# Mechanism Design with Spiteful Agents*

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**Abstract**

We study a mechanism-design problem in which spiteful agents strive to not only maximize their rewards but also, contingent upon their own payoff levels, seek to lower the opponents' rewards. Assuming either anonymity or efficiency, as well as individual rationality, we prove that a non-null incentive-compatible (IC) mechanism does not exist. We characterize the optimal spite-free mechanism showing it is a threshold mechanism with an ordering of the agents. Leveraging these findings, we partially extend our analysis to a problem with multiple items and copies. Overall, these results illuminate the challenges of auctioning items in the natural presence of other-regarding preferences.

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# 1 Introduction

On August 12, 2020, the Israeli Ministry of Communications announced the surprising results of the 5G spectrum auction held that year, in which three groups participated – Cellcom, Pelephone, and Partner. Although all telecommunications groups secured bandwidth bundles enabling 5G operations, the Cellcom group was required to pay 30% more than the Pelephone group despite receiving an inferior bundle.[1] That evening, the CEO of the Cellcom group tweeted an explanation for the seemingly poor outcome, stating: "We chose not to hurt the others, and they chose to hurt us; it's as simple as that!" Interestingly, the Cellcom group challenged the 5G spectrum auction in real time and appealed to the Administrative Court, claiming that the circumstances and auction design allowed for manipulation and spiteful bidding. The appeal was ultimately rejected.

The phenomenon of spiteful bidding is not unique to the Israeli 2020 spectrum auction. In the Swiss 2012 spectrum auction, Sunrise paid 34% more than Swisscom for an inferior bundle. Similarly, in the Austrian 2013 spectrum auction, revenues were much higher than expected due to highly aggressive bidding in the sealed-bid stage. This led Georg Serentschy, the Managing Director of the Telecommunications and Postal Services Division of the Austrian Regulatory Authority for Broadcasting and Telecommunications (RTR), to remark: "In the opinion of the regulatory authority, the price of EUR 2 billion, which was surprisingly high for us, is to be attributed to the consistently offensive strategy followed by the bidders." These auctions and public comments mark the starting point of our study: a mechanism design problem with spiteful agents.

This study builds upon the notion of spiteful agents—those who not only aim to maximize their rewards but also, given their own payoff levels, seek to minimize the rewards of their opponents. Such other-regarding preferences are quite natural, even expected, in scenarios where the competitive interaction among agents extends beyond a single auction. A prime example is obviously spectrum auctions, where participants also compete in the telecommunications market. However, this dynamic is not limited to telecommunications; it is equally applicable to

[1] The Cellcom group also paid 80% more than the Partner group, though the latter did not secure a superior bundle.

construction projects, retail businesses, the food industry, and any other economic sector where agents compete for advantage in external markets.

The main analysis considers a single-item, private-value auction where the agents strive to maximize their own payoffs, and *conditional on their own rewards*, jointly exhibit a spitefulness property. We refer to this property as *Spiteful Incentive Compatibility* (SIC), meaning that each agent prefers to reduce the payoffs of some other agents provided that no agent is made better off. We embed this property into the solution concept of a *Spite-Free Nash Equilibrium* (SNE), in which no player can increase their own payoff through a unilateral deviation, nor can they reduce the payoffs of others while keeping their own expected payoff fixed. Under these assumptions, we aim to characterize mechanisms that are Incentive compatible (IC) and Individually rational (IR) while admitting an SNE.

Our first and main result provides a characterization that links the aforementioned properties to threshold mechanisms. Formally, a threshold mechanism consists of an ordering of the players and an individual threshold value assigned to each player. Essentially, players are sequentially offered take-it-or-leave-it offers according to their designated threshold values and the predetermined ordering. The first player to accept the offer receives the item and pays their threshold value; if no player accepts, the item remains unallocated. We show that a mechanism is IC and IR with an SNE if and only if it is a threshold mechanism.

It is straightforward to see why threshold mechanisms satisfy the stated properties. Since payments are predetermined and offers are made sequentially, players have no ability to bid spitefully in a way that reduces others' expected payoffs without also decreasing their own. Figure 1 provides some visual intuition for this result by depicting a threshold mechanism in a two-agent setting. By contrast, the second-price mechanism does not satisfy these conditions, as non-winning bidders can spitefully raise their bids to reduce the expected payoff of the winning bidder. The more challenging part, however, is to show that *no other mechanism* satisfies the stated properties.

Next, we use the characterization to establish two impossibility results, both of which concern the *null mechanism*, under which the item is never allocated and no agent pays anything, regardless of the agents' bids. The first result shows that any IC, IR, and anonymous (i.e., Two-Agent Threshold Mechanism

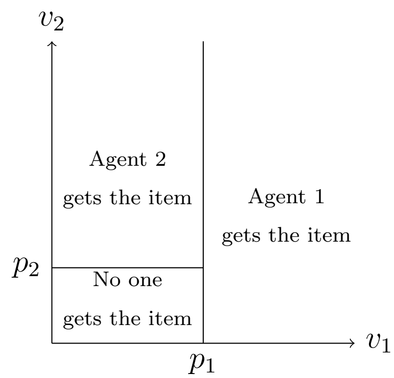


Figure 1: The figure illustrates the allocation and payments in a two-agent threshold mechanism, represented in the valuation plane. If agent 1 bids above her threshold $p_1$, she receives the item and pays $p_1$. Otherwise, the item is allocated to agent 2, provided that her bid exceeds $p_2$, in which case she pays $p_2$. It is immediate from this representation that neither agent can affect the other's expected payoff without forfeiting the possibility of obtaining the item.

symmetric) mechanism that admits an SNE must be the null mechanism. The second result shows that any IC, IR, and efficient mechanism[2] that admits an SNE must also be the null mechanism. The intuition behind both results is straightforward in light of our characterization. A symmetric threshold mechanism must be null in order to eliminate the role of the initial ordering of agents. Similarly, an efficient threshold mechanism must be null to eliminate the possibility that agents with lower valuations are ranked ahead of agents with higher valuations.

## 1.1 Related work

Morgan et al. (2003) derive symmetric equilibria for several types of common auction mechanisms when agents attach a disutility to the surplus of rivals.

Brandt and Weiß (2002); Brandt et al. (2007); Vetsikas and Jennings (2007) study the bidding behavior of spiteful agents who maximize a weighted difference of their own profit and their competitors' profit, and Maasland and Onderstal (2007) study the equilibrium behavior when losers care about the amount paid by the winner.

Janssen and Karamychev (2016); Janssen and Kasberger (2019) shows that truthful bidding is not an equilibrium in combinatorial auctions, and that agents' types are not fully revealed

[2]Efficiency here means that no losing agent bids above the winning agent.

in an efficient equilibrium.

Gretschko et al. (2016) discusses how spiteful bidding may affect agents' strategies in practice.

Sharma and Sandholm (2010) discusses the case that agents have a different extent of spitefulness.

Discussion of spiteful bidding in the context of experiments can be found in, e.g., Cooper and Fang (2008).

Spiteful behavior occurs in practice, as in the 2000 German 3G mobile phone spectrum license auction (see Brandt et al. (2007)).

### 1.2 The structure of the paper

The paper is organized as follows. Section 2 introduces the model and basic definitions. Section 3.1 presents the main characterization and two impossibility results. Section 4 analyzes the allocation of multiple items in the presence of spitefulness. Finally, Section 5 discusses optimal spite-free mechanisms.

## 2 The Model

We study the problem of assigning a single indivisible item among $n$ agents $I = \{1, \ldots, n\}$, where each agent $i \in I$ has a private valuation $v_i \in \mathbf{R}_+$ for the item. Let $V = \mathbf{R}_+$ denote the set of feasible valuations, and further assume that the reservation value of each agent, conditional on not receiving the item, is 0. A *bid* $b_i \in \mathbf{R}_+$ of agent $i$ is the agent's reported valuation, and a *bid profile* $b = (b_1, \ldots, b_n) \in \mathbf{R}_+^n$ is a tuple of bids, one for each agent. When relating to agent $i$, we follow the standard notation of $b = (b_i, b_{-i})$.

An *allocation* determines the assignment of the item. The set of allocations is $\mathcal{Z} = I \cup \{0\}$, so that the allocation $i \in \mathcal{Z}$ corresponds to the item being assigned to agent $i$, and the allocation 0 corresponds to the item being unassigned. Agent $i$'s *payment* is a non-negative real number. A *mechanism* $M = (A, P)$ consists of an allocation function $A : V^n \to \mathcal{Z}$ and a payment function $P : V^n \to \mathbf{R}_+^n$. The allocation function determines the allocation given a bid profile $b$, and the payment function determines the payments vector given $b$. For every agent $i$, the functions $A_i : V^n \to \{0, 1\}$ and $P_i : V^n \to \mathbf{R}_+$ describe the agent's assigned bundle (namely, $A_i(b) = 1$ if agent $i$ gets the item, and 0 otherwise) and payment, respectively.

For a given mechanism $M$, agent $i$'s utility function $u_i : V^n \times V \to \mathbf{R}_+$ is defined by

$$u_i(b; v_i) := v_i \cdot A_i(b) - P_i(b),$$

where $P_i(b)$ describes the agent's payment. In case agent $i$ reports the private valuation $b_i = v_i$ and to facilitate the exposition, we often use the notation $u_i(b)$ instead of $u_i(b; v_i)$. In case $b_i \neq v_i$, we say that agent $i$ *misreports* (his valuation).

Our use of $\mathbf{R}_+$ as the agents' action space involves some loss of generality compared to a more general, potentially multi-dimensional, action space. Nonetheless, this assumption is quite natural in practical settings where a single item is being allocated. Moreover, in Section 4, where we extend the model to the allocation of multiple items, the action space is generalized accordingly.

## 2.1 Simple properties of mechanisms

In this section we list several well-known properties of mechanisms, namely efficiency, anonymity, individual rationality, and incentive compatibility, that will be used throughout the analysis and characterization.

A mechanism $M = (A, P)$ is *anonymous* if its outcome is independent of the agents' indices. Formally, for every[3] permutation $\pi$ and every bid profile $b$,

$$\big(A_1(\pi(b)), \ldots, A_n(\pi(b))\big) = \pi\big(A_1(b), \ldots, A_n(b)\big), \quad \text{and} \quad P(\pi(b)) = \pi\big(P(b)\big).$$

A mechanism is *efficient* if, in equilibrium, the winning bidder's bid is at least as high as every other bid. Formally, for every non-zero bid profile $b \neq (0, \ldots, 0)$ such that $A(b) = i$ for some agent $i$, we have $b_i \geq b_j$ for every $j \in I$.

[3]We consider a permutation $\pi$ as a function from $\mathbf{R}_+^n$ to $\mathbf{R}_+^n$ such that for each coordinate $i$, there exists a unique coordinate $j$ with $\pi(x)_j = x_i$ for every vector $x$.

The mechanism is *Individually Rational* if every agent can secure the reservation value (normalized to 0) by bidding truthfully: $u_i(b; v_i) \geq 0$ for every agent $i$, every private valuation $v_i$, and every bid profile $b$ such that $b_i = v_i$. The mechanism is *Incentive Compatible* if the profile $b = v$ of truthful bids is a Nash equilibrium.

The following result, given in Lemma 1 below, lists standard properties of IR and IC mechanisms: (a) if the mechanism is IR, an agent who did not receive the object pays 0; (b) if the mechanism is IC and given the bids of all non-winning agents, the payment of the winning agent $i$ is independent of her bid, conditional on winning; and (c) if the mechanism is IR and IC, then any agent who receives the object under a given bid profile will also receives the object if she increases her bid; For proofs, see, e.g., Myerson (1981) or Krishna (2009).

**Lemma 1.** *1. Assume the mechanism $M = (A, P)$ is* IR. *Then, for any agent $i$ and any bid profile $b \in V^n$, if $A(b) \neq i$, then $P_i(b) = 0$.*

*2. Assume that $M$ is* IC. *Then, for any agent $i$ and bid profiles $(b_i, b_{-i})$ and $(b'_i, b_{-i})$, if $A_i(b_i, b_{-i}) = A_i(b'_i, b_{-i})$, then $P_i(b_i, b_{-i}) = P_i(b'_i, b_{-i})$.*

*3. Assume that $M$ is* IR *and* IC. *Then, for any agent $i \in I$, bid profile $b = (b_i, b_{-i}) \in V^n$, and bid $b'_i > b_i$, if $A(b) = i$, then $A(b'_i, b_{-i}) = i$.*

## 2.2 A notion of spitefulness

The main theme of this study concerns the property of spitefulness, for which we provide a formal definition incorporated in the solution concept. We say that a profile is a *Spite-Free Nash equilibrium* if it maximizes every agent's payoff and there is no unilateral deviation that maintains the same payoff level for the deviating agent, while weakly reducing the payoffs of *all* other agents and strictly reducing the payoffs of some. This notion is formally given in Definition 1 below.

**Definition 1.** *Fix a mechanism $M$. A profile $b$ is a* Spite-Free Nash equilibrium (SNE) *if, for any agent $i$, there is no bid profile $b' = (b'_i, b_{-i})$ such that either $u_i(b'; b_i) > u_i(b; b_i)$, or*

*$u_i(b'; b_i) = u_i(b; b_i)$ and $u_j(b'; b_j) \leq u_j(b; b_j)$ for all other agents $j \neq i$, with a strict inequality for at least one agent $j \neq i$.*

The SNE condition essentially assumes that agents first seek to maximize their own utility in equilibrium, and then prefer to deviate if doing so makes them (weakly) better off while reducing the payoffs of others. In this sense, it implicitly imposes lexicographic preferences: agents prioritize the maximization of their own expected payoffs, but, conditional on that, exhibit a form of spitefulness aimed at lowering their opponents' payoffs.

As with the standard Nash equilibrium and other solution concepts, the SNE specifies a particular action profile but does not describe how agents might converge to it. Implementing such an equilibrium can be non-trivial, since a spiteful deviation is aimed at reducing the expected payoffs of other agents, and these payoffs depend on their *private* valuations. Nevertheless, spiteful behavior is observed in practice, and in many cases, even a rough estimate of others' valuations may suffice to sustain such behavior.

We next present the main solution concept that we introduce, which is robust to spiteful bidding. A mechanism is spite-free incentive compatible if the profile where all agents bid truthfully is an SNE.

**Definition 2.** *A mechanism is* Spite-Free Incentive Compatible *(SIC) if the profile $b = v$ of truthful bids is an* SNE*.*

# 3 Characterizing a Spite-Free mechanism

In this section, we present the paper's main characterization result (Theorem 1). The section is organized into three parts. In Section 3.1, we define threshold mechanisms for a single indivisible good and characterize them as IR and SIC mechanisms. In Section 3.2, we present two impossibility results (Corollaries 1 and 2) that build on Theorem 1, showing that the only mechanism satisfying IR and SIC, and that is also either anonymous or efficient, is the null mechanism in which the item is never allocated.

### 3.1 Main result

We are now ready to present and prove the main result of the paper. To this end, we formally define *threshold mechanisms* in Definition 3 below. In general, a threshold mechanism consists of a sequence of take-it-or-leave-it offers, one for each agent, made according to a predetermined ordering of the agents and at exogenously fixed prices. The first agent who accepts her offer obtains the item at the stated price. More formally,

**Definition 3.** *A* threshold mechanism $M = (A, P)$ *is defined using a permutation* $R : I \to \{1, \ldots, n\}$ *on agents (i.e., a priority ranking), and a threshold* $t_i \in \mathbf{R}_+ \cup \{\infty\}$ *for each agent* $i \in I$*, such that for any bid profile* $b \in V^n$*:*

- *If there exists an agent* $j \in I$ *such that* $b_j > t_j$*, then* $A(b) = i$ *and* $P_i(b) = t_i$ *where* $i = \min\{R(j) : b_j \geq t_j\}$*.*
- *If* $b_j \leq t_j$ *for every agent* $j$*, then either* $A(b) = j$ *and* $P_j(b) = t_j$ *for some agent* $j \in I$ *satisfying* $b_j = t_j$*, or* $A(b) = 0$*.*

The permutation in Definition 3 specifies a priority ranking of the agents, with each agent assigned a threshold value for obtaining the object. If no agent truthfully bids above her assigned threshold, all agents receive zero ex-post payoffs, as in the case where the item remains unallocated. Furthermore, even when the priority ranking and thresholds are fixed, the threshold mechanism is not unique, since variations in the allocation rule may arise when agents do not bid strictly above their thresholds.

We now use the notion of a threshold mechanism to present the main result of the paper, stated in Theorem 1. The theorem asserts that a mechanism is IR and SIC if and only if it is a threshold mechanism. To build some intuition, consider the second-price mechanism. It is clearly IR and IC, yet not spite-free: the second-highest bidder can increase her bid to reduce the payoff of the winning agent. Such behavior is impossible under threshold mechanisms, since the priority ranking and payments are predetermined. This characterization captures the essence of the spitefulness property: the price paid by the winning bidder cannot depend on the actions of non-winning agents, as such dependence would allow them to act spitefully.

**Theorem 1.** *A mechanism is* IR *and* SIC *if and only if it is a threshold mechanism.*

The proof of Theorem 1 is extensive and is therefore deferred to Appendix A. It is based on a stronger notion of spitefulness, referred to as *Extreme SIC* (ESIC), in which agents not only bid truthfully to maximize their own payoffs in equilibrium, but also satisfy the following stricter (equilibrium) condition: there exists no unilateral deviation that leaves the deviator's payoff unchanged while *strictly* reducing the payoff of at least one other agent (even if such a deviation increases the payoffs of others). Clearly, every ESIC mechanism is also a SIC mechanism. We first characterize ESIC mechanisms and then extend the result to SIC mechanisms. See Appendix A for more details.

## 3.2 Efficient and Anonymous Spite-Free mechanisms

There are two immediate impossibility results that follow from Theorem 1. These results state that the only SIC and IR mechanism that sustains an additional condition of either efficiency, or anonymity, is the *null mechanism*, where, for each bid profile, all payments are zero and the item is never allocated to any agent. Formally, a mechanism $M$ is a null mechanism if $A(b) = 0$ and $P_i(b) = 0$ for each agent $i$ and every bid profile $b$. Alternatively, a null mechanism is a threshold mechanism where $t_i = \infty$ for every agent $i$.

Under the (threshold) null mechanism, the utility of each agent is always zero. Therefore, this mechanism is anonymous. Since under the null mechanism the object is never allocated, this mechanism is also efficient.

The first impossibility result, stated in Corollary 1, asserts that any anonymous, IR, and SIC mechanism must be a null mechanism. Given the characterization of IR and SIC mechanisms as threshold mechanisms, the intuition is immediate: once a priority ranking is fixed with some feasible payments, the mechanism can no longer be anonymous.

**Corollary 1.** *If a mechanism is* IR*, Anonymous and* SIC*, then it is the null mechanism.*

The second impossibility result, stated in Corollary 2, asserts that any efficient, IR, and SIC mechanism must also be a null mechanism. The intuition again follows directly from the characterization of IR and SIC mechanisms as threshold mechanisms: once (finite) thresholds and a priority ranking are fixed, the outcome need not be efficient, since agents with higher valuations may be ranked below lower-valued agents, implying that the winning bidder does not necessarily submit the highest bid.

**Corollary 2.** *If a mechanism is* IR*, efficient and* SIC*, then it is the null mechanism.*

Both Corollaries 1 and 2 follow directly from Theorem 1, and hence their proofs are omitted.

# 4 Multiple items setting

In the multiple items setting, we study the problem of assigning a finite set $S = \{a_1, a_2, \dots, a_K\}$ of items to the set of agents $I = \{1, 2, \dots, n\}$. A *bundle* $T_i \subseteq S$ for agent $i$ is a collection of items that can be allocated to her. The set of all bundles is $2^S$. For each agent $i \in I$, the valuation function $v_i : 2^S \to \mathbf{R}_+$ determines the worth agent $i$ has for each bundle. We assume that $v_i(\phi) = 0$ for each $i \in I$. The set of valuation functions is denoted by $V := \{v \in \mathbf{R}_+^{2^S} : v_i(\phi) = 0, \forall i \in I\}$.

A *bid* $b_i \in V$ of agent $i$ is the reported valuation of agent $i$. A *bid profile* $b = (b_1, \dots, b_n) \in V^I$ is a bid vector. An *allocation* is an assignment of bundles for each agent that satisfies the feasibility constraint $\bigcup_{i \neq j}(T_i \cap T_j) = \phi$, where $T_i, T_j \subseteq S$ are the assigned bundles of agents $i$ and $j$, respectively. The set of allocations is denoted by $\mathcal{Z}$. A *payment* for each agent is a non-negative real number. The mechanism and the utility of agents are defined similarly to the analogous concepts in the single item setup.

## 4.1 Threshold mechanisms for multiple items

In this section we define two families of mechanisms which generalize the threshold mechanism

to the setup of multiple items. As we will see, neither of these families is a SFIC, but when restricting the valuations of the agents, they are.

In the first family of mechanisms, items are assigned sequentially according to a process similar in threshold mechanism. According to her priority rankings, an agent is offered the item at a fixed threshold price. If that agent is not willing to pay that price, then the item is offered to the agent next in the priority order. If an agent accepts the offer, the next item is allocated in the similar fashion respecting the same priority ordering.

**Definition 4.** [Sequential Threshold Mechanism] A sequential threshold mechanism $M = (A, P)$ is defined using a (priority ranking) permutation $R : I \to \{1, \ldots, n\}$ on agents, and a threshold $t_i \in \mathbf{R}_+ \cup \{\infty\}$ for each agent $i \in I$. The items are assigned to agents sequentially in $K$ steps. At each step $k$,

the mechanism maintains for each agent $i$ a set $A_i^k \subseteq A$, which represents the set of items allocated to agent $i$ up to (including) step $k$, and a real number $P_i^k$, which represents the payment that agent $i$ has to pay for those items.

Let $A_i^0 = \phi$ and $P_i^0 = 0$ for each $i \in I$. Consider a bid profile $b \in V^n$. Step $k$ is as follows.

1. Let $S^k$ be the set of agents whose incremental valuation for item $a_k$ is greater than or equal to their threshold price:

$$S^k = \Big\{ j : b_j\Big(A_j^{k-1} \cup \{a_k\}\Big) - b_j\Big(A_j^{k-1}\Big) \geq t_j \Big\}.$$

2. If $S^k$ is empty, then the $a_k$ is not assigned to any agent. If $S^k$ is non-empty, then $a_k$ is assigned to agent $i$ in $S^k$ with highest priority. That is, $i = \min\{R(j) : j \in S^k\}$.

3. For every agent $i$ define the set $A_i^k$ as following. If $a_k$ is assigned to agent $j$, then $A_i^k = A_i^{k-1} \cup \{a_k\}$. If $a_k$ is not assigned to agent $j$, then $A_j^k = A_j^{k-1}$.

4. For every agent $i$ define the set $P_i^k$ as following. If $a_k$ is assigned to agent $i$, then $P_i^k = P_i^{k-1} + t_i$. If $a_k$ is not assigned to agent $j$, then $P_i^k = P_i^{k-1}$.

A restricted valuation domain is a setting where the set of valuations come from a set $V \subseteq \mathbf{R}_+^{2^S}$. $V$ is a *submodular items domain* if it contains only submodular valuations; that is, for each agent $i$, each $v_i \in V$, and each bundles $T_i, T_i' \subseteq S$, we have $v_i(T_i \cup T_i') \leq v_i(T_i) + v_i(T_i')$.

An *identical items domain* is a valuation domain, in which the valuations of agents does not depend on the identity of the items, that is, $v_i(T) = v_i(T')$ whenever $|T_i| = |T_i'|$. In this case, the set of valuations functions and set of bids is $V := \mathbf{R}_+^{\{1,\ldots,K\}}$.

An *identical submodular items domain* is combination of both the domain restrictions defined above.

**Proposition 1.** *In the identical submodular items domain, any sequential threshold mechanisms is SFIC.*

Sequential threshold mechanisms are not SFIC for general valuation functions. Following is the example of valuation functions in submodular items domain, but outside the identical items domain where a sequential threshold mechanism is not SFIC.

**Example 1.** *Let $S = \{a_1, a_2\}$. Consider an arbitrary sequential threshold mechanism $M$ given by priority order $R(1) = 1, R(2) = 2$, and thresholds $t_1$ for agent 1 and $t_2$ for agent 2. Consider a valuation function $v_1$ such that $v_1(\{a_1\}) = t_1$, $v_1(\{a_2\}) = t_1 + \epsilon$, and $v_1(\{a_1, a_2\}) = 2t_1 - \epsilon$, where $\epsilon$ is a small positive number.*

*On truthful reporting ($b_1 = v_1$), $M$ allocates item $a_1$ to agent 1, since $v_1(\{a_1\}) = t_1 \geq t_1$. The mechanism $M$ does not allocate item $a_2$ to agent 1, since $v_1(\{a_1, a_2\}) - v_1(\{a_1\}) < t_1$. Hence, $A_1(v_1, b_2) = \{a_1\}$, and the utility of agent 1 is 0. However, if agent 1 misreports the valuation of $a_1$ to $v_1'(\{a_1\}) = 0$, then $M$ will not allocate her $a_1$, since $v_1'(\{a_1\}) < t_1$. $M$ will allocate item $a_2$ to agent 1, since $v_1(\{a_2\}) = t_1 + \epsilon \geq t_1$. Hence, $A_1(v_1', b_2) = \{a_2\}$, and the utility of agent 1 is $\epsilon > 0$. Hence, agent 1 benefits by misreporting.* ◊

Following is the example of valuation functions in identical items domain, but outside the submodular items domain where a sequential threshold mechanism is not SFIC.

**Example 2.** *Let $S = \{a_1, a_2\}$. Consider an arbitrary sequential threshold mechanism $M$ given by priority order $R(1) = 1, R(2) = 2$, and thresholds $t_1$ for agent 1 and $t_2$ for agent 2. Consider a valuation function $v_1$ such that $v_1(\{a_1\}) = 0$, $v_1(\{a_2\}) = 0$ and $v_1(\{a_1, a_2\}) = 2t_1 + \epsilon$, where $\epsilon$ is a small positive number.*

*On truthful reporting, $M$ does not allocates item $a_1$ to agent 1, since $v_1(\{a_1\}) = 0 < t_1$. $M$ also does not allocate item $a_2$ to agent 1, since $v_1(\{a_2\}) = 0 < t_1$. Hence, $A_1(v_1, b_2) = \phi$, and the utility of agent 1 is 0. However, if agent 1 misreports the valuation of $a_1$ to $v_1'(\{a_1\}) = t_1$, then $M$ will allocate her $a_1$, since $v_1'(\{a_1\}) = t_1 \geq t_1$. $M$ will then also allocate item $a_2$ to agent*

*1, since* $v_1(\{a_1, a_2\}) - v_1(\{a_1\}) = (2t_1 + \epsilon) - t_1 = t_1 + \epsilon \geq t_1$*. Hence,* $A_1(v_1', b_2) = \{a_1, a_2\}$*, and the utility of agent 1 is* $\epsilon > 0$*. Hence, agent 1 benefits by misreporting.* $\diamondsuit$

In the second family of mechanisms, agents can choose a cluster of items that they want among the items which are not chosen by any other agent before. Agents can choose the clusters according to the priority ordering. Price is fixed, but depends on the cluster chosen.

**Definition 5.** [Cluster Threshold mechanism] A cluster threshold mechanism $M = (A, P)$ is defined using a (priority ranking) permutation $R : I \to \{1, \ldots, n\}$ on agents, and a threshold $t_i^T \in \mathbf{R}_+ \cup \{\infty\}$ for each agent $i \in I$ and subset $T \subseteq S$.

$S_i$ denotes the set of items available for agent $i$, and $A_i \subseteq S$ denotes the set of items allocated to agent $i$. Set $S_{R^{-1}(1)} = S$. We have,

$$A_i = \arg\max_{T \subseteq S_i} \left\{ b_i(T) - t_i^T \right\} \text{ and } S_{R^{-1}\left(R(i)+1\right)} = S_i \cup A_i.$$

A tie breaking order over $2^S$ is used to break ties. Payment for agent $i$ is $t_i^{A_i}$.

**Proposition 2.** *In the identical items domain, the cluster threshold mechanism with tie breaking order which picks an allocation with maximum number of items is SFIC.*

Following is the example of valuation functions outside the identical items domain where a cluster threshold mechanism is not SFIC.

**Example 3.** *Let* $S = \{a_1, a_2\}$*. Consider an arbitrary sequential threshold mechanism* $M$ *given by priority order* $R(1) = 1$, $R(2) = 2$*, and thresholds* $t_1(\{a_1\}) = 1, t_1(\{a_2\}) = 1, t_1(\{a_1, a_2\}) = 2$ *for agent 1 and* $t_2(\{a_1\}) = 1, t_2(\{a_2\}) = 1, t_2(\{a_1, a_2\}) = 2$ *for agent 2. The tie breaking rule chooses cluster* $\{a_1, a_2\}$ *over* $\{a_1\}$ *over* $\{a_2\}$*. Consider a valuation function* $v_1$ *such that* $v_1(\{a_1\}) = 1$, $v_1(\{a_2\}) = 1$, $v_1(\{a_1, a_2\}) = 1.5$*, and* $v_2(\{a_1\}) = 1$, $v_2(\{a_2\}) = 1.5$, $v_2(\{a_1, a_2\}) = 1.5$.

*On truthful reporting,* $M$ *allocates agent 1 cluster* $\{a_1\}$ *and agent 2 cluster* $\{a_2\}$*. The utility of agent 1 is 0 and the utility of agent 2 is 0.5. If agent 1 misreports the valuation of* $a_1$ *to* $0.5$*, then* $M$ *allocates agent 1 cluster* $\{a_2\}$ *and agent 2 cluster* $\{a_1\}$*. The utility of both agents is*

*0. Hence, agent 1 misreporting leads to a decrease in the utility of agent 2 while not harming agent 1.* $\Diamond$

## 4.2 Difficulty in the characterization

Although there are no non-trivial sequential threshold mechanisms or cluster threshold mechanisms which are SFIC for general valuation domains, there is abundance of SFIC mechanisms.

## 4.3 Necessary conditions

In this section we provide a necessary conditions for the mechanism to be individually rational and SFIC in general valuations domain.

The following lemma, which shows that the payment of agent $i$ depends only on $T_i$ and $b_{-i}$, is the analog of Lemma 1(2) for the multiple item setting.

**Lemma 2.** Assume that the mechanism $M$ is SFIC. For any agent $i$ and any bid profiles $(b_i, b_{-i})$ and $(b'_i, b_{-i})$, if $A_i(b_i, b_{-i}) = A_i(b'_i, b_{-i})$, then $P_i(b_i, b_{-i}) = P_i(b'_i, b_{-i})$.

*Proof.* Assume to the contrary that $A_i(b_i, b_{-i}) = A_i(b'_i, b_{-i}) = T_i \subseteq S$ and $P_i(b_i, b_{-i}) > P_i(b'_i, b_{-i})$. Then we have

$$u_i(b_i, b_{-i}; b_i) = b_i(T_i) - P_i(b_i, b_{-i}) < b_i(T_i) - P_i(b'_i, b_{-i}) = u_i\big((b'_i, b_{-i}); b_i\big).$$

Hence agent $i$ with type $v_i = b_i$ strictly benefits by misreporting to $b'_i$. This is a contradiction to the assumption that $M$ is SFIC. $\square$

Let $W_{-i}^{T_i} \subseteq V_{-i}$ be the set of all $b_{-i} \in W_{-i}^{T_i}$ for which there exists $b_i \in V_i$ with $A_i(b_i, b_{-i}) = T_i$. By Lemma 2, the payment function of agent $i$ when her allocation is $T_i$ can be viewed as a function $P_i^{T_i} : W_{-i}^{T_i} \to \mathbb{R}_+ \cup \{\infty\}$. Extend this function to $V_i$ be setting $P_i^{T_i}(b_{-i}) = \infty$ for $b_{-i} \notin W_{-i}^{T_i}$.

Theorem 2 below provide strong necessary conditions for a mechanism to be IR and SFIC.

**Theorem 2.** *Assume that a mechanism $M$ is SFIC and IR. Then the following hold for each agent $i$, each non-empty allocation $T_i \subseteq S$,*

*and each $(b_i, b_{-i}) \in V$:*

1. *$A_i(b_i, b_{-i}) = T_i$ whenever $b_i(T_i) - P_i^{T_i}(b_{-i}) > v_i(S_i) - P_i^{S_i}(b_{-i})$ for all $S_i \subseteq S, S_i \neq T_i$.*

2. *If $A_i(b_i, b_{-i}) = T_i$, then $b_i(T_i) - P_i^{T_i}(b_{-i}) \geq b_i(S_i) - P_i^{S_i}(b_{-i})$ for all $S_i \subseteq S, S_i \neq T_i$.*

3. *When there are only two agents, the function $P_i^{T_i} : V_{-i} \to \mathbb{R}$ has a finite range with at most $2^{K-|T_i|} + 1$ many elements.*

Parts 1 and 2 of Theorem 2 allow us to visualize the allocations for agent $i$ for the profiles in the set $V$ cross sectioned by fixing $b_{-i} \in V_{-i}$. For each agent $i$, $T_i \subseteq S$,

and $b_{-i} \in V_{-i}$,

define a set

$$V_i^{T_i}(b_{-i}) = \{b_i \in V_i : b_i(T_i) - P_i^{T_i}(b_{-i}) > b_i(S_i) - P_i^{S_i}(b_{-i}), \forall S_i \subseteq S, S_i \neq T_i\}.$$

Theorem 2 states that for each $b_i \in V_i^{T_i}(b_{-i})$, we have $A_i(b_i, b_{-i}) = T_i$. It follows from the definition that the sets $V_i^{T_i}(b_{-i})$ are mutually disjoint convex polytopes. Theorem 2 determines the allocation at all the bid vectors which are in the interior of one such polytope. However, it does not specify the allocation at the boundaries of these polytopes. Part 2 shows that when a bid vector lies in the intersection of the boundaries of two or more polytopes, the allocation could be any one of the sets whose polytope contains that bid vector.

The sets $V_i^{T_i}(b_{-i})$ indeed depend on the bid profile $v_{-i}$ of other agents. When there are only two agents, Part 3 of Theorem 2 shows that for each agent $i$ and non-empty allocation $T_i \subseteq S$, there exists a set $F_i^{T_i} \subseteq \mathbf{R}_+$ with $|F_{T_i}^i| = 2^{K-|T_i|} + 1$, such that $P_i^{T_i}(b_{-i}) \in F_i^{T_i}$ for each $v_{-i} \in V_{-i}$. Let $F_i = \prod_{T_i \subseteq S} F_i^{T_i}$. Hence, $|F_i| = \prod_{T_i \subseteq S} \left(2^{K-|T_i|} + 1\right) = \prod_{r=0}^{K} \left(2^{K-r} + 1\right)^{\binom{K}{r}}$.

A consequence of Part 3 of Theorem 2 is that it allows us to partition the set $V_{-i}$ into finitely many subsets, each one corresponding to a tuple in $F_i$, that is, $V_{-i} = \bigcup_{\alpha \in F_{-i}} V_\alpha$. A tuple $\alpha \in F_i$ matches each allocation $T_i \subseteq S$ to a particular payment value in $F_i^{T_i}$. Hence, for each $b_{-i}, b'_{-i} \in V_\alpha$ we have $P_i^{T_i}(b_{-i}) = P_i^{T_i}(b'_{-i}) \in F_i^{T_i}$ for each non-empty $T_i \subseteq S$, and thus, $V_i^{T_i}(b_{-i}) = V_i^{T_i}(b'_{-i})$ for each non-empty $T_i \subseteq S$.

Figure 2 illustrates these polytopes in the setting of 2 agents. The left-hand side diagram corresponds to the setting in which there are only two non-empty allocations, for instance, identical items domain with two items. In the blue region, which corresponds to $V_1^{T_1}(b_2)$, the allocation of agent 1 is $T_1$ (agent 1 obtains a only one item); in the green region, which corresponds to $V_1^{T'_1}(b_2)$, the allocation of agent 1 is $T'_1$ (agent 1 obtains a two item); and in the black region, which corresponds to $V_1^{\phi}(b_2)$, the allocation of agent 1 is $\phi$ (agent 1 does not obtain any item). The right-hand side diagram corresponds to the setting in which there are only three non-empty allocations, for instance, non-identical items domain with two items, $S = \{a_1, a_2\}$. In the blue region, which corresponds to $V_1^{T_1}(b_2)$, the allocation of agent 1 is $T_1$ (agent 1 obtains only $a_1$); in the green region, which corresponds to $V_1^{T'_1}(b_2)$, the allocation of agent 1 is $T'_1$ (agent 1 obtains only $a_2$); in the red region, which corresponds to $V_1^{T''_1}(b_2)$, the allocation of agent 1 is $T''_1$ (agent 1 obtains $a_1$ and $a_2$), and in the dark region $V_1^{\phi}(b_2)$, the allocation of agent 1 is $\phi$ (agent 1 does not obtain any item).

## 4.4 Proofs

In this subsection, we provide proofs for all the results in Section 4.

**Proof of Proposition 1** □

**Proof of Proposition 2** □

The proof requires a definition and three lemma.

**Definition 6.** *A mechanism $M$ is* monotone *if for any type vector $(v_i, v_{-i})$ such that $A_i(v_i, v_{-i}) = T_i$, and for any type $v'_i$ such that $v'_i(T_i) > v_i(T_i)$ and $v'_i(S_i) \leq v_i(S_i)$ for any $S_i \neq T_i$, we have $A_i(v'_i, v_{-i}) = T_i$.*

**Lemma 3.** *If $M$ is incentive compatible, then $M$ is monotone.*

*Proof.* Consider a type vector $(v_i, v_{-i})$ such that $A_i(v_i, v_{-i}) = T_i$, and type $v'_i$ such that $v'_i(T_i) > v_1(T_i)$ and $v'_i(S_i) \leq v_i(S_i)$ for every $S_i \neq T_i$. Let $T'_i = A_i(v'_i, v_{-i})$, and suppose by contradiction that$T'_i \neq T_i$. By assumption, $v'_i(T'_i) \leq v_i(T'_i)$.

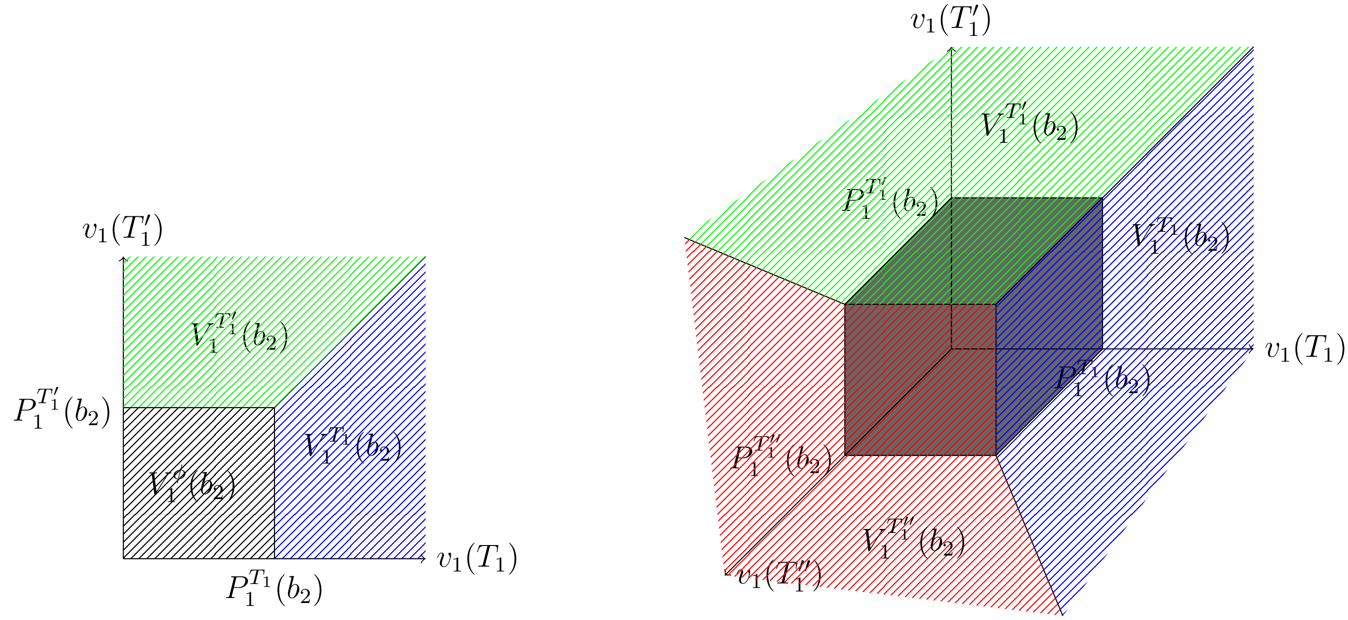


Figure 2: The partition of $V_1$ into convex polytopes depending on the allocation of agent 1 when agent 2's bid is fixed (see Lemma 4). The left figure illustrates the partition when there are 2 possible non-empty allocations and the right figure when there are 3 possible non-empty allocations. Coloured unbounded convex polytopes represent the sets with non-empty allocations (blue for $T_1$, green for $T_1'$ and red for $T_1''$) and the bounded polytope with darker (black) shed represents the set corresponding to the empty allocation.

Incentive compatibility of $M$ at $(v_i, v_{-i})$ implies that $u_i((v_i, v_{-i}); v_i) \geq u_i((v'_i, v_{-i}); v_i)$. That is,

$$v_i(T_i) - P_i^{T_i}(v_{-i}) \geq v_i(T'_i) - P_i^{T'_i}(v_{-i}) \geq v'_i(T'_i) - P_i^{T'_i}(v_{-i}). \tag{1}$$

Incentive compatibility of $M$ at $(v'_i, v_{-i})$ implies that $u_i((v'_i, v_{-i}); v'_i) \geq u_1((v_i, v_{-i}); v'_i)$. That is,

$$v'_i(T'_i) - P_i^{T'_i}(v_{-i}) \geq v'_i(T_i) - P_i^{T_i}(v_{-i}) > v_i(T_i) - P_i^{T_i}(v_{-i}). \tag{2}$$

Equations 1 and 2 contradict each other. Hence $T'_i = T_i$. □

Recall that $V_i^{T_i}(v_{-i}) = \{v_i \in V_i : v_i(T_i) - P_i^{T_i}(v_{-i}) \geq v_i(S_i) - P_i^{S_i}(v_{-i}) \forall S_i \subseteq S\}$.

**Lemma 4.** *Assume that a mechanism $M$ is SFIC and IR. Then for each agent $i$ and each non-empty allocation $T_i \subseteq S$,*

1. $A_i(v_i, v_{-i}) = T_i$ *if* $v_i(T_i) - P_i^{T_i}(v_{-i}) > v_i(S_i) - P_i^{S_i}(v_{-i}) \quad \forall S_i \subseteq S, S_i \neq T_i$.
2. *If* $A_i(v_i, v_{-i}) = T_i$ *then* $v_i(T_i) - P_i^{T_i}(v_{-i}) \geq v_i(S_i) - P_i^{S_i}(v_{-i}) \quad \forall S_i \subseteq S, S_i \neq T_i$.

*Proof.* First assume that $v_i \in \overline{V_i^{T_i}(v_{-i})}$. This implies that $P_i^{T_i}(v_{-i}) < \infty$. If possible, let $v_i(T_i) - P_i^{T_i}(v_{-i}) < v_i(S_i) - P_i^{S_i}(v_{-i})$ for some $S_i \neq T_i$. This inequality implies that $P_i^{S_i}(v_{-i}) < \infty$, and hence there exists some type $\widehat{v}_i$ such that $A_1(\widehat{v}_i, v_{-i}) = S_i$. Since $v_i \in \overline{V_i^{T_i}(v_{-i})}$, for every $\varepsilon > 0$, there exists $v'_i \in V_i^{T_i}(v_{-i})$ such that $|v_i - v'_i| < \varepsilon$. Choose $\varepsilon = \frac{1}{2}\Big(\big(v_i(S_i) - P_i^{S_i}(v_{-i})\big) - \big(v_i(T_i) - P_i^{T_i}(v_{-i})\big)\Big)$. The assumption $v_i(T_i) - P_i^{T_i}(v_{-i}) < v_i(S_i) - P_i^{S_i}(v_{-i})$ implies that $\varepsilon > 0$. Hence, we have

$$\begin{aligned} v'_i(S_i) - v'_i(T_i) &> \big(v_i(S_i) - \varepsilon\big) - \big(v_i(T_i) + \varepsilon\big) = \big(v_i(S_i) - v_i(T_i)\big) - 2\varepsilon \\ &= \big(v_i(S_i) - v_i(T_i)\big) - \big(v_i(S_i) - P_i^{S_i}(v_{-i})\big) + \big(v_i(T_i) - P_i^{T_i}(v_{-i})\big) \\ &= P_i^{S_i}(v_{-i}) - P_i^{T_i}(v_{-i}) \end{aligned}$$

This implies $v'_i(S_i) - P_i^{S_i}(v_{-i}) > v'_i(T_i) - P_i^{T_i}(v_{-i})$. Agent $i$ with valuation $v'_i$ can misreport to $\widehat{v}_i$ and obtain allocation $S_i$, thus contradicting the incentive compatibility. So, it is not possible that $v_i(T_i) - P_i^{T_i}(v_{-i}) < v_i(S_i) - P_i^{S_i}(v_{-i})$ for any $S_i \neq T_i$. Hence, $v_i \in \overline{V_i^{T_i}(v_{-i})}$ implies $v_i(T_i) - P_i^{T_i}(v_{-i}) \geq v_i(S_i) - P_i^{S_i}(v_{-i})$ for every $S_i \subseteq N$.

Now to prove the other direction, assume for some $v_i \in V_i$ that $v_i(T_i) - P_i^{T_i}(v_{-i}) \geq v_i(S_i) - P_i^{S_i}(v_{-i})$ for every $S_i \subseteq N$, and we will show that $v_i \in \overline{V_i^{T_i}(v_{-i})}$.

Consider a sequence $(V_i^I)_{n \geq 0}$ of types of agent $i$ defined as follows: For every $n \geq 0$, $V_i^I(T_i) = v_i(T_i) + \frac{1}{2^n}$ and $V_i^I(S_i) = v_i(S_i)$ for every $S_i \neq T_i$. For every $n \geq 0$, we have $V_i^I(T_i) - P_i^{T_i}(v_{-i}) > v_i(T_i) - P_i^{T_i}(v_{-i}) \geq v_i(S_i) - P_i^{S_i}(v_{-i})$ for every $S_i \subseteq N$. Since $V_i^{T_i}(v_{-i}) \neq \phi$, there exists $\widehat{v}_i \in V_i$ such that $A_i(\widehat{v}_i, v_{-i}) = T_i$. So, for every $n \geq 0$, we can conclude that $A_i(V_i^I, v_{-i}) = T_i$, otherwise agent $i$ with type $V_i^I$ can misreport to $\widehat{v}_i$ and obtain strictly higher utility with strictly preferred outcome $T_i$. So for every $n \geq 0$, we have $V_i^I \in V_i^{T_i}(v_{-i}) \subseteq \overline{V_i^{T_i}(v_{-i})}$. Since $\lim_{n \to \infty} V_i^I = v_i$, and $\overline{V_i^{T_i}(v_{-i})}$ is a closed set, we can conclude that $v_i \in \overline{V_i^{T_i}(v_{-i})}$. This completes the proof of claim 1 of the Lemma. □

**Lemma 5.** *Consider the setting of only two agents, that is, $I = \{1, 2\}$. Assume that $M$ is SFIC and IR. Then for each agent $i$ and every allocation $T_i \subseteq S$, the function $P_i^{T_i} : V_{-i} \to \mathbb{R}$ has finite range with at most $2^{K-|T_i|} + 1$ many elements.*

*Proof.* For the sake of contradiction, assume that $P_i^{T_i}$ takes strictly more than $2^{K-|T_i|}$ different values. Then for each $l = 1, \ldots, 2^{K-|T_i|} + 1$, there exists a type $v_{-i}^l \in W_{-i}^{T_i}$ of agent $-i$ such that the function $P_i^{T_i}$ takes distinct values for each type $v_{-i}^l$. That is, $P_i^{T_i}(v_{-i}^{l_1}) \neq P_i^{T_i}(v_{-i}^{l_2})$ for every $l_1 \neq l_2$.

Since $v_{-i}^l \in W_{-i}^{T_i}$, there exists a type $v_i^l$ of agent $i$ such that $A_i(v_i^l, v_{-i}^l) = T_i$ for $l = 1, \ldots, 2^{K-|T_i|} + 1$. Consider the type $\widehat{v}_i \in V_i$ of agent $i$ defined by $\widehat{v}_i(T_i) = m$ and $\widehat{v}_i(S_i) = 0$ for every $S_i \neq T_i$, where $m$ is an arbitrary real number that satisfies $m > v_i^l(T_i)$ for $l = 1, \ldots, 2^{K-|T_i|} + 1$. By monotonicity of $M$, we have $A_i(\widehat{v}_i, v_{-i}^l) = T_i$ for $l = 1, \ldots, 2^{K-|T_i|} + 1$.

Since the allocation for agent $i$ for the type vector $(\widehat{v}_i, v_{-i}^l)$ is $T_i$, each item from the set $S \setminus T_i$ is either is unallocated or allocated to one of the remaining $n-1$ agents. Hence there are $2^{|S \setminus T_i|} = 2^{K-|T_i|}$ different possible allocations for the remaining $n-1$ agents. Since there are $2^{K-|T_i|} + 1$ type vectors $(\widehat{v}_i, v_{-i}^l)$, there exists $l_1$ and $l_2$ such that $A_j(\widehat{v}_i, v_{-i}^{l_1}) = A_j(\widehat{v}_i, v_{-i}^{l_2})$.

The allocation for agent $-i$ is the same. That is, $A(\widehat{v}_i, v_{-i}^{l_1}) = A(\widehat{v}_i, v_{-i}^{l_2}) = (T_i, T_{-i})$ for some allocation $T_{-i}$ of agent $-i$. Since the type vector $(\widehat{v}_i, v_{-i}^{l_1})$ is a single perturbation of $(\widehat{v}_i, v_{-i}^{l_1})$ and both have the same allocations for both the agents, by Lemma ?? they are in the same component and so agent $i$ has the same payment. This contradicts the assumption that $P_i^{T_i}(v_{-i}^{l_1}) \neq P_i^{T_i}(v_{-i}^{l_2})$. □

# 5 Optimal Spite free mechanism - Example

Suppose there are $n$ agents whose private values are i.i.d. and uniformly distributed on $[0, 1]$. We will look for the spite free mechanism that maximizes the revenue of the seller.

Assume that the ranking of the agents is given by the identity permutation: agent $n$ is the highest, and agent 1 is the last in line. Denote by $t_i$ the threshold of agent $i$. The expected revenue is given by

$$\gamma(t_1, \ldots, t_n) = (1 - t_n)t_n + t_n\Big((1 - t_{n-1})t_{n-1} + t_{n-1}\big((1 - t_{n-2})t_{n-2} + t_{n-2}(\ldots(1 - t_1)t_1)\big)\Big).$$

Indeed, with probability $1 - t_n$ agent $n$ wins the object and pays $t_n$. Conditioned on the event that agent $n$ did not win the object, which occurs with probability $t_n$, we have the following: with probability $1 - t_{n-1}$ agent $n - 1$ wins the object and pays $t_{n-1}$, etc.

The term $t_1$ appears only in the last term, as $(1 - t_1)t_1$, and hence the optimal threshold for agent 1 is

$$t_1^* = \frac{1}{2}.$$

The term $t_2$ appears only in the event that agents $n, n - 1, \ldots, 3$ did not win the object. In that case, it appears as

$$(1 - t_2)t_2 + t_2(1 - t_1^*)t_1^* = t_2(1 + (t_1^*)^2 - t_2),$$

where the equality holds because $t_1^* = 1 - t_1^* = \frac{1}{2}$. The roots of this function are $t_2 = 0$ and $t_2 = 1 + (t_2^*)^2$, hence

$$t_2^* = \frac{1 + (1 - t_1^*)t_1^*}{2} = \frac{1 + (t_1^*)^2}{2}. \tag{3}$$

The term $t_3$ appears only in the event that agents $n, n-1, \ldots, 4$ did not win the object. In that case, it appears as

$$(1-t_3)t_3 + t_3\big((1-t_2^*)t_2^* + t_2^*(1-t_1^*)t_1^*\big) = t_3\big(1 - t_3 + (1-t_2^*)t_2^* + t_2^*(1-t_1^*)t_1^*\big).$$

This function is quadratic in $t_3$, one of its root is 0, and hence its maximum is attained at half the second root:

$$\begin{aligned}
t_*^3 &= \frac{1 + (1-t_2^*)t_2^* + t_2^*(1-t_1^*)t_1^*}{2} \\
&= \frac{(1-t_2^*) + t_2^* + (1-t_2^*)t_2^* + t_2^*(1-t_1^*)t_1^*}{2} \\
&= \frac{(1-t_2^*)(1+t_2^*) + t_2^*\big(1 + (1-t_1^*)t_1^*\big)}{2} \\
&= \frac{1-(t_2^*)^2}{2} + (t_2^*)^2 \\
&= \frac{1+(t_2^*)^2}{2},
\end{aligned}$$

where the fourth equality holds by Eq. (3).

We can continue recursively in an analogous manner, and obtain the recursive relation

$$t_i^* = \frac{1+(t_{i-1}^*)^2}{2}, \forall i.$$

As a conclusion, we obtain that the sequence $(t_n^*)_n$ increases. Since the unique positive root of the equation $t = \frac{1-t^2}{2}$ is $t = 1$, it follows that $\lim_{n\to\infty} t_n^* = 1$.

Denote by $\gamma_n$ the seller's revenue under $(t_i^*)_i$. The sequence $(\gamma_n)_n$ satisfies the recursive equation

$$\gamma_n = (1-t_n^*)t_n^* + t_n^*\gamma_{n-1}.$$

We argue that $\lim_{n\to\infty} \gamma_n = 1$. Indeed, for every $\varepsilon > 0$, if the seller sets all threshold to be equal to $1-\varepsilon$, then, as $n$ goes to infinity, the probability that at least one agent has a private value larger than $1-\varepsilon$ goes to 1, and hence $\lim_{n\to\infty} \gamma_n \geq 1-\varepsilon$.

Calculation using Google Sheets show that $\frac{1}{1-t_n}$ and $\frac{1}{1-\gamma_n}$ are both of the order of $\frac{n}{2}$. Moreover, the ratio between 1 minus the optimal revenue of the seller under any mechanism, and $1 - \gamma_n$, seems to converge to $\frac{1}{2}$. This means that the ratio-loss in the optimal revenue due to spite-freeness is asymptotically 2.

Can we have explicit formulas for $t_n^*$ and $\Gamma_n$? Can we compare $\gamma_n$ to the revenue under the optimal mechanism (with spite)? Aho and Sloane (1973) give an explicit solution for a sequence related to $t_n^*$, but I am not sure yet it is valid for our sequence.

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# Appendix A A notion of Extreme Spitefulness and the proof of Theorem 1

To prove our main result, we require an additional spitefulness notion, referred to as *Extreme Spite-Free Nash equilibrium* (ESNE) which allows agents to deviate if the deviating agent maintains the same utility level, while reducing the payoffs of other agents, but not necessarily all of them. That is, a profile is an ESNE if it maximizes every agent's payoff and there is no unilateral deviation which maintains the same payoff level for the deviating agent while strictly reducing the payoffs of *some* other agents.

**Definition 7.** *A profile $b$ is an* Extreme Spite-Free Nash equilibrium (ESNE) *if, for any agent $i$, there is no bid profile $b' = (b'_i, b_{-i})$ such that either $u_i(b'; b_i) > u_i(b; b_i)$, or $u_i(b'; b_i) = u_i(b; b_i)$ and $u_j(b'; b_j) < u_j(b; b_j)$ for some other agent $j \neq i$. Moreover, a mechanism is* Extreme Spite-Free Incentive Compatible (ESIC) *if the profile $b = v$ of truthful bids is an* ESNE*.*

There are two clear deductions from this definition. Notice that an ESIC mechanism is also a SIC mechanism, and in case of two agents ($n = 2$), the two spite-free notions of SIC and ESIC coincide. We shall first characterize the set of ESIC and IR mechanisms, showing they are threshold mechanisms, and later use this result to derive Theorem 1.

## A.1 A single perturbation path lemma

Before we proceed to characterize the set of ESIC and IR mechanisms, we define the notion of a *single perturbation path* and prove a supporting lemma for it. In general, a sequence of bid profiles $(b^r)_{r=0,1,\ldots,n}$ is a *single perturbation path* if bid profile $i$ in the sequence either coincides with bid profile $i+1$, or differs from it only in the bid of agent $i+1$. Formally,

**Definition 8.** *A sequence of bid profiles $(b^r)_{r=0,1,\ldots,n}$ is called a* single perturbation path *if for each $r = 0, 1, \ldots, n-1$, we have $b^r_k = b^{r+1}_k$ for each $k \neq r+1$.*

The following lemma shows that, given an ESIC and IR mechanism and conditional on a winning agent $i$, the payment of agent $i$ is independent of the bid profile.

**Lemma 6.** *Fix an* ESIC *and* IR *mechanism* $M$. *If* $A(b) = A(b') = i$ *for some agent* $i \in I$ *and bid profiles* $b, b' \in V^n$, *then* $P_i(b) = P_i(b')$.

*Proof.* The proof comprises three steps.

**Step 1:** Definitions.

Assume to the contrary that there exists an agent $i \in I$ and bid profiles $b, b' \in V^n$, such that $A(b) = A(b') = i$ and $P_i(b) > P_i(b')$. Define two bid profiles $c$ and $c'$ such that $c = (c_i, b_{-i})$ and $c' = (c'_i, b'_{-i})$, where $c_i = c'_i = \max\{b_i, b'_i\} + \epsilon$ for some $\epsilon > 0$. By Lemma 1(3), $A(c) = A(c') = i$, and by Lemma 1(2), $P_i(c) = P_i(b)$ and $P_i(c') = P_i(b')$. Therefore, $P_i(c) > P_i(c')$, and hence $u_i(c'; c_i) > u_i(c; c_i)$ accordingly.

Consider the single perturbation path $(c^r)_{r=0,\ldots,n}$ where $c^0 = c$ and $c^n = c'$. Note that $P_i(c^0) = P_i(c) > P_i(c') = P_i(c^n)$, and since $A(c^0) = A(c^n) = i$, we have $u_j(c^0; b_j) = u_j(c^n; b'_j) = 0$ for each agent $j \neq i$.

Define a set of agents $I_1 \subseteq I$ as $I_1 = \{r : u(c^{r-1}) \neq u(c^r)\}$, where $u(b) \in \mathbf{R}^n$ is the utility vector when each agent's bid in the bid profile $b$ matches the true valuation, and a set $I_2 \subseteq I$ as $I_2 = \{j : \exists r \text{ such that } u_j(c^r; c^r_j) \neq u_j(c^{r+1}; c^{r+1}_j)\}$. Agent $i$'s bid is constant throughout the single perturbation path, and therefore $c^{i-1} = c^i$, which implies that $i \notin I_1$. Since $u_i(c'; c_i) > u_i(c; c_i)$, $i \in I_2$.

**Step 2:** $|I_1| \geq |I_2|$.

Define a multi-value function $F : I_1 \to I_2$ as follows. For every $r \in I_1$ and every $j \in I_2$, set $j \in F(r)$ if and only if $u_j(c^{r-1}; c^{r-1}_j) \neq u_j(c^r; c^r_j)$. By Lemma 1(1), at most one agent has a strictly positive utility in any bid profile, and hence $|F(r)| \leq 2$ for every $r \in I_1$. For any $j \in I_2$ such that $j \neq i$, we have $u_j(c^0; b_j) = u_j(c^n; b'_j) = 0$ and $u_j(c^r; c^r_j) > 0$ for some $1 \leq r < n$, hence $|F^{-1}(j)| \geq 2$. In addition, the condition $u_i(c^0; c_i) < u_i(c^n; c_i)$ implies that $|F^{-1}(i)| \geq 1$. Therefore,

$$2|I_1| \geq \sum_{r \in I_1} |F(r)| = \sum_{j \in I_2} |F^{-1}(j)| \geq 1 + 2(|I_2| - 1) = 2|I_2| - 1,$$

where the first inequality holds since $|F(r)| \leq 2$ for every $r \in I_1$, and the second inequality holds since $|F^{-1}(i)| \geq 1$ and $|F^{-1}(j)| \geq 2$ for every $j \in I_2 \setminus \{i\}$. Thus, $|I_1| \geq |I_2|$, as claimed.

**Step 3:** Deriving a contradiction.

Because $i \notin I_1$ and $i \in I_2$, there exists an agent $j$ such that $j \in I_1$ and $j \notin I_2$. Since $j \notin I_2$, we have $u_j(c^{j-1}; c_j^{j-1}) = u_j(c^j; c_j^j)$, and since $j \in I_1$, we have $u_{j'}(c^{j-1}; c_{j'}^{j-1}) \neq u_{j'}(c^j; c_{j'}^j)$ for some agent $j'$. Hence, at either $c$ or $c'$, agent $j$ can misreport her bid to decrease the utility of agent $j'$ while keeping h, own utility fixed. This contradicts the ESIC property and concludes the proof. □

## A.2 Characterization of ESIC and IR mechanisms

In this subsection, we characterize the set of ESIC and IR mechanisms.

**Theorem 3.** *A mechanism is* IR *and* ESIC *if and only if it is a threshold mechanism.*

*Proof.* For each agent $i$, define a function $T_i : V^{n-1} \to \mathbf{R}_+ \cup \{\infty\}$ as follows. For any profile $b_{-i} \in V^{n-1}$, define

$$T_i(b_{-i}) = \inf\{b_i : A(b_i, b_{-i}) = i\},$$

and, if $A(b_i, b_{-i}) \neq i$ for all $b_i \in V_i$, then $T_i(b_{-i}) = \infty$. By Lemma 1(3), $A(b_i, b_{-i}) = i$ for every agent $i$, every bid profile $b_{-i}$ and any bid $b_i > T_i(b_{-i})$. The proof is divided into eight steps:

- In Step 1 we prove that the winner at bid profile $b$ pays $T_i(b_{-i})$.
- In Step 2 we prove that the amount the winner pays is some constant $t_i$, which depends only on her identity and not on the bid profile.
- In Step 3 we prove that if at bid profile $b_{-i}$ agent $i$ has a bid that makes her win, then for any bid profile $b'_{-i}$ that is dominated coordinatewise by $b_{-i}$, agent $i$ has a bid that makes her win.
- In Step 4 we prove that if agent $i$'s utility at bid profile $b$ is 0, then agent $i$'s utility is still 0 if agent $j \neq i$ lowers her bid.
- In Step 5 we prove that if no agent wins, then the bid of each agent $i$ is at most $t_i$.

- In Step 6 we prove that if $b$ and $b'$ are two bid profiles having the set of agents who bid at least $t_i$, and at $b'$ at least one agent $j$ bids strictly more than $t_j$, yield the same winner.
- In Step 7 we define a priority ranking $R$ among the agents.
- In Step 8 we finally prove that $M$ is a threshold mechanism with priority ranking $R$ and thresholds $(t_i)_{i\in I}$.

**Step 1:** For each agent $i$ and bid profile $b$, if $A(b) = i$ then $P_i(b) = T_i(b_{-i})$.

Suppose first that $A(b) = i$ and $P_i(b) > T_i(b_{-i})$ for some agent $i$ and bid profile $b$. Consider a bid $b'_i$ that satisfies $P_i(b) > b'_i > T_i(b_{-i})$. As mentioned above, $A(b'_i, b_{-i}) = i$, and by Lemma 1(2), $P_i(b'_i, b_{-i}) = P_i(b_i, b_{-i})$. Thus, $u_i((b'_i; b_{-i}); b'_i) = b'_i - P_i(b'_i, b_{-i}) < 0$, which contradicts the IR condition.

Suppose now that $A(b) = i$ and $P_i(b) < T_i(b_{-i})$ for some agent $i$ and bid profile $b$. Consider a bid $b'_i$ that satisfies $P_i(b) < b'_i < T_i(b_{-i})$. Since $b'_i < T_i(b_{-i})$, we have $A(b'_i, b_{-i}) \neq i$. However, $u_i(b; b'_i) = b'_i - P_i(b) > 0 = u_i((b'_i; b_{-i}); b'_i)$, which contradicts the IC condition. This concludes Step 1. $\diamond$

**Step 2:** For every agent $i$, there exists $t_i \in \mathbf{R}_+ \cup \{\infty\}$ such that for every bid profile $b_{-i} \in V^{n-1}$, either $T_i(b_{-i}) = t_i$ or $T_i(b_{-i}) = \infty$.

Assume, to the contrary, that there exists an agent $i$ and bid profiles $b_{-i}$ and $b'_{-i}$, such that $T_i(b_{-i}) < T_i(b'_{-i}) < \infty$. Lemma 1(3) implies that there is a bid $b_i > \max\{T_i(b_{-i}), T_i(b'_{-i})\}$ such that $A(b_i, b_{-i}) = A(b_i, b'_{-i}) = i$. By Step 1, $P_i(b_i, b_{-i}) = T_i(b_{-i}) < T_i(b'_{-i}) = P_i(b_i, b'_{-i})$, contradicting Lemma 6. $\diamond$

From now on we let $(t_i)_{i\in I}$ be the constants given by Step 2. Note that if $t_i = \infty$, agent $i$ never wins the object.

If $T_i(b_{-i}) = \infty$ for each $b_{-i} \in V^{n-1}$, or equivalently, if $A(b) \neq i$ for each $b \in V^n$, then we set $t_i = \infty$. In other words, for any agent $i \in I$, we have $t_i \in \mathbf{R}_+$ if and only if there exists a bid profile $b \in V^n$ such that $A(b) = i$.

**Step 3:** Fix a bid profile $b_{-i}$ such that $T_i(b_{-i}) = t_i < \infty$, and consider a bid profile $b'_{-i}$ such that $b'_j \leq b_j$ for every agent $j \neq i$. Then, $T_i(b'_{-i}) = t_i$.

Fix a bid $b_i > t_i$, so that $A(b_i, b_{-i}) = i$, $P_i(b_i, b_{-i}) = t_i$, and $u_i((b_i, b_{-i}); b_i) = b_i - t_i > 0$. We will show that $A(b_i, b'_{-i}) = i$. This will imply that $T_i(b'_{-i}) \leq T_i(b_{-i}) < \infty$, and therefore by Step 2 we will obtain that $T_i(b'_{-i}) = t_i$.

Assume then to the contrary that $A(b_i, b'_{-i}) \neq i$, and consider a single perturbation path $(b^r)_{r=0,1,\ldots,n}$, where $b^0 = (b_i, b_{-i})$ and $b^n = (b_i, b'_{-i})$. Let $r$ be the minimal index such that $A(b^r) = i$ and $A(b^{r+1}) \neq i$. By assumption, $b^r_{r+1} = b_{r+1} \geq b'_{r+1} = b^{r+1}_{r+1}$. If $A(b^{r+1}) = A(b'_{r+1}, b^r_{-(r+1)}) = r+1$, then by Lemma 1(3), $A(b^r) = A(b_{r+1}, b^r_{-(r+1)}) = r+1$, contradicting to condition $A(b^r) = i$. If $A(b^{r+1}) = j \notin \{r+1, i\}$, then given the bid profile $b^r$, agent $r+1$ can misreport $b'_{r+1}$, instead of $b_r$, thereby decreasing the utility of agent $i$. This contradicts the ESIC condition. Hence, $A(b_i, b'_{-i}) = i$, and by Step 2, $T_i(b'_{-i}) = t_i$. This concludes the proof of Step 3. $\diamond$

**Step 4:** Let $b$ be a bid profile, let $j$ be an agent, and let $b'_j < b_j$. If $u_i(b; b_i) = 0$ for every agent $i$, then $u_i((b'_j, b_{-j}); b_i) = 0$ for each agent $i$ (including $i = j$).

Let $b$, $j$, and $b'_j$ be as in the claim, and assume by contradiction that $u_i((b'_j, b_{-j}); b_i) \neq 0$ for some agent $i$. Since the mechanism is IR, it follows that $u_i((b'_j, b_{-j}); b_i) > 0$. By Lemma 1(1), $A(b') = i$, and hence by Lemma 1(3), $i \neq j$. But then at the bid profile $b'$, agent $j$ can misreport $b_j$, thereby lowering agent $i$'s payoff while not affecting her own payoff. Thus contradicts ESIC, and conclude the proof of Step 4.

**Step 5:** Let $b$ be a bid profile such that $A(b) = 0$. Then $b_i \leq t_i$ for every agent $i$.

Assume, to the contrary, that $A(b) = 0$ but there exists an agent $i$ such that $b_i > t_i$. By the definition of $t_i$, there exists a bid profile $b'$ such that $b'_i = b_i > t_i$ and $A(b') = i$. Define the bid profile $c$ as follows: $c_i = b_i > t_i$ and $c_j < \min\{b_j, b'_j, t_j\}$ for every $j \neq i$; in case $\min\{b_j, b'_j, t_j\} = 0$, set $c_j = 0$. Since $A(b) = 0$, the IR condition implies that $u_i(b; b_i) = 0$ for every agent $i$. By Step 4, applied recursively for all agents $k$ such that $b_k > c_k$, we have $u_i(c; b_i) = 0$. Consider a single perturbation path $(b^r)_{r=0,1,\ldots,n}$ where $b^0 = b'$ and $b^n = c$. Let $r$ be the first stage such that $A(b^r) = i$ and $A(b^{r+1}) \neq i$. By the construction of $c$ and Step 1, $u_i(b^r; b_i) = b_i - t_i > 0$ and $b^{r+1}_{r+1} = c_{r+1} < b'_{r+1} = b^r_{r+1}$. Lemma 1(3) implies that $A(b^{r+1}) \notin \{i, r+1\}$. Therefore, at the bid profile $b^r$, agent $r+1$ can misreport $c_{r+1}$ instead of $b^r_{r+1}$, and strictly decrease the utility of agent $i$ without affecting her own utility. This violates the ESIC property, and concliudes the proof of Step 5.

Fix every bid profile $b$ define

$$I_b = \{j : b_j \geq t_j\} \subseteq I.$$

This is the set of agents $j$ who bid at least $t_j$.

**Step 6:** For every two bid profiles $b$ and $b'$ such that (i) $I_b = I_{b'}$, (ii) $A(b) \neq 0$, and (iii) there exists an agent $j$ such that $b'_j > t_j$, we have $A(b) = A(b')$.

Assume, to the contrary, that the two bid profiles $b$ and $b'$ satisfy (i), (ii), and (iii) but $i = A(b) \neq A(b')$. By the definition of $t_i$, we have $i \in I_b = I_{b'}$. By Step 5, $A(b') \neq 0$. Denote $j = A(b')$. By increasing $b_i$ and $b'_j$ if necessary and by Lemma 1(3), we can assume w.l.o.g. that $u_j(b'; b'_j) = b'_j - t_j > 0$ and $u_i(b; b_i) = b_i - t_i > 0$. We can also assume that $b'_i = b_i$, hence $u_j(b'; b'_j) = b_j - t_j > 0$; indeed, otherwise, at bud profile $b'$ agent $i$ can misreport $b_i$, thereby decreasing the utility of agent $j$ without affecting her own utility.

Consider a single perturbation path $(b^r)_{r=0,1,\ldots,n}$ such that $b^0 = b$ and $b^n = b'$. We will show that for each $r = 0, 1, \ldots, n-1$, if $A(b^r) = i$ then $A(b^{r+1}) = i$, which contradicts the facts that $A(b^0) = A(b) = i$ and $A(b^n) = A(b') \neq i$. Assume then there exists $r < n$ such that $A(b^r) = i$ and $A(b^{r+1}) \neq i$. Since $b_i = b'_i$, we have $r+1 \neq i$. Since $I_b = I_{b'}$, whether $r+1 \notin I_b$ or $r+1 \in I_b$, at the bid profile $b^r$, when agent $r+1$ misreports $b^{r+1}_{r+1}$, her utility does not decrease, while the utility of agent $i$ strictly decreases (because $u_i(b^r; b_i) > 0 = u_i(b^{r+1}; b_i)$), contradicting the ESIC property. Thus, $A(b^{r+1}) = i$ as claimed. This concludes the proof of Step 6. $\diamond$

We are now ready to define the priority ranking $R$ that will be used in the definition of the threshold mechanism.

**Step 7:** A definition of a priority ranking $R : I \to \{1, \ldots, n\}$.

Let $I_\infty = \{j \in I : t_j = \infty\}$ be the set of agents who never win. To these agents, we arbitrarily assign ranking between $n - |I_\infty| + 1, \ldots, n$. That is, $R(I_\infty) = \{n - |I_\infty| + 1, \ldots, n\}$.

If $I_\infty = I$, we are done with the definition of $R$. Assume then that $I_\infty \neq I$, and suppose by induction that we already defined the $k$ agents who have the highest priority, for some $k = 0, 1, \ldots, n - |I_\infty| - 1$; that is, we already defined $R^{-1}(1), R^{-1}(2), \ldots, R^{-1}(k)$.

Denote by $B_k$ the set of all bid profiles $b$ that satisfy the following properties:

- $b_i < t_i$ for each $i$ such that $R(i) \leq k$: the bids of agents with high rank is low.
- $b_i \geq t_i$ for each $i$ such that $i \notin I_\infty \cup \{R^{-1}(1), R^{-1}(2), \ldots, R^{-1}(k)\}$, with at least one strict inequality. That is, agents who were not ranked yet have high bids.

By Step 6, for every $b, b' \in B_k$ we have $A(b) = A(b')$. By Step 5, it cannot be that $A(b) = 0$ for each $b \in B_k$. Since $b_i < t_i$ for every $i$ such that $R(i) \leq k$, we have $A(b) \notin \{R^{-1}(1), R^{-1}(2), \ldots, R^{-1}(k)\}$ for each $b \in B_k$. It follows that there is an agent $i_{k+1} \notin I_\infty \cup \{R^{-1}(1), R^{-1}(2), \ldots, R^{-1}(k)\}$ such that $A(b) = i_{k+1}$ for every $b \in B_k$. We define $i_{k+1}$ to be the next agent according to $R$:

$$R(i_{k+1}) = k + 1.$$

**Step 8:** $M$ is a threshold mechanism with the priority ranking function $R$ defined in Step 7 and the thresholds $(t_i)_{i \in I}$ given by Step 2.

We will show that there is a threshold mechanism $M^* = (A^*, P^*)$ with the priority ranking function $R$ and the thresholds $(t_i)_{i \in I}$ such that for every bid profile $b \in V^n$, $A(b)$ and $P(b)$ coincide with the agent and payments indicated by $M^*$.

By Steps 1 and 2, if $A(b) = i$ then $P^*(b) = t_i = P(b)$, while by Lemma 1(1), if $A(b) \neq i$, then $P^*(b) = 0 = P(b)$. Hence the payment function in $M$ coincides with the payment function of any threshold mechanism with the priority ranking function $R$ and the thresholds $(t_i)_{i \in I}$. We turn to handle the allocation function.

If $I_b = \emptyset$, that is, every agent $i$ bids below $t_i$, then the definition of $(t_i)_{i \in I}$ implies that $A(b) = 0$, Moreover, any threshold mechanism $M^* = (A^*, P^*)$ with the thresholds $(t_i)_{i \in I}$ will satisfy in this case $A^*(b) = 0$.

Assume from now on that $I_b \neq \emptyset$, and denote by $i^* \in I_b$ the agent with minimal ranking in

$I_b$ according to $R$:

$$R(i^*) < R(i), \quad \forall i \in I_b.$$

Then, for any threshold mechanism with the priority ranking function $R$ and the thresholds $(t_i)_{i \in I}$,

- If there is $j \in I_b$ such that $b_j > t_j$, then $A^*(b) = i^*$.
- Otherwise, $A^*(b) \in I_b \cup \{0\}$.

We will show that in this case $A(b)$ coincides with the above specifications. By the definition of $I_\infty$ we have $A(b) \notin I_\infty$, and since $M$ is IR we have $A(b) \in I_b \cup \{0\}$. Hence if $b_j = t_j$ for every $j \in I_b$, we have $A(b) \in I_b \cup \{0\}$. It therefore remains to handle the case that $b_j > t_j$ for at least one agent $j \in I_b$.

Assume then that this is the case. By Step 5, $A(b) \neq 0$. Denote $i = A(b)$. We need to show that $i = i^*$. By Step 1 and since $M$ is IR, we have $b_i \geq t_i$, so that $i \in I_b$. Therefore, if $i \neq i^*$, then $R(i^*) < R(i)$.

Because $R$ is a well-defined priority ranking, there is a bid profile $b^*$ such that (i) $A(b^*) = i^*$, and (ii) $b^*_j \geq t_j$ if and only if $R(j) \geq R(i^*)$ and $j \notin I_\infty$.

If $I_b = I_{b^*}$, then by Step 6 we have $i = A(b) = A(b^*) = i^*$, as needed. Otherwise, $I_b \subsetneq I_{b^*}$. Consider the bid profile $c$ where $c_j = b_j$ for every $j \in I_{b^*} \setminus I_b$, and $c_j = b^*_j$ otherwise. The bid profile $c$ is generated from $b^*$ by reducing the bid of every agent $j \in I_{b^*} \setminus I_b$ from $b^*_j$ to $b_j$. By Step 3, $A(c) = A(b^*) = i^*$, which implies that $I_c = I_b$ by construction. Thus, by Step 6 we have $i^* = A(c) = A(b) = i$, as needed. $\square$

## A.3 A supporting Proposition for at least 3 agents

**Proposition 3.** *Let mechanism $M$ be* IR *and* SIC *with $n \geq 3$. Then, there are no three distinct agents $i, j, k$ and bid profiles $b$, $b' = (b'_i, b_{-i})$, and $b'' = (b''_i, b_{-i})$, such that $b''_i > b'_i > b_i$, $u_j(b; b_j) > 0$, $u_k(b'; b'_k) > 0$, and $u_k(b''; b''_k) > 0$.*

*Proof.* **Step 0:** Preparations.

$$\begin{tikzcd}
b = (b_i, b_k, b_{-\{i,k\}}) \arrow[r, leftrightarrow, "i \text{ changes bid}"'] \arrow[d, leftrightarrow, "k \text{ changes bid}"] & (b'_i, b_k, b_{-\{i,k\}}) = b' \arrow[d, leftrightarrow, "k \text{ changes bid}"'] \\
c = (b_i, t_k, b_{-\{i,k\}}) \arrow[r, leftrightarrow, "i \text{ changes bid}"] & (b'_i, t_k, b_{-\{i,k\}}) = c'
\end{tikzcd}$$

Figure 3: The transitions between profiles $b, b', c$ and $c'$.

Assume to the contrary that there are three distinct agents $i, j, k$ and bid profiles $b$, $b' = (b'_i, b_{-i})$, and $b'' = (b''_i, b_{-i})$, such that $b''_i > b'_i > b_i$, $u_j(b; b_j) > 0$, $u_k(b'; b'_k) > 0$, and $u_k(b''; b''_k) > 0$.

Since $u_j(b; b_j) > 0$, we have $A(b) = j$, and since $u_k(b'; b_k) > 0$, we have $A(b') = k$. Let $t_k = T_k(b'_{-k})$, and consider the profiles $c = (t_k, b_{-k})$ and $c' = (t_k, b'_{-k}) = (t_k, b'_i, b_{-\{i,k\}})$. See Figure 3 for the mapping of transitions between these profiles.

Let $t'_i = T_i(c'_{-i})$.

**Step 1:** The agents' utilities at $c'$.

- If $A(c') = 0$, then by IR the utilities of all agents is 0.
- If $A(c') = k$, then Lemma 1(2) implies that the payment of agent $k$ is $T_k(b'_i, b_{-\{i,k\}}) = T_k(b'_{-k}) = t_k$, and hence $u_k(c'; c'_k) = 0$.
- If $A(c') = l \neq k$ and $u_l(c'; c'_l) > 0$, then at $c'$ agent $k$ can misreport $b_k$ (instead of $t_k$), and so the bid vector after the deviation is $b'$. Since $u_m(b'; b'_m) = 0 = u_m(c'; c'_m)$ for every $m \neq l$, and $u_l(c'; c'_l) > 0 = u_l(b'; b'_l)$, it follows that the mechanism is not SIC at $b'$, a contradiction. Hence, at the profile $c'$, either the item is unassigned, or the winner of the item gets 0 utility.

Thus, $u_m(c') = 0$ for all agents $m$.

**Step 2:** $t'_i = b'_i$.

Assume to the contrary that $t'_i \neq b'_i$. If $b'_i > t'_i$, then Lemma 1(3) implies that $A(c') = i$, and hence $u_i(c') = b'_i - t'_i > 0$, which contradicts Step 1. Hence $b'_i < t'_i$, and therefore by the definition of $t'_i$ we have $A(c') \neq i$. Since $b_i < b'_i$, by Lemma 1(3), we have $A(c) \neq i$.

We next argue that either $A(c) = 0$, or the winner $A(c)$ has utility 0 at $c$. Indeed, suppose that $l = A(c) \neq 0$ and $u_l(c; c_l) > 0$. We will derive a contradiction to the assumption that $M$ is SIC. Note that by the previous paragraph, $l \neq i$, and hence $c_l = c'_l$. Now, at bid profile $c$ agent $i$ can misreport from $b_i$ to $b'_i$, changing the bid profile to $c'$. Since $u_l(c; c_l) > 0 = u_l(c'; c'_l)$ and $u_m(c'; c_l) = 0 = u_m(c'; c_l)$ for each $m \neq A(c)$, we reach a contradiction to the SIC assumption.

We are now ready to derive the contradiction to the conclusion that $b'_i < t'_i$. Suppose that at the bid profile $b$, agent $k$ misreports from $b_k$ to $t_k$, resulting in the profile $c$. The utility of agent $j$ strictly decreases, while the utility of every other agent remains the same, contradicting the assumption that $M$ is SIC.

**Step 3:** The end of the proof.

The argument in Step 2 did not use the bid vector $b''$. Hence, an analogous argument to that provided in Step 2 shows that $b''_i = t'_i$. But then $b'_i = t'_i = b''_i$, which contradicts the assumption that $b''_i > b'_i$. □

## A.4 Proof of Theorem 1

*Proof.* If $n = 2$, then the result follows directly from Theorem 3 given in Appendix A, so assume that $n \geq 3$. One direction is straightforward. Fix a threshold mechanism $M$ and a profile $b$. Assume that agent $i$ bids truthfully in the sense that $b_i = v_i$. If agent $i$ wins the item, then $b_i \geq t_i$ and $u_i(b; v_i) = v_i - t_i = b_i - t_i \geq 0$. Otherwise, $i \neq A(b)$ and $u_i(b; v_i) = 0$. Thus, the mechanism is IR.

Next, we prove that the mechanism is SIC. Fix the truthful profile $b = v$. If $v_i < t_i$ for every agent $i$, then all agents have a utility of 0 given $b = v$, and there are no profitable deviations. Otherwise, assume that some agents bid at least there private thresholds, and let $i = A(b)$ be the winning agent. For every agent $j \neq i$ such that $v_j = b_j \geq t_j$, it follows that $R(j) > R(i)$, so a unilateral deviation of this agent $j$ does not affect the allocation and payment of agent $i$, or any other agent. Moreover, for every agent $j \neq i$ such that $v_j = b_j < t_j$, any deviation that changes the allocation of item implies that this agent $j$ wins the item with a negative utility of $v_j - t_j < 0$. So, the mechanism is SIC.

We turn the prove that every IR and SIC mechanism is a threshold mechanism. Assume, by contradiction, that there exists an IC and IR mechanism $M$ which is not a threshold mechanism. By Theorem 3, $M$ is not IC under Extreme SNE. Since $M$ is not ESIC, there exists a bid profile $b = (b_1, \ldots, b_n)$, an agent $i$, and a bid $b'_i \neq b_i$ such that either (i) $u_i((b'_i, b_{-i}); b_i) > u_i(b; b_i)$ or (ii) $u_i((b'_i, b_{-i}); b_i) = u_i(b; b_i)$ and there exists an agent $j$ such that $u_j((b'_i, b_{-i}); b_j) < u_i(b; b_j)$. Since $M$ is SIC, (i) cannot hold, and hence (ii) holds. Moreover, since$M$ is SIC, there exists an agent $k \neq i, j$ such that $u_k((b'_i, b_{-i}); b_k) > u_k(b; b_k)$. Without loss of generality, assume that $b_i < b'_i$.

Let $b' = (b'_i, b_{-i})$. Since $M$ is IR, we have $u_j(b; b_j) > u_j((b'_i, b_{-i}); b_j) \geq 0$. This implies that $A(b) = j$. Similarly, $u_k((b'_i, b_{-i}); b_k) > u_k(b; b_k) \geq 0$, which implies that $A(b'_i, b_i) = k$.

If there exists a bid $b''_i$ such that $u_l((b''_i, b_{-i}); b_l) = 0$ for all agents $l$, then $(b''_i, b_{-i})$ is a deviation from $(b'_i, b_{-i})$ that contradicts SIC. Thus, for every $b''_i \notin \{b_i, b'_i\}$, either $A(b''_i, b_{-i}) = i$ or the winner $A(b''_i, b_{-i})$ has a strictly positive utility.

Let $t_i = T_i(b_{-i})$. By definition and Lemma 1(3), for any $b''_i > t_i$ we have $A(b''_i, b_{-i}) = i$, and for any $b''_i < t_i$ we have $A(b''_i, b_{-i}) \neq i$. Hence, $b_i < b'_i \leq t_i$. If $b'_i < t_i$, then $(b'_i, t_i)$ is an open (non-empty) interval, and there exist two distinct bids $b''_i, b'''_i \in (b'_i, t_i)$ that yield the same winner: $A(b''_i, b_{-i}) = A(b'''_i, b_{-i})$. Because this winner has a strictly positive utility, we get a contradiction to Proposition 3, so we conclude that $b_i < b'_i = t_i$.

Since $t_i = b'_i$ and $u_k((b'_i, b_{-i}); b_k) > 0$, at the profile $(b'_i, b_{-i})$ agent $i$ can misreport a bid higher than $b'_i$, so that in the new profile agent $i$ wins the item and pays $t_i$ (as in Step 1 of Theorem 3), so her utility remain zero. In this case, the utility of agent $k$ strictly decreases, and the utility of every other agent remains 0, contradicting the SNE assumption. □