Artificial intelligence and factor substitutability

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Abstract

This paper develops an endogenous growth model to investigate how advances in artificial intelligence (AI) affect long-term economic development by altering the substitutability between capital and labour. Departing from traditional models where technology enhances the factors' productivities, this framework features a distinct capacity of AI to affect the elasticity of substitution itself. We analyse the endogenous R&D allocation between human capital, machine intelligence and physical capital productivity. There are two qualitatively different long-run equilibria. In the first one, physical capital and labour remain permanent technological complements, and research is always directed to human capital enhancement. In this equilibrium output grows sub-exponentially, and machine intelligence is never improved. In the alternative equilibrium with explosive output growth, research ultimately shifts towards advancing machine intelligence beyond human capabilities, driving the economy into a regime where physical capital substitutes for labour. These results offer novel theoretical underpinnings for understanding the macroeconomic implications of AI as a general-purpose technology whose nature and transformative effects are qualitatively distinct from previous technological innovations in human history.

Keywords: Artificial Intelligence, Endogenous Growth, Technological Change, R&D, Elasticity of Substitution.

JEL Classification: O31, O32, O33, O41

1. Introduction

Traditional growth theories—from the Solow–Swan paradigm to endogenous R&D frameworks—yield either sustained exponential expansion or convergence to a balanced-growth path. These trajectories are grounded in the observed exponential pattern of economic development since the Industrial Revolution.

However, advances in artificial intelligence introduce a qualitatively new channel: AI can transform the elasticity of substitution between labour and capital, rather than merely augmenting one factor or the other. Previously, progress was perceived as technological improvements augmenting either labour or capital, whereas AI represents a qualitatively different form of progress: it can alter the substitutability between labour and capital, thus modifying the nature of the production function.

As (Hanson, 2008) observes, "Machine intelligence on a human level, if not higher, would do nicely. Its arrival could produce a singularity." Moreover, he further explains why AI can drive a new growth pattern: "The next radical jump in economic growth seems more likely to come from something that has a profound effect on everything, because it addresses the one permanent shortage in our entire economy: human time and attention."

To clarify this concept, the term "singularity" is widespread in the AI literature. A commonly used classification, proposed by (Aghion et al., 2018), distinguishes between Type I—an accelerating growth rate that remains finite—and Type II—a divergence of output to infinity in finite time. By contrast, (Hanson, 2008) uses the term to denote a breakdown in prior economic trends.

A key insight is that, as machines grow more intelligent, they can perform an increasing range of tasks formerly exclusive to humans. Building on these insights, this paper contributes to the existing literature by developing an endogenous growth model in which the elasticity of substitution between capital and labour is linked to the relative cognitive abilities of machines and humans. Technological progress can take multiple forms in this framework—AI-driven advances that alter substitutability, in addition to conventional labour- or capital-augmenting improvements. The allocation of R&D effort is endogenously determined by profit-maximising decisions, and the impact of AI on substitutability hinges on the intelligence gap between humans and machines. If machines become more advanced, they can perform tasks traditionally undertaken by humans, implying that capital increasingly substitutes for labour. Conversely, if humans maintain cognitive superiority, certain tasks remain exclusive to human capability, necessitating a complementary interaction between labour and capital.

The paper proceeds as follows: the next Section reviews existing approaches to modelling AI in the growth literature and the growth impacts they postulate. Section 3 formalises the intelligence-dependent elasticity framework and investigates its dynamics. Section 4 calibrates the model to plausible AI trajectories, examines transitional paths, and conducts sensitivity analysis that are discussed in Section 5. Section 6 concludes.

2. Literature review

2.1. Modelling AI

There are two traditions in the growth literature to account for the technological impact on economic dynamics, namely endogenous and exogenous growth models. Various channels of this impact result in different output trajectories summarized in Fig.1. We start our review with the class of models where technology is endogenously determined.

a. Endogenous growth models

(Aghion et al., 2018) developed an endogenous growth model specifying output and ideas production functions as follows:

$$\dot{A} = A^{\phi}F(.)$$

In the baseline model, AI impacts the R&D sector by altering factor productivity (see Fig. 1), enabling capital to substitute for labour across a wider range of tasks. This effect corresponds to a productivity indicator increasing from 0 to 1 in newly automated tasks within idea production. Specifically, F(.) is expressed as:

$$F = \left(\int_0^1 X_i^{\rho} di\right)^{\frac{1}{\rho}}$$
$$X_i = \begin{cases} L, \text{if not automated} \\ K, \text{if not automated} \end{cases}$$

Assuming symmetrical allocation of capital and labour across tasks implies capital and labour usage per automated and non-automated tasks as $\frac{K}{\beta}$ and $\frac{L}{1-\beta}$, respectively, with β indicating the proportion of automated tasks. Aggregating these gives:

$$F = \left[\beta\left(\frac{K}{\beta}\right)^{\rho} + (1-\beta)\left(\frac{L}{1-\beta}\right)^{\rho}\right]^{\frac{1}{\rho}} = \left(\beta^{1-\rho}K^{\rho} + (1-\beta)^{1-\rho}L^{\rho}\right)^{\frac{1}{\rho}} = F(BK, HL),$$
$$\text{where} \quad B = \beta^{\frac{1-\rho}{\rho}}, H = (1-\beta)^{\frac{1-\rho}{\rho}}$$

Thus, automation disperses capital more widely, working analogously to labour-augmenting technological progress.

Under incomplete automation—where new knowledge still requires research effort—the growth trajectory might explode if the fraction of tasks automated and the contribution of the existing stock of knowledge are sufficiently large. Assuming the production function and idea accumulation are defined as:

$$Y = A^{\omega}K^{\alpha}((1-s_H)HL)^{1-\alpha}$$
$$\dot{A} = K^{\beta}(s_H HL)^{\lambda}A^{\phi};$$

Growth explodes if $\gamma = \frac{\omega}{1-\alpha} \cdot \frac{\beta}{1-\phi} > 1$, while a balanced growth path (BGP) is reached if $\gamma = 1$; otherwise, the economy converges to a steady state. If full automation is achieved, it suffices to have $\phi > 0$ a singularity.

(Trammell & Korinek, 2023) provide an overview of (Agrawal et al., 2018), a model in which idea creation results from the combination of two existing ideas. There is limited access to aggregate knowledge, and a researcher can combine only A^{Φ}, giving $2^{A^{\Phi}}$ potential combinations, and the idea generation function takes the form:

$$\dot{A} = \theta \ln(2)\, A^{\Phi}$$

The authors state that AI increases accessibility to existing knowledge; therefore, researchers can combine more ideas that raises ϕ, corresponding to an increase in the elasticity with respect to existing knowledge. The outcome of this model is standard: the achieved trajectory is determined by comparing ϕ with 1.

(Besiroglu et al., 2024) use the approach of (Jones, 1995) for knowledge accumulation:

$$\dot{A} = D(s_k K)^{\beta}(s_H HL)^{\lambda}A^{\Phi}$$

They argue that adopting deep learning models in the research process increases R&D efficiency, mitigating the bias–variance trade-off and making the industry more capital-intensive. Hence, the impact is modelled as an increase in the new idea elasticity with respect to capital, $\beta' > \beta, \beta' + \lambda' > \beta + \lambda$, where the condition $\beta + \phi < 1$ is imposed, restricting the growth trajectory to exponential with a permanently increased rate.

(Hanson, 2001) views AI as a perfect substitute for labour in final goods production, entering as a new factor. The labour force is decomposed as $L = H + R$, where H denotes human labour and R – machine intelligence. R is endogenously determined in the profit-maximisation problem when deciding total investment allocation. In this endogenous setup, the stock of knowledge follows a learning-by-doing assumption and is proportional to past output, while the real cost of computers diminishes over time as a negative function of the existing computer capital stock (M):

$$A \propto \left[\int_{-\infty}^{t} Y(s)\, ds\right]^{\vartheta}$$

$$P \propto \left[\int_{-\infty}^{t} M(s)\ ds\right]^{-\psi}$$

The resulting trajectory depends on parameter calibration (population growth rate, factor elasticities, learning-by-doing effects) and can be exponential if there is positive labour force growth or even exhibit explosive growth.

(Nordhaus, 2021) analyses AI's impact by modelling capital as more substitutable for labour in final goods production; indeed, the elasticity of substitution rises above 1, leading to asymptotic convergence to the AK model. This allows for endogenous growth via capital accumulation and potentially explosive growth given technological improvements over time.

A similar endogenous growth trajectory—one that asymptotically resembles the AK model and allows for growth in the absence of diminishing returns—was proposed by (Berg et al., 2018) in the corner case of perfect elasticity of substitution ($\sigma_2 = \infty$) between robotic capital and labour:

$$Y = n\left[a^{\frac{1}{\sigma_1}}K^{\frac{\sigma_1-1}{\sigma_1}} + (1-a)^{\frac{1}{\sigma_1}}V^{\frac{\sigma_1-1}{\sigma_1}}\right]^{\frac{\sigma_1}{\sigma_1-1}},\ \text{where } V = \left[e^{\frac{1}{\sigma_2}}L^{\frac{\sigma_2-1}{\sigma_2}} + (1-e)^{\frac{1}{\sigma_2}}(bR)^{\frac{\sigma_2-1}{\sigma_2}}\right]^{\frac{\sigma_2}{\sigma_2-1}}$$

Although the modelled impact of AI on final goods production is increased robot productivity (b), it was initially set to 0; hence, it effectively becomes a new factor of production in the newly formed nest of the CES function. Robot productivity is projected to reach a steady state, and the accumulation of capital and robots is endogenously determined by the user's optimisation problem (UMP). In the long run, total factor productivity is assumed to be constant; therefore, growth stabilises at a constant rate.

(Korinek & Suh, 2024) extend the analysis by allowing AI to affect both the R&D and final goods sectors. Their task-based model is similar to (Aghion et al., 2018), with the impact modelled as increased capital productivity in newly automated tasks. The difference is that both the idea and output tasks can be automated simultaneously. Knowledge accumulation then takes the form:

$$\log \dot{A} = \log A^{\phi} + \int \log x\,(i)\ d\Gamma(i)$$

Here $x(i)$ is the idea production of the i_{th} task, and $\Gamma(i)$ is the cumulative distribution of research tasks in complexity space, with the most difficult performed solely by labour. The fraction of automated tasks is assumed to grow over time, and if there are no limits to automation, it will surpass the level at which the singularity is reached.

(Ray & Mookherjee, 2022) present another multi-sector approach, assuming machine intelligence perfectly substitutes labour as a new factor. The labour component transforms similarly to (Hanson, 2001) representation: $L = H + vR$, additionally accounting for robots' productivity factor (v). (Trammell & Korinek, 2023) review this model, simplifying it by defining two sectors (final goods and robotics). Under the introduced assumption of full depreciation, the robotics sector can be treated analogously to R&D in earlier models. They present the case where robots substitute ($v = 1$) humans in final goods and $v \in (0,1)$ in robotics production:

$$Y = F\big((1-s_K)K, (1-s_H)HL + (1-s_R)R\big)$$
$$R = G(s_K K, s_H HL + v s_R R)$$
$$s_i - \text{Share of factor i allocated to robotics production}$$

Under certain values of v, it is optimal to allocate capital and robots to reproducing the latest technologies, moving the economy to a balanced growth path; otherwise, if the conditions on v are not met, growth stagnates.

b. Exogenous growth models

Exogenous growth models serve as baselines for the endogenous models discussed above, where assumptions such as a constant rate of automation are imposed. Many task-based frameworks for final goods production use the form:

$$Y = A\,F(.)$$

Here, $F(.)$ is defined as in the previous subsection. Such a setup is used by (Aghion et al., 2018; Korinek & Suh, 2024), where the exogenous rate of automation progress affects a factor's productivity, $1 - \beta = e^{-\zeta t}$, or progress of an automation index in task-complexity space, and operates similarly, while capturing the fact that more complex tasks are more difficult to automate. (Acemoglu & Restrepo, 2019) additionally analyse the possibility that new tasks are created while old ones are replaced. Generally, the growth trajectory follows a balanced growth path unless there is a binding fixed factor, which leads to convergence. Saturation also occurs in (Berg et al., 2018), where robotic capital imperfectly substitutes labour and cannot overcome subexponential limits imposed by AI productivity boundaries.

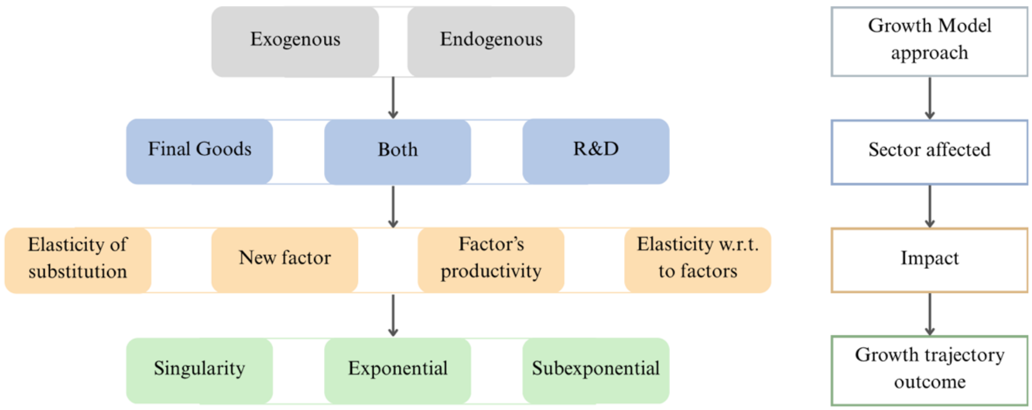


Figure 1. Conceptual Map of AI Modelling in Growth Literature

2.2. Capital-skill complementarity

Capital-skill complementarity was initially proposed and studied in the manufacturing sector by (Griliches, 1969). It highlights how capital investment benefits skilled labour to a greater extent compared to unskilled workers. Further, the idea was explored by (Krusell et al., 2000), who showed that in the US, the demand for skilled labour increases with capital accumulation, pushing skill premiums upward. (Lindquist, 2005) applied a similar framework in Sweden and detected strong empirical support for capital-skill complementarity as a crucial contributor to wage inequality. In India, this phenomenon was not present until the 1990s because of the strict government planning and licensing requirements. However, after the 1991 regulatory reforms, the alleviation of controls boosted output growth and allowed the easier adoption of imported technologies, giving rise to statistically significant capital-skill complementarity (Berman et al., 2005). Additionally, (Papageorgiou & Chmelarova, 2005) explored a nonlinear relationship, showing that capital-skill complementarity is strongest in middle-income countries, while in developed and low-income countries, the evidence is weaker, suggesting that it is not a permanent feature.

However, recent empirical research suggests that AI-driven technologies contribute to the weakening of this effect, evidenced by a decline in capital-skill complementarity after the 2000s (Castex et al., 2022) and an aggregate increase in the capital-labour elasticity of substitution (Ialenti & and Pialli, 2024). The notion of the "great reversal", introduced by (Beaudry et al., 2016), refers to the peak in cognitive tasks in the 1990s, followed by a decline in the 2000s, explained by the maturation of information technologies and investments in automation, which initially required skilled labour for development and implementation but later became self-sustaining innovations. Consequently, highly educated workers moved down the occupational ladder, displacing less skilled workers.

3. Model

3.1. Setup

The general model used in the existing literature to capture the impact of AI is a CES production function:

$$Y = (a(AK)^{\rho} + (1-a)(HL)^{\rho})^{\frac{1}{\rho}}$$

Analyses of AI's influence have taken the form of shifts in ρ (Nordhaus, 2021), (Trammell & Korinek, 2023), or in factor efficiencies A (capital-augmented progress) and H (labour-augmented progress) (Aghion et al., 2018). However, this modelled impact has typically been treated as an exogenous jump or growth rate. This paper fills that gap by modelling AI progress as an endogenous decision to improve specific capital properties, particularly the ability to substitute for human labour.

The CES production function has a few corner properties:

1. $\sigma \to 0 \Rightarrow Y \to min(AK, HL)$
2. $\sigma \to 1 \Rightarrow Y \to (AK)^a (HL)^{1-a}$
3. $\sigma \to \infty \Rightarrow Y \to aAK + (1-a)HL$

Let substitutability depend on the relative "smartness" of machines and humans. The simplest way to express this mathematically is $\sigma(\gamma, H) = \frac{\gamma}{H}$, where σ – elasticity of substitution between K and L; H and γ are the levels of **human capital** (education level and cognitive skills) and **machine intelligence** (advancements in artificial intelligence), respectively. The critical threshold $H = \gamma$ represents the natural transition point between two economic regimes. At this threshold, human and machine cognitive abilities are equally effective, marking a structural shift in the economic environment—from predominantly complementary interactions between humans and machines to predominantly substitutable interactions. The degree of complementarity, therefore, directly depends on human cognitive superiority: high $\frac{H}{\gamma}$ implies stronger complementarity, corresponding to empirical evidence on capital–skill complementarity. Plugging this relation into the baseline CES function yields:

$$Y = \left(a(AK)^{1-\frac{H}{\gamma}} + (1-a)(HL)^{1-\frac{H}{\gamma}} \right)^{\frac{1}{1-\frac{H}{\gamma}}}$$

- When humans are relatively smarter ($H > \gamma$), the economy is in the **"complements regime",** where capital and labour inputs primarily complement each other. In this regime, the elasticity $\frac{\gamma}{H}$ is less than 1.
- When machines surpass human intelligence ($H < \gamma$), the economy transitions into the **"substitutes regime",** where the elasticity $\frac{\gamma}{H}$ exceeds 1.
- Equal intellectual abilities $H = \gamma$ push the economy into the **"Cobb-Douglas corner".**

In the considered economy, the total population consists of two groups: raw labour (L), employed exclusively in the production sector, and scientists (S) working in R&D. For simplicity, both groups are constant in size, consist of homogeneous agents, and are normalised to unity ($L = S = 1$). Additionally, labour cannot move between sectors.

Agents save a constant, exogenously determined fraction s of their income. The capital accumulation equation is defined by: $\dot{K} = sY - \delta K$, where δ is the exogenous depreciation rate of capital.

In the R&D sector, scientists can improve any of the three variables: A, H, or γ: $\dot{J} = S,\ J \in \{A, H, \gamma\}$, which implies diminishing marginal returns to R&D over time: $\frac{\dot{J}}{J} \to 0$ when $J \to \infty$.

Scientists' allocation among research areas is made endogenously. Assume that the R&D department of a price-taking firm—operating in both the output and R&D markets—chooses its allocation area to maximise current profit:

$$\Pi = F(AK, H, \gamma) - w_L L - rK - w_S S_J \longrightarrow \max_{J \in \{A,H,\gamma\}}$$

The marginal cost of allocation in each area is identical and equals the scientists' wage, w_S, while the marginal benefit of allocating to an area $J \in \{A, H, \gamma\}$is $\frac{\partial F(AK,H,\gamma)}{\partial J}$. The profit-maximising choice of J is therefore the one that yields the greatest increase in output.

Without loss of generality, assume that at any given time, all scientists allocate their efforts exclusively to a single research direction. Relaxing this assumption would smooth the trajectories of H, γ, and A over time but would not affect the model's qualitative results, such as its long-run relationships.

Thus, it is useful to define the relative contribution ratio of any two variables i, j as $R(i,j) = \frac{\partial F}{\partial i} / \frac{\partial F}{\partial j}$. Scientists direct their efforts toward a variable i only if it yields a strictly higher marginal contribution compared to both alternative variables. Mathematically, this requires: $R(i,j) > 1$ & $R(i,k) > 1$.

3.2. Transition to "substitutes regime"

The economy moves from complements to substitutes regime whenever near the threshold where human and machine intelligence are nearly equal ($H \gtrsim \gamma$), two conditions become true simultaneously:

1. At the margin, improving machine intelligence (γ) raises output more than improving human capital (H).
2. Machine-intelligence (γ) R&D contributes more to output than further improvements in capital efficiency (A).

Proposition 1 (Transition conditions)

The transition to substitutes regime occurs when the following conditions hold:

1. $\lim\limits_{\gamma \to H} R(H,\gamma) < 1 \Rightarrow a \left(ln \frac{AK}{H} \right)^2 > 1$

2. $\lim\limits_{\gamma \to H} R(\gamma, A) > 1 \Rightarrow a < 1 - \frac{2H}{A\left(ln \frac{AK}{H} \right)^2}$

(See the Appendix for the proof of this and other results.)

3.3. Complementarity trap

Definition: a stable state with an eternal R&D focus on human capital enhancements, such that labour and capital will never become substitutes.

This trap arises because the economy has little incentive to switch to more substitutable factors. Contributing reasons include:

1. a low output elasticity with respect to capital.
2. a low capital level when the saving rate is small or depreciation is high.
3. a low capital efficiency and machine intelligence.

The second two reasons affect the economy's short-run chance for transition via impact on necessary conditions for transition, while the first determines trap avoidance in the long run.

Lemma 1
In a complementarity trap, the production function takes the form of $Y = H$.

Proof: As H continuously increases, the CES function converges to the Leontief specification – $\min(AK, H)$. If $AK < H$, then the output depends exclusively on AK, and researchers allocate efforts to improve capital efficiency. This implies entering the AK model regime, where A continuously improves, leading to endogenous growth and eventually pushing AK above H.

Proposition 2 (Escape from the trap)
A trap is avoided when $-\frac{1}{\ln(1-a)} < 1 \Rightarrow a > -\frac{1}{e} + 1 \approx 0.63.$

In the trap, capital stabilises at the steady-state level defined by $\dot{K} \approx sH - \delta K^* = 0 \Rightarrow K^* \approx \frac{s}{\delta} H$ and $\frac{s}{\delta} A > 1$ following Lemma 1. When output elasticity with respect to capital is small, the comparative-benefit ratio $R(H,\gamma)$ remains above 1, indicating a permanent research direction in favour of human capital—even when it is already high—while improvements in machine efficiency are not considered, since the capital component is formally absent from the production function in this setting.

3.4. No trap economy

Lemma 2 (Perfect complements or substitutes).
Any complementarity level other than perfect is unsustainable.

Proof: Assume, for contradiction, that $0 < \hat{\sigma} < 1$ holds over an extended period. Due to diminishing returns and the properties of the CES function, capital stabilises at a steady state with a linear dependence on H (denote $K = b(A) \cdot H$, where b is a monotonically increasing function of A). Since $\sigma = \frac{\gamma}{H}$ maintaining fixed elasticity requires simultaneous improvements—i.e., switching research focus—between both H and γ , or alternatively, conducting R&D to improve capital efficiency A, which also leaves elasticity unchanged. The CES function is the weighted average of AK and H inputs, when $\rho < 0$ a significant weight is put on the smaller of the AK, H components.

1. If capital efficiency improves, AK starts to grow faster than H, incentivising enhancements in machine intelligence to relax complementarity and better combine abundant capital with scarce labour.

2. If only H and γ improved in parallel:

    - If $Ab \neq 1, \gamma$ improvements facilitate the relative decrease in the $\min(AK, H)$ component's weight, increasing the weight of max(AK, H). When $Ab \neq 1$, the gap $|AK - H|$ increases with H, so weight rebalancing appears desirable, or in the case of $Ab < 1$ an increase in the value of the smaller component by increasing A.
    - If $Ab = 1 \implies AK = H$, therefore, weight rebalancing appears senseless, while an increase in inputs has a positive contribution. At some point, when $K > 1$ (which eventually occurs irrespective of initial b, when H increases), investments in A rather than H will be more beneficial, because it is not accompanied by the negative impact of greater weight on the smaller input. Finally, $Ab > 1$ leads to a switch to γ enhancements in the future.

Therefore, no stable complementarity level persists once the trap is escaped. Neither a fixed elasticity below nor equal to one is sustainable. When the first but not the second condition for transition holds, scientists focus on machine efficiency until the second condition is met, with improvements in A reinforcing this shift and triggering the transition to the substitutes regime.

3.5. Economy after transition

Consider an economy able to maintain its capital stock as developments continue, so it is not reduced by depreciation or insufficient savings (evolution in the opposite case is left to the discussion section). Endogenous growth leads to continuous capital accumulation, which discourages investment in human capital; in fact, increasing human capital may reduce output because it raises complementarity between a relatively large capital stock and scarce labour. This claim can be deduced from the limit of the ratio of the relative contribution of H and γ enhancements: $\lim_{K\to\infty} R(H, \gamma) = -\frac{\gamma}{H} < 0$.

In the long run, unbounded capital accumulation induces R&D decision-makers to allocate efforts solely between machine intelligence and efficiency. Equilibrium requires indifference between expanding γ and A, achieved by maintaining a certain balance.

Proposition 3 (Long-Run Indifference)
In the long run $\lim_{K\to\infty} R(\gamma, A) = -\frac{HA \ln a}{(\gamma - H)^2} = 1$ implies $A = -\frac{(\gamma - H)^2}{H \ln a}$, where H is the stabilised level of human capital during the transition phase.

It is evident that excessive enhancements in γ or A create incentives for the opposite improvement, thereby restoring balance.

4. Calibrations and sensitivity analysis

This section provides illustrations of the economy's development and the evolution of H, γ and A over time for various initial parameters. For dynamics modelling, the discrete version of the model is used.

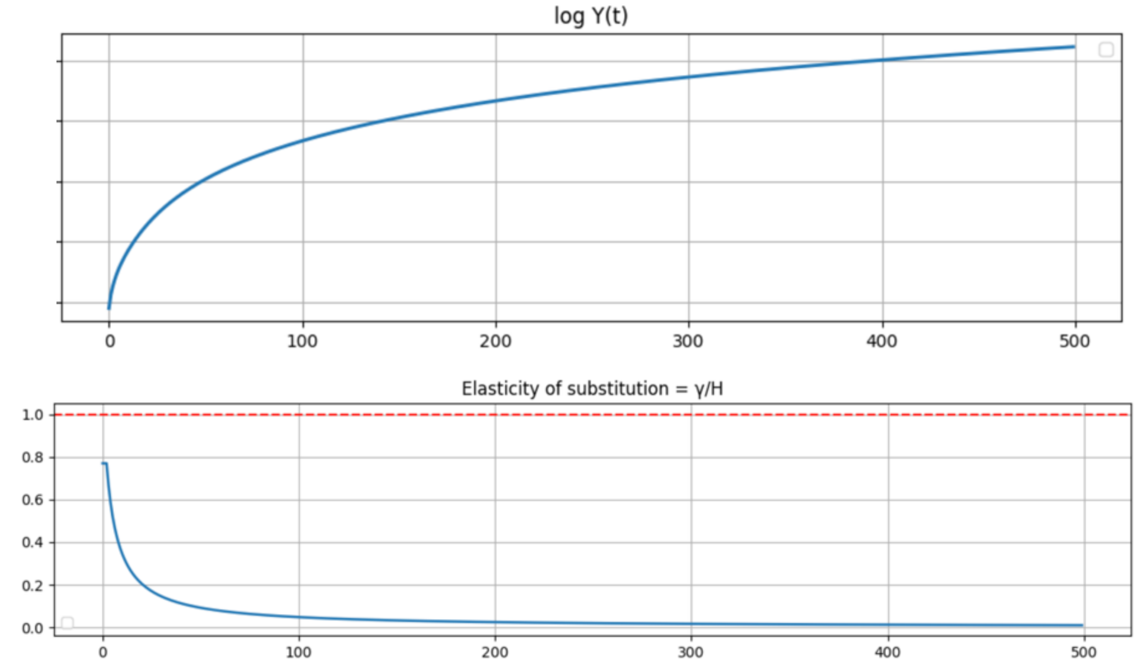
The upper graph displays the natural logarithm of output, while the lower graph depicts the elasticity of substitution measured as $\frac{\gamma}{H}$ and is interpreted as **"cognitive ratio".**

A red dashed line on the lower graph marks the threshold where human and machine intelligence are equal ($H = \gamma \Rightarrow \frac{\gamma}{H} = 1$). This chart also reveals scientists' allocation over time: a decreasing ratio indicates improvements in human capital; an increasing gap indicates advances in machine intelligence; and a constant gap signifies that scientists are focused entirely on enhancing machine efficiency.

Two general cases are illustrated:

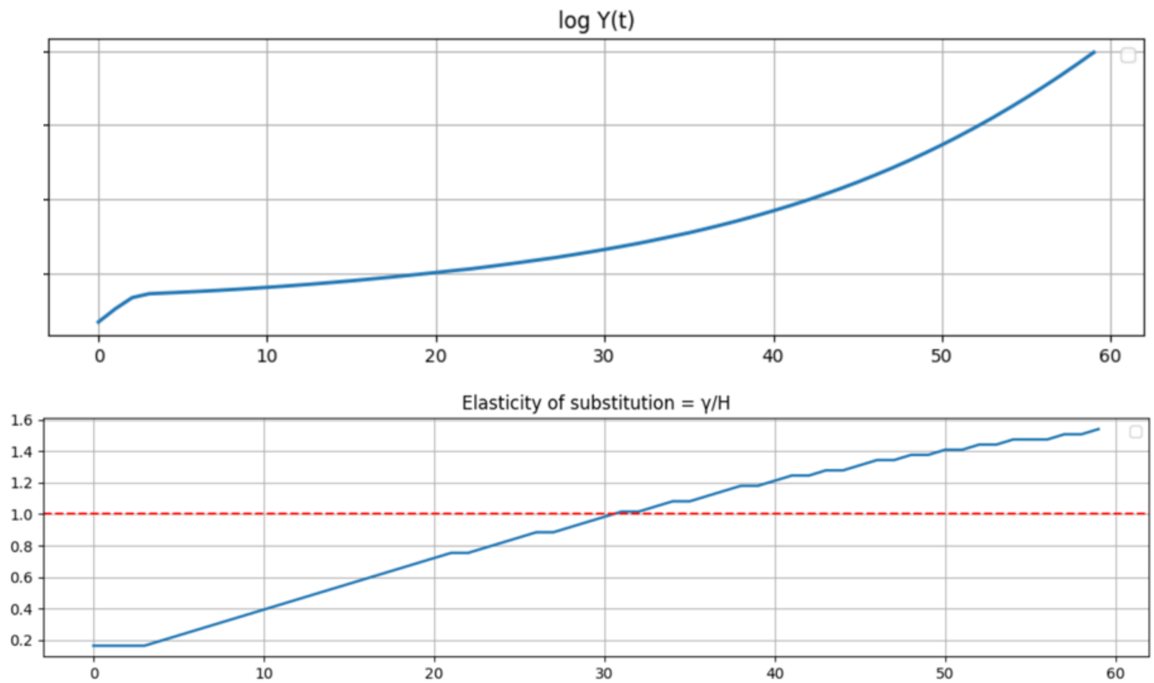
1. **Complementarity trap:** short-run fluctuations appear to be insufficient to overcome the trap.
2. **Ultimate sustained substitutes regime:** following Proposition 2, the economy has sufficient output elasticity with respect to capital, allowing it to avoid the trap in the long run.

Complementarity trap



$H_0 = 6.5, \gamma_0 = 5, A_0 = 1, K_0 = 7, a = 0.3, s = 0.25, \delta = 0.05$

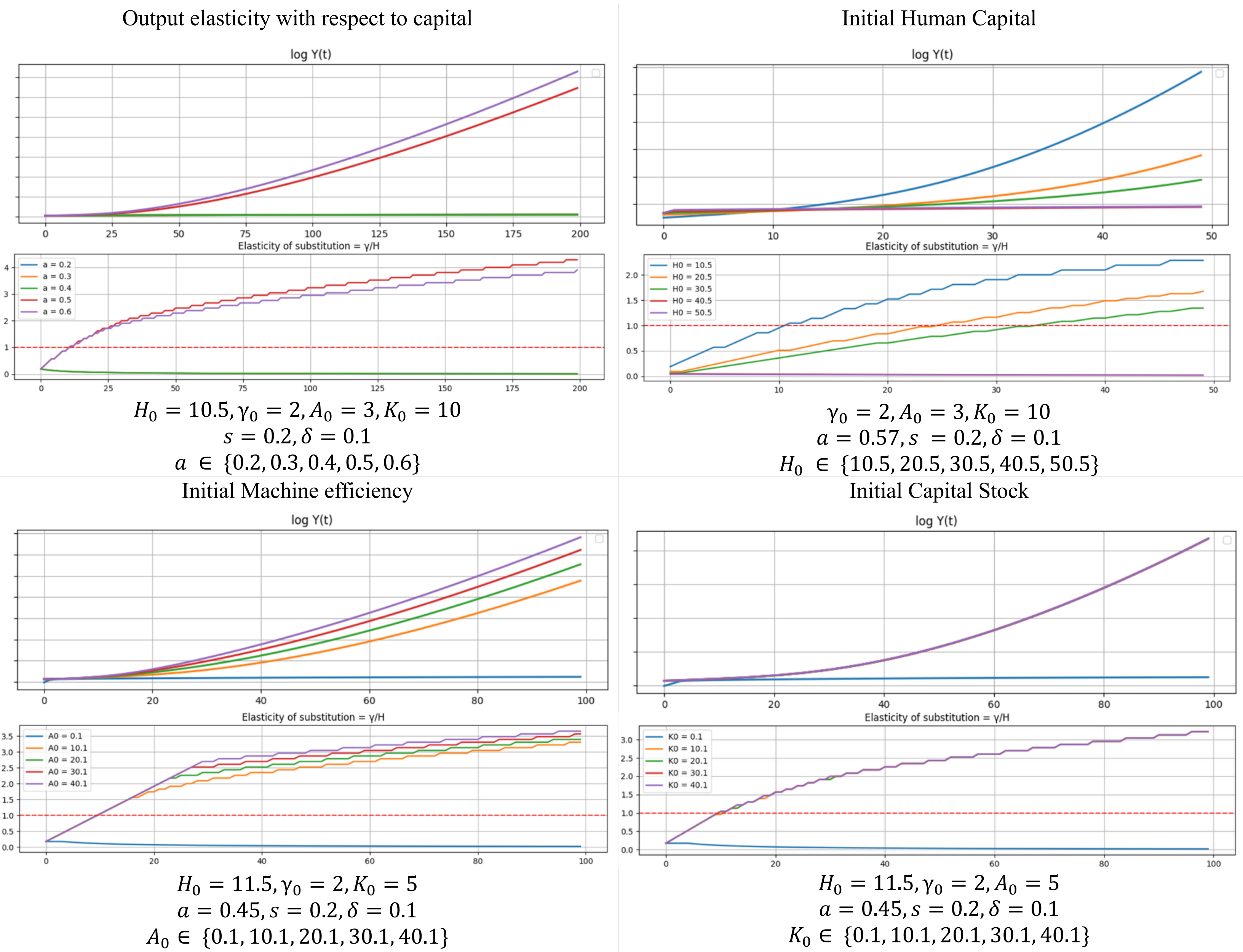
Ultimate sustained substitutes regime



$H_0 = 30.5, \gamma_0 = 5, A_0 = 1, K_0 = 5, a = 0.635, s = 0.25, \delta = 0.05$

Figure 2. Baseline Calibration Scenarios: Complementarity Trap vs. Substitutes Regime

To perform sensitivity analysis we make additional calibrations, examining variations in the economic conditions. The scenarios below illustrate the impact of variations in output elasticity with respect to capital, initial human capital, machine efficiency, and capital stock.



Output elasticity with respect to capital: $H_0 = 10.5, \gamma_0 = 2, A_0 = 3, K_0 = 10$; $s = 0.2, \delta = 0.1$; $a \in \{0.2, 0.3, 0.4, 0.5, 0.6\}$

Initial Human Capital: $\gamma_0 = 2, A_0 = 3, K_0 = 10$; $a = 0.57, s = 0.2, \delta = 0.1$; $H_0 \in \{10.5, 20.5, 30.5, 40.5, 50.5\}$

Initial Machine efficiency: $H_0 = 11.5, \gamma_0 = 2, K_0 = 5$; $a = 0.45, s = 0.2, \delta = 0.1$; $A_0 \in \{0.1, 10.1, 20.1, 30.1, 40.1\}$

Initial Capital Stock: $H_0 = 11.5, \gamma_0 = 2, A_0 = 5$; $a = 0.45, s = 0.2, \delta = 0.1$; $K_0 \in \{0.1, 10.1, 20.1, 30.1, 40.1\}$

Figure 3. Parameter Sensitivity in Escaping the Complementarity Trap

This analysis predominantly concerns the transition to the substitutes regime when the condition for escaping the complementarity trap is not satisfied (insufficient output elasticity with respect to capital). Hence, the analysis aims to identify how each parameter affects the likelihood of transition and the long-run trajectory of elasticity and output.

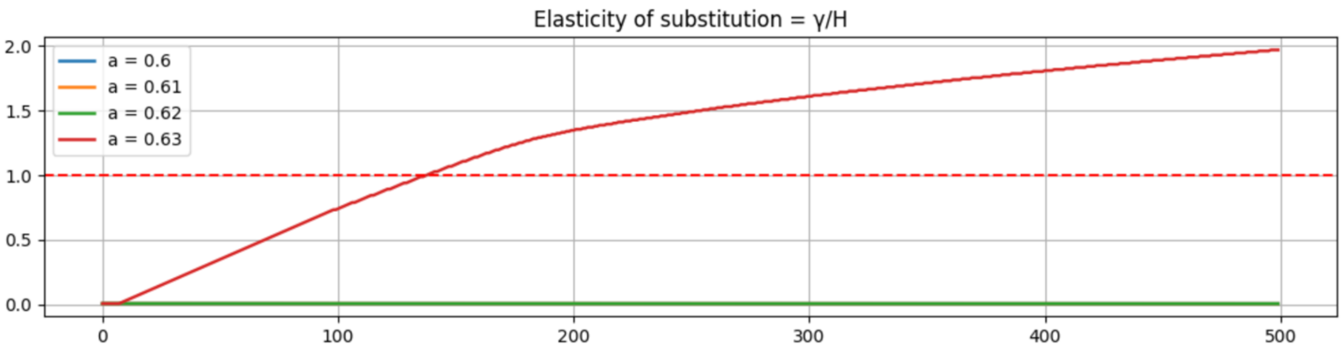
Other parameter sensitivities are provided in the Appendix. The effects of saving and depreciation rates are qualitatively similar to those of K, since they operate via the same mechanism. The influence of various levels of initial machine intelligence also looks analogous to the initial human capital level.

5. Discussion

This section briefly discusses multiple-regime switches during economic development. The calibrations showed that the initial capital stock positively influences the likelihood of transition from a regime where capital complements labour to one in which it substitutes for labour; this is also evident from the formal two necessary conditions for the transition. Consider an economy initially near the threshold, where transition occurs due to a very high capital stock.

If capital stock is very high—far above the Cobb-Douglas specification steady state level (under no technological or human capital progress)—both conditions for transition to substitutes are met. However, rapid capital depreciation can undermine this, pushing the economy back near the threshold. With lower capital, scientists may shift focus to improving human capital, returning the economy to the complements regime.

The economy's future trajectory depends on whether it satisfies the complementarity trap avoidance condition. If satisfied, after sufficient human capital accumulation, maintaining or increasing complementarity becomes undesirable, triggering a transition to the substitutes regime without further regime switches. The sensitivity test of Proposition 2 is demonstrated below.



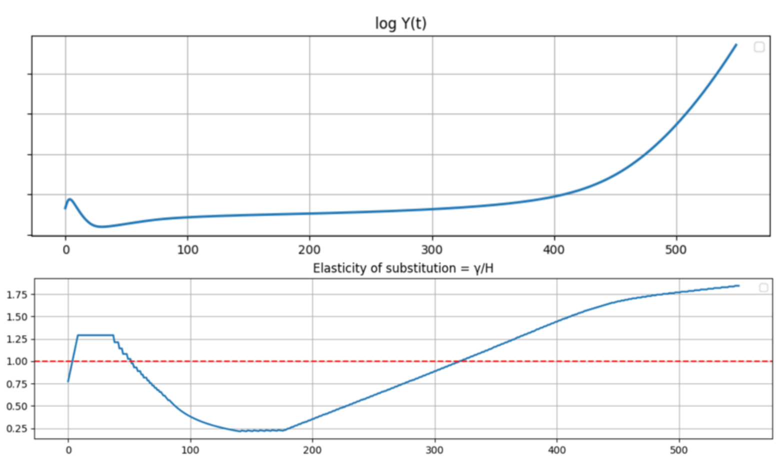
$$H_0 = 125.5, \gamma_0 = 1, K_0 = 1, A_0 = 1$$
$$s = 0.2, \delta = 0.1$$
$$a \in \{0.6, 0.61, 0.62, 0.63\}$$

Figure 4. Threshold Condition Test: Proposition 2

For economies failing to avoid the trap, initially high capital stock may enable escape, where slow capital decline or rapid reversal to accumulation supports sustained low incentives for human capital enhancement, as shown in the calibrations. However, if this economy transitions back to the complements regime, it will never reach the threshold again and will converge to a high degree of complementarity.

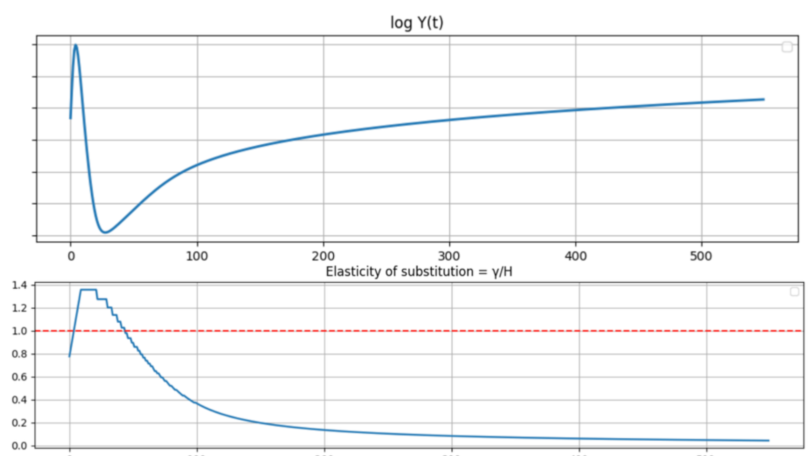
The following scenarios illustrate cases where the trap is avoided with regime switches and where it is not avoided, missing the chance for sustained substitutability.

Multiple switches



$$H_0 = 15.5, \gamma_0 = 12, A_0 = 10, K_0 = 100000$$
$$a = 0.7, s = 0.01, \delta = 0.5$$

A missed chance



$$H_0 = 15.5, \gamma_0 = 12, A_0 = 10, K_0 = 100000$$
$$a = 0.6, s = 0.01, \delta = 0.5$$

Figure 5. Non-Unique Regime Transitions: Scenarios with Reversible and Irreversible Shifts

Apart from the possible model transition paths, a critical limitation of the analysis lies in the ad hoc assumption regarding the specific functional form linking the elasticity of substitution to the relative cognitive abilities of humans and machines, expressed as the ratio $\frac{H}{\gamma}$. Instead, various alternative formulations could exhibit similar corner behaviour—where elasticity approaches zero when humans are far more intelligent than machines and infinity in the opposite case—with a threshold approximately located where machines surpass human intelligence. Additionally, the threshold could be specified as a human intelligence level plus a certain premium, capturing the initial dominance of humans and their efforts to prevent undesirable outcomes associated with machines gaining control.

Furthermore, it is important to account for the broader impact of AI on the R&D sector, including the potential for machines to self-train or to generate spillover effects that enhance human capital and create feedback loops. This latter assumption is plausible given the nature of AI as a widely accessible tool. If machines acquire knowledge not initially possessed by humans, likely, such knowledge will later be assimilated by humans without requiring specialised R&D investment to increase human capital.

Finally, accounting for expectations of the future AI development trajectory driven by government programmes or assimilated foreign open-source innovations can result in a different outcome of R&D decisions and the final equilibrium. In particular, an expectations-driven outcome concerning technology adoption was introduced by (Badhwar & Das, 2014), an economy with initially unfavourable conditions – high wealth inequality – can still avoid a "low technology trap" if optimistic business expectations regarding aggregate technological progress persist.

6. Conclusion

This paper has developed a theoretical prognostic model that introduces a novel perspective on technological progress by capturing a newly emergent feature of capital—intelligence—distinct from past innovations, which primarily enhanced efficiency. The framework employs a constant-elasticity-of-substitution (CES) production function, linking the elasticity of substitution to the superiority of human relative to machine intellect. By endogenising the changes in the elasticity of substitution between capital and labour over time, the model explains how historical growth trajectories can break down and illustrates the mechanism that enables a new regime of accelerated expansion. As predicted, once machines become sufficiently intelligent to substitute for human labour, the economy gains the potential for a Type I singularity—an accelerating growth rate. This result aligns with the existing predictions that a breakdown occurs when smarter robots than people prevail.

The analysis also derives closed-form threshold conditions under which profit-maximising R&D allocation shifts away from human-capital deepening towards machine-capital improvements, thereby establishing a substitutable relationship between the two factors. However, this transition to a substitutes regime occurs only if capital already possesses certain qualitative characteristics—such as a sufficiently high output elasticity or advanced levels of efficiency and intelligence. Moreover, agents' saving decisions influence the likelihood of switching growth patterns: in the absence of adequate saving behaviour, the economy remains trapped in a "complementarity trap", where immediate returns to human-ability enhancements always dominate, even if humans were infinitely intelligent. Under such a trap, growth follows a sub-exponential trajectory, and the long-run promise of a singularity is overlooked. In fact, once R&D is continuously directed toward machine intelligence rather than human-capital accumulation, the model converges in the long run to an AK-type outcome in which the technology parameter A rises without bound—yielding superior growth irrespective of the economy's initial characteristics, such as output elasticities or efficiency levels.

Broadly speaking, two categories of long-run outcomes emerge—either humans become infinitely smart, while machine intellect is not enhanced, resulting in sub-exponential growth, driven by a permanent focus on human-capital improvements.

The alternative scenario is a continuous rise in robots' intellectual abilities combined with a constancy of human capital; here, the economy attains a singularity, with R&D balanced between machine intelligence and efficiency gains. Within these broad categories, a variety of transitional dynamics can occur: for example, an initial concentration on capital enhancements that persists indefinitely, or oscillations in the relative levels of humans' and machines' cognitive abilities persist, leading to temporary switches of the economy between complementarity and substitutability regimes before settling on a long-run trajectory.

Although the model is purely theoretical and prognostic—clarifying the channels through which AI may transform long-run growth—it does not yield precise quantitative forecasts. Instead, it identifies the principal determinants of R&D investment and outlines possible future trajectories of the economy, contingent on the evolving relationship between human and machine intelligence. Future research could enrich these predictions by endogenising the saving rate, incorporating safety-driven constraints on machine-intelligence improvements, or modelling R&D allocation decisions that balance short-term returns against long-run prospects.

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Appendix

1 Partial derivatives

Production function:

$$F(AK,H,\gamma)=\left(a\,(AK)^{1-\frac{H}{\gamma}}+(1-a)\,H^{1-\frac{H}{\gamma}}\right)^{\frac{1}{1-\frac{H}{\gamma}}}.$$

1.1 Partial derivative w.r.t. H

$$F(AK,H,\gamma)=\left(a\,(AK)^{1-\frac{H}{\gamma}}+(1-a)\,H^{1-\frac{H}{\gamma}}\right)^{\frac{1}{1-\frac{H}{\gamma}}},\quad F'_H=\frac{\partial F}{\partial H}.$$

$$T_1=1-\frac{H}{\gamma},\quad T_2=a\,(AK)^{T_1}+(1-a)\,H^{T_1},\quad T'_{1,H}=-\frac{1}{\gamma}.$$

By log-diff. w.r.t. H:

$$\frac{\partial\ln F}{\partial H}=\frac{T_1\,\frac{T'_{2,H}}{T_2}-(\ln T_2)\,T'_{1,H}}{T_1^2},$$

hence

$$F'_H=F\,\frac{1}{T_1^2}\Big[T_1\,\frac{T'_{2,H}}{T_2}+\frac{\ln T_2}{\gamma}\Big].$$

Compute

$$T'_{2,H}=a\,(AK)^{T_1}\Big(-\tfrac{1}{\gamma}\ln(AK)\Big)+(1-a)\,H^{T_1}\Big(-\tfrac{1}{\gamma}\ln H+\frac{T_1}{H}\Big).$$

$$\boxed{F'_H=\frac{\big[a(AK)^{1-\frac{H}{\gamma}}+(1-a)\,H^{1-\frac{H}{\gamma}}\big]^{\frac{1}{1-\frac{H}{\gamma}}}}{(1-\frac{H}{\gamma})^2}\Bigg[(1-\tfrac{H}{\gamma})\,\frac{-\frac{a(AK)^{1-\frac{H}{\gamma}}}{\gamma}\ln(AK)+(1-a)H^{1-\frac{H}{\gamma}}\big(-\frac{1}{\gamma}\ln H+\frac{1-\frac{H}{\gamma}}{H}\big)}{a(AK)^{1-\frac{H}{\gamma}}+(1-a)H^{1-\frac{H}{\gamma}}}+ \\ +\ \frac{1}{\gamma}\ln\!\big[a(AK)^{1-\frac{H}{\gamma}}+(1-a)\,H^{1-\frac{H}{\gamma}}\big]\Bigg].}$$

1.2 Partial derivative w.r.t. A

$$F'_A=\frac{\partial F}{\partial A},\quad T_1=1-\frac{H}{\gamma},\quad T_2=a\,(AK)^{T_1}+(1-a)\,H^{T_1}.$$

By the chain rule,

$$F'_A=\frac{1}{T_1}\,T_2^{\frac{1}{T_1}-1}\,T'_{2,A}.$$

Since only the first term depends on A,

$$T'_{2,A}=\frac{\partial}{\partial A}\big[a\,(AK)^{T_1}\big]=a\,(AK)^{T_1}\,\frac{T_1}{A}.$$

$$\boxed{F'_A=\frac{a\,(AK)^{T_1}}{A}\,T_2^{\frac{1}{T_1}-1}=\frac{a\,(AK)^{1-\frac{H}{\gamma}}}{A}\,\big[a\,(AK)^{1-\frac{H}{\gamma}}+(1-a)\,H^{1-\frac{H}{\gamma}}\big]^{\frac{1}{1-\frac{H}{\gamma}}-1}}.$$

1.3 Partial derivative w.r.t. γ

$$F'_\gamma=\frac{\partial F}{\partial\gamma},\quad T_1=1-\frac{H}{\gamma},\quad T_2=a\,(AK)^{T_1}+(1-a)\,H^{T_1},\quad T_{1,\gamma}=\frac{\partial T_1}{\partial\gamma}=\frac{H}{\gamma^2}.$$

By log-diff. w.r.t. γ:

$$\frac{\partial\ln F}{\partial\gamma}=\frac{\partial}{\partial\gamma}\Big(\tfrac{1}{T_1}\ln T_2\Big)=\frac{T_1\,\frac{T_{2,\gamma}}{T_2}-(\ln T_2)\,T_{1,\gamma}}{T_1^2},$$

hence

$$F'_\gamma=F\,\frac{1}{T_1^2}\Big[T_1\,\frac{T_{2,\gamma}}{T_2}-T_{1,\gamma}\,\ln T_2\Big].$$

Compute

$$T_{2,\gamma} = \frac{\partial}{\partial\gamma}\big[a\,(AK)^{T_1} + (1-a)\,H^{T_1}\big] = a\,(AK)^{T_1}\,T_{1,\gamma}\ln(AK) + (1-a)\,H^{T_1}\,T_{1,\gamma}\ln H.$$

Substitute $T_{1,\gamma} = H/\gamma^2$:

$$T_{2,\gamma} = \frac{H}{\gamma^2}\big[a\,(AK)^{T_1}\ln(AK) + (1-a)\,H^{T_1}\ln H\big].$$

$$\boxed{\begin{aligned} F'_\gamma = {} & \frac{H}{\gamma^2\left(1-\frac{H}{\gamma}\right)^2}\Big[a\,(AK)^{1-\frac{H}{\gamma}} + (1-a)\,H^{1-\frac{H}{\gamma}}\Big]^{\frac{1}{1-\frac{H}{\gamma}}} \times \\ & \times \left[\left(1-\tfrac{H}{\gamma}\right)\frac{a\,(AK)^{1-\frac{H}{\gamma}}\ln(AK) + (1-a)\,H^{1-\frac{H}{\gamma}}\ln H}{a\,(AK)^{1-\frac{H}{\gamma}} + (1-a)\,H^{1-\frac{H}{\gamma}}} - \ln\Big[a\,(AK)^{1-\frac{H}{\gamma}} + (1-a)\,H^{1-\frac{H}{\gamma}}\Big]\right].\end{aligned}}$$

2 Limits

2.1 $\lim_{\gamma\to H} R(H,\gamma)$

Begin with the exact expressions:

$$F'_H = F\,\frac{1}{T_1^2}\Big[T_1\,\frac{T'_{2,H}}{T_2} + \frac{\ln T_2}{\gamma}\Big], \qquad F'_\gamma = F\,\frac{1}{T_1^2}\Big[T_1\,\frac{T'_{2,\gamma}}{T_2} - T'_{1,\gamma}\,\ln T_2\Big],$$

where

$$T_1 = 1-\frac{H}{\gamma}, \quad T_2 = a\,(AK)^{T_1} + (1-a)\,H^{T_1}, \quad T'_{1,\gamma} = \frac{H}{\gamma^2}.$$

Hence

$$R(H,\gamma) = \frac{F'_H}{F'_\gamma} = \frac{\gamma^2}{H}\,\frac{T_1\,\frac{T'_{2,H}}{T_2} + \frac{1}{\gamma}\ln T_2}{T_1\,\frac{T_3}{T_2} - \ln T_2},$$

with

$$T_3 = a\,(AK)^{T_1}\ln(AK) + (1-a)\,H^{T_1}\ln H.$$

Step 1: Expand $T_2(T_1)$ for small T_1
Using

$$c^{T_1} = 1 + T_1\,\ln c + \tfrac{T_1^2(\ln c)^2}{2} + O(T_1^3),$$

Write

$$T_2(T_1) = a\,(AK)^{T_1} + (1-a)\,H^{T_1} = 1 + T_1\,T_4 + T_1^2\,T_5 + O(T_1^3),$$

where

$$T_4 = a\,\ln(AK) + (1-a)\,\ln H, \qquad T_5 = \tfrac{1}{2}\Big[a(\ln(AK))^2 + (1-a)(\ln H)^2\Big].$$

Hence

$$\ln T_2(T_1) = T_1\,T_4 + T_1^2\Big(T_5 - \tfrac{T_4^2}{2}\Big) + O(T_1^3).$$

Step 2: Approximate $\frac{T'_{2,H}}{T_2}$ and $\frac{T_3}{T_2}$. Replace $\frac{1}{\gamma}$ by $\frac{1}{H}$

$$T'_{2,H} = a\,(AK)^{T_1}\Big(-\tfrac{1}{H}\ln(AK)\Big) + (1-a)\,H^{T_1}\Big(-\tfrac{1}{H}\ln H + \frac{T_1}{H}\Big) =$$

$$= -\frac{a}{H}\ln(AK) - \frac{a}{H}T_1(\ln(AK))^2 + (1-a)\left(-\frac{1}{H}\ln(H) + \frac{T_1}{H} - \frac{T_1}{H}(\ln(H))^2\right) + O(T_1^2)$$

$$\frac{1}{T_2} = 1 - T_1\,T_4 + T_1^2(T_4^2 - T_5) + O(T_1^3)$$

$$\frac{T'_{2,H}}{T_2} = -\frac{1}{H}T_4 + T_1(-\frac{2T_5}{H} + \frac{1-a}{H} + \frac{T_4^2}{H}) + O(T_1^2)$$

and

$$T_3 = a\ln(AK)(1+T_1\ln(AK)) + (1-a)\ln(H)(1+T_1\ln(H)) + O(T_1^2) = T_4 + 2T_1T_5 + O(T_1^2)$$

$$\frac{T_3}{T_2} = T_4 - T_1T_4^2 + 2T_1T_5 + O(T_1^2)$$

Step 3: Expand numerator and denominator brackets of the second part

Numerator bracket:

$$N(T_1) := T_1\,\frac{T'_{2,H}}{T_2} + \frac{1}{\gamma}\,\ln T_2 = T_1\Big(-\frac{1}{H}T_4 + T_1(-\frac{2T_5}{H} + \frac{1-a}{H} + \frac{T_4^2}{H})\Big) + \frac{1}{H}\Big(T_1T_4 + T_1^2(T_5 - \tfrac{T_4^2}{2}) + O(T_1^3)\Big) =$$

$$= \frac{T_1^2}{H}(1 - a - (T_5 - \frac{T_4^2}{2})) + O(T_1^3)$$

Subtracting:

$$\begin{aligned}
T_5 \;-\; \frac{T_4^2}{2} &= \frac{1}{2}\Big[a(\ln(AK))^2 + (1-a)(\ln H)^2\Big] \;-\; \frac{1}{2}\Big[a^2(\ln(AK))^2 + 2\,a(1-a)\,\ln(AK)\,\ln H + (1-a)^2(\ln H)^2\Big] \\
&= \frac{1}{2}\Big[\,a(\ln(AK))^2 - a^2(\ln(AK))^2 \;+\; (1-a)(\ln H)^2 - (1-a)^2(\ln H)^2 \;-\; 2\,a(1-a)\,\ln(AK)\,\ln H\Big] \\
&= \frac{1}{2}\Big[\,a(1-a)\,(\ln(AK))^2 \;+\; (1-a)a\,(\ln H)^2 \;-\; 2\,a(1-a)\,\ln(AK)\,\ln H\Big] \\
&= \frac{a(1-a)}{2}\Big[\,(\ln(AK))^2 + (\ln H)^2 - 2\,\ln(AK)\,\ln H\Big] \\
&= \frac{a(1-a)}{2}\,\big(\ln(AK) - \ln H\big)^2 = \frac{a(1-a)}{2}\,\Big(\ln\tfrac{AK}{H}\Big)^2.
\end{aligned}$$

Thus

$$N(T_1) = \frac{T_1^2}{H}(1 - a - \frac{a(1-a)}{2}\,\Big(\ln\tfrac{AK}{H}\Big)^2) + O(T_1^3),$$

since $\gamma \to H$.

Denominator bracket:

$$D(T_1) := T_1\,\frac{T_3}{T_2} - \ln T_2 = T_1T_4 + T_1^2\Big(2T_5 - T_4^2\Big) - \Big(T_1T_4 + T_1^2(T_5 - \tfrac{T_4^2}{2})\Big) + O(T_1^3) = T_1^2(T_5 - \tfrac{T_4^2}{2}) + O(T_1^3) =$$

$$T_1^2\frac{a(1-a)}{2}\,\Big(\ln\tfrac{AK}{H}\Big)^2 + O(T_1^3)$$

Step 4: Compute the limit

$$R(H,\gamma) = \frac{\gamma^2}{H}\,\frac{N(T_1)}{D(T_1)} \sim \frac{\gamma^2}{H}\,\frac{\frac{T_1^2}{H}(1 - a - \frac{a(1-a)}{2}\Big(\ln\frac{AK}{H}\Big)^2)}{T_1^2(\frac{a(1-a)}{2}\Big(\ln\frac{AK}{H}\Big)^2)} = \frac{1 - a - \frac{a(1-a)}{2}\Big(\ln\frac{AK}{H}\Big)^2}{\frac{a(1-a)}{2}\Big(\ln\frac{AK}{H}\Big)^2}.$$

$$\boxed{\lim_{\gamma\to H} R(H,\gamma) = \frac{2}{a\,\big(\ln\frac{AK}{H}\big)^2} - 1.}$$

2.2 $\lim_{\gamma\to H} R(\gamma, A)$

Using the notation and expansions from the previous subsection:

$$F'_\gamma = F\,\frac{1}{T_1^2}\Big[T_1\,\frac{T_{2,\gamma}}{T_2} - T_{1,\gamma}\,\ln T_2\Big], \quad F'_A = \frac{a\,(AK)^{T_1}}{A}\,T_2^{\frac{1}{T_1}-1},$$

It was shown that when $\gamma \to H$

$$F'_\gamma \sim F\,\frac{1}{HT_1^2}\Big[T_1^2\frac{a(1-a)}{2}\,\Big(\ln\tfrac{AK}{H}\Big)^2\Big] = F\,\Big[\frac{a(1-a)}{2H}\,\Big(\ln\tfrac{AK}{H}\Big)^2\Big]$$

$$\lim_{\gamma\to H}\frac{F'_\gamma}{F'_A} = \frac{F\,\Big[\frac{a(1-a)}{2H}\,\Big(\ln\frac{AK}{H}\Big)^2\Big]}{\frac{a\,(AK)^{T_1}}{A}\,T_2^{\frac{1}{T_1}-1}} = \frac{T_2\,\Big[\frac{a(1-a)}{2H}\,\Big(\ln\frac{AK}{H}\Big)^2\Big]}{\frac{a\,(AK)^{T_1}}{A}}$$

and $(AK)^{T_1} \to 1,\ T_2 \to 1$. Therefore,

$$\boxed{\lim_{\gamma\to H} R(\gamma, A) = \frac{A(1-a)}{2H}\,\Big(\ln\frac{AK}{H}\Big)^2.}$$

2.3 $\lim_{H\to\infty} R(H,\gamma)$

Starting from

$$\frac{F'_H}{F'_\gamma} = \frac{\gamma^2}{H}\,\frac{T_1\,\frac{T'_{2,H}}{T_2} + \frac{1}{\gamma}\ln T_2}{T_1\,\frac{T_3}{T_2} - \ln T_2},$$

with

$$T_1 = 1 - \frac{H}{\gamma},\quad T_2 = a\,(AK)^{T_1} + (1-a)\,H^{T_1},\quad T_3 = a\,(AK)^{T_1}\ln(AK) + (1-a)\,H^{T_1}\ln H.$$

1. Substitute $K = \frac{sH}{\delta}$, so $AK = (sA/\delta)\,H$. Then

$$(AK)^{T_1} = \left(\tfrac{sA}{\delta}\right)^{T_1} H^{T_1},$$

and

$$T_2 = H^{T_1}\big[a(sA/\delta)^{T_1} + (1-a)\big] \sim (1-a)\,H^{T_1},\quad T_3 \sim (1-a)\,H^{T_1}\ln H.$$

2. Hence

$$\frac{T_3}{T_2} \sim \ln H,\qquad \ln T_2 = T_1\ln H + \ln(1-a) + o(1).$$

3. As $H\to\infty$, $T_1 \sim -H/\gamma$. Compute

$$\frac{T'_{2,H}}{T_2} = \frac{\partial_H T_2}{T_2} = \frac{d}{dH}\ln T_2 = -\frac{1}{\gamma}\ln H - \frac{1}{\gamma} + o(1).$$

4. Numerator:

$$T_1\Big(-\tfrac{1}{\gamma}\ln H - \tfrac{1}{\gamma}\Big) + \tfrac{1}{\gamma}\big(T_1\ln H + \ln(1-a)\big) = \frac{H}{\gamma^2} + \frac{1}{\gamma}\ln(1-a) + o(1).$$

Denominator:

$$T_1\ln H - \big(T_1\ln H + \ln(1-a)\big) = -\ln(1-a) + o(1).$$

5. Taking the limit,

$$\frac{F'_H}{F'_\gamma} \sim \frac{\gamma^2}{H}\,\frac{\frac{H}{\gamma^2} + \frac{1}{\gamma}\ln(1-a)}{-\ln(1-a)} \;\longrightarrow\; -\frac{1}{\ln(1-a)}.$$

$$\boxed{\lim_{H\to\infty} R(H,\gamma) = -\frac{1}{\ln(1-a)}.}$$

2.4 $\lim_{K\to\infty} R(H,\gamma)$

Notation:

$$T_1 = 1 - \frac{H}{\gamma},\quad T_2 = a\,(AK)^{T_1} + (1-a)\,H^{T_1},\quad T_3 = a\,(AK)^{T_1}\ln(AK) + (1-a)\,H^{T_1}\ln H.$$

Since

$$\frac{F'_H}{F'_\gamma} = \frac{\gamma^2}{H}\,\frac{T_1\,\frac{T'_{2,H}}{T_2} + \frac{1}{\gamma}\ln T_2}{T_1\,\frac{T_3}{T_2} - \ln T_2},$$

analyze as $K\to\infty$:

1. Asymptotics of T_2, T_3:

$$(AK)^{T_1}\to\infty,\quad H^{T_1}\text{ bounded} \implies T_2 \sim a\,(AK)^{T_1},\quad T_3 \sim a\,(AK)^{T_1}\ln(AK).$$

Hence

$$\frac{T_3}{T_2} \sim \ln(AK),\quad \ln T_2 \sim \ln a + T_1\ln(AK).$$

2. Evaluate $T'_{2,H}/T_2$:

$$\frac{T'_{2,H}}{T_2} = \partial_H \ln T_2 \sim \frac{d}{dH}\big(\ln a + T_1\ln(AK)\big) = T'_{1,H}\,\ln(AK) = -\frac{1}{\gamma}\ln(AK),$$

where $T'_{1,H} = \frac{\partial T_1}{\partial H} = -\frac{1}{\gamma}$.

3. Numerator and denominator:
Numerator:

$$T_1 \frac{T'_{2,H}}{T_2} + \frac{1}{\gamma} \ln T_2 \sim T_1 \Big(-\tfrac{1}{\gamma} \ln(AK)\Big) + \frac{1}{\gamma}\big(\ln a + T_1 \ln(AK)\big) = \frac{1}{\gamma} \ln a.$$

Denominator:

$$T_1 \frac{T_3}{T_2} - \ln T_2 \sim T_1 \ln(AK) - \big(\ln a + T_1 \ln(AK)\big) = -\ln a.$$

4. Limit:

$$\frac{F'_H}{F'_\gamma} \sim \frac{\gamma^2}{H} \, \frac{\frac{1}{\gamma} \ln a}{-\ln a} = -\frac{\gamma}{H}.$$

$$\boxed{\lim_{K\to\infty} R(H,\gamma) = -\frac{\gamma}{H}.}$$

2.5 $\lim_{K\to\infty} R(\gamma, A)$

$$F'_\gamma = F \, \frac{1}{T_1^2} \Big[T_1 \, \frac{T'_{2,\gamma}}{T_2} - T'_{1,\gamma} \, \ln T_2\Big], \qquad F'_A = \frac{a\,(AK)^{T_1}}{A} \, T_2^{\frac{1}{T_1}-1},$$

so

$$\frac{F'_\gamma}{F'_A} = \frac{A\,T_2}{a\,(AK)^{T_1}} \, \frac{T_1 \, \frac{T'_{2,\gamma}}{T_2} - T'_{1,\gamma} \, \ln T_2}{T_1^2},$$

with

$$T_1 = 1 - \frac{H}{\gamma}, \quad T'_{1,\gamma} = \frac{H}{\gamma^2}, \quad T_2 = a\,(AK)^{T_1} + (1-a)\,H^{T_1}.$$

As $K \to \infty$, $(AK)^{T_1} \to \infty$ while H^{T_1} remains bounded, so

$$T_2 \sim a\,(AK)^{T_1}, \quad \ln T_2 \sim \ln a + T_1 \ln(AK), \quad \frac{T'_{2,\gamma}}{T_2} = \frac{\partial}{\partial\gamma} \ln T_2 = T'_{1,\gamma} \, \ln(AK) + o(1).$$

Hence the inner numerator is

$$T_1 \, \frac{T'_{2,\gamma}}{T_2} - T'_{1,\gamma} \, \ln T_2 \sim T_1 T'_{1,\gamma} \ln(AK) - T'_{1,\gamma}\big(\ln a + T_1 \ln(AK)\big) = -T'_{1,\gamma} \, \ln a.$$

Since $T'_{1,\gamma} = H/\gamma^2$ and $T_1 = (\gamma - H)/\gamma$, it follows that

$$\boxed{\lim_{K\to\infty} R(\gamma, A) = -\frac{A\,H \, \ln a}{(\gamma - H)^2}.}$$

3 Proposition Proofs

3.1 Proposition 1. Transition conditions

The first condition: $\lim_{\gamma\to H} R(H,\gamma) < 1$
Using the result from (2.1)

$$\frac{2}{a\,\big(\ln \frac{AK}{H}\big)^2} - 1 < 1 \implies \boxed{1 < a \left(\ln \frac{AK}{H}\right)^2}$$

The second condition: $\lim_{\gamma\to H} R(\gamma, A) > 1$
Using the result from (2.2)

$$\frac{A(1-a)}{2H} \left(\ln \frac{AK}{H}\right)^2 > 1 \implies 1 - a > \frac{2H}{A\Big(\ln \frac{AK}{H}\Big)^2} \implies \boxed{a < 1 - \frac{2H}{A\Big(\ln \frac{AK}{H}\Big)^2}}$$

3.2 Proposition 2. Escape from the trap

Escape condition: $\lim_{H \to \infty} R(H, \gamma) < 1$
Using the result from (2.3):

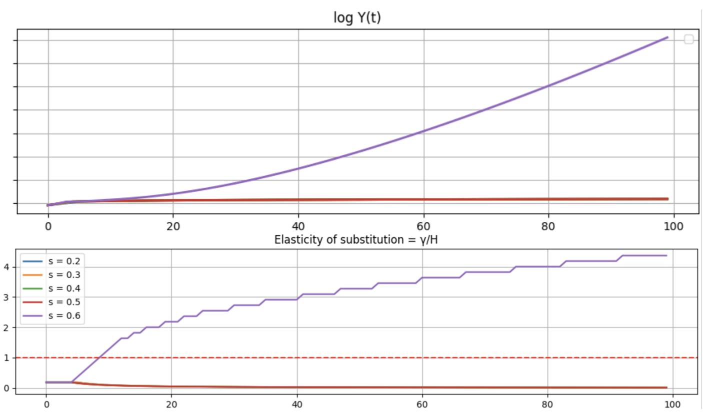
$$-\frac{1}{\ln(1-a)} < 1 \implies -1 > \ln(1-a) \implies \frac{1}{e} > 1 - a \implies \boxed{a > -\frac{1}{e} + 1}$$

3.3 Proposition 3. Long-Run Indifference

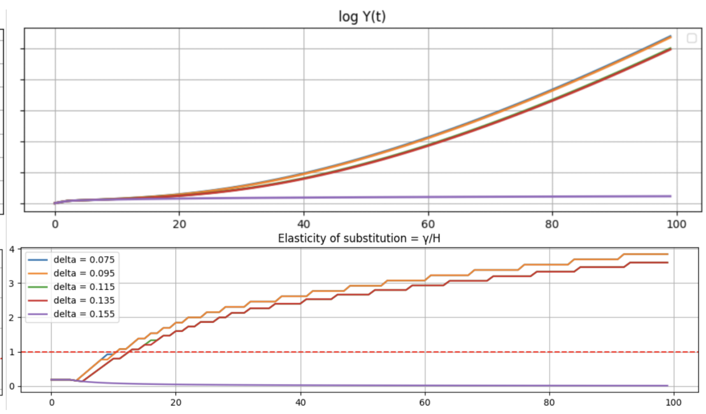
Long-run indifference: $\lim_{K \to \infty} R(\gamma, A) < 1$ Using the result from (2.5):

$$-\frac{A\,H\,\ln a}{(\gamma - H)^2} = 1 \implies \boxed{A = -\frac{(\gamma - H)^2}{H\,\ln a}}$$

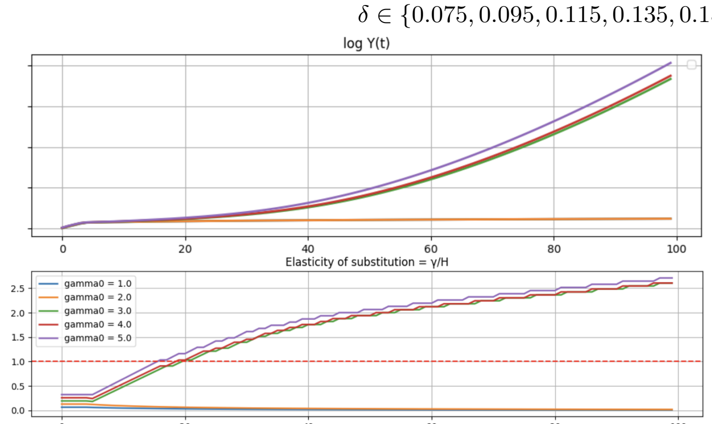
4 Sensitivity analysis



(a) **Saving rate impact**
$a = 0.45, \delta = 0.1, H_0 = 5.5, \gamma_0 = 1, A_0 = 1, K_0 = 0.1$
$s \in \{0.2, 0.3, 0.4, 0.5, 0.6\}$



(b) **Depreciation rate impact**
$a = 0.5, s = 0.2, H_0 = 5.5, \gamma_0 = 1, A_0 = 1, K_0 = 1$
$\delta \in \{0.075, 0.095, 0.115, 0.135, 0.155\}$



(c) **Initial machine intelligence**
$a = 0.5, s = 0.2, \delta = 0.075, H_0 = 15.5, A_0 = 1, K_0 = 1$
$\gamma_0 \in \{1, 2, 3, 4, 5\}$

Figure 1: Sensitivity analysis continued