R)}$$

$$V_2^P = \frac{7 - S_1(3-\tau_P) - 2\tau_P}{(3-\tau_P)(7-(6-\tau_P)\tau_P)}$$

For $\tau_R = \tau_P = \tau$,

$$V_2^R - V_2^P = \frac{(\tau-4)(\tau-2)\tau(S_1(\tau-3) - 2\tau + 7)}{(\tau-3)((\tau-6)\tau+7)((\tau-5)\tau+7)}$$

which is < 0 for $\tau \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_{1\tau}$

It also follows that

$$\frac{\partial V_2^R}{\partial \tau} \leq 0 \leq \frac{\partial V_2^P}{\partial \tau}$$

for $\tau \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_{1\tau}$ and $S_1 \leq S_1^{V_2}$

$$0 < \frac{\partial V_2^R}{\partial \tau} \leq \frac{\partial V_2^P}{\partial \tau}$$

for $\tau \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_{1\tau}$ and $S_1 > S_1^{V_2}$

(iv) From lemma 1 and 4, we get

$$AQ^R = \frac{7 - S_1^2 + 2S_1\tau_R - 5\tau_R}{2(7-(5-\tau_R)\tau_R)^2} + \frac{S_1(-S_1(\tau_R-4) + \tau_R - 7) - 1}{2(7-(5-\tau_R)\tau_R)} + S_1 - \frac{2}{3-\tau_R} + 1$$

$$AQ^P = \frac{-S_1^2(\tau_P-3)^4 + S_1(\tau_P(\tau_P(\tau_P(2\tau_P-23)+91)-135)+49)(\tau_P-3)}{2(\tau_P-3)((\tau_P-6)\tau_P+7)^2} + \frac{\tau_P(\tau_P(\tau_P(2(\tau_P-13)\tau_P+121)-244)+217)-98}{2(\tau_P-3)((\tau_P-6)\tau_P+7)^2}$$

For $\tau_R = \tau_P = \tau$,

$$AQ^R - AQ^P = \frac{\tau(S_1(\tau-3) - 2\tau + 7)(S_1(\tau-2)(2\tau-7)(\tau-3)^3}{2(\tau-3)((\tau-6)\tau+7)^2((\tau-5)\tau+7)^2} - \frac{\tau(\tau(\tau((\tau-16)\tau+104)-337)+539)+343)}{2(\tau-3)((\tau-6)\tau+7)^2((\tau-5)\tau+7)^2}$$

which < 0 for $S_1 < {S_1}^{AQ}$ and > 0 for $S_1 \in ({S_1}^{AQ}, \bar{S}_{1\tau})$ where

$$ {S_1}^{AQ} = \frac{\tau(\tau(\tau((\tau-16)\tau+104)-337)+539)-343}{(\tau-3)^3(\tau-2)(2\tau-7)} $$

.

(v) From lemma 1 and 4, we get

$$ \begin{aligned} QD^R &= S_1 + V_1^R - V_2^R \\ &= S_1 + \frac{7+S_1(3-\tau_R)-\tau_R(\tau_R^2-7\tau_R+15)}{(3-\tau_R)(7-(5-\tau_R)\tau_R)} - (\frac{(7-S_1(3-\tau_R)-2\tau_R)(1-\tau_R)}{(3-\tau_R)(7-(5-\tau_R)\tau_R)}) \\ &= \frac{S_1(\tau_R-3)^2+(\tau_R-2)\tau_R}{(\tau_R-5)\tau_R+7} \end{aligned} $$

$$ \begin{aligned} QD^P &= S_1 + V_1^P - V_2^P \\ &= S_1 + \frac{7+S_1(3-\tau_P)-\tau_P^3+8\tau_P^2-17\tau_P}{(3-\tau_P)\,(7-(6-\tau_P)\tau_P)} - (\frac{7-S_1(3-\tau_P)-2\tau_P}{(3-\tau_P)(7-(6-\tau_P)\tau_P)}) \\ &= \frac{2S_1+\tau_P-7}{(\tau_P-6)\tau_P+7} + S_1 + 1 \end{aligned} $$

For $\tau_R = \tau_P = \tau$,

$$ \begin{aligned} QD^R - QD^P &= \frac{S_1(3-\tau)^2-(2-\tau)\tau}{7-(5-\tau)\tau} - (1+S_1-\frac{7-2S_1-\tau}{7-(6-\tau)\tau}) \\ &= \frac{(3-\tau)\tau(7-S_1(3-\tau)-2\tau)}{(7-(6-\tau)\tau)(7-(5-\tau)\tau)} \end{aligned} $$

which is > 0 for $\tau \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_{1\tau}$

From lemma 1 and 4, we get

$\frac{\partial QD^R}{\partial\tau} - \frac{\partial QD^P}{\partial\tau} = \frac{S_1-3}{(\tau-5)\tau+7} + \frac{4(S_1(\tau-3)-2\tau+7)}{((\tau-6)\tau+7)^2} + \frac{S_1(\tau-4)-\tau+7}{((\tau-5)\tau+7)^2} + \frac{1}{(\tau-6)\tau+7}$

which is > 0 for $\tau \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_{1\tau}$

(vi) From proposition 1 and 2, we get

$$ P_1^R = \frac{7+S_1(3-\tau_R)(1-\tau_R)-\tau_R}{7-(5-\tau_R)\tau_R} $$

$$P_1^P = \frac{(7 + S_1(3-\tau_P)(1-\tau_P) - 3\tau_P)}{(7-(6-\tau_P)\tau_P)}$$

For $\tau_R = \tau_P = \tau$,

$$P_1^R - P_1^P = -\frac{(\tau-1)\tau(S_1(\tau-3) - 2\tau + 7)}{((\tau-6)\tau+7)((\tau-5)\tau+7)}$$

which is > 0 for $\tau \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_{1\tau}$

From proposition 1 and 2, we get

$$P_2^R = \frac{7 - S_1(3-\tau_R) - 2\tau_R}{7-(5-\tau_R)\tau_R}$$

$$P_2^P = \frac{(7 - S_1(3-\tau_P) - 2\tau_P)}{(7-(6-\tau_P)\tau_P)}$$

For $\tau_R = \tau_P = \tau$,

$$P_2^R - P_2^P = -\frac{\tau(-S_1(\tau-3) + 2\tau - 7)}{((\tau-6)\tau+7)((\tau-5)\tau+7)}$$

which is < 0 for $\tau \in [0, \frac{1}{2})$ and $S_1 \leq \bar{S}_{1\tau}$

(vii) From Lemma 1 and 2

$$AP^R = \frac{-S_1^2(\tau-2)(\tau-3)^2 + 2S_1(\tau-2)\tau(\tau-3)^2 + \tau(-2(\tau-13)\tau - 91) + 98}{2((\tau-5)\tau+7)^2}$$

$$AP^P = \frac{-S_1^2(\tau-2)(\tau-3)^2 + 2S_1(\tau-4)(\tau-2)\tau(\tau-3) + \tau(-6(\tau-8)\tau - 119) + 98}{2((\tau-6)\tau+7)^2}$$

For $\tau_R = \tau_P = \tau$,

$AP^R - AP^P = \frac{(\tau-2)\tau(S_1(\tau-3)-2\tau+7)(S_1(\tau-3)(\tau-2)(2\tau-7)+\tau(49-2\tau((\tau-9)\tau+26)))}{2((\tau-6)\tau+7)^2((\tau-5)\tau+7)^2}$

which is < 0 for $S_1 < {S_1}^{AP}$ and > 0 for $S_1 > {S_1}^{AP}$,

where ${S_1}^{AP} = \frac{2\tau^4 - 18\tau^3 + 52\tau^2 - 49\tau}{2\tau^3 - 17\tau^2 + 47\tau - 42}$

□

Proof. This is proof of Lemma 8

(i) $$\pi_1^R = \frac{(\tau_R-4)\big(S_1^2(\tau_R-2)(\tau_R-3)^2+2S_1(\tau_R-2)(\tau_R((\tau_R-7)\tau_R+15)-7)(\tau_R-3)+\tau_R(28-(\tau_R-4)\tau_R((\tau_R-8)\tau_R+17))-98\big)}{2(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2}$$

$$\pi_1^P = \frac{S_1^2((\tau_P-7)\tau_P+8)(\tau_P-3)^2-\tau_P(\tau_P(\tau_P(\tau_P((\tau_P-14)\tau_P+70)-126)-72)+483)}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2} + \frac{392+2S_1(\tau_P(\tau_P(\tau_P((\tau_P-14)\tau_P+73)-171)+171)-56)(\tau_P-3)}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2}$$

It may be checked that for $S_1 \le \bar{S}_{1\tau} \Rightarrow \pi_1^R \ge \pi_1^P$.

(ii) Using Lemma 2 and 5

$$\Pi^R = \frac{2S_1\tau_R((\tau_R-6)\tau_R+11)((\tau_R-5)\tau_R+5)(\tau_R-3)-S_1^2(\tau_R((\tau_R-7)\tau_R+19)-16)(\tau_R-3)^2}{2(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2} - \frac{\tau_R(\tau_R(\tau_R(\tau_R((\tau_R-12)\tau_R+45)+5)-446)+1071)-784}{2(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2}$$

$$\Pi^P = \frac{2S_1\tau_P(\tau_P((\tau_P-9)\tau_P+25)-19)(\tau_P-3)^2-S_1^2(\tau_P((\tau_P-8)\tau_P+21)-16)(\tau_P-3)^2}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2} - \frac{\tau_P(\tau_P(\tau_P(\tau_P((\tau_P-10)\tau_P+14)+175)-839)+1393)-784}{2(\tau_P-3)^2((\tau_P-6)\tau_P+7)^2}$$

For $S_1 < \bar{S}_{1\tau}$, $\Pi^R > \Pi^P$.

□

Proof. This is proof of lemma 9

(i) From Lemma 3 and 6, we get

$$CS^P - CS^R = \frac{\tau(S_1(\tau-3)-2\tau+7)\big(S_1(\tau-2)(2\tau-7)(\tau-3)^2+\tau(\tau(-4\tau(\tau((\tau-15)\tau+91)-287)-1993)+1813)-686\big)}{4(\tau-3)((\tau-6)\tau+7)^2((\tau-5)\tau+7)^2}$$

which is ≥ 0 for $\tau \in [0, \frac{1}{2})$ and $S_1 \le \bar{S}_{1\tau}$

(ii) From Lemma 3 and 6, we get

$$W^R = \frac{2S_1(\tau_R(\tau_R(2\tau_R((\tau_R-12)\tau_R+58)-275)+312)-147)(\tau_R-3)}{4(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2} + \frac{\tau_R\left(-2\tau_R^3+10\tau_R^2+\tau_R-28\right)-49-S_1^2(\tau_R(\tau_R(2\tau_R-15)+44)-41)(\tau_R-3)^2}{4(\tau_R-3)^2((\tau_R-5)\tau_R+7)^2}$$

$W^P = \frac{1}{4}\left(\frac{S_1^2(5-2\tau_P)+4S_1(\tau_P-3)-1}{(\tau_P-6)\tau_P+7} + \frac{2S_1(S_1(3-2\tau_P)+5\tau_P-7)-6\tau_P+7}{((\tau_P-6)\tau_P+7)^2} + 4S_1 - \frac{1}{(\tau_P-3)^2}\right)$

We put $\tau_R = \tau_P = \tau$, we get:

$$W^R - W^P = \frac{\tau(S_1(\tau-3)-2\tau+7)(S_1(\tau-3)(\tau-2)(\tau(\tau(2(\tau-12)\tau+115)-250)+189)}{4(\tau-3)^2((\tau-6)\tau+7)^2((\tau-5)\tau+7)^2}$$
$$+ \frac{\tau(\tau(\tau(2(\tau-21)\tau+297)-934)+1323)-686)}{4(\tau-3)^2((\tau-6)\tau+7)^2((\tau-5)\tau+7)^2}$$

For
$S_1 < S_1^W$, $W^R < W^P$ and for
$S_1 > S_1^W$, $W^R > W^P$ where

$$S_1^W = \frac{\tau(\tau(\tau(-2(\tau-21)\tau-297)+934)-1323)+686}{(\tau-3)(\tau-2)(\tau(\tau(2(\tau-12)\tau+115)-250)+189)}$$

□

7.5 List of cutoffs on S_1

1. $\bar{S}_1 = \frac{7}{3}$
2. $\bar{S}_1{}^R = \frac{7-2\tau_R}{3-\tau_R}$
3. $\bar{S}_1{}^P = \frac{7-2\tau_P}{3-\tau_P}$
4. $\bar{S}_{1\tau} = \frac{7-2\tau}{3-\tau}$
5. $S_1^{V_2^R} = \frac{\tau_R(\tau_R(2(\tau_R-9)\tau_R+49)-28)-35}{(\tau_R-3)^2((\tau_R-2)\tau_R-2)}$
6. $S_1^{P_1^R} = \frac{(\tau_R-14)\tau_R+28}{(\tau_R-8)\tau_R+13}$
7. $S_1^{AP^R} = \frac{34-14\tau_R}{5(\tau_R-7)\tau_R+55} - \sqrt{\frac{((\tau_R-5)\tau_R+7)^2(\tau_R(2(\tau_R-7)\tau_R+35)-31)}{(\tau_R-3)(\tau_R-1)^2((\tau_R-7)\tau_R+11)^2}} + \frac{6}{5-5\tau_R} + 2$
8. $S_1{}^{\pi_1^R} = \frac{4\tau_R^5-57\tau_R^4+323\tau_R^3-904\tau_R^2+1242\tau_R+((\tau_R-5)\tau_R+7)\sqrt{(\tau_R-2)(\tau_R(\tau_R(\tau_R(\tau_R(5\tau_R-69)+379)-1038)+1421)-779)-6}}{(\tau_R-3)^2(\tau_R((\tau_R-9)\tau_R+24)-19)}$
9. $S_1^{AQ^P} = \frac{-\tau_P((\tau_P-8)(\tau_P-6)\tau_P+2)+\frac{((\tau_P-6)\tau_P+7)\sqrt{\tau_P(\tau_P(\tau_P(\tau_P((\tau_P-22)\tau_P+251)-1596)+5483)-9462)+6385}}{\tau_P-3}+161}{4(3(\tau_P-6)\tau_P+29)}$
10. $S_1^{P_1^P} = \frac{3-\tau}{(\tau-4)\tau+5} + \frac{3}{2}$

11. $S_1^{AP^P} = \frac{3\tau^4-30\tau^3+117\tau^2-220\tau-((\tau-6)\tau+7)\sqrt{\tau(3\tau((\tau-8)\tau+27)-136)+93}+168}{(\tau-3)(\tau((\tau-7)\tau+21)-23)}$

12. ${S_1}^{\pi_1^P} = -\sqrt{\frac{((\tau_P-6)\tau_P+7)^2(\tau_P(\tau_P(\tau_P(\tau_P(\tau_P(\tau_P((\tau_P-20)\tau_P+172)-834)+2514)-4874)+5987)-4270)+1348)}{(\tau_P-3)^4(\tau_P(\tau_P(2\tau_P-21)+60)-47)^2}} - \frac{51\tau_P^2-522\tau_P+597}{188-4\tau_P(\tau_P(2\tau_P-21)+60)} - \frac{\tau_P}{2} - \frac{12}{\tau_P-3} + \frac{2}{(\tau_P-3)^2} + \frac{7}{4}$

13. ${S_1}^{\Pi_R} = \frac{2\tau_R^6-36\tau_R^5+273\tau_R^4-1103\tau_R^3+2448\tau_R^2-2745\tau_R+1155-((\tau_R-5)\tau_R+7)\sqrt{\tau_R(\tau_R(\tau_R(\tau_R(\tau_R((55-2\tau_R)\tau_R-582)+3198)}}{(\tau_R-3)^2(\tau_R(\tau_R((\tau_R-9)\tau_R+36)-61)+27)}$

14. ${S_1}^{\Pi^P} = -\frac{4(\tau_P+3)}{3(\tau_P-6)\tau_P+45} - \sqrt{\frac{((\tau_P-6)\tau_P+7)^2(\tau_P(\tau_P(\tau_P((\tau_P-15)\tau_P+89)-257)+366)-204)}{(\tau_P-3)^4(\tau_P-1)((\tau_P-6)\tau_P+15)^2}} - \frac{2}{3(\tau_P-3)} + 3$

15. ${S_1}^{AQ} = \frac{\tau(\tau(\tau((\tau-16)\tau+104)-337)+539)-343}{\tau-3)^3(\tau-2)(2\tau-7)}$

16. ${S_1}^{AP} = \frac{2\tau^4-18\tau^3+52\tau^4 9\tau}{2\tau^3-17\tau^2+47\tau-42}$

17. $S_1^W = \frac{\tau(\tau(\tau(-2(\tau-21)\tau-297)+934)-1323)+686}{(\tau-3)(\tau-2)(\tau(\tau(2(\tau-12)\tau+115)-250)+189)}$

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