# How much is there to distribute? *

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**Abstract**

This paper provides an axiomatic justification to the mean and the median as criteria for assessing "how much there is to distribute" of a cardinally and ordinally measurable variable respectively. The axioms satisfied by the criteria are the anonymity and Pareto principles as well as the property that an inversion in the scale of measurement of the variable leads to an inversion of the ranking of distributions. Mean and median rankings appear essentially to be the only orderings that satisfy the three properties when the latter is applied to a cardinal and an ordinal scale respectively.

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# 1 Introduction

When can we say that there is "more to distribute" in one distribution of an attribute than in another? If the attribute is cardinally measurable, like income, consumption, or wealth, a common answer is that there is more to distribute when the *mean* is larger. The mean is - indeed - the most widely used indicator of a distribution's "central tendency". It is so important that most concerns about reducing inequalities in the distribution of a cardinally measurable attribute are expressed in a way that are mean-preserving. One of the most important transfer principle satisfied by most inequality indices states that a transfer of a given quantity of the attribute from a relatively well-endowed agent to a relatively less well-endowed agent should always be seen as inequality reducing. Transfers of this kind are called *Pigou-Dalton transfers*. They incorporate, in their very definition, the requirement that they preserve the mean of the distribution[1].

The definition of "how much there is to distribute" is less clear when the distributions involve an attribute, like health status or declared happiness, whose meaningful measurement is only believed to be *ordinal*. As very well-argued by Allison and Foster (2004) for health status and by Bond and Lang (2019) for declared happiness, the mean of the distribution of an ordinally measurable attribute can *not* meaningfully serve as an indicator of the distribution's "central tendency". The reason for this is clear. The ranking of two distributions based on their means may not be invariant with respect to changes in the units of measurement of the attribute that are consistent with its ordinal character. If there are three increasingly ordered levels of declared happiness, "low", "normal" and "high" say, then any numbering of those levels consistent with this ordering can serve as an ordinal scale. Consider two distributions of five agents into these three levels of happiness. In the first distribution, three agents are in the low category and the two others are in the high category. In the second distribution all five agents are in the middle category. The ranking of these two distributions by their mean

[1] A Pigou-Dalton transfer produces indeed the converse of a "mean-preserving-spread" (see e.g. Rotshchild and Stiglitz (1970)) on the distribution of the attribute.

crucially depends upon the particular scale used to assign happiness levels to the categories. If one uses the scale which assigns number 1, 2 and 3 to the respective three categories, the mean will be larger in the second distribution (2) than in the first (9/5). However if one uses the square of those three numbers as indicators of happiness levels, one would obtain the reverse conclusion that the mean is larger in the first distribution (21/5) than in the second (4).

A notion of central tendency that may seem more appropriate to an ordinally measurable attribute is the *median*, defended by Allison and Foster (2004), Apouey (2007) and Chen, Oparina, Powdthavee, and Srisuma (2022) among others. It is indeed clear that the ranking of distributions based on their median does not depend upon the particular numerical scale used to measure the attribute's quantities. In the above example, the second distribution, where the "normal" category is the median, would be ranked above the first, where the "low" category is the median, irrespective of the numerical scale used to measure happiness. But the ranking of distributions based on their median is not the only one that is independent from the numerical scale. Other rankings with this property are those based on the mode, the positional dictatorship rules (see e.g. d'Aspremont and Gevers (1977), Gevers (1979) or Blackorby, Donaldson, and Weymark (1984)) or criteria based on some pre-specified quantile of the distribution (the median being only one of them). Identifying the proper criterion for appraising the central tendency of a distribution of an ordinal variable is all the more important as this criterion - if identified - would be the natural reference point with respect to which *inequalities* in the distribution of an ordinal attribute could be appraised. Allison and Foster (2004) have proposed for example a notion of "spread away from the median" as a natural definition of increase in inequality in the distribution of an ordinal variable. A "spread away from the median" is any change in the distribution that transfers probability mass away from the median either at the right, at the left or on both side of the median while preserving it. They have therefore strongly endorsed the median as the natural reference. Cowell and Flachaire (2017), who use a different approach consisting in measuring the difference between a given distribution and a specific equal distribution in which the probability mass is concentrated in one reference point, have considered in turns different such reference points, including the median and the maximal value of the variable. Mendelson (1987) adopted a more general attitude of defining dispersion as any spread away from a given quantile, without requiring the quantile to be the median. Yet he leaves unanswered the question of which quantile should chosen.

Along the lines of Hammond (1976), Gravel, Magdalou, and Moyes (2021) (see also Bennia, Gravel, Magdalou, and Moyes (2022)) have proposed *Hammond transfers* as the analogues, for distributions of an ordinal attribute, to Pigou-Dalton transfers as undisputable instances of inequality reduction. A Hammond transfer is like a Pigou-Dalton transfer, but without the requirement - meaningless for an ordinal variable - that what is taken away from the donor be equal to what is given to the receiver. While a Hammond transfer provides a plausible definition of "elementary inequality reduction" applicable to distributions of an ordinal attribute, it does not preserve any known notion of central tendency. Gravel, Magdalou, and Moyes (2019) have actually shown that the fact of going from a distribution of a continuous ordinal attribute to another by a finite sequence of Hammond transfers is equivalent to the dominance of the first distribution by the second by *both* the lexicographic extension of the maximin criterion *and* the lexicographic extension of the minimax criterion. As a consequence of this result, for any position of an agent in a distribution of the ordinal variable, one can construct a Hammond transfer that will change the quantity of the ordinal variable received by the agent in that position. There is thus no position in the distribution that is systematically preserved by a Hammond transfer. Hammond transfers can also be applied to cardinally measurable attributes. Pigou-Dalton transfers are, after all, nothing else than mean-preserving Hammond transfers. The question thus arises: what is the notion of central tendency that should be preserved by Hammond transfers when applied to distributions of an ordinal attribute? And, after all, why taking the mean as the natural notion of central tendency for distributions of a cardinally meaningful attribute ?

This paper provides an axiomatic answer to these questions. The answer assumes that any definition of "how much there is to distribute" takes the form of an *anonymous* and *Paretian ordering* of distributions of the attribute's quantities. The approach also sometimes - but not always - considers the *separability* requirement, widely discussed in the classical social choice literature (e.g. by Blackorby, Donaldson, and Weymark (1984), d'Aspremont and Gevers (1977) and Maskin (1978)), that agents who receive the same quantity of the attribute in two distributions should have no say in their ranking. The key unifying principle considered by the approach is the (much) less noticed *reversal consistency* requirement that the ranking of two distributions be reversed when the numerical scale used to measure the attribute quantities is reversed. Imagine indeed that one is interested in comparing distributions of peoples' exposure to pollution in Delhi and Kolkata and that the comparison concludes that overall pollution is *higher* in Delhi than in Kolkata. It would seem quite natural then to require that the conclusion should also be that the air quality is *lower* in Delhi than in Kolkata. While we believe such a reversal consistency to be a natural requirement for comparisons of distributions based on "how much there is to distribute", we do not believe it to be appropriate for comparisons based "how equally it is distributed". If inequalities in people's exposures to air pollution are, say, higher in Delhi than in Kolkata, this should not entail that inequalities in access to clean air be lower in Delhi than in Kolkata. Quite to the contrary, it could be held that inequality comparisons should *not* be affected by an inversion of the scale of measurement of the attribute variable, and such an invariance requirement has actually been proposed (see e.g. Bosmans (2016), Chakravarty, Chattopadhyay, and d'Ambrosio (2015), Clarke, Gerdtham, Johannesson, Bingefors, Smith, and Oswald (2002), Erreygers (1990), Lambert and Zheng (2011), de la Vega and Aristondo (2012), Permanyer (2016), Yalonetsky (2022) and Abul-Naga and Yalonetsky (2024)) as a property that inequality rankings of distributions should satisfy.

The precise formulation of the reversal consistency requirement depends upon the assumed measurability of the attribute. Following the classical "informational basis" approach of social choice theory (see e.g. Blackorby, Donaldson, and Weymark (1984), d'Aspremont and Gevers (1977), Gevers (1979), Sen (1977b), Sen (1977a) and Maskin (1978)), we define cardinal and ordinal measurability as suitable invariance requirements on the ranking of distributions. Cardinal measurability for example requires the ranking to be invariant with respect to any *increasing affine* transformation applied to the attribute's quantities. Accordingly, we define *cardinal reversal consistency* as the requirement that the ranking of two distributions be reversed if a *decreasing* affine transformation is applied to the indicator. We then show that the ranking of distributions based on their mean is the only anonymous and separable ordering that satisfies the Pareto principle and cardinal reversal consistency. This result can be seen as a significant generalization of the classical Maskin (1978) characterization of utilitarianism in a welfarist framework. Indeed, ranking vectors based on their mean is formally equivalent - with a fixed population - to ranking them based on their sum. Maskin (1978) characterization of utilitarianism uses separability, a stronger version of the Pareto axiom, anonymity, cardinal measurability and continuity. Our characterization does not use any continuity, but replaces cardinal measurability by the stronger cardinal reversal consistency.

Ordinal measurability demands that the ranking of two distributions be invariant with respect to any increasing transformation of the attribute's quantities. In accordance with this notion, we define *ordinal reversal consistency* as the requirement that the ranking of two distributions be reversed when any decreasing transformation is applied to the attribute's quantities. Since the ranking of distributions based on the mean, which does not satisfy ordinal reversal consistency, is the only separable anonymous ordering that satisfies the Pareto principle and cardinal reversal consistency - a weaker requirement than ordinal reversal consistency - it follows that there are *no* separable Paretian anonymous orderings that satisfy ordinal reversal consistency. We then drop the separability requirement and restrict our attention to anonymous and Paretian orderings that satisfy ordinal reversal consistency. We show that the only ordering of distributions with a unique median that satisfies those properties is precisely the ordering induced by this median. However, we also show that there are no anonymous orderings of distributions with two medians that satisfy ordinal reversal consistency and Pareto. Hence, when unique, the median happens to be "the" natural criterion for appraising "how much there is to distribute" of an ordinally measurable attribute, as is the mean in the case of a cardinally measurable attribute when an additional separability condition is imposed. To the very best of our knowledge, this characterization of the median is novel, even though rankings of finite sets of objects that are equivalent to their median element have been characterized by Nitzan and Pattanaik (1984) using a duality axiom that is actually somewhat related to ordinal reversal consistency.

This leaves open the question: what are the anonymous orderings of distributions that satisfy Pareto and cardinal reversal consistency but not necessarily separability? Both the mean and the median rankings of distributions satisfy those properties. So do, in fact, all rankings of distributions that are strictly consistent with some *symmetric rank-dependent weighted averages* of the attributes quantities received by agents. A symmetric rank-dependent weighted average of a list of numbers is a weighted average of those numbers in which the weights depend upon the ranks of those numbers and are identical for any two numbers that occupy the same rank when initiated from either the bottom or the top. For example, both the lowest and the highest numbers in a list have the same rank - namely one - when calculated from the bottom or from the top. The mean is obviously a member of this class in which all weights are equal. The ranking of distributions based on their median is also a member of this class if the number of individuals is odd. Indeed a median-based ranking of lists of numbers is nothing but a rank-dependent weighted average of those numbers that puts a weight of 1 on the median element of the list and a weight of 0 on all other ranks. This weighting scheme is symmetric if the number of individuals is odd, because every rank calculated from the top or from the bottom would receive an equal weight of 0 and the unique middle element would receive a weight of 1. If there are two medians, then the symmetric average of the two medians would also be a symmetric rank-dependent weighted average. Other symmetric rank-dependent weighted average rankings are the symmetric average of their best and their worst components. This paper also provides a characterization of all ranking of distributions that are strictly consistent with some symmetric rank-dependent weighted average as the only anonymous and Paretian orderings of distribution that satisfy a strengthening of cardinal reversal consistency that we call, after Gevers (1979), almost fully cardinal reversal consistency. This axiom requires the ranking of two distributions to be reversed when (possibly) different negative affine transformations assigning the same slope to everyone are applied to individuals' well beings in the case where those affine transformation reverse the order of the individuals in the distribution. This axiom is stronger than cardinal reversal consistency, but is weaker than ordinal reversal consistency. This result leaves, however, open the question of identifying the class of all anonymous orderings that satisfy Pareto and cardinal reversal consistency. This class certainly contains rankings that are strictly consistent with symmetric rank-dependent weighted average. Does it contain anything else ? The question remains open.

The remaining of the paper is as follows. The next section describes the framework and the properties imposed on the ranking of distributions. Section 3 states, proves and discusses the results and section 4 concludes.

# 2 The framework

We are interested in comparing distributions of an observable attribute such as income, health or education between a given number, $n$ say, of agents[2] on the basis of "how much there is to distribute". Any distribution is depicted as a list $\mathbf{y} = (y_1, ..., y_n)$ of $n$ real numbers - interpreted as a column vector - where, for $i = 1, ..., n$, $y_i$ denoted the quantity of the attribute received by agent $i$ in distribution $\mathbf{y}$. For any distribution $\mathbf{y} \in \mathbb{R}^n$, we denote by $\mathbf{y}_{(.)}$ its increasingly ordered permutation defined by:

$$\mathbf{y}_{(.)} = \boldsymbol{\pi}.\mathbf{y}$$

for some $n \times n$ permutation matrix $\boldsymbol{\pi}$ such that $y_{(i)} \leq y_{(i+1)}$ for all $i = 1, ..., n-1$. For any real number $a$, we denote by $\mathbf{a}^n$ the $n$-tuple whose

[2] All results of this paper can be extended to distributions involving a variable collection of individuals if the Dalton (1920) replication axiom is added to the other fixed population properties discussed herein.

components are all $a$. Distributions are compared by means of an ordering[3] $\succsim$ on $\mathbb{R}^n$ (with asymmetric and symmetric factors $\succ$ and $\sim$ respectively). Requiring the appraisal of "how much there is to distribute" to be made by an ordering is clearly significant. For one thing, it rules out the possibility that it be made by an incomplete binary relation such as the component-wise quasi-ordering of $n$-tuples - sometimes referred to as the Pareto quasi-ordering - or its anonymous Suppes (1966) extension.

The very definition of "how much there is to distribute" depends upon the measurability of the attribute. We consider two types of measurability: ordinal, and cardinal. Borrowing from the classical social choice literature (see e.g. d'Aspremont and Gevers (1977), Gevers (1979), Sen (1977b) and Sen (1977a)), we define those types of measurability as invariance requirements imposed on the ranking of distributions with respect to specific changes in the scale of measurement of the attribute, the stronger the invariance, the less precise the measurement.

The least precise of the measurements considered herein is the ordinal one, defined as follows.

**Definition 1** *The ordering* $\succsim$ *on* $\mathbb{R}^n$ *assumes ordinal measurability of the attribute if for any two distributions* $(y_1, ..., y_n)$ *and* $(z_1, ..., z_n) \in \mathbb{R}^n$ *and any increasing function* $f : A \longrightarrow B$ *for some subsets* $A$ *and* $B$ *of real numbers such that* $y_i \in A$ *and* $z_i \in A$ *for* $i = 1, ..., n,$ *one has* $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (f(y_1), ..., f(y_n)) \succsim (f(z_1), ..., f(z_n))$

A ranking of distributions that assumes ordinal measurability is thus unaffected by any change in the units of measurement of the attribute that preserves the ordering of the attribute's quantities among the different agents. Well-known examples of binary relations on $\mathbb{R}^n$ that assume ordinal measurability of the attribute are the *positional dictatorship* rankings discussed by Gevers (1979), Roberts (1980) and Blackorby, Donaldson, and Weymark (1984), among many others. A ranking $\succsim$ of the elements of $\mathbb{R}^n$ is called a positional dictatorship if there is a position $i \in \{1, ..., n\}$ such that for every $\mathbf{y}$

[3] An ordering is a reflexive, complete and transitive binary relation.

and $\mathbf{z} \in \mathbb{R}^n$, $y_{(i)} > z_{(i)} \Longrightarrow \mathbf{y} \succ \mathbf{z}$. Hence, a positional dictatorship ranking of distributions is based on the quantity of the attribute held by the agent who occupies some specific position in the ordering of the agents' attribute quantities. Examples of positional dictatorship rankings are the Maxi-Min and the Lexi-Min rules, discussed notably by Hammond (1976) and d'Aspremont and Gevers (1977), where the agent with the lowest quantity of attribute "dictates" his/her preference upon the society. Observe that positional dictatorships only restrict the ranking in cases where the attribute quantity of the agent in the relevant position differs in the two considered distributions (the higher the quantity, the better the distribution). Positional dictatorships do not restrict in any way the ranking of distributions that provide the agent in the relevant position with the same attribute quantity. For example both the Maxi-Min and the Lexi-Min rankings are positional dictatorships based on the worst position.

Another example of positional dictatorship - examined more closely herein - is the one based on the median position. For an odd $n$, the median position is $\frac{n+1}{2}$. For an even $n$, we distinguish between the lower median position $\frac{n}{2}$ and the upper median position $\frac{n}{2}+1$. The median may thus not be unique if $n$ is even. However, for $n$ even, we say that the median is unique if $y_{(\frac{n}{2})} = y_{(\frac{n}{2}+1)}$. Let $\mathbb{M}^n \subseteq \mathbb{R}^n$ denote the set of distributions with a unique median. Hence $\mathbb{M}^n = \mathbb{R}^n$ if $n$ is odd but $\mathbb{M}^n$ is a proper subset of $\mathbb{R}^n$ if $n$ is even. For any distribution $\mathbf{y} \in \mathbb{M}^n$, we denote by $m(\mathbf{y})$ its unique median. Hence $m(\mathbf{y}) = y_{(\frac{n+1}{2})}$ if $n$ is odd and $m(\mathbf{y}) = y_{(\frac{n}{2})} = y_{(\frac{n}{2}+1)}$ if $n$ is even. We denote by $\succsim^{med}$ the median ordering of $\mathbb{M}^n$ defined, for every two distributions $\mathbf{y}$ and $\mathbf{z} \in \mathbb{M}^n$, by:

$$\mathbf{y} \succsim^{med} \mathbf{z} \Longleftrightarrow m(\mathbf{y}) \geq m(\mathbf{z})$$

An example of an ordering of distributions in $\mathbb{R}^n$ that does *not* assume ordinal measurability of the attribute is the ordering $\succsim^{sum}$ defined by:[4]

$$\mathbf{y} \ \succsim^{sum} \ \mathbf{z} \Longleftrightarrow \sum_{i=1}^{n} y_i \geq \sum_{i=1}^{n} z_i$$

As noticed by Allison and Foster (2004) and Bond and Lang (2019), this ranking does not assume ordinal measurability of the attribute because if $n = 2$ for example, it would consider that $(3,3) = (9^{1/2}, 9^{1/2})$ is equivalent to $(2,4) = (4^{1/2}, 16^{1/2})$ but would not consider that $(9,9)$ is equivalent to $(4,16)$ even thought the function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by $f(x) = x^{1/2}$ is increasing.

While the ordering $\succsim^{sum}$ does not assume ordinal measurability of the attribute quantity, it does assume cardinal measurability. There are at least three notions of cardinal measurability of differing strength, all satisfied by $\succsim^{sum}$, that can be formulated.

The least precise of the three is the following, which corresponds to what classical social choice theory (see e.g. Blackorby, Donaldson, and Weymark (1984)) calls "Cardinal-Unit comparability".

**Definition 2** *The ordering $\succsim$ on $\mathbb{R}^n$ assumes Cardinal-Unit (CU) measurability of the attribute if for any $(y_1, ..., y_n)$ and $(z_1, ..., z_n) \in \mathbb{R}^n$ and any list of $n + 1$ real numbers $a_1, ..., a_n$ and $b$ with $b > 0$ one has $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (a_1 + by_1, ..., a_n + by_n) \succsim (a_1 + bz_1, ..., a_n + bz_n)$.*

A ranking of distributions thus assumes CU measurability of the attribute if it is invariant with respect to any change in the attribute's quantities obtained by means of increasing affine functions. Observe however that, contrary to ordinal measurability, the invariance requirement associated to CU measurability does *not* impose that the same transformation be applied to all agents. The only restriction imposed on the increasing affine functions is that all agents' quantities be multiplied by the same positive number $b$. However, agents may be assigned different values of the intercepts $a_i$. The

[4] In the fixed population setting of this paper, comparing vectors of quantities based on the sum of their components is clearly equivalent to comparing them based on their average.

invariance of the ranking with respect to the multiplication of all agents attribute quantities by the same positive $b$ implies that the gains and losses in those quantities between two distributions are comparable across agents. However, the requirement for the ranking to be invariant with respect to the addition of different (possibly negative) numbers to the attribute quantities of the different agents means that those quantities are not comparable across agents within a distribution. Because of this, it cannot be said that the invariance requirement associated to CU measurability is more or less demanding than that associated to ordinal measurability or, equivalently, that CU measurement is more or less precise than ordinal measurement. The two kinds of measurement are simply not comparable in terms of their precision.

The second considered notion of cardinal measurability, called *Almost Fully Cardinal* measurability by Gevers (1979), is more precise than both ordinal and CU measurability. It is defined as follows.

**Definition 3** *The ordering* $\succsim$ *on* $\mathbb{R}^n$ *assumes Almost Fully Cardinal (AFC) measurability of the attribute if for any* $(y_1, ..., y_n)$ *and* $(z_1, ..., z_n) \in \mathbb{R}^n$ *and any list of* $n+1$ *real numbers* $a_1, ..., a_n$ *and* $b$ *with* $b > 0$ *for which there exists an increasing function* $f : A \longrightarrow B$ *for some subsets* $A$ *and* $B$ *of real numbers satisfying, for* $i = 1, ..., n$, $y_i \in A$, $z_i \in A$, $f(y_i) = a_i + by_i \in B$ *and* $f(z_i) = a_i + bz_i \in B$, *one has* $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (a_1 + by_1, ..., a_n + by_n) \succsim (a_1 + bz_1, ..., a_n + bz_n)$.

The invariance requirement underlying AFC measurability is weaker than that associated to CU measurability because it is not assumed to hold for all real numbers $a_1, ..., a_n$ and $b$ with $b > 0$ applied to the two distributions. It is only required to hold for those real numbers that preserve the orderings of the attribute quantities of the individuals within each distribution. The invariance requirement associated to AFC measurability can be seen as the intersection of those associated to Ordinal and CU measurability, and is therefore weaker, as an invariance requirement, than either of them.

An even more precise cardinal notion is the following *Fully Cardinal* measurability.

**Definition 4** *The ordering* $\succsim$ *on* $\mathbb{R}^n$ *assumes Fully Cardinal (FC) measurability of the attribute if for any* $(y_1, ..., y_n)$ *and* $(z_1, ..., z_n) \in \mathbb{R}^n$ *and any list pair of real numbers* $a$ *and* $b$ *with* $b > 0$ *one has* $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (a + by_1, ..., a + by_n) \succsim (a + bz_1, ..., a + bz_n)$.

Hence a ranking of distributions that assumes FC measurability of the attribute is invariant only when the same increasing affine transformation is applied to the attribute quantities of all agents. As indicated above, it is easy to check that the ordering $\succsim^{sum}$ considers the attribute to be cardinally measurable for any of the three notions just defined (it suffices to verify that it satisfies the stronger CU measurability). There are however rankings of distributions commonly believed to assume cardinal measurability of the attribute that do not satisfy all three invariance requirements. One of them is the ranking of two distributions induced by the fact of going from one to the other by means of a single Pigou-Dalton transfer. Let us denote by $\succsim^{PD}$this ranking defined on $\mathbb{R}^n$ as follows.

**Definition 5** *(Pigou-Dalton transfer)* $(y_1, ..., y_n) \succsim^{PD} (z_1, ..., z_n)$ *if and only if there are two individuals* $i$ *and* $j$ *and a positive real number* $\Delta$ *such that:*
*(i)* $y_h = z_h$ *for all* $h \notin \{i, j\}$ *and,*
*(ii)* $z_i < z_i + \Delta = y_i \leq y_j = z_j - \Delta < z_j$.

As is well-known, the transitive closure of $\succsim^{PD}$ is nothing else than the Lorenz dominance quasi-ordering applied to distributions with the same mean. We leave to the reader the task of verifying that $\succsim^{PD}$ assumes AFC measurability (and therefore also FC measurability) of the attribute. However, it does not assume CU measurability, as shown in the following example.

**Example 1** *Assume* $n = 2$ *and consider* $(y_1, y_2) = (2, 3)$ *and* $(z_1, z_2) = (1, 4)$. *Hence* $(y_1, y_2) \succsim^{PD} (z_1, z_2)$. *However, contrary to what is required by CU measurability, it is not the case that* $(2y_1, 2y_2 - 6) = (4, 0) \succsim^{PD} (2z_1, 2z_2 - 6) = (2, 2)$.

We now turn to the properties that an ordering $\succsim$ of distributions on the basis of "how much there is to distribute" can plausibly satisfy when accounting for those various possible measurability of the attribute. The following anonymity principle, requiring that the "individuals' names don't matter", seem to be rather natural for this purpose.

**Axiom 1** ***Anonymity.*** *For any* $(y_1, ..., y_n) \in \mathbb{R}^n$ *and any one-to-one function* $i : \{1, ..., n\} \to \{1, ..., n\}$ $(y_1, ..., y_n) \sim (y_{i(1)}, ..., y_{i(n)})$.

The anonymity axiom is not specifically tailored to the issue of appraising "how much there is to distribute". Indeed, rankings of distributions based on inequality such as the just discussed transitive closure of $\succsim^{PD}$ are also typically required to satisfy anonymity.

A requirement that is somewhat more closely related to appraising "how much there is to redistribute" is the Pareto principle, defined as follows.

**Axiom 2** ***Pareto.*** *For any* $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ *such that* $y_i \geq z_i$ *for all* $i \in \{1, ..., n\}$, $(y_1, ..., y_n) \succsim (z_1, ..., z_n)$ *and for any* $(y_1, ..., y_n)$, $(z_1, ..., z_n)$ $\in \mathbb{R}^n$ *such that* $y_i > z_i$ *for all* $i \in \{1, ..., n\}$, $(y_1, ..., y_n) \succ (z_1, ..., z_n)$.

It is indeed hardly disputable that an increase in the attribute's quantity received by everyone is indicative of an increase of "how much there is to distribute". Observe that our formulation of the Pareto principle is the somewhat weak one that leads to a strict ranking of the two distributions only when every agent records a strict increase in his/her endowment of the attribute. In the classical social choice literature, several results have been obtained with the stronger Pareto requirement that no reduction in the well-being of everyone and a strict increase in the well-being of someone is sufficient for obtaining a strict ranking.

Another often invoked property of a ranking of distributions, again not specifically related to the definition of "how much there is to distribute", is the separability requirement that agents who are, in the words of Sen (1970), "unconcerned" by a move from one distribution to another - in the sense of not experiencing themselves any change in their situation - should have no say in the ranking of the two distributions. Such a say should be left to those who do experience changes. We formally stated as follows this separability axiom, discussed in the classical social literature by Sen (1970), d'Aspremont and Gevers (1977), Maskin (1978) and Blackorby and Donaldson (1982) among many others.

**Axiom 3** ***Separability***. *For any group $G \subset \{1, ..., n\}$, and any four distributions $\mathbf{y}$, $\mathbf{z}$, $\mathbf{y\prime}$ and $\mathbf{z\prime} \in \mathbb{R}^n$ such that $y_g = z_g$ and $y'_g = z'_g$ holds for all $g \in G$ and $y_i = y_i^{'}$ and $z_i = z_i^{'}$ for all $i \in \{1, ..., n\} \backslash G$ $\mathbf{y} \succsim \mathbf{z} \Longleftrightarrow \mathbf{y}' \succsim \mathbf{z}'$.*

Separability is undoubtedly a strong requirement that is, for example, violated by the median ordering $\succsim^{med}$. Yet, it is satisfied by the ordering $\succsim^{sum}$ and by many criteria for comparing distributions, be it on the basis of "how much there is to distribute" or of "how unequal" the distribution is.

While the three preceding axioms have been widely discussed in the literature, the following, that require the ordering to satisfy *reversal consistency* with respect to the unit of measurement of the attribute's quantities has not. Yet it seems quite tightly connected to the appraisal of "how much there is to distribute". We may indeed want to measure the attribute quantities positively (e.g. more affluence is better) or negatively (less poverty is better). Similarly, we may want to measure the quality of the environment positively (a cleaner environment is better) or negatively (more pollution is worse). If what we are after is a criterion for appraising "how much there is to distribute", it would seem natural that the ranking of two distributions of a variable measured positively be the opposite of the ranking of two distributions of a variable measured negatively.

It seems therefore natural to impose one of the following properties of *reversal consistency* whose exact formulation depends upon the assumed measurability of the attribute.

**Axiom 4** ***Ordinal Reversal Consistency***. *For any $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ and any strictly decreasing function $f : A \longrightarrow B$ for some subsets $A$ and $B$ of real numbers such that $y_i \in A$ and $z_i \in A$ for $i = 1, ..., n$, one has $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (f(y_1), ..., f(y_n)) \precsim (f(z_1), ..., f(z_n))$.*

**Axiom 5** ***CU Reversal Consistency****. For any* $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ *and any list of* $n+1$ *real numbers* $a_1, ..., a_n$ *and* $b$ *with* $b < 0$ *one has* $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (a_1 + by_1, ..., a_n + by_n) \precsim (a_1 + bz_1, ..., a_n + bz_n)$.

**Axiom 6** ***AFC Reversal Consistency.*** *For any* $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ *and any list of* $n+1$ *real numbers* $a_1, ..., a_n$ *and* $b$ *with* $b < 0$ *for which there exists a decreasing function* $f : A \longrightarrow B$ *for some subsets* $A$ *and* $B$ *of real numbers satisfying, for* $i = 1, ..., n$, $y_i \in A$, $z_i \in A$, $f(y_i) = a_i + by_i \in B$ *and* $f(z_i) = a_i + bz_i \in B$, *one has* $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (a_1 + by_1, ..., a_n + by_n) \precsim (a_1 + bz_1, ..., a_n + bz_n)$.

**Axiom 7** ***FC Reversal Consistency****. For any* $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ *and any two real numbers* $a$ *and* $b$ *with* $b < 0$ *one has* $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (a + by_1, ..., a + by_n) \precsim (a + bz_1, ..., a + bz_n)$.

Ordinal Reversal Consistency has been used under the name of *duality* in the somewhat different context of ranking finite sets of objects by Nitzan and Pattanaik (1984). Requirements that a ranking of distributions be restricted in a certain way when the measurement of the attribute quantities is reversed have also been discussed in the context of inequality measurement by many authors, including Bosmans (2016), Chakravarty, Chattopadhyay, and d'Ambrosio (2015), Clarke, Gerdtham, Johannesson, Bingefors, Smith, and Oswald (2002), Erreygers (1990), Lambert and Zheng (2011), de la Vega and Aristondo (2012), Permanyer (2016), Yalonetsky (2022) and Abul-Naga and Yalonetsky (2024), Yalonetsky (2022) and Abul-Naga and Yalonetsky (2024). In this latter context, it is often required, to the contrary, that the ranking of distributions be invariant to an inversion in the measurement of the attribute. For example, it can be easily checked that the Pigou-Dalton binary relation $\succsim^{PD}$ examined above is invariant with respect to either an AFC or an FC inversion in the measurement of the variable. Hence, $\succsim^{PD}$does not satisfy neither AFC nor FC Reversal Consistency. Such an invariance makes sense for a notion of inequality concerned with the dispersion of the attribute's quantities received by the agents away from the distribution' central tendency. However, it does not make sense for a ranking based on "how much there is to distribute".

A first thing to notice about these reversal consistency axioms is that they entail their associate ordinal or cardinal measurability notions of Definitions 1- 4, as established in the following proposition.

**Proposition 1** *If an ordering $\succsim$ of $\mathbb{R}^n$ satisfies Ordinal Reversal Consistency, then it assumes ordinal measurability of the attribute. Similarly, if $\succsim$ satisfies CU, AFC and FC Reversal Consistency, then it assumes the attribute to be, respectively, CU, AFC and FC measurable.*

***Proof.*** *Suppose that $\succsim$ is an ordering of $\mathbb{R}^n$ that satisfies Ordinal Reversal Consistency. Hence, for any two distributions $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ and any strictly decreasing function $f : A \longrightarrow B$ for some subsets $A$ and $B$ of real numbers such that $y_i \in A$ and $z_i \in A$ for $i = 1, ..., n$, one has $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (f(y_1), ..., f(y_n)) \precsim (f(z_1), ..., f(z_n)) \Leftrightarrow (-f(y_1), ..., -f(y_n)) \succsim (-f(z_1), ..., -f(z_n))$. Hence $\succsim$ assumes ordinal measurability of the attribute as per Definition 1 since a function $f : A \longrightarrow B$ is decreasing (increasing) if and only if $-f$ is increasing (decreasing). Similarly, assume that $\succsim$ is an ordering of $\mathbb{R}^n$ that satisfies CU Reversal Consistency Consistency. Then, for any $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ and any list of $n+1$ real numbers $a_1, ..., a_n$ and $b$ with $b < 0$ one has $(y_1, ..., y_n) \succsim (z_1, ..., z_n) \Longleftrightarrow (a_1 + by_1, ..., a_n + by_n) \precsim (a_1 + bz_1, ..., a_n + bz_n) \Longleftrightarrow (a_1 - by_1, ..., a_n - by_n) \succsim (a_1 - bz_1, ..., a_n - bz_n)$. Hence $\succsim$ assumes CU measurability of the attribute as per Definition 2. The argument for the two other notions of cardinal measurability is similar.* ■

In the next section, we show that cardinal and ordinal inversion consistency imposed on an anonymous ordering of $\mathbb{R}^n$ that satisfies the Pareto axiom characterize the rankings $\succsim^{sum}$ and $\succsim^{med}$ respectively, the latter when the distributions have a unique median and the former when the ordering is also required to satisfy separability. We also show that there are no anonymous orderings of $\mathbb{R}^n$ that satisfy Pareto and Ordinal reversal consistency when there are two medians. We finally show, without separability, that there are quite a few anonymous orderings of $\mathbb{R}^n$ that satisfy weak Pareto and Almost Fully Cardinal reversal consistency including $\succsim^{sum}$ and $\succsim^{med}$(when the median is unique) and the ranking of distributions based on the symmetric average of the two median elements (when the median is not unique). As a matter of fact, the Paretian and Anonymous orderings of $\mathbb{R}^n$ that satisfy AFC reversal consistency are all the rankings based on a rank-dependent weighted average of the $n$-tuples whose weights are symmetric with respect to the ranks defined from above, and from below. *A fortiori*, since AFC Reversal Consistency is stronger than FC reversal consistency, this suggests that there are also quite a few not necessarily separable anonymous orderings of $\mathbb{R}^n$ that satisfy FC Reversal consistency (even though we are not able to identify exactly what these are). Non surprisingly, if we strengthen the reversal consistency requirement to CU reversal consistency, we can reproduce Milnor (1954) classical result - also used in d'Aspremont and Gevers (1977) - and conclude that $\succsim^{sum}$ is the only anonymous ordering of $\mathbb{R}^n$ that satisfies Pareto and CU Reversal Consistency.

# 3 Results

We actually start with this latter result and characterize the ordering $\succsim^{sum}$ as the only ordering of $\mathbb{R}^n$ that satisfies Anonymity, Weak Pareto and CU Reversal Consistency. We provide the proof for completeness even though it can be found in Milnor (1954) (Theorem 2, p. 53) and d'Aspremont and Gevers (1977) (Theorem 3, p. 203) thanks to Proposition 1.

**Theorem 1** *An ordering $\succsim$ of $\mathbb{R}^n$ satisfies Anonymity, Pareto and CU Reversal Consistency if and only if $\succsim = \succsim^{sum}$.*

***Proof.*** *We leave to the reader the easy task of verifying that $\succsim^{sum}$ is an ordering of $\mathbb{R}^n$ that satisfies Anonymity, Pareto and Cardinal Unit reversal Consistency. To go in the other direction, consider an ordering $\succsim$ of $\mathbb{R}^n$ that satisfies Anonymity, Pareto and CU reversal Consistency. Let us first show that any two distributions $(y_1, ..., y_n)$, $(z_1, ..., z_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i$ should be equivalent. Indeed, by Anonymity $(y_1, ..., y_n) \sim (y_{(1)}, ..., y_{(n)})$ and $(z_1, ..., z_n) \sim (z_{(1)}, ..., z_{(n)})$. Since by Proposition 1, $\succsim$ assumes CU measurability of the attribute as per Definition 2, it follows that $(y_1, ..., y_n) \sim (y_{(1)}, ..., y_{(n)}) \succsim (z_{(1)}, ..., z_{(n)}) \sim (z_1, ..., z_n) \Leftrightarrow (y_{(1)} - \min(y_{(1)}, z_{(1)}), ..., y_{(n)} -$*

$\min(y_{(n)}, z_{(n)})) \succsim (z_{(1)} - \min(y_{(1)}, z_{(1)}), ..., z_{(n)} - \min(y_{(n)}, z_{(n)}))$. *Each of the distribution* $(y_{(1)}-\min(y_{(1)}, z_{(1)}), ..., y_{(n)}-\min(y_{(n)}, z_{(n)}))$ *and* $(z_{(1)}-\min(y_{(1)}, z_{(1)}), ..., z_{(n)}-\min(y_{(n)}, z_{(n)}))$ *is equivalent to its respective ordered permutation thanks to Anonymity again. Reapplying the same operation of subtracting from each component of the two ordered permutation their minimal value leads eventually, because the sum of the components of the two vectors is the same, to the conclusion that* $\mathbf{0}^n \succsim \mathbf{0}^n \iff (y_1, ..., y_n) \succsim (z_1, ..., z_n)$ *thanks to the transitivity of* $\succsim$. *The statement* $(y_1, ..., y_n) \sim (z_1, ..., z_n)$ *then follows at once from the reflexivity of* $\succsim$. *Applying Pareto and transitivity to this conclusion that* $(y_1, ..., y_n) \sim (z_1, ..., z_n)$ *for any two* $(y_1, ..., y_n)$ *and* $(z_1, ..., z_n)$ $\mathbb{R}^n$ *such that* $\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} z_i$ *gives the required conclusion that* $(y_1, ..., y_{(n)}) \succsim (z_1, ..., z_n) \iff \sum_{i=1}^{n} y_i \geq \sum_{i=1}^{n} z_i$. ■

We observe that Theorem 1 implies that CU measurability and CU Reversal Consistency are equivalent requirements when imposed on an anonymous and Paretian ordering of $\mathbb{R}^n$ since they both characterize the ordering $\succsim^{sum}$ in this setting. As we now show, such an equivalence does not hold if we weaken reversal consistency from CU to AFC. In order to study this case, we first define the following family of orderings of $\mathbb{R}^n$ that we refer to as rank-dependent weighted average orderings.

**Definition 6** *Rank-dependent weighted average ordering. An ordering* $\succsim$ *of* $\mathbb{R}^n$ *is said to be a weighted average ordering if there are n rank-dependent non-negative weights* $w_1, ..., w_n$ *summing to* 1 *such that, for every two* $\boldsymbol{y}$ *and* $\mathbf{z} \in \mathbb{R}^n$, $\iff \sum_{i=1}^{n} w_i y_{(i)} > \sum_{i=1}^{n} w_i z_{(i)} \Longrightarrow \mathbf{y} \succ \mathbf{z}$.

Hence, rank-dependent weighted average orderings are those that are compatible - in the strict sense - with the strict ranking of two $n$-tuples induced by comparing some weighted average of the values of those $n$-tuples with the weights being a function of the *rank of the values*, and not of the *values themselves*. Observe that when $n$ is odd, the median ordering $\succsim^m$ is a member of this class for which $w_i = 0$ for all $i \neq (n+1)/2$ and

$w_{(n+1)/2} = 1$. The sum ordering is also a member of this class for which $w_i = 1/n$ for all $i$. Observe also that two distribution with the same rank-dependent weighted average are not required to be considered equivalent. For example both the Maxi-Min and the Lexi-Min rankings of $n$-tuples are rank-dependent weighted average rankings based on a weight of 1 attached to the lowest number (one of the lowest if there are ties) and a weight of zero on all others.

The next theorem actually focuses on the subclass of *symmetric* rank-dependent weighted average ordering defined as follows.

**Definition 7** *A rank dependent weighted average ordering* $\succsim$ *of* $\mathbb{R}^n$ *is said to be symmetric if its* $n$ *rank-dependent non-negative weights* $w_1, ..., w_n$ *summing to* 1 *satisfy* $w_i = w_{n+1-i}$ *for all* $i$.

Hence, a symmetric rank-dependent weighted average ordering is based on rank-dependent weighting scheme that assigns equal weights to the top and the bottom positions in a distribution, equal weights to the second top and the second bottom positions,etc. It can of course assign equal weights to all. If $n$ is even, the ranking defined by comparing elements of $\mathbb{R}^n$ on the basis of the symmetric average of their two median elements in position $n/2$ and $n/2+1$ (and assigning zero weights to all other position $i$) would be a member of this class. The following theorem, that rides heavily on Theorem 4 in Gevers (1979), establishes that symmetric rank-dependent weighted average orderings of $\mathbb{R}^n$ are the only Paretian and Anonymous orderings that satisfy AFC Reversal consistency.

**Theorem 2** *An ordering* $\succsim$ *of* $\mathbb{R}^n$ *satisfies Anonymity, Pareto and AFC Reversal Consistency if and only if* $\succsim$ *is a symmetric rank-dependent weighted average ordering.*

***Proof.*** *We leave to the reader the task of verifying that any symmetric rank-dependent weighted average ordering satisfies Anonymity, Pareto and AFC Reversal Consistency. In the other direction, let* $\succsim$ *be an ordering of* $\mathbb{R}^n$ *satisfying Anonymity, Pareto and AFC Reversal Consistency. We notice that, thanks to Proposition 1,* $\succsim$ *assumes AFC measurability of the*

*attribute. Hence, by Theorem 4 in Gevers (1979), there exists a set of rank-dependent weights* $w_1, ..., w_n$ *summing to* 1 *such that for any two distributions* $(y_1, ..., y_n)$ *and* $(z_1, ..., z_n)$ *in* $\mathbb{R}^n$*, one has (thanks to anonymity and transitivity)* $(y_1, ..., y_n) \sim (y_{(1)}, ..., y_{(n)}) \succ (z_{(1)}, ..., z_{(n)}) \sim (z_1, ..., z_n)$ *whenever* $\sum_{i=1}^{n} w_i y_{(i)} > \sum_{i=1}^{n} w_i z_{(i)}$*. Let us show that any such set of rank-dependent weights that is not symmetric leads to a violation AFC Reversal consistency. For this sake, consider any non-symmetric set of rank dependent weights* $w_1, ..., w_n$*. By definition, this non-symmetric set of weights must be such that there is a position* $i_0 \in \{1, ..., n\}$ *such that* $w_{i_0} \neq w_{n+1-i_0}$*. Assume first that* $i_0 \neq 1$*. Without loss of generality, suppose that* $w_{i_0} > w_{n+1-i_0}$*. Consider then a distribution* $(y_1, ..., y_n) \in \mathbb{R}^n$ *such that* $y_{(i_0)} > y_{(i_0-1)}$ *and* $y_{(n+1-i_0)} < y_{(n+2-i_0)}$ *and define the distribution* $(z_1, ..., z_n) \in \mathbb{R}^n$ *by:*
$z_{(i)} = y_{(i)}$ *for all positions* $i \notin \{i_0, i_{n+1-i_0}\}$,
$z_{(i_0)} = y_{(i_0)} - \varepsilon$ *and,*
$z_{(n+1-i_0)} = y_{(n+1-i_0)} + \varepsilon.$
*for* $\varepsilon = \min(y_{(i_0)} - y_{(i_0-1)}, y_{(n+2-i_0)} - y_{(n+1-i_0)}) > 0$*. Since* $w_{i_0} > w_{n+1-i_0}$*, we must have that* $(y_1, ..., y_n) \succ (z_1, ..., z_n)$ *for the rank-dependent weighted average ordering* $\succsim$ *induced by the set of weights* $w_1, ..., w_n$*. Consider now the two ordered distributions* $(\widehat{y}_{(1)}, ..., \widehat{y}_{(n)}) = (-y_{(n)}, ..., -y_{(1)})$ *and* $(\widehat{z}_{(1)}, ..., \widehat{z}_{(n)}) = (-z_{(n)}, ..., -z_{(1)})$*. We observe that for all* $i \in \{1, ..., n\}$, $\widehat{y}_{(i)} = -y_{(n+1-i)}$ *and* $\widehat{z}_{(i)} = -z_{(n+1-i)}$ *so that* $(\widehat{y}_{(1)}, ..., \widehat{y}_{(n)})$ *and* $(\widehat{z}_{(1)}, ..., \widehat{z}_{(n)})$ *only differ by the positions* $i_0$ *and* $n+1-i_0$*. Hence, we have* $\sum_{i=1}^{n} w_i(\widehat{y}_{(i)} - \widehat{z}_{(i)}) = \varepsilon[w_{i_0} - w_{n+1-i_0}] > 0$ *so that the ordering the rank-dependent weighted average ordering* $\succsim$ *induced by the set of weights* $w_1, ..., w_n$ *contradicts AFC reversal consistency. The argument for the case where* $i_0 = 1$ *is the same but without any constraint imposed on the strictly positive real number* $\varepsilon$*.* ■

Since Gevers (1979) obtains a characterization of the whole family of rank dependent weighted average orderings - and not only those who are symmetric - with AFC measurability, anonymity and Pareto, we thus conclude from Theorem 2 that AFC Reversal Consistency is a strictly stronger requirement than AFC measurability when imposed on an anonymous and

Paretian ordering of $\mathbb{R}^n$.

What are the anonymous orderings of $\mathbb{R}^n$ that satisfy Pareto and the weaker, but much more natural in our view, FC Reversal Consistency ? We do not know the answer to this question. For sure all symmetric rank-ordered weighted average rankings of Theorem 2 belong to it.

However, we know the class of all *separable* and anonymous orderings of $\mathbb{R}^n$ that satisfy FC Reversal Consistency and Pareto. As it turns out, this class consists in a single member - $\succsim^{sum}$- as established in the following theorem.

**Theorem 3** *An ordering* $\succsim$ *on* $\mathbb{R}^n$ *satisfies Anonymity, Pareto, Separability and Cardinal Reversal Consistency if and only if* $\succsim = \succsim^{sum}$.

***Proof.*** *We leave to the reader the task of verifying that the ordering* $\succsim^{sum}$ *on* $\mathbb{R}^n$ *satisfies Anonymity, Pareto, Separability and FC Reversal Consistency. To prove the other implication, consider any ordering* $\succsim$ *on* $\mathbb{R}^n$ *satisfying Anonymity, Pareto, Separability and Cardinal Reversal Consistency and define, from that ordering, the binary relation* $\succsim^{n-j}$ *on* $\mathbb{R}^{n-j}$ *for* $j = 1, ...n-1$ *and any* $\mathbf{y}$ *and* $\mathbf{z} \in \mathbb{R}^{n-j}$ *by:*

$$\begin{aligned}
\mathbf{y} &\succsim {}^{n-j}\mathbf{z} \Longleftrightarrow \exists G \subset \{1, ..., n\} \text{ satisfying } \#G = j \text{ and } \widehat{\mathbf{y}} \text{ and } \widehat{\mathbf{z}} \in \mathbb{R}^n \text{ such that:} \\
\widehat{y}_i &= \widehat{z}_i \text{ for all } i \in G, \\
\widehat{y}_h &= y_h \text{ and } \widehat{z}_h = z_h \text{ for all } h \in \{1, ..., n\} \backslash G \text{ and} \\
\widehat{\mathbf{y}} &\succsim \widehat{\mathbf{z}}.
\end{aligned}$$

*Since* $\succsim$ *is an ordering of* $\mathbb{R}^n$ *that satisfies separability, each such binary relation* $\succsim^{n-j}$ *is an ordering of* $\mathbb{R}^{n-j}$ *who does not depend upon the particular set* $G$ *used in its definition. It can be seen easily that if* $\succsim$ *is, as assumed, separable and anonymous, then so is* $\succsim^{n-j}$ *on its domain of definition. If* $\succsim$ *satisfies the Pareto principle, then* $\succsim^{n-j}$ *satisfies the weak Pareto requirement that* $y_i \geq z_i$ *for all* $i \in \{1, ..., n-j\}$ *implies* $\mathbf{y} \succsim^{n-j} \mathbf{z}$. *However,* $\succsim^{n-j}$ *does* ***not*** *necessarily satisfy the additional Pareto requirement that* $y_i > z_i$ *for all* $i \in \{1, ..., n-j\}$ *implies* $\mathbf{y} \succ^{n-j} \mathbf{z}$. *We now show that* $\succsim^{n-j}$ *satisfies FC Reversal Consistency on its domain of definition. By contradiction, suppose* *it does not and, therefore, that there are two distributions* $\mathbf{y}$ *and* $\mathbf{z} \in \mathbb{R}^{n-j}$ *such that* $(y_1, ..., y_{n-j}) \succsim^{n-j} (z_1, ..., z_{n-j})$ *and some real numbers* $b < 0$ *and* $a$*, such that* $(by_1 + a, ..., by_{n-j} + a) \succ^{n-j} (bz_1 + a, ..., bz_{n-j} + a)$*. For any group* $G \subset \{1, ..., n\}$*, we write the vector* $\mathbf{y} \in \mathbb{R}^n$ *as* $\mathbf{y} = (\mathbf{y}^G; \mathbf{y}^{G^C})$ *for the vectors* $\mathbf{y}^G \in \mathbb{R}^{\#G}$ *and* $\mathbf{y}^{G^C} \in \mathbb{R}^{n-\#G}$ *defined by* $\mathbf{y}_i^G = y_i$ *for all* $i \in G$ *and* $\mathbf{y}_h^G = y_h$ *for all* $h \in \{1, ..., n\} \backslash G$*. By definition of* $\succsim^{n-j}$*,* $(y_1, ..., y_{n-j}) \succsim^{n-j} (z_1, ..., z_{n-j})$ *if and only if there is a group* $G \subset \{1, ..., n\}$ *with* $\#G = j$ *and some* $\widehat{\mathbf{y}}$ *and* $\widehat{\mathbf{z}} \in \mathbb{R}^n$ *satisfying* $\widehat{y}_i = \widehat{z}_i$ *for all* $i \in G$*,* $\widehat{y}_h = y_h$ *and* $\widehat{z}_h = z_h$ *for all* $h \in \{1, ..., n\} \backslash G$ *such that* $(\widehat{y}^G; (y_1^{G^C}, ..., y_{n-j}^{G^C})) \succsim (\widehat{y}^G; (z_1^{G^C}, ..., z_{n-j}^{G^C}))$ *and* $(\widehat{y}^G; (by_1^{G^C} + a, ..., by_{n-j}^{G^C} + a)) \succ (\widehat{y}^G; (bz_1^{G^C} + a, ..., bz_{n-j}^{G^C} + a)$*. Since* $\succsim$ *is separable, we must have also* $((b\widehat{y}_1^G + a, ..., b\widehat{y}_{\#G}^G + a); (by_1^{G^C} + a, ..., by_{n-j}^{G^C} + a)) \succ ((b\widehat{y}_1^G + a, ..., b\widehat{y}_{\#G}^G + a); (z_1^{G^C}, ..., z_{n-j}^{G^C}))$ *which contradicts FC Reversal Consistency imposed on* $\succsim$*. We now prove by reverse induction that for any* $j \in \{1, ..., n-1\}$*, if a distribution* $\mathbf{y} \in \mathbb{R}^{n-j}$ *is such that* $\sum_{i=1}^{n-j} y_i = 0$ *then* $\mathbf{y} \sim^{n-j} \mathbf{0}^{n-j}$*. Since the ordering* $\succsim^{n-j}$ *is anonymous and satisfies FC Reversal Consistency, it is invariant thanks to Proposition 1 to applying any strictly increasing common affine transformation to all the components of the two compared vectors. Hence, we have that* $(y_1, ..., y_{n-j}) \sim^{n-j} (y_{(1)}, ..., y_{(n-j)}) \sim^{n-j} (\sum_{i=1}^{n-j} y_i, ..., \sum_{i=1}^{n-j} y_i) \Leftrightarrow (y_{(1)} - \sum_{i=1}^{n-j} y_i, ..., y_{(n-j)} - \sum_{i=1}^{n-j} y_i) \sim^{n-j} \mathbf{0}^{n-j}$*. Hence, proving that* $\mathbf{y} \sim^{n-j} \mathbf{0}^{n-j}$ *for any* $\mathbf{y} \in \mathbb{R}^{n-j}$ *such that* $\sum_{i=1}^{n-j} y_i = 0$ *amounts to proving that* $(y_1, ..., y_{n-j}) \sim^{n-j} (\frac{1}{n-j} \sum_{i=1}^{n-j} y_i, ..., \frac{1}{n-j} \sum_{i=1}^{n-j} y_i)$*. The equivalence is trivial if* $j = n - 1$*. Suppose by induction that for all* $k \in \{1, ..n - 1\}$*, it has been established that if* $\sum_{i=1}^{k} y_i = 0$ *then* $\mathbf{y} \sim^k \mathbf{0}^k$*. Let us show that if* $\sum_{i=1}^{k+1} y_i = 0$ *holds, then* $\mathbf{y} \sim^{k+1} \mathbf{0}^{k+1}$*. By contradiction, suppose there is a distribution* $\mathbf{y} \in \mathbb{R}^{k+1}$*, such that* $\sum_{i=1}^{k+1} y_i = 0$ *and* $\mathbf{y} \succ^{k+1} \mathbf{0}^{k+1}$*. Since* $\succsim^{k+1}$*is reflexive,* $\mathbf{y} \neq \mathbf{0}^{k+1}$*. Since* $\mathbf{y}$ *is such that* $\sum_{i=1}^{k+1} y_i = 0$ *we must have* $y_{(1)} < 0 < y_{(k+1)}$*. If we multiply the distribution* $\mathbf{y}$ *by* $-1$*, it follows from FC Reversal Consistency that* $\mathbf{0}^{k+1} \succ^{k+1} (-y_1, ..., -y_{k+1}) \sim^{k+1} (-y_{(1)}, ..., -y_{(k+1)})$ *(using anonymity and transitivity). Consider then multiplying the distribution* $(-y_{(1)}, ..., -y_{(k+1)})$ *by* $-\frac{y_{(1)}}{y_{(k+1)}} > 0$*. Since* $\succsim^{k+1}$ *is invariant with respect to increasing affine transformations by virtue of Proposition 1, we have* $\mathbf{0}^{k+1} \succ^{k+1} (\frac{y_{(1)}y_{(1)}}{y_{(k+1)}}, \frac{y_{(2)}y_{(1)}}{y_{(k+1)}}, ..., y_{(1)})$*. Observe that the the components of the distribution* $(\frac{y_{(1)}y_{(1)}}{y_{(k+1)}}, \frac{y_{(2)}y_{(1)}}{y_{(k+1)}}, ..., y_{(1)})$ *also sum to zero. Compare* *now the two zero-sum distributions* $\mathbf{y}$ *and* $(\frac{y_{(1)}y_{(1)}}{y_{(k+1)}}, \frac{y_{(2)}y_{(1)}}{y_{(k+1)}}, ..., y_{(1)})$*. The two distributions have the component* $y_{(1)}$ *in common while the sub-distributions distributions* $(y_2, ..., y_k)$ *and* $(\frac{y_{(1)}y_{(1)}}{y_{(k+1)}}, ..., \frac{y_{(1)}y_{(k)}}{y_{k+1}})$ *have the same mean (or equivalently the same sum). Hence, by the induction hypothesis, anonymity and transitivity* $(y_2, ..., y_k) \sim^k (\frac{y_{(1)}y_{(1)}}{y_{(k+1)}}, ..., \frac{y_{(1)}y_{(k)}}{y_{(k+1)}})$*. By separability, we conclude that that* $\mathbf{y} \sim^{k+1} (\frac{y_{(1)}y_{(1)}}{y_{(k+1)}}, ..., y_{(1)})$ *which is a contradiction. We invite the reader to verify using the same reasoning that assuming instead* $\mathbf{0}^{k+1} \succ^{k+1} (y_1, ..., y_{k+1})$ *would also lead to a contradiction. Hence we have establish that for any two distributions* $\mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *with the same sum,* $\mathbf{y} \sim \mathbf{z}$*. Consider now two distributions* $\mathbf{y}$ *and* $\mathbf{z} \in \mathbb{R}^n$ *such that* $\sum_{i=1}^n y_i > \sum_{i=1}^n z_i$*. We just proved that* $\mathbf{y} \sim (\sum_{i=1}^n y_i, ..., \sum_{i=1}^n y_i)$ *and* $\mathbf{z} \sim (\sum_{i=1}^n z_i, ..., \sum_{i=1}^n z_i)$*. From the Pareto axiom, we conclude that* $(\sum_{i=1}^n y_i, ..., \sum_{i=1}^n y_i) \succ (\sum_{i=1}^n z_i, ..., \sum_{i=1}^n z_i)$ *and, by transitivity, that* $\mathbf{y} \succ^n \mathbf{z}$*. Hence, for any two distributions* $\mathbf{y}$ *and* $\mathbf{z} \in \mathbb{R}^n$*,* $\mathbf{y} \succsim \mathbf{z} \Longleftrightarrow \sum_{i=1}^n y_i \geq \sum_{i=1}^n z_i$ *and this completes the proof.* ■

Theorem 3 can be seen as a (significant) reformulation of Maskin (1978) classical characterization of utilitarianism. Maskin's characterization was using, along with anonymity, separability and the requirement that the ranking be an ordering, continuity, a strong version of the Pareto principle, and FC measurability. Theorem 3 rides on a weaker version of the Pareto principle, dispenses entirely with continuity and replaces FC measurability by the stronger FC reversal consistency.

Since the ordering $\succsim^{sum}$ does not satisfy Ordinal Reversal consistency, we immediately conclude that there are no anonymous and separable orderings of $\mathbb{R}^n$ that satisfy Pareto and ordinal reversal consistency. If however we abandon the requirement for the ranking of distribution to be separable, then, as stated in the following theorem, we obtain the median as the unique anonymous and Paretian ordering of distributions with a unique median that satisfies ordinal reversal requirement.

**Theorem 4** *An ordering* $\succsim$ *of* $\mathbb{M}^n$ *satisfies Anonymity, Weak Pareto and Ordinal Reversal Consistency if and only if* $\succsim = \succsim^{med}$*.*

***Proof.*** *We leave to the reader the task of verifying that* $\succsim^{med}$ *satisfies Anonymity, Pareto and Ordinal Reversal Consistency. In the other direction, let* $\succsim$ *be an ordering of* $\mathbb{M}^n$ *that satisfies Anonymity, Pareto and Ordinal Reversal Consistency. We first show that for any distribution* $\mathbf{y} \in \mathbb{M}^n$, $\mathbf{y}$ *is equivalent to the distribution* $(m(\mathbf{y}), ..., m(\mathbf{y})) \in \mathbb{M}^n$. *To do so, suppose by contradiction that* $\mathbf{y} \succ (m(\mathbf{y}), ..., m(\mathbf{y}))$. *By Anonymity and transitivity, this amounts to assuming that* $(y_{(1)}, ..., y_{(n)}) \succ (m(\mathbf{y}), ..., m(\mathbf{y}))$. *Using Ordinal reversal consistency, we conclude that* $(-m(\mathbf{y}), ..., -m(\mathbf{y})) \succ (-y_{(1)}, ..., -y_{(n)})$. *Note that* $-y_{(n)} \leq ... \leq -y_{(1)}$ *and* $y_{(1)} \leq ... \leq y_{(n)}$. *Hence we can construct an increasing function* $f$ *such that* $f(-y_{(n)}) = y_{(1)} \leq ... \leq f(-y_{(1)}) = y_{(n)}$. *The function* $f$ *is such that for any* $i \in \{1, ..., n\}$, $f(-y_{(i)}) = y_{(n-i+1)}$. *Since by Proposition 1,* $\succsim$ *is invariant to any strictly increasing transformation, it follows that,* $(f(-m(y)), ..., f(-m(y))) \succ^n (f(-y_{(1)}), ..., f(-y_{(n)})) = (y_{(n)}, ..., y_{(1)}) \sim (y_{(1)}, ..., y_{(n)})$ *(by anonymity). If* $n$ *is odd,* $f(-m(y)) = f(-y_{(\frac{n+1}{2})}) = y_{(\frac{n+1}{2})} = m(y)$. *If* $n$ *is even,* $f(-m(y)) = f(-y_{(\frac{n}{2})}) = y_{(\frac{n}{2}+1)}$. *However, since the median of* $y$ *is unique, we have* $y_{(\frac{n}{2}+1)} = m(y)$. *Hence we can conclude that* $(f(-m(\mathbf{y})), ..., f(-m(\mathbf{y}))) = (m(\mathbf{y}), ..., m(\mathbf{y}))$. *But this contradict the initial assumption that* $(y_{(1)}, ..., y_{(n)}) \succ (m(\mathbf{y}), ..., m(\mathbf{y}))$. *We leave to the reader the task of verifying with the same reasoning that* $(m(\mathbf{y}), ..., m(\mathbf{y})) \succ (y_{(1)}, ..., y_{(n)})$ *also leads to a contradiction. Hence we must have that* $(m(\mathbf{y}), ..., m(\mathbf{y})) \sim (y_{(1)}, ..., y_{(n)})$. *Combining this result with Transitivity and Pareto will lead us to the required conclusion that* $m(\mathbf{y}) \geq m(\mathbf{z}) \Longleftrightarrow \mathbf{y} \succsim \mathbf{z}$ *for any two distributions* $\mathbf{y}$ *and* $\mathbf{z}$ *in* $\mathbb{M}^n$. ■

The importance of the restriction for the median to be unique is somewhat significant for this result however. Indeed, as shown in the following theorem, there does not exist any ordering on $\mathbb{R}^n \backslash \mathbb{M}^n$ that satisfies Anonymity, Pareto and Ordinal Reversal Consistency.

**Theorem 5** *There does not exist any ordering* $\succsim$ *on* $\mathbb{R}^n \backslash \mathbb{M}^n$ *that satisfies Anonymity, Pareto and Ordinal Reversal Consistency.*

***Proof.*** *Suppose by contradiction that there is an ordering* $\succsim$ *on* $\mathbb{R}^n \backslash \mathbb{M}^n$ *satisfying Anonymity, Pareto and Ordinal Reversal Consistency. Consider* *some distribution* $\mathbf{y} \in \mathbb{R}^n \backslash \mathbb{M}^n$ *with two different medians (since* $\mathbf{y} \notin \mathbb{M}^n$*). Having two distinct medians can only happen if* $n$ *is even. Hence we have* $y_{(\frac{n}{2})} \neq y_{(\frac{n}{2}+1)}$*. Using the same succession of transformations as those used in the proof of Theorem 4, it is possible to conclude that* $\mathbf{y}$ *is equivalent to the distribution* $\mathbf{y}' = (y'_{(1)}, ..., y'_{(n)})$ *where for any* $i \in \{1, ..., \frac{n}{2}\}$, $y'_{(i)} = y_{(\frac{n}{2})}$ *and for any* $i \in \{\frac{n}{2}+1, ..., n\}$, $y'_{(i)} = y_{(\frac{n}{2}+1)}$*. Define* $\mathbf{z} \in \mathbb{R}^n$ *to be such that* $z_{(\frac{n}{2})} < y_{(\frac{n}{2})}$ *and* $z_{(\frac{n}{2}+1)} > y_{(\frac{n}{2}+1)}$*. By reapplying the same sequence of transformations than the one used in the Theorem 4, we arrive at the conclusion that* $\mathbf{y}' \sim \mathbf{z}'$ *where for any* $i \in \{1, ..., \frac{n}{2}\}$, $z'_{(i)} = z_{(\frac{n}{2})}$ *and for any* $i \in \{\frac{n}{2}, ..., n\}$, $z'_{(i)} = z_{(\frac{n}{2}+1)}$*. It is then possible to construct a distribution, say* $\mathbf{x} \in \mathbb{R}^n$*, such that* $z_{(\frac{n}{2})} < x_{(\frac{n}{2})} < y_{(\frac{n}{2})}$ *and* $z_{(\frac{n}{2}+1)} < x_{(\frac{n}{2}+1)}$*. From Pareto,* $\mathbf{x} \succ \mathbf{z}$*. However, when comparing* $\mathbf{x}$ *and* $\mathbf{y}$*, since the median rankings are not unanimous, we have* $\mathbf{x} \sim \mathbf{y}$ *and, for the same reasons,* $\mathbf{y} \sim \mathbf{z}$*. But this either contradicts the transitivity of* $\succsim$ *or the strict ranking* $\mathbf{x} \succ \mathbf{z}$*.* ■

Is the limitation provided to Theorem 4 by Theorem 5, which only concerns distributions involving an even number of agents, empirically significant ? Not very much if the compared distributions of the attribute involve a large number of agents whose attribute quantities can take finitely many values. We show in that case in Appendix A that the probability of observing a sample of independent observations on the attribute's quantities of an even number of agents with more than one median vanishes when the number of agents becomes arbitrarily large. Hence, it seems that Theorem 4 is somewhat generic and that the median is really the natural definition of "how much there is to distribute" of an ordinally measurable attribute.

# 4 Conclusion

The main conclusion of this paper is that if one accepts the prerequisite that any ordering of distributions of an attribute between a given collection of agents on the basis of "how much there is to distribute" should satisfy the Pareto principle, be anonymous and be consistent with the reversal in the measurement of the attribute, then one is led to the conclusion that the ordering should be based on the mean if the attribute is assumed to be either CU measurable or separable and FC measurable, and on the median if the attribute is assumed to be ordinally measurable and each compared distribution has only one median. Results are somewhat less tight if one assumes AFC cardinal measurability of the attribute. In this case, the set of all symmetric rank-dependent weighted average orderings emerge as possibilities, a set which contains the mean, the unique median (if the number of individuals is odd) and the symmetric average of the two medians (if the number of individuals is even. This set also contains symmetric averages of the lowest and the highest levels of well-being in a distributions. The class of orderings that satisfy Fully Cardinal Reversal Consistency but not separability is even larger than this class, but we do not know anything about it.

These results suggest, for sure, that the median is a highly natural definition of "how much there is to distribute" of an ordinally measurable attribute. This immediately suggests an important avenue for future research in ordinal inequality measurement: that of obtaining an implementable criterion for verifying when a distribution has been obtained from another by a finite sequence of median preserving Hammond transfers. Gravel, Magdalou, and Moyes (2021) (and more directly Gargani (2025)) have identified an easily implementable criterion, the intersection of two dominances, that is equivalent to the fact of going from the dominated to the dominating distribution by a finite sequence of Hammond transfers. Yet many of these transfers do not preserve the median, and therefore cannot be considered to capture pure equalization in an ordinal setting. One could think, of course, of applying the intersection of the two dominance proposed in Gravel, Magdalou, and Moyes (2021) to distributions with the same median. However, as shown in Gargani (2025), there are examples of situations where the intersection of the two dominances of Gravel, Magdalou, and Moyes (2021) is observed between two distributions with the same median but where it is not possible to go from the doubly dominated to the doubly dominant distribution only by median preserving Hammond transfers. Some non-median preserving Hammond transfers may be required in the process. Hence, the identification of an implementable criterion that coincides with the possibility of going from a distribution to another by median preserving transfers is an important step in the research agenda.

Another step would be to identify precisely the class of all anonymous orderings of distributions that satisfy Pareto and FC reversal consistency only. We leave this for future research.

# A Appendix

We show that the impossibility identified in Theorem 5 may not be empirically significant if it applied to reasonably large sample of discrete survey data. Consider indeed a typical such survey data $(x_1, ..., x_n) \in \{1, ..., l\}^n$ where $x_i$ denote the observed quantity of attribute received by individual $i$ in the survey, with this attribute quantity taking any value in the finite set $\{1, ..., l\}$ of categories ordered from the worst to the best. For any category $k \in \{1, ..., l\}$, we denote by $p_k^x = \#\{i \in \{1, ..., n\} : x_i = k\}/n$ the discrete probability of having someone falling in category $k$. As in Theorem 5, we restrict attention to surveys with an even number $n$ of individuals. For any category $k \in \{1, \ldots, l-1\}$, let $C_k = \#\{i \in \{1, ..., n\} : x_i \leq k\}$ denote the number of individuals in the sample whose attribute quantity do not exceed $k$. The median is not unique whenever there is a category $k \in \{1, \ldots, l-1\}$ such that $C_k = n/2$. We want to show that the probability $P\left(\bigcup_{k=1}^{l-1}\{C_k = n/2\}\right)$ to have a non-unique median vanishes as $n$ gets large.

We first observe that, for every category $k \in \{1, \ldots, l-1\}$, $C_j \sim$Bin$(n, q_k)$ where $q_k = p_1^x + \cdots + p_k^x$. Indeed, if we define, for every $i \in \{1, ..., n\}$ and $k \in \{1, ..., l-1\}$ the variable:

$$Y_i^k = \mathbf{1}_{\{x_i \leq k\}} = \begin{cases} 1 & \text{if } x_i \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

we have that $Y_i^k \sim \text{Bernoulli}(q_k)$. Since the trials $Y_1^k, \ldots, Y_n^k$ are independent and $C_k = \sum_{i=1}^n Y_i^k$, it follows that $C_k \sim \text{Bin}(n, q_k)$. We next observe that:

$$P\left(\bigcup_{k=1}^{l-1}\{C_k = n/2\}\right) \; \leq \; (l-1)\binom{n}{n/2}2^{-n}.$$

Indeed, by Booles inequality, $P\left(\bigcup_{k=1}^{l-1}\{C_k = n/2\}\right) \; \leq \; \sum_{k=1}^{l-1} P(C_k = n/2)$. Since $C_k \sim \text{Bin}(n, q_k)$, we have that $P(C_k = n/2) = \binom{n}{n/2}q_k^{n/2}(1-q_k)^{n/2}$. Since this probability is maximized at $q_k = 1/2$, it follows that $P\left(\bigcup_{k=1}^{l-1}\{C_k = n/2\}\right) \leq (l-1)\binom{n}{n/2}2^{-n}$.

We finally conclude that

$$\lim_{n\to\infty} P\left(\bigcup_{k=1}^{l-1}\{C_k = n/2\}\right) \; = \; 0.$$

Indeed, applying Stirling approximation, $n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$, we have $\binom{n}{n/2}2^{-n} \approx \sqrt{\frac{2}{\pi n}}$. Hence, $\lim_{n\to\infty}(l-1)\binom{n}{n/2}2^{-n} = \lim_{n\to\infty}(l-1)\sqrt{\frac{2}{\pi n}} \; = \; 0$. Hence, the probability of having two medians becomes negligible when $n$ is large.

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