Majority Rule Under Peer-influence*

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Abstract

In the standard collective choice theoretic framework, individuals are assumed to have intrinsic preferences that are not influenced by their peers. We introduce peer-influence in preference formation and study its implication for the Majority rule. In particular, we revisit the McGarvey (1953) conjecture which states that actual minimum number of individuals necessary to express all possible social preference over n elements under majority rule is approximately n. We show, when individual preferences are strict quasi-ordering, it is possible to represent all social preferences as an outcome of majority rule for a society with same number of individuals as there are alternatives. We further establish that the above result may not hold true in the presence of peer-influence. The outcomes of majority rule are constrained by several factors including the societal structure, population size, and the degree of incompleteness in individual preferences. In some societies, we show that certain preferences may never be realized under majority rule, while in others, the reverse may hold true.

JEL Classification Codes: D11, D71, D91.
Keywords: Network; Peer-influence; Adaptive Preference; Majority Rule.

1. Introduction

The standard framework in collective choice theory (Arrow (1951, 1963)) assumes that individuals in a community or a society are autonomous. They have intrinsic preferences that are not influenced by social or peer interactions. However, this assumption is contrary to what we experience in reality. In actual communities or groups individuals often influence each other through various forms of interactions. This can include adopting similar tastes, opinion, social norm or even political and moral beliefs. Empirical evidence supports the existence of peer-influence in group activities,[1] indicating that individuals

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[1]Mas, Alexandre and Enrico Moretti (2009), for example, have shown that productivity of a worker depends on the productivity of co-workers in the same team. Zimmerman (2003) confirms peer influence

are, in fact, not fully autonomous. In this paper, we shall explore some implications of peer influence in preference formation for the majority rule.

Preference relation formed under peer-influence as discussed above is a case/example of adaptive preference. Given a society, adaptive preferences can be formed in many different ways. For example, individuals can take a majoritarian view in the sense that if she cannot decide between alternative x and y, she can make a decision based on the preferences of the majority of people in her neighborhood. Another more *careful* individual may form opinion between x, y by adopting the preferences of her neighbours whenever there is a consensus in her neighbourhood on the preference between x and y. We call it an unanimity based peer influence (UPI). Motivation for UPI may be drawn from the social phenomenon known as *homiphily.* In a society people tend to be similar to the people around them. Therefore, it is reasonable to expect that the opinions of individuals will not differ significantly from those of their friends.[2] This way of formation of adaptive preference is somewhat similar to the process introduced in Cuhadaroglu (2017) in the context of revealed preference theory.

In this paper, we represent a society by an undirected network of individuals and identify their peers by their neighbours in the network. We consider UPI as a channel of peer-influence. As said before, under UPI individual experiences peer-influence only when he is indecisive about some alternative, this is to say, his(her) preference is incomplete and there is unanimity among the neighbour around the preference.[3] Thus, in our model adaptive preference relation of an individual is an extension of her intrinsic (initial) preference relation as the UPI helps to remove incompleteness in the initial preference relation. We call it her peer-influenced relation. Another simplification that we make in our analysis is that the peer-influenced relation is transitive. This is for the requirement that individuals maintain their rationality even under peer influence. We illustrate UPI with the following example.

Example 1.1. Consider a society with 5 individual, $N = \{1, 2, 3, 4, 5\}$, represented by the following graph (Fig 1). Let $X = \{x, y, z\}$ be the set of alternatives available to them.

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in academic outcomes of the students using a natural experiment. Calvo-Armengol et al. (2009) have shown how structural properties of friendship networks affect individual outcomes in education. Glaeser et al. (1996), explains high variance in crime rates across time and space by social interactions.

[2]Choice behaviours based on homophily have been captured in diverse contexts. See, for example, Jackson (2020), Easley and Kleinberg (2010), Borah et al. (2018).

[3]We concede that in the real world, in rare cases, individuals may change their opinions based on peer influence (a case when peer influence occurs even when the intrinsic preference relations is complete), and a more general analysis would allow for this possibility; however, in the formal analysis in this paper, we abstract from that possibility.

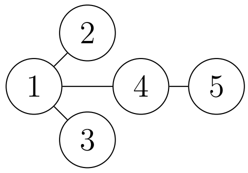


Fig 1

Individuals have the following initial preferences.

$R'_1 = \{(z,x)\}$, $R'_2 = \{(x,y),(x,z)\}$, $R'_3 = \{(x,y)\}$, $R'_4 = \{(x,y),(x,z)\}$, $R'_5 = \{(z,x)\}$.

We obtain peer-influenced preference relations of the individuals under UPI. Individual 1 is indecisive over x, y and y, z. Furthermore, all his(her) neighbours prefer x to y. Under UPI she adopts the preference xPy. For y, z there is no consensus. Thus, 1's peer-influenced preference relation becomes $R_1 = zP_1xP_1y$ in view of transitivity. For rest of the individuals, the peer-influenced preference relations are as follows: $R_2 = \{(x,y),(x,z)\}$, $R_3 = zP_3xP_3y$, $R_4 = \{(x,y),(x,z)\}$, $R_5 = zP_5xP_5y$. It is evident from the example that the ability of any individual to influence others significantly depends on the structure of the society.

After we obtain a profile of peer-influenced relations of the individual from the initial preferences, majority rule is applied on it to determine the social preference.

In the context of autonomous individuals, McGarvey (1953) has shown that, given an arbitrary reflexive and complete preference relation, over a set of n elements, a group of individuals exists with strict individual preference orderings such that the group preference relation as determined by the majority rule is the given preference relation. In other words, any reflexive and connected social preference can be shown as an outcome of majority rule for some profile of individual preferences. He has also given a conjecture that the actual minimum number of individuals necessary to express all possible social (group) preference over n elements is approximately n.

We first provide an extension of McGarvey (1953)'s result by showing that the same result can be obtained when individuals, instead of strict ordering, possess strict quasi-ordering[4]. Additionally, we show that all reflexive and connected social preference with n alternatives can be expressed as an outcome of the majority rule for a society of n individuals with quasi-ordering preferences. This addresses McGarvey (1953) conjecture under the assumption of strict quasi-ordering preference.

However, we also highlight a contrast by establishing that, in the presence of unanimity-based peer influence, the above result may not hold true. Whether a social preference can be represented by majority rule depends on several factors, including societal structure, population size, and the degree of incompleteness in individual preferences. We argue, in some societies all social preferences cannot be represented as outcomes of majority

[4]A strict quasi-ordering is a non-empty, incomplete but transitive and asymmetric preference relation over the set of alternatives.

rule, while in others, the reverse is true. Restrictions imposed by these factors shape the outcomes of majority rule.

The role of peer-influence in the formation of adaptive preference has been well documented in the context of individual choices relating to revealed preference theory (Chambers et al. (2023), Kashaev et al. (2021), Fershtman et al.(2018), Blume et al. (2015), Borah et al. (2018), Brock et al. (2001)). The literature on peer-influence in collective choice theory is scanty. Sprumont (2000), Ray and Zhou (2001) and Keshav and Pal (2023) analyze collective choice where individuals interact in game theoretic set-up to determine social outcomes. There also, the individual preferences over outcomes are independent of social interactions. Jain (2014) investigates the implications for social choice when some individuals do not possess intrinsic preferences and instead relate their preferences to social preferences by conforming or nonconforming. The adaptive preference thus formed, based on social preference, differs from UPI.

This paper is divided into 6 sections. In section 2 we discuss the model. Section 3 contains the benchmark results and their extensions. Section 4 contains the main results under peer-influence. Section 5 discusses how population size and the completeness of preferences impact social outcomes under majority rule. Section 5 concludes the paper.

2. The Model

Let N be a set of n individuals and X be a finite set alternatives, $|X| \geq 3$. We represent a society or a community of n individuals by an undirected connected graph $G(N, g)$, where g is an $n \times n$ real-valued symmetric matrix where $g_{ij} \in \{1, 0\}$ is the entry in the i^{th} row and j^{th} column of the matrix g. g_{ij} takes value 1 if and only if individual i is connected to individual j and 0 otherwise. g is generally referred to as an adjacency matrix. Let A^n be the set of all $n \times n$ adjacency matrices. Thus, $G(N, g), g \in A^n$ contains information about a society where individuals have mutual connections with each other. For an individual i, $N_i = \{j \mid g_{ij} = 1\}$ denotes the set of neighbours or peers of i.

Unlike the standard models in collective choice theory, we assume that individuals are non-autonomous. Their preferences are influenced by interactions with peers. To allow peer-influence in preference formation, we assume that individuals are partially decisive. In other words, we assume that instead of a strict ordering, individuals' *initial* (intrinsic) preferences are non-empty, incomplete but transitive and asymmetric (strict quasi-ordering) over X.[5] They consistently form a complete preference to the extent possible by interacting with their peers. The preference relation thus obtained we call it a peer-influenced relation. We denote by R'_i the initial preference relation of individual i and by R_i the peer influenced relation of i. As said, R'_i's are a stick quasi-ordering but

[5]The assumption of strict preferences of individuals is common in the literature related to incentive theory, voting etc.

not a strict ordering. T is the set of all non-empty, transitive and asymmetric preference relations on X; $R'_i \in T$. For consistency we also require $R_i \in T$. P and I respectively denote the asymmetric and symmetric parts of a preference relation.[6]

While there may exist many forms of peer-influence in preference formation, in this paper we introduce a particular form which we call unanimity based peer-influence (UPI).

Unanimity based peer influence (UPI): In our model, peer-influence occurs only for the pairs of alternatives over which the individual is indecisive in her initial preference. We assume that for the pairs of alternatives for which an individual is indecisive, she looks up to preferences of her immediate neighbours in the community. Whenever possible, she forms a preference relation that is a transitive extension[7] of her initial preference by adopting the preference of her neighbours over those pairs of alternatives for which there is an unanimity (agreement) among them. In case when there is no unanimity among the neighbours or the transitive extension of the initial preference does not exist, the individual adheres to her initial preference relation. We refer to this process of peer-influence in preference formation as a unanimity based peer-influence (UPI). The preference relation thus obtained under UPI we call it a peer-influenced relation of the concerned individual. For example, if individual i lacks completeness over the alternatives x, y and all her immediate neighbours in the community prefer x over y, then under UPI she adopts their preference between x and y to form an extended transitive preference relation whenever it is possible. Peer-influenced preference relation, therefore, depends not only on the profile of initial preference of the individuals, $R' = (R'_1, R'_2, ...R'_n)$, but also on the community structure $G(N, g)$.

Formally, we define a peer-influenced relation based on UPI for an individual i with the help of a function, f_i. We call f_i a peer-influence function of individual i. We define $f_i : A^n \times T^{n-1} \times T \to T$ as follows:

$$f_i(R'_i, R'_{-i}, g) = R_i, \text{ such that } \begin{cases} (i)\ R_i \text{ is a transitive extension of } R'_i, \qquad \text{and} \\ (ii)\ \forall x, y \in X \text{ if } (x, y), (y, x) \notin R'_i, \text{ then} \\ (x, y) \in R_i \leftrightarrow [\forall j \in N_i(G(N, g)) : (x, y) \in R'_j, \text{ and} \\ N_i(G(N, g)) \neq \emptyset]. \end{cases}$$

$f_i(R'_i, R'_{-i}, g) = R_i = R'_i$, if there does not exist R_i satisfying (i) and (ii).

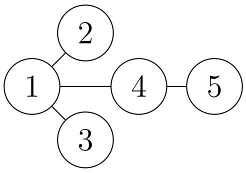
Given a network represented by g and a profile of initial preferences R', f_i generates a transitive preference relation for individual i, R_i considering the preferences of her

[6]For brevity we write P_i instead of $P(R_i)$ and P'_i instead of $P(R'_i)$.

[7]A preference relation R_1 is said to be a transitive extension of R_2 if and only if $R_2 \subseteq R_1$ and $P(R_2) \subseteq P(R_1)$, and also R_1 is transitive.

neighbours. We call R_i a peer-influenced relation of individual i obtained from R' under $G(N, g)$. We require R_i to satisfy transitivity to ensure that individuals are consistent in formation of peer-influenced preference.

Example 2.1. *Consider a society with 5 individual,* $N = \{1, 2, 3, 4, 5\}$*, as shown in Example 1.1. Let* $X = \{x, y, z\}$ *be the set of alternatives available to them.*



Individuals have the following initial preference.

$R'_1 = \{(z, x)\}$, $R'_2 = \{(x, y), (x, z)\}$, $R'_3 = \{(x, y)\}$, $R'_4 = \{(x, y)\}$, $R'_5 = \{(z, x)\}$.

The peer-influenced relations now can be obtained from $(R'_1, R'_2, R'_3, R'_4, R'_5)$ *following* f_i*:*
$R_1 = zP_1xP_1y$, $R_2 = \{(x, y), (x, z)\}$, $R_3 = zP_3xP_3y$, $R_4 = zP_4xP_4y$, $R_5 = zP_5xP_5y$.

Example 2.2. *Consider the same graph given in Fig 1. Let the initial preference of the individuals over* $X = \{x, y, z, w\}$ *now be:*

$R'_1 = \{(z, x)\}$, $R'_2 = R'_3 = \{(x, y), (y, z)(x, z), (x, w)\}$, $R'_4 = xP'_4yP'_4z$, $R'_5 = xP'_5yP'_5z$.

The peer-influenced relations now can be obtained from $(R'_1, R'_2, R'_3, R'_4, R'_5)$ *following* f_i*:*
$R_1 = \{(z, x)\}$, $R_2 = R_3 = \{(x, y), (y, z)(x, z), (x, w)\}$, $R_4 = xP_4yP_4z$, $R_5 = xP_5yP_5z$.
Notice, for individual 1, peer-influenced preference relation is same as the initial preference due to R_1 *not satisfying property (i) in the function* f_i*.*

Remark 1. *If* $g = \theta$*, where* θ *is a null matrix, then* $R'_i = R_i$, $\forall i \in N$*. This means, when the graph is scattered i.e. there is no connection between any two individual, no peer-influence occurs and we are back to the assumption autonomous individuals.*

Remark 2. *In case of three alternatives, if everyone in the neighbourhood of* i *has a preference 'x preferred over y' in their initial preferences, and* i *is undecided over* x, y*, then the peer influenced preference relation of* i *also includes* x *preferred over* y*. This may not hold true for more than three alternatives. See Remark 1.*

Majority rule: Given a society $G(N, g)$, and R', we obtain a peer-influenced profile of individual preference relations $(R_1, R_2, ..., R_n) = R$ following the function f_i. Let $N(xP_iy)$ be the number of individuals in the profile R, who prefer x to y.

Given a society $G(N, g)$, with C being the set of all reflexive and connected preference relations over X, we formally define a majority rule under peer influence, $f^g : T^n \to C$ as follows:

$$(\forall (R'_1, R'_2, ..., R'_n) \in T^n)(\forall x, y \in X)[x\mathcal{R}'y \leftrightarrow N(xP_iy) \geq N(yP_ix)],$$

where $\mathcal{R}' = f^g(R_1', R_2', ..., R_n')$ is the social preference relation corresponding to the profile $(R_1', R_2', ..., R_n')$. For any two alternatives x and y the majority rule generates an outcome $xP(\mathcal{R}')y$, $xI(\mathcal{R}')y$, $yP(\mathcal{R}')x$ according as $N(xP_iy) \gtreqless N(yP_ix)$.

Notice, when $g = \theta$, in view of Remark 1, f^θ becomes the standard majority rule without peer influence.

3. Benchmark Results

McGarvey (1953) has shown that, given an arbitrary reflexive and complete preference relation, over a set of n elements, a group of individuals exists with strict individual preference orderings such that the group preference relation as determined by the majority rule is the given preference relation. He has also given a conjecture that the actual minimum number of individuals necessary to express all possible social preference over n elements is approximately n.

We first prove a simple extension of the result by showing that the same result holds ture if individuals have non-empty, incomplete but transitive and asymmetric preference relations instead of strict ordering. Additionally, we show that under this case all social preferences can be expressed as an outcome of majority rule for a society with same number of individuals as there are alternatives.

Theorem 1. *Given an arbitrary reflexive and complete preference relation, over a set X of any n elements, a group of n individuals exists with strict quasi-ordering preferences such that the group preference as determined by the majority rule is the given preference relation.*

Proof. Let $\mathcal{R}'$ be any arbitrary reflexive and complete social preference relation with n alternatives. We proceed by showing that there exists a profile of preferences of n individuals that generates the desired social preference.

We construct a profile of individual preferences according as whether $P(\mathcal{R}') \neq \emptyset$ or not.

Suppose, $P(\mathcal{R}') \neq \emptyset$.

We introduce a binary relation Q_i on X for individual i, $i \in N$, as follows:

$$Q_i = \{(x_i, x_j) \mid x_j \in X, j \neq i\}.$$

Given $\mathcal{R}'$, construct another binary relation of an individual i, V_i in the following way:

$$V_i = Q_i \cap \mathcal{R}'.$$

Notice, some V_i's are non-empty since $\mathcal{R}' \neq \emptyset$. Furthermore, V_i's are transitive.
Let $V_j \neq \emptyset$. Now construct a profile of individual preferences $(R_1, R_2, ...R_n)$ as follows:

$$\begin{aligned} R_i &= V_i \text{ if } V_i \neq \emptyset \\ R_i &= V_j \text{ if } V_i = \emptyset \end{aligned}$$

It may be verified that the profile of preference $(R_1, R_2, ...R_n)$ thus constructed generates the social preference $\mathcal{R}'$ under majority rule.

If $P(\mathcal{R}') = \emptyset$ then we construct a profile of preferences of n individuals based on whether n is even or odd.

Case 1: Let n be even and $x, y \in X$. Construct a profile of preference by assigning the preference xPy to the half of the individuals in the society, and assign yPx to the rest. This preference profile generates the desired social outcome under the majority rule.

Case 2: Let n be odd and $x, y, z \in X$, $|X| \geq 3$. Construct a profile of preference by assigning xPy, xPz to individual 1, yPx to individual 2, zPx to individual 3, and to the half of the remaining individuals xPy, and yPx to the rest. This preference profile generates the desired social outcome under the majority rule.

□

Corollary 1. *Given an arbitrary reflexive and complete preference relation, over a set X of any n elements, a group of $m(\geq n)$ individuals exists with strict quasi preference orderings such that the group preference as determined by the majority rule is the given preference relation.*

Proof. Proof is similar to Theorem 1. □

4. Majority Rule Under Peer-influence: Results

Here we explore some implications of peer-influence for the majority rule. While peer-influence does affect the outcomes of the majority rule as demonstrated in the previous examples, the degree of this influence varies based on societal structure. The applicability of Theorem 1 and Corollary 1, thus, depends on the structure of the society.

We first show that there exists a society for which Theorem 1 and Corollary holds true in the presence of UPI. We then show that the same results do not hold if we consider a different societal structure.

Theorem 2. *There exist a society $G(N, g)$ such that for any reflexive and complete social preference relation, over a set X of any n elements, there is a group of n individuals with strict quasi-ordering preferences such that the group preference as determined by the majority rule in the presence of UPI is the given preference relation.*

Proof. We consider $G(N, g)$ to be a completely connected graph and show that the statement holds true. Let $\mathcal{R}'$ be a reflexive and complete social preference relation with n alternatives. We proceed according to whether $P(\mathcal{R}') \neq \emptyset$ or not.

Suppose $P(\mathcal{R}') \neq \emptyset$. We consider three mutually exhaustive cases: (i)$|P(\mathcal{R}')| = 1$, (ii) $|P(\mathcal{R}')| = 2$, and (iii) $|P(\mathcal{R}')| \geq 3$.

Case: (i)$|P(\mathcal{R}')| = 1$.

Construct a profile of individual preference by assigning each of n individuals exactly one pair of strict preference from $P(\mathcal{R}')$ such that the set $P(\mathcal{R}')$ gets exhausted. Since $|P(\mathcal{R}')| = 1$ it is immediate that everyone is assigned the same preference. Thus there is no Unanimity based peer-influence (UPI). This yields $R_i = R_i'$ for all $i \in N$. With this preference profile, majority rule generates the desired social relation.

Case: (i)$|P(\mathcal{R}')| = 2$.

We proceed under this case by considering whether $P(\mathcal{R}')$ transitive or not. For transitive $P(\mathcal{R}')$, construct a profile of individual preference by assigning $P(\mathcal{R}')$ to each of n individuals. Since everyone is assigned the same preference. Thus there is no UPI. This yields $R_i = R_i'$ for all $i \in N$. Majority rule generates the desired social relation for this profile of peer-influenced relation.

For the case when $P(\mathcal{R}') = 2$ and it is intransitive we consider two mutually exhaustive cases: (a) $n = 3$; (b) $n > 3$.

Let $P(\mathcal{R}') = \{(x, y), (y, z)\}$.

Case (a): $n = 3$.

Assign the preferences in the following way: $xP_1'y$, $xP_2'y$, and $yP_3'x$, $yP_3'z$. Since the society $G(N, g)$ is completely connected, every individual is a neighbour of everyone else. This implies that for the above configuration neighbours of any individual do not agree on a pair of alternatives over which the individual is indecisive. Thus there is no UPI for any individual and the desired social preference is obtained under majority rule.

Case (b): $n > 3$.

For $n = 2k + 1$, k is a positive integer, construct a profile of individual preference by assigning each of k individuals exactly one same strict preference from $P(\mathcal{R}')$ and assign the other strict preference from $P(\mathcal{R}')$ to $k + 1$ individuals. Since every individual is a neighbour of everyone else, it follows that for the above configuration all the neighbours of any individual do not have the same preference over any pair of alternatives. Thus there is no UBI for any individual and the desired social preference is obtained under majority rule.

For $n = 2k$, construct a profile of individual preference by assigning each of k individuals exactly one same strict preference from $P(\mathcal{R}')$ and assign the other strict preference from $P(\mathcal{R}')$ to k individuals. Notice, for the above configuration of the profile of preferences of the individuals, neighbours of any individuals do not have same preference over any pair of alternatives. It follows that there is no UBI for any individual and the desired social preference is obtained under majority rule for the above profile.

Case (ii): $|P(\mathcal{R}')| \geq 3$.

Recall Theorem 1. Construct a profile of individual preference following the same procedure shown in Theorem 1. It follows that from this configuration that all the neighbours of any individual do not have the same preference over any pair of alternatives. Since the society $G(N, g)$ is completely connected, every individual is a neighbour of everyone else. Thus there is no UPI for any individual and the desired social preference is obtained under majority rule.

If $P(\mathcal{R}') = \emptyset$ then we construct a profile of preferences of n individuals based on whether n is even or odd.

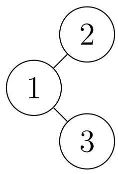
Case 1: Let n be even and $x, y \in X$. Construct a profile of preference by assigning the preference xPy to the half of the individuals in the society, and assign yPx to the rest. This preference profile generates the desired social outcome under the majority rule.

Case 2: Let n be odd and $x, y, z \in X$, as $|X| \geq 3$. Construct a profile of preference by assigning xPy, xPz to individual 1, yPx to individual 2, zPx to individual 3, and to the half of the remaining individuals xPy, and yPx to the rest. This preference profile generates the desired social outcome under the majority rule. □

In Theorem 2, we have demonstrated that in a closely connected society, represented by a fully connected network, Theorem 1 holds true. Notive, in a cohesive (dense) society, an individual's ability to influence others is diminished, as they are likely to encounter a diverse range of opinions. However, this is not the case in a minimally connected or non-cohesive society with a smaller number of individuals.[8] Our next theorem addresses this distinction. We show there exists a society with n individual and a social preference relation such that it is not possible to represent the social preference as an outcome of majority rule. This draws a contrast to Theorem 1 and thereby McGarvey (1953) conjecture.

Theorem 3. *There exist a society $G(N, g)$ and a social preference relation $\mathcal{R}$ of m elements such that it is not possible to obtain $\mathcal{R}$ as an outcome of majority rule in presence of peer-influence for any profile of preferences of m individuals, where the individual preferences are strict quasi-orderings.*

Proof. We shall establish our claim by considering $G(N, g)$ to be a star network with $X = \{x, y, z\}$ and $N = \{1, 2, 3\}$.



First, we prove that if $R'_2 = R_2$, then R'_1 and R'_2 have at least one common indeterminate pair (the same applies to individual 3). Proof is by contradiction. Suppose that individuals 1 and 2 have a common indeterminate pair. Since R'_2 is incomplete, let $(x, y) \notin R'_2$. If $(x, y) \notin R'_1$, then the proof is complete trivially. So, let $(x, y) \in R'_1$, then by remark, 2 $(x, y) \in R_2$ and $R'_2 \neq R_2$ contradiction.

[8] A minimally connected society (network) is a graph where removing any single edge will disconnect the graph.

The further proof proceeds in three cases:

Case 1: $R_2' = R_2$, $R_3' = R_3$ and $R_1' = R_1$

By the above proof, there is at least one indeterminate pair between 1 and 2 and between 1 and 3. If this pair is the same, then that would lead to social indifference, a contradiction; thus, the pairs must be different. That means 1 has only one determinate preference in R_1'. Let it be a preference between x and $y(xP_1'y$ or $yP_1'x)$. Since $R_2' = R_2$, $R_3' = R_3$, the pair must be determinate for 2 and 3 as well (otherwise 1 will have peer influence on 2 and 3, and their preferences would change). Social preference $\mathcal{R} : x\mathcal{P}y$, means that at least one between 2 and 3 have x preferred over y, say $xP_2'y$. Now, if 2 is indeterminate over the other two pairs, then social preference will have indifferences because 3 can have determinate preference over at most two pairs, one of which is x and y, and 1 has preference over x and y only. Thus, R_2' admits only one of (x, z) or (z, y). In that case, social preference cannot be $z\mathcal{P}x$ or $y\mathcal{P}z$ respectively.

Case 2: $R_2' \neq R_2$ or $R_3' \neq R_3$

Since 2 and 3 are in a symmetric position, say $R_2' \neq R_2$, and a similar argument applies otherwise. For 2, the peer influence must have been from 1. Thus, they will have the same preference over at least one pair post influence. Social preference dictates it must be one of $(x, y), (y, z), (z, x)$. Let it be (x, y), without loss of generality. $xP_1'y \Rightarrow \sim (yP_1'z) \wedge \sim (zP_1'x)$. Now consider the following possibilities for pre-peer influence preferences of 2 (there must be one of the following as pre-peer influence cannot be empty), all of them lead to contradiction-

- If $yP_2'z$ then with peer influence (xP_2z). This with $\sim (zP_1'x)$ implies $\sim (z\mathcal{P}x)$.
- If $(xP_2'z)$ then, $\sim (zP_1'x)$ implies $\sim (z\mathcal{P}x)$.
- If $zP_2'x$ then with peer influence (zP_2y). This with $\sim (yP_1'z)$ implies $\sim (y\mathcal{P}z)$.
- If $(zP_2'y)$ then, $\sim (yP_1'z)$ implies $\sim (y\mathcal{P}z)$.

Case 3: $R_2' = R_2$, $R_3' = R_3$ and $R_1' \neq R_1$

Since 1 will have peer influence only if both 1 and 2 have the same preference over at least one pair, it must be one of $(x, y), (y, z), (z, x)$ following social preference. Without loss of generality, let it be (x, y). Rest of the proof proceeds as case 2.

$xP_j'y \Rightarrow \sim (yP_j'z) \wedge \sim (zP_j'x), j \in \{2, 3\}$,

Now consider the following possibilities for pre-peer influence preferences of 1 (there must be one of the following as pre-peer influence cannot be empty), all of them lead to contradiction-

- If $yP_1'z$ then with peer influence (xP_1z). This with $\sim (zP_j'x)$ implies $\sim (z\mathcal{P}x)$.
- If $(xP_1'z)$ then, $\sim (zP_j'x)$ implies $\sim (z\mathcal{P}x)$.

- If zP'_1x then with peer influence (zP_1y). This with $\sim(yP'_jz)$ implies $\sim(y\mathcal{P}z)$.
- If (zP'_1y) then, $\sim(yP'_jz)$ implies $\sim(y\mathcal{P}z)$.

□

The result stated in Theorem 2 also can be obtained with four individuals and three alternative (proof is left to the reader).

5. Incomplete preference, population size and social outcome: A discussion

Social outcomes under the majority rule are often sensitive to the size of the population, the degree of incompleteness of individual preferences and societal structure. The impact of societal structure on shaping social outcomes has already been demonstrated through the contrasting results presented in Theorem 2 and Theorem 3. To further illustrate the importance of population size and the degree of incompleteness of individual preferences, we will examine the findings outlined in Theorem 3 and show how these results are sensitive to both population size and the degree of incompleteness in individual preferences.

Theorem 3 establishes that a cyclical social preference cannot be obtained with three alternatives and three individuals if the society has a star architecture. We show that in the same society a cyclical social preference can emerge if we vary the population size.

To this end, consider a star society with five individuals, where individual 1 is at the center. Let the initial preferences of the individuals be as follows:

$R'_1=\{(x,y)\}, R'_2=\{(z,x)\},\ R'_3=\{(y,z)\}, R'_4=\{(z,x),(y,x)\}, R'_5=\{(y,z),(y,x)\}$. The peer-influenced relations obtained from $(R'_1,R'_2,R'_3,R'_4,R'_5)$ are as follows: $R_1=\{(x,y)\},\ R_2=zP_2xP_2y,\ R_3=xP_3yP_3z,\ R'_4=\{(z,x),(y,x)\},\ R'_5=\{(y,z),(y,x)\}$. Notice that this peer-influenced preference relation generates a cyclical social preference $xP(\mathcal{R})y,\ yP(\mathcal{R})z,\ zP(\mathcal{R})x$.

To demonstrate that the social outcome may be sensitive to the degree of incompleteness, we consider the initial preferences of the individuals to be *minimally complete.* We define a *minimally complete* preference relation as follows:

Definition 1. *Preference relation of individual i, R'_i is said to be minimally complete if there exists only one pair of distinct alternatives $x,y\in X$ such that $R'_i=\{(x,y)\}$.*

Notice that a minimally complete preference relation is a strict quasi-ordering but the converse is not true. Individuals may have minimally complete preference for a number of reasons. It may arise when individuals have very little information about the alternatives or the alternatives have countervailing properties leading to incomparability, and so on.

We show, if the individuals have minimally complete initial preferences then some social preferences cannot be realized in certain situation under the majority rule with varying population size.

It is important to mention here that under normal circumstances when there is no peer-influence this assumption of minimally complete preference of the individuals does not prohibit the majority rule from generating any social outcome (See Theorem 1). In other words, with this assumption it is possible to obtain any social preference under majority rule when peer-influence is absent.. However the result may alter when we introduce minimally complete preference in the presences of peer-influence.

The following theorem holds true.

Theorem 4. *Let the society $G(N, g)$ be a star network and the preferences of the individuals be minimally complete over $X = \{x, y, z\}$. If the person who is at the centre of the star network has the preference $R'_i \in \{xP'_iy, yP'_iz, zP'_ix\}$ then it is not possible to obtain $xP(\mathcal{R})y$, $yP(\mathcal{R})z$, $zP(\mathcal{R})x$ as an outcome of majority rule in presence of peer-influence for any profile of preferences of $n(\geq 3)$ individuals.*

Proof. Let the society $G(N, g)$ be a star network and the preferences of the individuals be minimally connected. Let the individual 1 be at the centre of $G(N, g)$ and have the preference $R'_1 = \{xP'_1y\}$. Since there are three alternatives and preferences of the individuals are minimally complete, any individual i has exactly one of these six preferences as their initial preference: $\{xP'_iy, yP'_ix, yP'_iz, zP'_iy, zP'_ix, xP'_iz\}$. Given xP'_1y, peer-influenced preference of an individual, therefore, belongs to
$\{xP_iy, yP_ix, xP_iyP_iz, \{zP_iy, xP_iy\}, zP_ixP_iy, \{xP_iz, xP_iy\}\}$.

Let R be a profile of peer-influenced relations derived from the initial profile R'. Let n_1, n_2, n_3, n_4, n_5, and n_6 be the number of individuals who have preferences xP_iy, yP_ix, $xP_iyP_iz, \{zP_iy, xP_iy\}, zP_ixP_iy, \{xP_iz, xP_iy\}$ respectively in the profile R. Notice $yP(\mathcal{R})z$ entails $N(yP_iz) > N(zP_iy)$.
$N(yP_iz) > N(zP_iy) \rightarrow n_3 > n_4 + n_5$.
Similarly, $zP(\mathcal{R})x$ entails $N(zP_ix) > N(xP_iz)$.
$N(zP_ix) > N(xP_iz) \rightarrow n_5 > n_3 + n_6$
Combining these two inequalities we get,
$n_3 + n_5 > n_4 + n_5 + n_3 + n_6$
$\rightarrow 0 > n_4 + n_6$. It is a contradiction.

Let $R'_1 = \{yP'_1z\}$. Given yP'_1z, peer-influenced preference of an individual, therefore, belongs to
$\{xP_iyP_iz, \{yP_ix, yP_iz\}, \{yP_iz\}, \{zP_iy\}, \{xP_iz, yP_iz\}, yP_izP_ix\}$.

Let R be a profile of peer-influenced relations derived from the initial profile R'. Let m_1, m_2, m_3, m_4, m_5, and m_6 be the number of individuals who have preferences xP'_iyP_iz, $\{yP_ix, yP_iz\}, yP_iz, zP_iy, \{xP_iz, yP_iz\}, yP_izP_ix$. respectively in the profile R.

Notice $xP(\mathcal{R})y$ entails $N(xP_iy) > N(yP_ix)$.
$N(xP_iy) > N(yP_ix) \to m_1 > n_2 + m_6$.
Similarly, $zP(\mathcal{R})x$ entails $N(zP_ix) > N(xP_iz)$.
$N(zP_ix) > N(xP_iz) \to m_6 > m_1 + m_5$
Combining these two inequalities we get,
$m_1 + m_6 > n_2 + m_6 + m_1 + m_5$
$\to 0 > m_2 + m_5$. It is a contradiction.

Let $R'_1 = \{zP'_1x\}$. Given zP'_1x, peer-influenced preference of an individual, therefore, belongs to
$\{zP_ixP_iy, \{yP_ix, zP_ix\}, yP_izP_ix, \{zP_iy, zP_ix\}, xP_iz, zP_ix\}$.

Let R be a profile of peer-influenced relations derived from the initial profile R'. Let k_1, k_2, k_3, k_4, k_5, and k_6 be the number of individuals who have preferences zP_ixP_iy, $\{yP_ix, zP_ix\}$, $yP_izP_ix, \{zP_iy, zP_ix\}, xP_iz, zP_ix$. respectively in the profile R. Notice $xP(\mathcal{R})y$ entails $N(xP_iy) > N(yP_ix)$.
$N(xP_iy) > N(yP_ix) \to k_1 > k_2 + k_3$.
Similarly, $zP(\mathcal{R})x$ entails $N(yP_iz) > N(zP_iy)$.
$N(yP_iz) > N(zP_iy) \to k_3 > k_1 + k_4$
Combining these two inequalities we get,
$k_1 + k_3 > k_2 + k_3 + k_1 + k_4$
$\to 0 > k_2 + k_4$. It is a contradiction.

□

6. Conclusion

Preferences of the individuals are often swayed by the interactions with the peers in a social network. In this paper we have explored a particular channel of peer-influence in a social network, which we call unanimity based peer influence (UPI) and study its impact on the social outcome under majority rule. We show that social outcome under majority rule in presence of UPI among many things depends on structure of social network, population size and the level of incompleteness of the preference relations. For some society, it is possible that some preferences are never realized under majority rule. Additionally, whether social outcome under majority rule satisfies any rationality condition such as transitivity is also governed by the aforementioned factors. A profile of individual preference that generates a transitive social preference in absence of peer-influence may not do so under UPI. Recall Example 2.1. The initial preferences of the individuals are as follows:
Individuals have the following initial preference.

$R'_1 = \{(z, x)\}$, $R'_2 = \{(x, y), (x, z)\}$, $R'_3 = \{(x, y)\}$, $R'_4 = \{(y, x), (y, z)\}$, $R'_5 = \{(z, x)\}$. In absence of any peer-influence, this profile of preferences generates a transitive social

preference: $zPxPy$. Now introduce UPI. The peer-influenced relations are as follows: $R_1 = zPxPy$, $R_2 = \{(x, y), (x, z)\}$, $R_3 = zPxPy$, $R_4 = R_5 = yPzPx$.
The social preference is a cycle - xPy, yPz, and zPx.

An interesting exercise which can be taken up as an extension of this paper is to investigate the necessary and sufficient conditions for transitive, quasi-transitive or acyclic social preference under majority rule in the presence of UPI.

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