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A Stochastic Bubble Forest

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Abstract

We consider an economy with overlapping generations of relatively patient consumers who live for two periods. There is within-cohort heterogeneity in old-age endowments that depends on an aggregate state. That state is independent and identically distributed across generations. We assume consumers in their old age cannot be forced to give up any real resources. A stationary equilibrium in which state-contingent claims can be sold against the collateral of a single safe bubble security is efficient. The same allocation is also an equilibrium outcome in an economy with a sufficient number of stochastic bubble securities that can be traded subject to collateral constraints. When consumers cannot sell any securities short at all, the same efficient allocation can be implemented with a stochastic bubble forest: a continuum of Lucas trees that bear no fruit, with prices that evolve stochastically. Dynamic spanning is a potential rationale for the existence of distinct bubble assets.

1 Introduction

We describe an economy in which a proliferation of bubble securities can help risk sharing. Our economy has overlapping generations of two types of consumers who live for two periods. For each type, the endowments of young consumers are the same across generations. But the old-age endowments of the two types of consumers depend on an aggregate state that takes on two values. The distribution of that state is independent and identical across dates. Old consumers cannot be forced to give up any of their endowments. This means that risk sharing is possible only with some kind of trade across the generations.

*This is a report on work in progress. We welcome any comments or suggestions.

If consumers are sufficiently patient, then there is a constrained efficient allocation that is stationary and involves consumers in every generation giving up consumption when they are young in exchange for possibly state-dependent consumption when they are old. This allocation can be implemented by having competitive intermediaries sell state-contingent claims against the collateral of a bubble security that trades at a strictly positive price, which is the same every period. Young consumers can use these state contingent claims to target their savings for states in which their old-age endowments are particularly low.

Similarly, there is also a decentralization in which some consumers hold the safe bubble security and sell state contingent consumption in one of the two states, up to the safe collateral value of the bubble security they hold. That is, individual consumers are acting as intermediaries. In both these decentralizations, not only the allocation of consumption but also the equilibrium prices are stationary.

There are also decentralizations of the stationary constrained efficient allocation with non-stationary price processes. Suppose there are two bubble securities with prices that depend on the aggregate state—we will call this a stochastic bubble orchard.[1] Suppose further that a young consumer can sell a security short as long as that short position is fully collateralized by a long position in the other bubble security. More precisely, the value of the portfolio of this consumer must be non-negative in both of the two possible states that will be realized when the consumer is old.[2] One can imagine that consumers hold their bubble securities in brokerage accounts that allow for fully collateralized short sales of all types of securities that can be held in the account.

We show how to construct price processes for these bubble securities so that consumers face the exact same budget set as they do when trading state contingent claims with intermediaries. In our construction, the aggregate market value of the two stochastic bubbles is the same as the value of the safe bubble security in the intermediary decentralization. And the two bubble price processes are martingales under the risk-neutral probabilities generated by the state-contingent prices of that intermediary decentralization.

It is easy to see that short sales will be used in equilibrium when there are binding collateral constraints. In the special case in which the collateral constraints do not bind and the aggregate endowment is safe, there is perfect risk sharing and the risk neutral probabilities are just the actual probabilities for the aggregate state. So the two bubble price processes must be positive martingales that are bounded above by the price of the

[1]This terminology was inspired by "The Lucas Orchard" of Martin [2013]. The orchard will have more than two trees when there are more than the two aggregate states we have assumed here.

[2]Some authors use the term solvency constraints for the type of state-contingent collateral constraints we rely on here (Hindy [1995]).

safe bubble in the intermediary decentralization. Such martingales must converge almost surely. If it is the case that both types of consumers trade away from their endowments, then we can use this to prove that some consumers will eventually to have to sell bubble securities short. That is, short sales may well be necessary even if collateral constraints never bind.

This raises the question: what if the enforcement technology of our economy is so limited that the collateral constraints in the three decentralizations we have described so far cannot be relied upon? We show that the constrained efficient allocation can be implemented with a continuum of bubble securities —a stochastic bubble forest— that cannot be sold short at all. We construct an equilibrium in which the aggregate market value of the continuum of bubble securities is constant. But the prices of a fraction of these bubble securities collapse in unison, and those of the remaining bubble securities appreciate. The equilibrium price process of every individual bubble security follows a martingale under the risk-neutral probabilities and consumers can assemble portfolios of these securities with payoffs that are contingent only on the aggregate state. In this economy, enforcement only needs to allow for contemporaneous exchanges of goods and securities and prevent outright theft.

We now have four possible decentralizations of the stationary constrained efficient allocation of our economy. In each of these, there are bad equilibria. Autarky is always one of them. The scope for multiple equilibria expands as the enforcement regime weakens. In the intermediary economy, there is only one stationary allocation that is constrained efficient. In the decentralization with a stochastic bubble orchard, there are many risk-neutral martingale decompositions of the safe aggregate bubble that implement the stationary constrained efficient allocation. But, depending on parameters, there may also be an equilibrium in which the relative price of the two bubble securities is constant. This means markets are incomplete. Consumers simply earn a zero return on all their savings and the resulting consumption allocations may no longer be constrained efficient. The same outcome is possible in the stochastic forest economy. Furthermore, in that economy, there are, at every point in time, many ways to partition the set of securities still trading at a positive price.

The amount of coordination of beliefs required for equilibria in the stochastic bubble orchard and forest economies to lead to good outcomes is considerable. In the intermediary decentralization, consumers only have to believe that the price of the bubble security tomorrow is the same as it is today. Conditional on that belief, the required coordination is imposed by market clearing conditions. Of course, there is a large literature that studies the incentives of those who must, in a richer setting, operate intermediaries of the type that our decentralization relies upon.

The model we have described is motivated by the proliferation of cryptographic bubble securities that has been observed since the discovery of the blockchain technology. The price volatility of many of these securities seems to have led to the development of stablecoins, which are meant to trade at a fixed price against government-supplied currencies. The introduction of stablecoins that are not backed by anything is economically equivalent to counterfeiting. Stablecoins that are backed by government securities are best viewed as modern incarnations of money market mutual funds. Our interest is, instead, in cryptocurrencies such as bitcoin and ether whose prices have turned out to be quite volatile in terms of US dollars, not unlike the government supplied currencies of countries other than the United States. Our theory provides a possible rationale for these volatile bubble securities, even though that rationale is perhaps not what their originators intended.

Some Related Literature Our paper is a direct descendant of the exchange rate indeterminacy paper of Kareken and Wallace [1981]. In their two-country overlapping generations economy, useless pieces of paper issued by the two countries can have value, as in Samuelson [1958]. Focusing on deterministic equilibria, they show that the nominal exchange rate is indeterminate when consumers everywhere can trade in both currencies. Building on this idea, Manuelli and Peck [1990] construct a large class of stochastic exchange rate processes for a two-country overlapping generations economy. Garratt and Wallace [2018] emphasize that these exchange rate indeterminacy results fully apply in the context of a one-country overlapping generations economy with government money and bitcoin. In these papers, the implicit assumption is that within-generation risk-sharing is already perfect. As a result, stochastic bubble prices do not add risk-sharing opportunities. Schilling and Uhlig [2019] use an environment similar to that of the Townsend [1980] turnpike economy to model the coexistence of government money and bitcoin. In their model, too, the implicit assumption is that risk-sharing within type (that is, even or odd consumers) is perfect. We give an example of an economy in which the stochastic price processes of distinct bubble securities can be used to improve risk sharing, a possibility that was alluded to but not investigated in Ravikumar and Wallace [2002, p.21].

Levine [1991] describes an environment with an aggregate shock in which hidden information limits risk sharing. He shows that a single bubble security (interpreted as money) with a stochastic price process can give consumers more opportunities to share risk. The government can implement perfect risk sharing by selecting the appropriate rate at which it transfers new money to consumers. Our environment also limits risk sharing opportunities but has no lump-sum government transfers. Instead, we exploit the inherent indeterminacy of the relative prices of different bubble securities to give consumers the hedging opportunities they need.

Multiple equilibria are common in random matching models of decentralized exchange. Matsuyama, Kiyotaki, and Matsui [1993] and Ravikumar and Wallace [2002] show that a single currency may rule out inferior equilibria in a random matching environment with different groups of agents who meet at different rates. Kocherlakota and Krueger [1999] and Kiyotaki and Moore [2003] give examples of economies in which multiple currencies are useful, though not for the risk-sharing reasons highlighted in our paper.

There are strong incentives for individual agents in our economy to add to the supply of bubble securities. We have abstracted from this. A differentiation of beliefs about distinguishable components of the existing supply of bubble securities is enough to implement efficient allocations. Fernández-Villaverde and Sanches [2019] explicitly model the costly "mining" of competing cryptocurrencies.

2 The Economy

The economy has overlapping generations of consumers who live for two periods. Historical time is indexed by $t \in \{0, 1, \ldots\}$. In every generation, there are two types of consumers, denoted by $j \in \{K, L\}$. They could be the owners of various types of inelastically supplied factors of production. The measure of type-j consumers is $\mu_j \in (0, 1)$. There is one consumption good at every date. At all times, the initial endowments of young consumers of type-j are $Y_{y,j} > 0$. The endowments of type-j consumers who are old at time t are $Y_{o,j}(\eta_t) > 0$, where $\eta_t \in \{0, 1\}$ is an aggregate state that is realized at time t. The aggregate states are `iid` over time, and the probabilities of the states 0 and 1 are π_0 and $\pi_1 = 1 - \pi_0 \in (0, 1)$, respectively.

Everyone has the same preferences over consumption when young, C_y, and consumption in old age $C_o(\eta)$ when the aggregate state is $\eta \in \{0, 1\}$. These preferences are given by

$$\mathcal{U}(C_y, C_o(\cdot)) = (1 - \beta) \ln (C_y) + \beta \sum_{\eta \in \{0,1\}} \pi_\eta \ln(C_o(\eta)),$$

where $\beta \in (0, 1)$. Relative to the standard overlapping generations economy, our main assumption is that consumers in their old age cannot be forced to give up any amount of their endowments of consumption. This means that we can restrict attention to consumption allocations $\{C_{y,j}, C_{o,j}(\cdot)\}_{j\in\{K,L\}}$ that satisfy

$$C_{o,j}(\eta) \geq Y_{o,j}(\eta), \quad \eta \in \{0,1\}, \tag{1}$$

for both $j \in \{K, L\}$. These are simple lower bounds that restrict the consumption set relative to the more common $\mathbb{R}^3_+$.[3]

The constraint (1) means that any Pareto improvements over autarky must involve some reallocation of consumption across the generations. We are interested in versions of our economy in which trade across the generations can lead to Pareto improvements, taking into account the constraint (1). We are going to describe three distinct market structures for this economy that lead to the same stationary allocation, an allocation that is Pareto efficient subject to (1).

3 A Related Two-Period Storage Economy

To clearly understand the properties of our overlapping generations economy, it helps to first consider a very simple two-period economy that has a stochastic physical storage technology.[4] Consumer preferences and endowments are exactly as in the overlapping generations economy. And consumers in the second period cannot be forced to hand over any part of their second-period endowments. So the constraint on feasible consumption allocations (1) applies. The physical storage technology is linear with gross returns $\{R_\eta\}_{\eta\in\{0,1\}} \subset \mathbb{R}_{++}$.

3.1 Chartered Intermediaries

In this economy, assume that it is possible to charter intermediaries that can buy and store consumption in the first period by selling state contingent claims backed by the value of their stored consumption in the second period. That is, consumption stored by chartered intermediaries can serve as collateral. There is free entry into setting up chartered intermediaries.[5]

[3] Clearly, we make no attempt at generality. The simple structure of our model makes for easy proofs. In our two-period overlapping generations economy, the constraints (1) can be viewed as special cases of the constraints used in Kehoe and Levine [1993], Kocherlakota [1996], and Alvarez and Jermann [2000].

[4] The development of the overlapping generations economy starts in Section 4.

[5] We could let individual consumers store consumption and sell contingent claims against the collateral of stored consumption. This requires the additional assumption that that old consumers can be forced to give up stored consumption even though they cannot be forced to give up their old-age endowments of consumption.

Write $\{q_\eta\}_{\eta\in\{0,1\}}$ for the state-contingent prices, denominated in units of period-one consumption, of claims that deliver one unit of consumption in a particular state in the second period. Everyone takes these prices as given. Intermediaries maximize the profits from selling state contingent claims $X(\cdot)$ and storing A units of consumption,

$$\max_{X(\cdot),A\geq 0}\left\{\sum_{\eta\in\{0,1\}} q_\eta X(\eta) - A : X(\eta) \leq R_\eta A, \quad \eta\in\{0,1\}\right\}.$$

There can be no profits in any equilibrium. The state contingent claims prices must therefore satisfy

$$\sum_{\eta\in\{0,1\}} q_\eta R_\eta \leq 1. \tag{2}$$

If the storage technology is used in equilibrium, then profits have to be zero. In that case, (2) must hold with equality.

3.2 The Consumer Problem

Type-j consumers face the sequence of budget constraints

$$C_{y,j} + \sum_{\eta\in\{0,1\}} q_\eta X_j(\eta) \quad \leq \quad Y_{y,j}, \tag{3}$$

$$C_{o,j}(\eta) \quad \leq \quad Y_{o,j}(\eta) + X_j(\eta), \qquad 0 \leq X_j(\eta), \qquad \eta\in\{0,1\}. \tag{4}$$

Since utility is strictly increasing, these inequalities will always hold with equality. We can therefore restrict attention to the present-value budget constraint

$$C_{y,j} + \sum_{\eta\in\{0,1\}} q_\eta C_{o,j}(\eta) \leq Y_{y,j} + \sum_{\eta\in\{0,1\}} q_\eta Y_{o,j}(\eta) \tag{5}$$

together with the two inequalities given in (1). The optimal choices of a type-j consumer can be characterized by the first-order conditions,

$$\frac{1-\beta}{C_{y,j}} \times q_\eta \geq \frac{\beta\pi_\eta}{C_{o,j}(\eta)}, \qquad \text{w.e. if } C_{o,j}(\eta) > Y_{o,j}(\eta), \qquad \eta\in\{0,1\}, \tag{6}$$

together with the constraints (1) and (5).

The optimal consumer choices are functions of the state contingent prices $\{q_\eta\}_{\eta\in\{0,1\}}$. Because of the constraint (1), consumers can only be net buyers of consumption in the two states in the second period. It follows that individual utilities are weakly decreasing in the state contingent prices $\{q_\eta\}_{\eta\in\{0,1\}}$.

Explicit solutions to the consumer problem are reported in Appendix 1. The optimal consumption choices are continuous in $\{q_\eta\}_{\eta\in\{0,1\}} \subset \mathbb{R}_{++}$. As expected from the fact that utility is logarithmic, the optimal consumption expenditures are piecewise linear in $[q_0 Y_{o,j}(0), q_1 Y_{o,j}(1)]/Y_{y,j}$, on domains that are defined by linear inequalities. An example of these domains is shown in Figure 1.

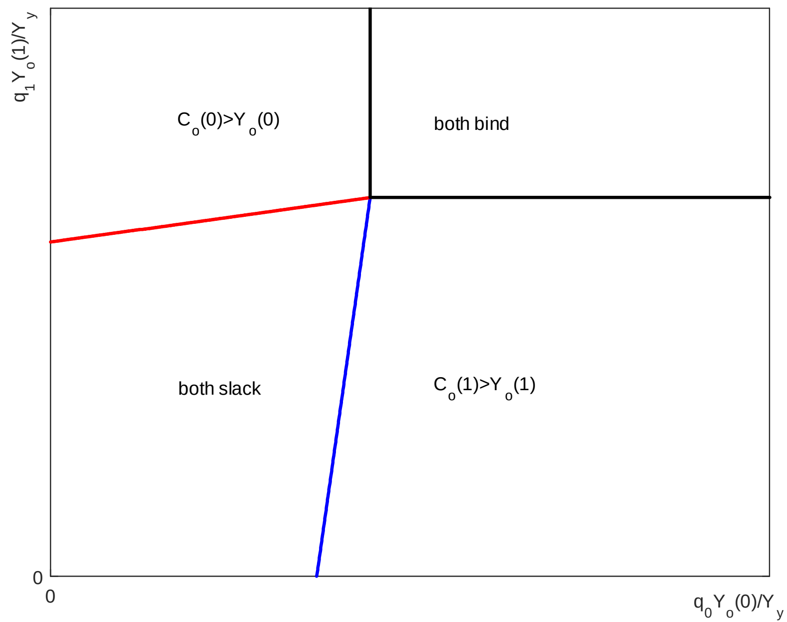

Figure 1 *The regions where the decision rules are linear.*

Intuitively, the constraint (1) in state η tends to bind for a consumer who has a large amount of wealth in that state. And $C_{o,j}(\eta) = Y_{o,j}(\eta)$ for both $\eta \in \{0,1\}$ when $\min\{q_0, q_1\}$ is large enough. In the interior of each of these domains, the demand curves satisfy the gross substitutes property

$$\frac{\partial C_{y,j}}{\partial q_\eta} \geq 0, \qquad \frac{\partial C_{o,j}(\eta)}{\partial q_\eta} \leq 0, \qquad \frac{\partial C_{o,j}(\kappa)}{\partial q_\eta} \geq 0, \qquad \kappa \neq \eta, \qquad \eta \in \{0,1\}.$$

Furthermore, $\lim_{q_\eta \downarrow 0} C_{o,j}(\eta) = \infty$ for both $\eta \in \{0,1\}$.

3.3 Competitive Equilibria

A competitive equilibrium is defined by prices $\{q_\eta\}_{\eta\in\{0,1\}}$ that satisfy (2), consumer choices determined by (1) and (5)-(6), and an aggregate level of storage $A \geq 0$ that satisfies

$$R_\eta A = \sum_{j\in\{K,L\}} \mu_j \left(C_{o,j}(\eta) - Y_{o,j}(\eta)\right), \qquad \eta \in \{0,1\}. \tag{7}$$

In addition, (2) must hold with equality if $A > 0$. Walras' law says that A will be equal to the aggregate savings in period one.

Suppose there is an equilibrium with $A = 0$. This means that the state prices are such that $(1-\beta)q_\eta/Y_{y,j} \geq \beta\pi_\eta/Y_{o,j}(\eta)$ for both $\eta \in \{0,1\}$ and both $j \in \{K, L\}$. Because of the lower bounds (1), increasing the state prices can only reduce the budget set of consumers. Therefore, if state prices are such that (2) holds as a strict inequality, then the same allocation of consumption will be an equilibrium allocation when state prices are raised in such a way that (2) holds as an equality. In other words, it is possible to restrict attention to prices for which (2) holds with equality when determining equilibrium allocations.

This observation allows us to determine the set of equilibrium allocations by finding prices q_0 and q_1 so that

$$\frac{1}{R_0}\sum_{j\in\{K,L\}}\mu_j\left(C_{o,j}(0)-Y_{o,j}(0)\right)=\frac{1}{R_1}\sum_{j\in\{K,L\}}\mu_j\left(C_{o,j}(1)-Y_{o,j}(1)\right) \tag{8}$$

together with

$$\sum_{\eta\in\{0,1\}} q_\eta R_\eta = 1, \tag{9}$$

for consumption choices $\{C_{y,j}, C_{o,j}(\cdot)\}_{j\in\{K,L\}}$ that solve (1) and (5)-(6). Given the solution to the consumer problem, this gives us two equilibrium conditions than must be solved for two state prices. The linear restriction (9) can be used to eliminate one of these state prices, and then (8) must be solved for the remaining state price.

Lemma 1 *The two-period storage economy has at least one competitive equilibrium. The equilibrium allocation is unique and Pareto efficient.*

Proof: The standard proof can be used to show that competitive equilibria are Pareto optimal. The dependence of the solution $[C_{y,j}, C_{o,j}(0), C_{o,j}(1)]$ to the consumer problem on $q_1 = (1-q_0R_0)/R_1$ implies that the difference between the left- and right-hand sides of (8) is a function a function q_0. Write this function as $f(q_0)$. Then $f(q_0)$ is very large and positive if q_0 is close to zero, and very large and negative if q_1 is very close to 0. The function $f(\cdot)$ is continuous, and so there must be an equilibrium. The function $f(\cdot)$ is also monotone. If the A implied by (7) is strictly positive, then someone is not at both corners. In turn, that means that the slope of $f(q_0)$ is not zero at an equilibrium q_0. The monotonicity then implies that there can only be one equilibrium with $A > 0$. If the A implied by (7) is zero, then no trade is an equilibrium. If there were also an equilibrium with $A > 0$, then the consumers who trade in the equilibrium with $C_{o,j}(\eta) > Y_{o,j}(\eta)$ for some $\eta \in \{0,1\}$ must be strictly better off than in the $A = 0$ equilibrium. Consumers who do not trade cannot be worse off. This contradicts the Pareto optimality of equilibria.■

It will be useful to know under what conditions equilibrium storage is strictly positive.

Lemma 2 *The equilibrium allocation implies strictly positive storage if and only if the economy satisfies*

$$\frac{\beta}{1-\beta}\sum_{\eta\in\{0,1\}}\pi_\eta R_\eta \max_{j\in\{K,L\}}\left\{\frac{Y_{y,j}}{Y_{o,j}(\eta)}\right\}>1. \tag{10}$$

Proof: By Lemma 1, there is a unique level of equilibrium storage A. Suppose $A>0$. This implies $R_\eta A>0$ for both $\eta\in\{0,1\}$. Therefore, for each $\eta\in\{0,1\}$, there must be at least one consumer type j for whom $(1-\beta)q_\eta/Y_{y,j}<\beta\pi_\eta/Y_{o,j}(\eta)$. Together with the fact that (2) has to hold with equality because $A>0$, this implies (10). Conversely, suppose $A=0$, and hence $R_\eta A=0$ for both $\eta\in\{0,1\}$. This implies $(1-\beta)q_\eta/Y_{y,j}\geq\beta\pi_\eta/Y_{o,j}(\eta)$ for both $\eta\in\{0,1\}$ and both $j\in\{K,L\}$. Together with (2), this contradicts (10).■

When returns change from $\{R_\eta\}_{\eta\in\{0,1\}}$ to $\{R'_\eta\}_{\eta\in\{0,1\}}$, a Pareto improvement relative to the equilibrium for $\{R_\eta\}_{\eta\in\{0,1\}}$ is a possibility only if $R'_\eta>R_\eta$ for at least one $\eta\in\{0,1\}$. Otherwise the equilibrium for $\{R_\eta\}_{\eta\in\{0,1\}}$ would not be efficient. Also, if the equilibrium savings in the equilibrium for $\{R_\eta\}_{\eta\in\{0,1\}}$ are strictly positive, then $R'_\eta\leq R_\eta$ and strictly for at least one $\eta\in\{0,1\}$ means that someone must be worse off in the economy with returns $\{R'_\eta\}_{\eta\in\{0,1\}}$ than in the equilibrium for $\{R_\eta\}_{\eta\in\{0,1\}}$.

4 Overlapping Generations With Intermediaries

We now return to the overlapping generations economy. Assume the initial old are endowed with $D>0$ units of a bubble security that is not a claim to any real resources. The price of this security at time $t\in\{0,1,\ldots\}$ is $s_t\geq 0$. As in the two-period storage economy, we assume it is possible to charter intermediaries that sell state contingent claims to buy assets in competitive markets. Here, these intermediaries buy the bubble security, and young consumers buy state contingent claims sold by these intermediaries. Our maintained assumption is that the bubble security holdings of chartered intermediaries can serve as collateral.

4.1 Bubble Securities as Collateral

We restrict attention to price trajectories of the form $s_t=S_{t-1}(\eta_t)$, where $S_{t-1}(\cdot)$ is a function that depends on the entire history $\{\eta_a\}_{a=-\infty}^{t-1}$. This abstracts from sunspot equilibria.

Write $Q_t(\eta)$ for the time-t price of a claim that delivers to the old consumer who holds it at time $t+1$ one unit of consumption when the time-$t+1$ state is $\eta \in \{0,1\}$. Intermediaries that enter at time t settle their claims at time $t+1$ and exit. New intermediaries enter every period. Intermediaries at time t solve

$$\max_{x(\cdot)\geq 0} \left\{ \sum_{\eta\in\{0,1\}} Q_t(\eta)x(\eta) - s_t : x(\eta) \leq S_t(\eta), \eta \in \{0,1\} \right\}. \tag{11}$$

That is, an intermediary that buys one unit of the bubble security can sell claims contingent on the realization of η_{t+1} up to the value of the bubble security at time $t+1$. In any equilibrium, the resulting profits cannot be positive, and therefore

$$\sum_{\eta\in\{0,1\}} Q_t(\eta)S_t(\eta) \leq s_t. \tag{12}$$

In any equilibrium with $s_t > 0$, the inequality (12) will have to be an equality, or else intermediaries will not enter the market. Market clearing requires that $S_t(\eta)D$ is equal to the face value of the claims contingent on the event $\eta_{t+1} = \eta$ purchased by consumers who are young at time t.

4.2 Stationary Equilibria

We are interested in stationary equilibria with prices of the form

$$S_t(\eta) = s, \quad Q_t(\eta) = q_\eta, \quad \eta \in \{0,1\}. \tag{13}$$

Given the state prices q_0 and q_1, young consumers in this economy face the exact same decision problem as in the two-period storage economy. Their decision rules are again determined by (1) and (5)-(6).

If $s = 0$, then the state contingent prices must be so that young consumers do not want to trade. If $s > 0$, then intermediaries hold the aggregate bubble $sD > 0$ and consumers must be willing to buy the state contingent claims sold by intermediaries to finance their holdings of the bubble security. Furthermore, (12) has to hold with equality at the prices (13). Since the bubble price is constant, this forces $q_0 + q_1 = 1$. That is, the price of the implied pure discount bond must be equal to 1. The risk-free rate is zero.

There is always a no-bubble equilibrium, with $s = 0$. Because of the constraints (1), it implies autarky. Trade is only possible in stationary equilibria if s is positive. In that case, intermediaries buy the bubble security from old consumers and sell state contingent claims to young consumers. In the next period intermediaries either continue by issuing new state contingent claims or by selling their bubble security holdings to other intermediaries. Either way, their decision problem is exactly that of intermediaries in the two-period storage economy with returns $R_0 = R_1 = 1$. In light of the positive storage condition (10), this motivates the following assumption about the parameters of the overlapping generations economy,

$$\frac{\beta}{1-\beta} \sum_{\eta \in \{0,1\}} \pi_\eta \max_{j \in \{K,L\}} \left\{ \frac{Y_{y,j}}{Y_{o,j}(\eta)} \right\} > 1. \tag{CC}$$

This condition is the opposite of the high-interest rate assumption adopted by Alvarez and Jermann [2000]. We now have the following proposition.

Proposition 1 *The economy with competitive intermediaries has a stationary equilibrium in which $sD > 0$ if and only if the economy satisfies (CC). There can be at most one equilibrium with $sD > 0$.*

Proof: Suppose there is a stationary equilibrium with $s > 0$. Then the implicit storage returns are $R_\eta = 1$ for both $\eta \in \{0, 1\}$. Applying (10) for these returns gives (CC). Conversely, suppose (CC) holds, and hence (10) as well for $R_\eta = 1$ in both states $\eta \in \{0, 1\}$. By Lemma 1, the implicit storage economy has a unique equilibrium allocation. By Lemma 2, that allocation has positive storage. In the overlapping generations economy, this implies $sD > 0$.■

4.2.1 An Equivalent Market Structure

In the equilibrium described in Proposition 1, an individual consumer who holds a claim on s units of consumption contingent on the event $\eta_{t+1} = 0$ is effectively long one unit of the bubble security and short s units of consumption contingent on the event $\eta_{t+1} = 1$. Rather than having an intermediary hold one unit of the bubble security and selling s units of state-contingent consumption in the two possible states to (at least) two different consumers, one can have an individual consumer hold one unit of the bubble security and sell s units of consumption contingent on the event $\eta_{t+1} = 1$ to other consumers.

Such an arrangement can work, and will respect the constraint (1), if there are brokerages that can keep track of the individual consumer holdings of the bubble security and their promises to deliver state-contingent consumption at the next date. These brokerages are not counterparties but do serve to enforce collateral constraints. We will say more about brokerages below.

4.2.2 Pareto Efficiency

A planner in the overlapping generations economy can choose a sequence of aggregate transfers from young to old consumers $\{b_t\}_{t=0}^{\infty}$. At date t, b_t can be a function of $\{\eta_s\}_{s=0}^{t}$. Within each generation, the planner can assign consumption subject to the resources available, and the constraints (1). Because of these constraints, the b_t have to be non-negative at all times. And these b_t can never exceed the contemporaneous aggregate endowment of the young consumers.

In the equilibrium of Proposition 2, we have $b_t = b > 0$. Given this b, an important feature of the equilibrium is that the consumption allocation for consumers born at date t only varies with the η_{t+1} that will be realized in their old age. It does not depend on the history $\{\eta_s\}_{s=0}^{t}$. In particular, there is no risk sharing across generations. We will prove that the planner cannot do better. That is, there is no trajectory $\{b_t\}_{t=0}^{\infty}$, possibly stochastic, that constitutes a Pareto improvement.

To prove this, it suffices to show that there is no deterministic Pareto improving trajectory $\{b_t\}_{t=0}^{\infty}$. If it were really the case that there is a stochastic Pareto improvement, then there would also have to be a deterministic Pareto improvement. To see this, suppose that $\{b_t\}_{t=0}^{\infty}$ is a Pareto improvement with $b_{t+1} = B_t(\eta_{t+1})$ for some t, where $B_t(\cdot)$ is a function of $\{\eta_s\}_{s=0}^{t}$ that satisfies $B_t(1) > B_t(0)$. By definition, the continuation $\{b_s\}_{s=t+1}^{\infty}$ cannot make consumers born at any $s \geq t+1$ worse off than in the equilibrium allocation, for either of the two possible realizations of $\eta_{t+1} \in \{0,1\}$. Therefore, following each of the possible realizations of $\eta_{t+1} \in \{0,1\}$, the planner could use either the planned continuation for the event $\eta_{t+1} = 0$ or the planned continuation for the event $\eta_{t+1} = 1$. Both continuations are such that nobody born at any $s \geq t+1$ will be worse off than in the equilibrium allocation. Therefore, the planner can use the continuation that starts with $B_t(1)$ following both possible realizations of $\eta_{t+1} \in \{0,1\}$. Since $B_t(1) > B_t(0)$, this will actually make some consumers born at t better off. The result is a new Pareto improvement $\{b'_s\}_{s=0}^{\infty}$. Starting from this new Pareto improvement, we can repeat this procedure at $t+1$. And so on. The result is a deterministic Pareto improvement.[6]

Based on these considerations, we can now prove the desired result.

Proposition 2 *Assume the economy satisfies (CC). Then the resulting stationary equilibrium with $sD > 0$ is Pareto efficient.*

Proof: We have already argued that we can restrict attention to potentially Pareto improving allocations that are deterministic. Conjecture that the deterministic trajectory

[6] The same procedure can be used to disprove conjectured Pareto improvements that depend on sunspot states.

$\{b_t\}_{t=0}^{\infty}$ is a Pareto improvement. Recall that in two-period storage economies, $R'_\eta < R_\eta$ for both $\eta \in \{0,1\}$ implies that for at least one type of consumer welfare is lower in the R'_η economy than in the R_η economy. Therefore, if $b_{t+1} < b_t$ for some t, then at least someone in the generation born at time t must be worse off than in the stationary equilibrium. So it must be that $b_{t+1} \geq b_t$ for all $t \in \{0, 1, \ldots\}$. Since the initial old cannot be made worse off, we can infer that $b_t \geq b$ for all $t \in \{0, 1, \ldots\}$.

Given these constraints, the only way that someone in some generation $T \in \{0, 1, \ldots\}$ can be made strictly better off without hurting others is if $b_{T+1} > b_T \geq b$. So $\{b_t\}_{t=0}^{\infty}$ is a non-decreasing sequence that will eventually be in (b, ∞). Feasibility requires that this sequence is bounded above by the aggregate endowment of young consumers. Therefore, this sequence converges to some limit $b_\infty \geq b_{T+1} > b$. But that would mean that the two-period storage economy has a storage level $b_\infty > b$ with utilities that are no less for any type of consumer than they are at the storage level b. A strict convex combination of these two allocations would contradict the two-period Pareto optimality result of Lemma 1.■

4.3 Two Very Familiar Benchmarks

Consumers in our economy face complete markets subject to non-negativity constraints. It is interesting to compare this economy with two familiar benchmarks.

4.3.1 Incomplete Markets and no Borrowing

If there are no intermediaries that can sell state contingent claims against the collateral of a bubble security, and assuming that consumers cannot themselves use the bubble security as collateral, then there may still be a stationary equilibrium in which consumers directly trade a bubble security across the generations. In such an equilibrium, markets are incomplete. And consumers can still not borrow, because of (1).

As before, $s = 0$ implies autarky. If $s > 0$, young consumers can save at a safe zero rate of return. At least some of them are willing to do so if and only if

$$\frac{\beta}{1-\beta} \max_j \left\{ \sum_{\eta \in \{0,1\}} \pi_\eta \times \frac{Y_{y,j}}{Y_{o,j}(\eta)} \right\} > 1. \tag{IM}$$

To construct the resulting incomplete markets equilibrium, simply solve the relaxed first-order conditions

$$\frac{1-\beta}{Y_{y,j} - b_j} = \beta \sum_{\eta \in \{0,1\}} \frac{\pi_\eta}{Y_{o,j}(\eta) + b_j}, \qquad j \in \{K, L\}$$

for $b_K \in (-\infty, \infty)$ and $b_L \in (-\infty, \infty)$. The solutions are well defined and unique. Then define the price of the bubble security via

$$sD = \sum_{j \in \{K,L\}} \mu_j \max\{0, b_j\}.$$

Because of condition (IM), at least one of the two b_j will be strictly positive, and hence sD will be strictly positive. If condition (IM) fails, then both b_j are zero or negative. In that case, the constraint (1) says that both consumers will choose not to trade. This implies a zero demand for the bubble security, and hence s must be zero.

4.3.2 Frictionless Complete Markets

For an alternative benchmark, suppose young consumers can sell state contingent claims against their old-age endowments. So the constraint (1) does not apply. The result is an entirely standard overlapping generations economy with complete markets. In each generation, there is a representative consumer. Autarky is again a possible equilibrium. There is a stationary equilibrium with a strictly positive bubble if and only if

$$\frac{\beta}{1-\beta} \sum_{\eta \in \{0,1\}} \pi_\eta \times \frac{\sum_{j \in \{K,L\}} \mu_j Y_{y,j}}{\sum_{j \in \{K,L\}} \mu_j Y_{o,j}(\eta)} > 1. \tag{CM}$$

This is the familiar requirement that the slope of the offer curve of the representative agent at autarky must be less than 1.

4.3.3 Comparing Conditions for a Strictly Positive Bubble

The following lemma allows us to compare the necessary and sufficient conditions for a strictly positive bubble across market structures.

Lemma 3 *The conditions (CC), (IM), and (CM) are related via*

$$\sum_{\eta \in \{0,1\}} \pi_\eta \max_{j \in \{K,L\}} \left\{ \frac{Y_{y,j}}{Y_{o,j}(\eta)} \right\} \geq \max_j \left\{ \sum_{\eta \in \{0,1\}} \pi_\eta \times \frac{Y_{y,j}}{Y_{o,j}(\eta)} \right\} \geq \sum_{\eta \in \{0,1\}} \pi_\eta \times \frac{\sum_{j \in \{K,L\}} \mu_j Y_{y,j}}{\sum_{j \in \{K,L\}} \mu_j Y_{o,j}(\eta)}. \tag{14}$$

And these inequalities can be strict.

The maximization on the left-hand side of the first inequality is simply a relaxed version of the maximization on the right-hand side. As detailed in the appendix, the second inequality is a consequence of the fact that the maximum of a convex and continuous function on a convex and compact domain is attained at an extreme point of that domain.

The left-hand sides of (CC), (IM), and (CM) can be interpreted to be the equilibrium prices of one-period discount bonds in the autarky equilibria that emerge when the price of the bubble security is zero.[7] So the inequalities in Lemma 3 say that the no-bubble interest rates are ranked: the no-bubble interest rate is lowest in the intermediary economy, highest in the frictionless complete markets economy, and in between these two extremes in the economy with (1) and incomplete markets. The idea that incomplete markets equilibria tend to result in low interest rates relative to frictionless complete markets equilibria is of course extremely familiar. The fact that the autarky equilibrium in an intermediary economy produces an even lower interest rate arises because consumers can better target their non-negative savings across states of the world than they can in the incomplete markets economy.

A corollary is that an economy may well have a stationary intermediary equilibrium with a strictly positive bubble, but no stationary incomplete markets equilibrium with a strictly positive bubble. Or those two market structures could both admit stationary equilibria with a strictly positive bubble, while a frictionless complete markets environment does not.

In this last scenario, an immediate corollary is that the initial old consumers are better off in the intermediary and incomplete markets equilibria with a strictly positive bubble than they are in the frictionless complete markets equilibrium. Because the equilibrium in the latter economy is Pareto efficient, it must be that the young in the other two economies are worse off.

5 The Bubble Orchard

Consider again the overlapping generations economy in which old consumers cannot be forced to hand over any part of their endowments. So consumption allocations must satisfy (1). Assume the economy satisfies (CC), so that Proposition 2 delivers a Pareto efficient equilibrium if there are intermediaries or consumers who can sell state contingent claims against the collateral of a safe bubble security. It turns out that state contingent claims are not needed once we introduce multiple bubble securities and consider bubble prices that evolve stochastically over time.

[7] In the complete markets case, "autarky" means autarky for the representative agent. That is, there is perfect risk-sharing within every generation but no trade across the generations.

5.1 Two Stochastic Bubble Securities

Suppose there are two distinct bubble securities that are traded directly by consumers. Write $D > 0$ for the aggregate supply of one bubble security, and $B > 0$ for the aggregate supply of the other. The initial old hold the aggregate supply of both. To be concrete, interpret D to be the supply of nominal government debt of a government that runs a balanced budget and sets its nominal interest rate equal to zero. And interpret B to be a fixed supply of bitcoin.

At time t, the price of nominal government debt is $s_t \geq 0$ and the price of bitcoin is $p_t \geq 0$, both in units of consumption. We restrict attention to price processes $\{s_t\}_{t=0}^{\infty}$ and $\{p_t\}_{t=0}^{\infty}$ that are of the form $s_{t+1} = S_t(\eta_{t+1})$ and $p_{t+1} = P_t(\eta_{t+1})$, where $S_t(\cdot)$ and $P_t(\cdot)$ are non-negative functions that only depend on $\{\eta_s\}_{s=-\infty}^{t}$. That is, date-$t+1$ prices do not depend on sunspots. They only depend on the history up to and including date t, and on the realization of η_{t+1}.

5.1.1 Brokerages and Collateral Constraints

There are no intermediaries that sell state contingent claims to consumers. Instead, we assume there are brokerages that can facilitate collateralized short sales of bubble securities. Young consumers at time t can choose to go short in one of the bubble securities as long as the market value of their portfolio of bubble securities at time $t+1$ is non-negative for both of the possible realizations of η_{t+1}. In other words, we assume that consumers are able to use one bubble security as collateral for a short position in the other. These collateral constraints ensure that consumption allocations again respect the constraint (1) that old consumers can never be forced to give up any of their old-age endowments.

To be more precise, type-j young consumers at time t can choose current consumption $C_{y,j,t}$, state-contingent old-age consumption $C_{o,j,t+1}(\cdot)$, and holdings of $D_{j,t}$ units of government debt and $B_{j,t}$ units of bitcoin subject to the sequence of budget constraints

$$C_{y,j,t} + s_t D_{j,t} + p_t B_{j,t} \leq Y_{y,j}, \tag{15}$$

$$C_{o,j,t+1}(\eta) \leq Y_{o,j}(\eta) + S_t(\eta) D_{j,t} + P_t(\eta) B_{j,t}, \quad \eta \in \{0,1\}, \tag{16}$$

and subject to the collateral constraints

$$0 \leq S_t(\eta) D_{j,t} + P_t(\eta) B_{j,t}, \quad \eta \in \{0,1\}. \tag{17}$$

We interpret $B_{j,t} < 0$, say, as a promise of consumer j to deliver $-B_{j,t}$ units of bitcoin at $t+1$. The collateral constraint (17) ensures that consumer j can sell enough government debt at $t+1$ to fulfil this promise. The brokerages in which consumers hold their bubble securities are assumed to be able to enforce these constraints without acting as a counterparty. Other consumers are the counterparties, and they treat promises to deliver bitcoin as equivalent to bitcoin because such promises are secured by (17).

Market clearing now requires that aggregate consumption matches aggregate endowments, and that consumers are willing to hold the aggregate supply of bubble securities. We again restrict attention to equilibria that give rise to stationary consumption allocations. As in the economy with intermediaries, one such equilibrium is autarky, with the two bubble securities each trading at a zero price.

5.1.2 The Incomplete Markets Allocation Again

If we assume that the economy satisfies (IM), then another possible equilibrium is simply one in which both bubble securities prices are positive and constant over time, with $s_t/p_t \in (0,\infty)$. The resulting matrix of returns is singular. This leads to the incomplete markets equilibrium described earlier. With two bubble securities, we now have a version of the Kareken and Wallace [1981] indeterminacy: the constant relative price s_t/p_t can be anything. Along the lines of Manuelli and Peck [1991], one can generalize this by letting $s_t D = \xi_t b$ and $p_t B = (1-\xi_t)b$, where $b > 0$ is the aggregate value of the bubble in the incomplete markets equilibrium, and $\{\xi_t\}_{t=0}^{\infty}$ is a martingale in $(0,1)$ that only depends on a sunspot. In such an equilibrium, individual consumers can simply hold the safe market portfolio of bubble securities. The sunspot martingale adds no risk-sharing opportunities and creates risk that is perfectly diversifiable.

5.1.3 Implementing Efficient Stationary Allocations

Now assume that the economy satisfies (CC). The stationary Pareto efficient equilibrium consumption allocation we obtained for the intermediary economy is also an equilibrium consumption allocation in an economy with two bubble securities in which the collateral constraints (17) are enforced. Let $\{q_\eta\}_{\eta\in\{0,1\}} \subset (0,1)$ be the equilibrium state-contingent prices defined by the equilibrium given in Proposition 2. Since $q_0 + q_1 = 1$, these state prices are also risk-neutral probabilities. Also, let $b > 0$ be the aggregate market value of the bubble security in Proposition 2.

Fix some $\xi_0 \in (0,1)$ and let $\xi_t = \Xi_{t-1}(\eta_t)$ for all $t \in \mathbb{N}$, where $\Xi_t(\cdot)$ is defined by

$$\Xi_t(\eta) = \xi_t + \begin{cases} -d_t & \text{if } \eta = 0 \\ u_t & \text{if } \eta = 1 \end{cases} . \tag{18}$$

Here, d_t and u_t are functions of $\{\eta_s\}_{s=0}^t$ that are almost completely arbitrary. The only requirement is that they satisfy

$$d_t \in (0, \xi_t), \quad u_t \in (0, 1-\xi_t), \quad q_0 d_t = q_1 u_t. \tag{19}$$

The upper and lower bounds on d_t and u_t ensure that $\{\xi_t\}_{t=0}^\infty \subset (0,1)$. Write $\mathbb{E}_t[\cdot]$ for the conditional expectation calculated using the risk-neutral probabilities $\{q_\eta\}_{\eta\in\{0,1\}}$. The construction of $\{\xi_t\}_{t=0}^\infty$ given in (18)-(19) ensures that

$$\mathbb{E}_t\left[\xi_{t+1}\right] = \mathbb{E}_t\left[\Xi_t(\eta_{t+1})\right] = \sum_{\eta\in\{0,1\}} q_\eta \Xi_t(\eta) = \xi_t.$$

In other words, $\{\xi_t\}_{t=0}^\infty \subset (0,1)$ is a martingale under the risk neutral probabilities—a risk-neutral martingale. A concrete example of d_t and u_t that satisfy (19) is

$$\begin{bmatrix} d_t \\ u_t \end{bmatrix} = \min\left\{\frac{1-\xi_t}{q_0}, \frac{\xi_t}{q_1}\right\} \times \alpha \begin{bmatrix} q_1 \\ q_0 \end{bmatrix} \tag{20}$$

where $\alpha \in (0,1)$ is a constant parameter.

Now take the bubble prices to be $s_t D = \xi_t b$ and $p_t B = (1-\xi_t)b$ for all $t \in \{0,1,\ldots\}$. This means that the price functions that appear in (16) and (17) are $S_t(\cdot)D = \Xi_t(\cdot)b$ and $P_t(\cdot)B = (1-\Xi_t(\cdot))b$ for all $t \in \{0,1,\ldots\}$. The matrix of date-$t+1$ payoffs implied by (18) is then

$$\begin{bmatrix} S_t(0) & P_t(0) \\ S_t(1) & P_t(1) \end{bmatrix} = b \begin{bmatrix} \xi_t - d_t & 1-\xi_t+d_t \\ \xi_t+u_t & 1-\xi_t-u_t \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & B \end{bmatrix}^{-1}. \tag{21}$$

As expected, the determinant $-(d_t+u_t)b^2/(DB) < 0$ of the matrix on the right-hand side is non-zero. So markets are complete at every date and in every state. Because of this, the closed convex cone of $D_{j,t}$ and $B_{j,t}$ that satisfy (17) can be used to generate any state-contingent payoff vector $X_{j,t}(\cdot) = S_t(\cdot)D_{j,t} + P_t(\cdot)B_{j,t}$ in $\mathbb{R}^2_+$. For such payoff vectors, the risk-neutral martingale property of $\{\xi_t\}_{t=0}^\infty$ implies

$$\sum_{\eta\in\{0,1\}} q_\eta X_{j,t}(\eta) = \mathbb{E}_t\left[S_t(\eta_{t+1})D_{j,t} + P_t(\eta_{t+1})B_{j,t}\right] = s_t D_{j,t} + p_t B_{j,t}.$$

Therefore, the budget set defined by (15)-(16) is the same as the budget set (3)-(4) in the intermediary economy. This proves our next proposition.

Proposition 3 *Assume the economy satisfies (CC). If consumers can hold bubble securities in brokerage accounts subject to the collateral constraints (17), then the economy has an equilibrium* *with the same consumption allocation as the Pareto efficient allocation constructed in Proposition 2.*

If condition (IM) holds, then our economy has three distinct stationary equilibria: autarky, an incomplete markets equilibrium, and an equilibrium that delivers the Pareto optimal allocation of Proposition 2. In contrast to what happens in the intermediary economy, there are now two equilibria with a strictly positive aggregate bubble. Interestingly, if the economy satisfies (CC) but not (IM), then there is no stationary equilibrium with a single bubble security that trades at a constant and strictly positive price. The only stationary equilibrium with a strictly positive aggregate bubble that survives is the one described in Proposition 3.

5.2 Are Short Sales Necessary?

If there is only one type of consumer, then there are no opportunities for sharing risk. Conditions (CC) and (IM) are the same, and the stationary equilibrium in which consumers hold a safe bubble security leads to a Pareto efficient allocation. Consumers do not go short in bubble securities. The same conclusion applies when, in this economy, there are multiple stochastic bubble securities subject to short-sale constraints, as in Manuelli and Peck [1991]. And it applies when (1) binds in both states for one of the two types of consumers.

Alternatively, if the economy satisfies (CC) and the equilibrium allocation described in Proposition 2 implies that the constraints (1) bind for some consumers in exactly one state, then some of the collateral constraints (17) will also bind. That then immediately implies that short sales are necessary in the economy with two stochastic bubbles.

Here we show that even if the collateral constraints (17) never bind, short sales may well be necessary in an economy with two stochastic bubbles. Consider the example of an economy in which the old-age endowments are risky for both types of consumers, but not in the aggregate. Suppose further that $C_{o,j}(\eta) > Y_{o,j}(\eta)$ for both $\eta \in \{0,1\}$ and $j \in \{K, L\}$ in the intermediary allocation of Proposition 2. That is, the constraints (1) do not bind. This means that consumers in a given generation will perfectly share risk in the intermediary economy. Their old-age consumption will be risk-free. It follows that the state prices must be $q_\eta = \pi_\eta$ for both $\eta \in \{0,1\}$. Risk-neutral probabilities are actual probabilities. All of this means that the economy must satisfy not just (CC) or (IM) but also (CM).

In a version of this economy that has two stochastic bubble securities, it may well be the case that early generations of consumers do not take short positions in either bubble asset. The ex post returns can easily be different enough to allow perfect risk sharing with strictly positive portfolio weights. But $\{\xi_t\}_{t=0}^{\infty} \subset (0,1)$ is now a martingale under the actual probabilities $\{\pi_\eta\}_{\eta\in\{0,1\}}$. Such martingales must converge almost surely. As we are now going to show, this implies that, almost surely, future generations will eventually have to go short in one of the two bubble securities.

To prove this, let $\mathcal{R}_t$ be the return matrix for consumers who are young at time t. From (21), since $[s_t D, p_t B] = [\xi_t, 1-\xi_t]b$,

$$\mathcal{R}_t = \begin{bmatrix} S_t(0)/s_t & P_t(0)/p_t \\ S_t(1)/s_t & P_t(1)/p_t \end{bmatrix} = \begin{bmatrix} 1-\frac{d_t}{\xi_t} & 1+\frac{d_t}{1-\xi_t} \\ 1+\frac{u_t}{\xi_t} & 1-\frac{u_t}{1-\xi_t} \end{bmatrix}. \tag{22}$$

Type-j consumers at time t then choose $[s_t D_{j,t}, p_t B_{j,t}]$ to solve

$$\begin{bmatrix} C_{o,j}(0) - Y_{j,o}(0) \\ C_{o,j}(1) - Y_{j,o}(1) \end{bmatrix} = \mathcal{R}_t \begin{bmatrix} s_t D_{j,t} \\ p_t B_{j,t} \end{bmatrix}, \tag{23}$$

where $\{C_{o,j}(\cdot)\}_{j\in\{K,L\}}$ is the equilibrium consumption allocation for the intermediary economy. Because we are assuming that risk sharing is perfect, this consumption allocation is of the form

$$\begin{bmatrix} C_{o,j}(0) \\ C_{o,j}(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} C_{o,j} \tag{24}$$

for some $C_{o,K} > 0$ and $C_{o,L} > 0$.

As already noted, the construction (18)-(19) means that the return matrix $\mathcal{R}_t$ is always non-singular. Its determinant is $-(u_t + d_t)/[(1-\xi_t)\xi_t] < 0$. The bounds on the up and down increments u_t and d_t imply that all the entries of $\mathcal{R}_t$ are strictly positive. Non-negative portfolio weights for the two bubble securities therefore define a cone of payoffs that is a proper subset of $\mathbb{R}^2_{++} \cup \{0\}$. Payoffs in $\mathbb{R}^2_{++}$ but outside that cone require short positions. Solving (23) for $[s_t D_{j,t}, p_t B_{j,t}]$ and using (22) as well as (24) gives

$$\begin{bmatrix} s_t D_{j,t} \\ p_t B_{j,t} \end{bmatrix} = a_t \begin{bmatrix} -1 \\ 1 \end{bmatrix} [Y_{o,j}(1) - Y_{o,j}(0)] + \begin{bmatrix} \xi_t \\ 1-\xi_t \end{bmatrix} z, \tag{25}$$

where the coefficients a_t and z are given by

$$a_t = \frac{(1-\xi_t)\xi_t}{u_t + d_t}, \qquad z = C_{o,j} - \sum_{\eta\in\{0,1\}} \pi_\eta Y_{j,o}(\eta).$$

So $-a_t > 0$ is the determinant of $\mathcal{R}_t^{-1}$. This coefficient will be large if the return matrix is close to singular. Our assumption that $Y_{o,j}(0) \neq Y_{o,j}(1)$ and that the aggregate endowment is safe implies that the $Y_{o,j}(1) - Y_{o,j}(0)$ are of opposite signs for $j \in \{K, L\}$. So the first term in (25) will have large negative entries for some $j \in \{K, L\}$ if the return matrix is close to singular. Since we are assuming that $C_{o,j}(\eta) > Y_{o,j}(\eta)$ for both $\eta \in \{0, 1\}$ and both $j \in \{K, L\}$, the coefficient z that determines the scale of the second term in (25) is strictly positive. This term will have a positive near-zero entry if ξ_t is close to 0 or 1. In that case, the first term of (25) can produce negative solutions for $s_t D_{j,t}$ or $p_t B_{j,t}$ even if $\mathcal{R}_t$ is not close to singular.

To find out whether it is possible avoid short sale constraints, we need to know more precisely what can be said about the coefficients $\{a_t\}_{t=0}^{\infty} \subset \mathbb{R}_{++}$ and the martingale $\{\xi_t\}_{t=0}^{\infty} \subset (0, 1)$. The martingale convergence theorem implies that $\{\xi_t\}_{t=0}^{\infty}$ converges almost surely to a random variable ξ_∞. That random variable must take values in $[0, 1]$. The triangle inequality implies that the martingale increments $\{\xi_{t+1} - \xi_t\}_{t=0}^{\infty}$ converge to zero almost surely. Together with $\xi_{t+1} - \xi_t \in \{u_t, -d_t\}$ this implies $\min\{u_t, d_t\} \to 0$ almost surely. But $\max\{u_t, d_t\} / \min\{u_t, d_t\} = \max\{\pi_0/\pi_1, \pi_1/\pi_0\}$. So $\max\{u_t, d_t\}$ also converges to zero almost surely. It follows that $u_t + d_t$ converges to zero almost surely.

Using the fact that $u_t + d_t = u_t/\pi_0 = d_t/\pi_1$ together with $d_t \in (0, \xi_t)$ and $u_t \in (0, 1 - \xi_t)$ one can verify that

$$a_t = \frac{(1 - \xi_t)\xi_t}{u_t + d_t} > \max\{\xi_t \pi_0, (1 - \xi_t)\pi_1\} \geq \pi_0 \pi_1.$$

For trajectories that lead to $\xi_\infty \in (0, 1)$, the fact that $u_t + d_t$ converges to zero almost surely implies that a_t grows without bound. So the first term in (25) will dominate, resulting in eventual short positions. On the other hand, for trajectories that lead to $\xi_\infty \in \{0, 1\}$, the second term in (25) will have a row that converges to zero.[8] And the fact that $a_t \geq \pi_0 \pi_1$ then ensures that the first term eventually dominates, because it does not become small. Again, this implies eventual short positions.

We can assemble these results to prove the following proposition.

Proposition 4 *Assume the economy satisfies (CC) and that $\sum_{j \in \{K,L\}} \mu_j Y_{o,j}(\eta)$ does not vary with $\eta \in \{0, 1\}$, while $Y_{o,j}(\eta)$ does. Assume that (1) does not bind for the intermediary consumption allocation of Proposition 2. In the equilibrium with two stochastic bubble securities that implements this allocation, almost surely, consumers have to sell bubble securities short eventually.*

[8]In the example (20), $a_t = \max\{\xi_t \pi_0, (1 - \xi_t)\pi_1\} / \alpha$. Since $u_t + d_t \to 0$ almost surely, this forces $\xi_\infty \in \{0, 1\}$.

6 The Bubble Forest

Suppose now that nobody can sell contingent claims against the collateral of a safe bubble, or sell one type of bubble security short against the collateral of another bubble security. Proposition 4 shows that this can be restrictive even when the constraints (1) do not bind. Here we show that, for all economies that satisfy condition (CC), the efficient allocation described in Propositions 2 and 3 can be implemented with a continuum of stochastic bubble securities that are subject to short-sale constraints.

Take there to be a unit interval of bubble securities indexed by $\omega \in [0, 1)$. The aggregate supply of these securities is given by a uniform density on $[0, 1)$ with height $D > 0$. The price of bubble security $\omega \in [0, 1)$ at time t is denoted by $s_{\omega,t}$. Consumers can abandon any of their holdings of bubble securities at any time. So the prices of these bubble securities must be non-negative at all times.

6.1 The Price Processes

Let $b > 0$ denote the aggregate market value of the safe bubble in the Pareto efficient equilibrium described in Propositions 2 and 3. Prices for the economy with a continuum of bubble securities can then be constructed as follows. At $t = 0$, the bubble securities prices are $s_{\omega,0}D = b$ for all $\omega \in [0, 1)$. Over time, the prices of these securities are going to drop to zero, randomly, in a way that depends only on the partial histories of $\{\eta_t\}_{t=0}^{\infty}$. Write $[\underline{\omega}_t, \overline{\omega}_t) \subset [0, 1)$ for the interval of bubble securities with strictly positive prices at time t. So $[\underline{\omega}_0, \overline{\omega}_0) = [0, 1)$. At time $t + 1$, the prices of these securities are given by

$$s_{\omega,t+1} = \frac{s_{\omega,t}}{q_0} \times \begin{cases} 1 & \text{if } \eta_{t+1} = 0 \\ 0 & \text{if } \eta_{t+1} = 1 \end{cases}, \quad \omega \in [\underline{\omega}_t, q_1\underline{\omega}_t + q_0\overline{\omega}_t),$$

$$s_{\omega,t+1} = \frac{s_{\omega,t}}{q_1} \times \begin{cases} 0 & \text{if } \eta_{t+1} = 0 \\ 1 & \text{if } \eta_{t+1} = 1 \end{cases}, \quad \omega \in [q_1\underline{\omega}_t + q_0\overline{\omega}_t, \overline{\omega}_t).$$

In other words, the event $\eta_{t+1} = 0$ causes the prices of all securities indexed by $(\omega - \underline{\omega}_t) / (\overline{\omega}_t - \underline{\omega}_t) \leq q_0$ to rise by a factor $1/q_0 > 1$. All other prices drop to zero. Conversely, the event $\eta_{t+1} = 1$ causes the prices of all securities with $(\overline{\omega}_t - \omega) / (\overline{\omega}_t - \underline{\omega}_t) < q_1$ to rise by a factor $1/q_1 > 1$. And again, all other prices drop to zero. An example of how these prices evolve is given in Figure 2.

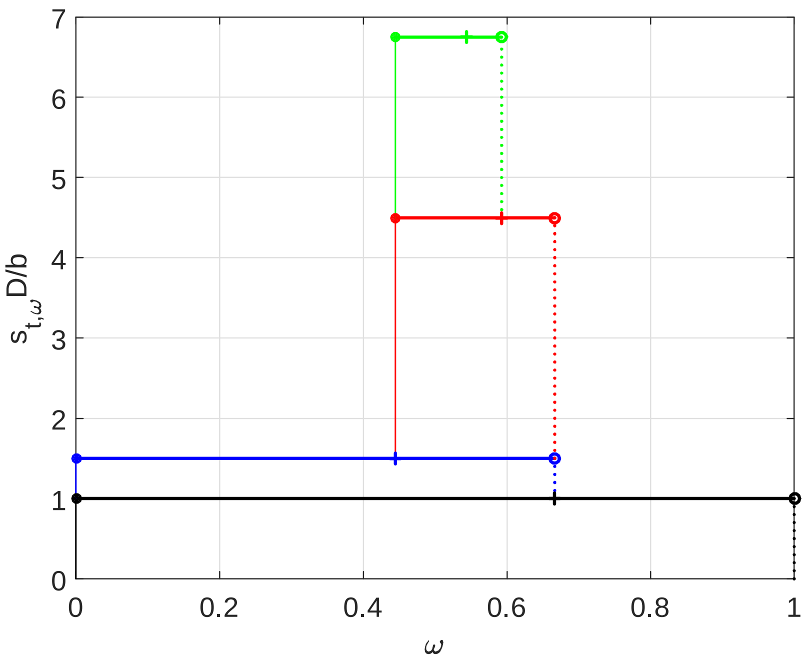

Figure 2 *Prices for the partial history* $(\eta_1, \eta_2, \eta_3) = (0, 1, 0)$ *when* $q_0 = 2/3$.

One implication is that the interval $[\underline{\omega}_t, \overline{\omega}_t)$ of bubble securities with strictly positive prices at time t shrinks to

$$[\underline{\omega}_{t+1}, \overline{\omega}_{t+1}) = \begin{cases} [\underline{\omega}_t, q_1\underline{\omega}_t + q_0\overline{\omega}_t) & \text{if } \eta_{t+1} = 0 \\ [q_1\underline{\omega}_t + q_0\overline{\omega}_t, \overline{\omega}_t) & \text{if } \eta_{t+1} = 1 \end{cases}$$

at time $t+1$. From t to $t+1$, this reduces the length $\overline{\omega}_t - \underline{\omega}_t$ of the interval with positive prices to $\overline{\omega}_{t+1} - \underline{\omega}_{t+1} = q_{\eta_{t+1}} (\overline{\omega}_t - \underline{\omega}_t)$. Starting from $\overline{\omega}_0 - \underline{\omega}_0 = 1$, this gives

$$\overline{\omega}_t - \underline{\omega}_t = \prod_{s=1}^{t} q_{\eta_s},$$

for all $t \in \mathbb{N}$. This certainly implies $\overline{\omega}_t - \underline{\omega}_t \leq (\max\{q_0, q_1\})^t$ and hence $\lim_{t\to\infty} (\overline{\omega}_t - \underline{\omega}_t) = 0$ for every possible trajectory $\{\eta_t\}_{t=0}^{\infty}$. So the interval of bubble securities with strictly positive prices becomes vanishingly small over time.

Since $s_{\omega,0}D = b$ for all $\omega \in [0, 1)$, and since prices that do not drop to zero from t to $t+1$ grow by a common factor $1/q_{\eta_{t+1}} > 1$, we have $s_{\omega,t+1} = s_{\omega,t}/q_{\eta_{t+1}}$ for all $\omega \in [\underline{\omega}_{t+1}, \overline{\omega}_{t+1})$ and $s_{\omega,t+1} = 0$ otherwise. Since $s_{\omega,0}D = b$, an immediate implication is that $(\overline{\omega}_t - \underline{\omega}_t)\, s_{\omega,t}D = b$ for all $\omega \in [\underline{\omega}_t, \overline{\omega}_t)$ and $s_{\omega,t+1} = 0$ otherwise. That is, every security with a positive price at time t has the same price and the market capitalization of all securities is b,

$$\int_0^1 s_{\omega,t} D \mathrm{d}\omega = \int_{\underline{\omega}_t}^{\overline{\omega}_t} s_{\omega,t} D \mathrm{d}\omega = b.$$

Over time, the market capitalization of all bubble securities becomes concentrated on an ever decreasing subset of the initial unit measure of securities. In Figure 2, the size of the areas under the black, blue, red, and green lines is always 1.

6.1.1 Martingales Under $\{q_\omega\}_{\omega\in\{0,1\}}$

The recursion $s_{\omega,t+1} = s_{\omega,t}/q_{\eta_{t+1}}$ for all $\omega \in [\underline{\omega}_t, \overline{\omega}_t)$ also implies

$$\mathbb{E}_t\left[s_{\omega,t+1}\right] = q_0 \times \frac{s_{\omega,t}}{q_0} + q_1 \times 0 = q_0 \times 0 + q_1 \times \frac{s_{\omega,t}}{q_1} = s_{\omega,t}.$$

All prices outside $[\underline{\omega}_t, \overline{\omega}_t)$ are zero and remain zero, which trivially ensures $\mathbb{E}_t\left[s_{\omega,t+1}\right] = s_{\omega,t}$. So the price process for every bubble security $\omega \in [0, 1)$ is a non-negative martingale under the risk neutral probabilities. It will be a martingale under the actual probabilities if and only if $q_\eta = \pi_\eta$.

6.2 Assembling State Contingent Claims

Recall that $s_{\omega,t}D/b = 1/\left(\overline{\omega}_t - \underline{\omega}_t\right)$ for $\omega \in \left[\overline{\omega}_t - \underline{\omega}_t\right)$ and $\overline{\omega}_{t+1} - \underline{\omega}_{t+1} = q_{\eta_{t+1}}\left(\overline{\omega}_t - \underline{\omega}_t\right)$ at all $t \in \{0, 1, \ldots\}$. This implies

$$\begin{aligned} X_t(0) &= \frac{1}{b}\int_{\underline{\omega}_t}^{q_1\underline{\omega}_t + q_0\overline{\omega}_t} s_{\omega,t+1} D \mathrm{d}\omega = \begin{cases} 1 & \text{if } \eta_{t+1} = 0 \\ 0 & \text{if } \eta_{t+1} = 1, \end{cases} \\ X_t(1) &= \frac{1}{b}\int_{q_1\underline{\omega}_t + q_0\overline{\omega}_t}^{\overline{\omega}_t} s_{\omega,t+1} D \mathrm{d}\omega = \begin{cases} 0 & \text{if } \eta_{t+1} = 0 \\ 1 & \text{if } \eta_{t+1} = 1. \end{cases} \end{aligned}$$

Observe that $X_t(\eta) = 1$ if $\eta_{t+1} = \eta$ and zero otherwise. This says that the aggregate supply of bubble securities can be decomposed into two state-contingent claims: one that pays 1 if and only if $\eta_{t+1} = 0$ and another that pays 1 if and only if $\eta_{t+1} = 1$. The time-t price the state contingent claim $X_t(0)$ is

$$\mathbb{E}_t\left[\frac{1}{b}\int_{\underline{\omega}_t}^{q_1\underline{\omega}_t + q_0\overline{\omega}_t} s_{\omega,t+1} D \mathrm{d}\omega\right] = \frac{1}{b}\int_{\underline{\omega}_t}^{q_1\underline{\omega}_t + q_0\overline{\omega}_t} \mathbb{E}_t\left[s_{\omega,t+1} D\right] \mathrm{d}\omega = \frac{1}{b}\int_{\underline{\omega}_t}^{q_1\underline{\omega}_t + q_0\overline{\omega}_t} s_{\omega,t} D \mathrm{d}\omega = q_0$$

because the length of the interval $[\underline{\omega}_t, q_1\underline{\omega}_t + q_0\overline{\omega}_t)$ is $q_0\left(\overline{\omega}_t - \underline{\omega}_t\right)$ and $s_{\omega,t}D/b = 1/\left(\overline{\omega}_t - \underline{\omega}_t\right)$. Similarly the date-$t$ price of $X_t(1)$ is q_1.

Given these bubble prices, consumers can use portfolios with non-negative weights to select any payoff vector $[X_t(0), X_t(1)] \in \mathbb{R}_+^2$. And the prices of these portfolios will be same as the prices of these payoff vectors in the intermediary economy. So consumers have the same sets of feasible consumption allocations as in the economy with competitive intermediaries.

Proposition 5 *Assume the economy satisfies (CC). Then the economy with a stochastic bubble forest has an equilibrium that implements the efficient allocation of Propositions 2 and 3.*

In the equilibrium we have described, whenever bubble securities are traded from one generation to the next, they all trade at the same price. One possible interpretation of this environment is that there is really only one bubble security. One can say that the initial supply of the one bubble security comes in the form of a measure of "tokens" with serial numbers $\omega \in [0, 1)$. The density of tokens with the same serial number is D. Tokens with different serial numbers can now be distinguished. The equilibrium beliefs at time t are then that the prices of some tokens will appreciate while other will collapse to zero, depend on the realization of η_{t+1}.

7 Conclusion

In an overlapping generations economy with risky old-age endowments and consumers who cannot be forced to give up resources in their old age, constrained efficient risk sharing is an equilibrium in an economy with collateral constraints and sufficiently many stochastic bubble securities to span the underlying set of states of the world. Our results can be extended to an economy in which there are also Lucas trees that produce real dividends for an extended period of time, but not forever. In such an economy, stocks trading on the stock market could be securities backed by both a Lucas tree and a bubble asset. Those stocks can then eventually become pure bubble securities.

The constrained efficient allocation can also be implemented in an economy in which consumers cannot go short in any security. This implementation requires a continuum of stochastic bubble securities. Compared to the economy with collateral constraints, this requires minimal enforcement but a maximal coordination of beliefs. As Woodford [1990] explained in the context of a monetary economy with multiple equilibria, learning can lead to a coordination of beliefs that results in sunspot equilibria. Whether it is possible for consumers to learn to coordinate on the risk sharing beliefs that we have used is the subject of ongoing research.

There is a wide range of inefficient equilibria that are possible in the economy considered in this paper, including autarky. In Amol and Luttmer [2024], we show that the government can use off-equilibrium fiscal policies to ensure that the price of unbacked nominal government debt is uniquely determined, as long as the government can also prevent other bubble securities from being traded. That result can be extended to the intermediary economy in this paper. Whether off-equilibrium fiscal policy might also be able to eliminate inefficient equilibria when there are multiple stochastic bubble securities is an open question.

A The Consumer Decision Rules

Given strictly positive q_0 and q_1, the consumer decision rules are

$$C_y = \min\left\{Y_y, (1-\beta)\min\left\{\frac{Y_y + q_0 Y_o(0)}{1-\beta\pi_1}, \frac{Y_y + q_1 Y_o(1)}{1-\beta\pi_0}, Y_y + \sum_{\kappa\in\{0,1\}} q_\kappa Y_o(\kappa)\right\}\right\},$$

together with

$$\begin{aligned} q_0 C_o(0) &= \max\left\{q_0 Y_o(0), \beta\pi_0 \min\left\{\frac{Y_y + q_0 Y_o(0)}{1-\beta\pi_1}, Y_y + \sum_{\kappa\in\{0,1\}} q_\kappa Y_o(\kappa)\right\}\right\}, \\ q_1 C_o(1) &= \max\left\{q_1 Y_o(1), \beta\pi_1 \min\left\{\frac{Y_y + q_1 Y_o(1)}{1-\beta\pi_0}, Y_y + \sum_{\kappa\in\{0,1\}} q_\kappa Y_o(\kappa)\right\}\right\}. \end{aligned}$$

It is not difficult to verify that this satisfies (1) and (5)-(6). These decision rules imply Figure 1.

B Proof of Lemma 3

Let $\Delta = \{(d_K, d_L) \in \mathbb{R}_+^2 : d_K + d_L = 1\}$ and define $f : \Delta \to \mathbb{R}$ by

$$f(d) = \sum_{\eta\in\{0,1\}} \frac{\pi_\eta}{\sum_j d_j Y_{o,j}(\eta)/Y_{y,j}}.$$

The unit simplex Δ is convex and compact, and f is continuous on Δ because $Y_{o,j}(\eta)/Y_{y,j} > 0$ for both $\eta \in \{0,1\}$ and $j \in \{K, L\}$. Let $E(\Delta) = \{(1,0),(0,1)\}$. These are the extreme points of Δ. The fact that f is convex implies $\max_{d\in\Delta} f(d) = \max_{d\in E(\Delta)} f(d)$.

Consider the weights $w_j = \mu_j Y_{y,j} / \sum_i \mu_i Y_{y,i}$, $j \in \{K, L\}$. Then

$$\sum_{\eta \in \{0,1\}} \pi_\eta \times \frac{\sum_j \mu_j Y_{y,j}}{\sum_j \mu_j Y_{o,j}(\eta)} = f(w).$$

On the other hand,

$$\max_j \left\{ \sum_{\eta \in \{0,1\}} \pi_\eta \times \frac{Y_{y,j}}{Y_{o,j}(\eta)} \right\} = \max_{d \in E(\Delta)} \left\{ \sum_{\eta \in \{0,1\}} \frac{\pi_\eta}{\sum_j d_j Y_{o,j}(\eta)/Y_{y,j}} \right\} = \max_{d \in E(\Delta)} f(d).$$

The second inequality in (14) therefore says that $\max_{d \in E(\Delta)} f(d) \geq f(w)$. This is true because $w \in \Delta$.

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