Leveraging Firm Entry Policy to Drive Innovation, Growth, and Income Distribution in the Presence of Entry Threats and Rent-Seeking

Abstract

This paper presents an analytical model that investigates the dynamics of rent-seeking, innovation, and entry policies in a two-sector economy characterized by skilled and unskilled labor. The model explores how incumbent firms in an intermediate goods sector react to the threat of new entrants and how rent-seeking behavior influences innovation and economic productivity. A key feature of the model is the role of a policymaker who sets firm entry policies and responds to bribes offered by incumbent firms seeking to restrict market entry.

The analysis distinguishes between advanced and backward incumbent firms. Advanced firms, which operate at the frontier of technological productivity, choose to innovate to retain their competitive position in response to entry threats. In contrast, backward firms face higher barriers to innovation and are more likely to bribe policymakers to deter new competition. The magnitude of the bribes depends on the difference in profits with and without entry threats, as well as the costs of innovation.

The model highlights how rent-seeking by backward firms distorts market competition, leading to suboptimal innovation and lower aggregate productivity in the skilled sector. Policymakers, balancing between maximizing bribes and addressing wage inequality, face conflicting incentives. If a policymaker prioritizes welfare, they may restrict entry to reduce wage inequality, thereby lowering competitive pressures and innovation. Alternatively, a policymaker focused on maximizing bribes may encourage higher entry threats, fostering innovation but exacerbating income inequality.

This paper contributes to the literature on rent-seeking and economic growth by providing a nuanced understanding of how firm behavior, entry policies, and innovation are interlinked, with important implications for labor markets and income distribution. The model provides insights into the broader economic consequences of rent-seeking behavior and entry regulation, emphasizing the need for balanced policies that encourage innovation while minimizing economic distortions caused by rent-seeking activities.

1 Introduction

The dynamics of economic growth, income distribution, and innovation in the presence of market entry threats and rent-seeking behavior remain pivotal topics in contemporary economic research. This paper builds on the foundational insights of Aghion and Howitt's seminal work on creative destruction and endogenous growth, extending the analysis to explore how firm entry policies interact with rent-seeking and technological innovation within a dual-sector economy. By examining the interaction between incumbent firms, policymakers, and potential entrants, this study provides a nuanced perspective on how market structures and policy incentives shape long-term economic trajectories.

The literature on growth and innovation highlights the dual role of market competition in fostering technological progress while influencing income distribution. Aghion and Howitt (1992) introduced the Schumpeterian growth framework, where innovation arises from creative destruction, emphasizing the role of firm dynamics and market entry. Subsequent extensions have examined the impact of policy distortions, such as lobbying and rent-seeking, on innovation incentives. In particular, Aghion and Howitt (2009) underscored the importance of balancing competitive pressures and market incentives to achieve optimal growth. Key differences between this study and the work of Aghion and Howitt lie in the explicit modeling of a dual-sector economy and the incorporation of rent-seeking behavior as a distortionary force that impacts the innovation incentives of advanced and backward firms differently. This paper also extends the analysis by endogenizing policymaker decisions, highlighting how bribes from incumbent firms can influence entry policies.

Further advancements in the literature have explored the broader implications of creative destruction and entry dynamics. For instance, Grossman and Helpman (1991) emphasized the role of knowledge spillovers in enhancing innovation incentives, while Acemoglu and Robinson (2006) analyzed how institutional factors mediate the relationship between innovation and growth. Aghion et al. (2005) investigated the relationship between competition, innovation, and firm-level productivity, providing insights into how policy distortions can stifle innovation. Rodrik and Wacziarg (2005) argued that democratic transitions can result in economic volatility but also highlight how democracies reduce the extent to which existing oligarchies prevent market entry. Additionally, Acemoglu et al. (2005) explored the nuanced relationship between democracy and economic development, finding that democracy's benefits on growth vary across sectors, particularly those closer to the technological frontier. These studies, including Aghion, Alesina, and Trebbi (2007), which highlighted the role of democracy in reducing entry barriers and promoting innovation, collectively inform the theoretical framework of this paper. The aim is to bridge the gap between endogenous growth theory and the political economy of rent-seeking.

The model posits an economy characterized by two distinct sectors: an unskilled sector producing homogenous goods with labor as the sole input, and a skilled sector producing differentiated goods using skilled labor and intermediate inputs. The introduction of entry threats creates a complex dynamic where incumbent firms, depending on their proximity to the technological frontier, either innovate to maintain their competitive position or engage in rent-seeking to deter new entrants. Advanced firms, operating at the productivity frontier, face incentives to innovate when entry threats increase, fostering technological progress but potentially exacerbating income inequality. Conversely, backward firms, constrained by their distance from the frontier, are more likely to bribe policymakers, thereby distorting competition and reducing aggregate productivity.

The results presented in this paper reveal several critical insights. First, entry threats serve as a double-edged sword, encouraging innovation among advanced firms while discouraging backward firms from undertaking similar efforts. Second, the analysis highlights the central role of policymakers in determining economic outcomes, as their decisions on entry regulation significantly influence both aggregate productivity and income distribution. Finally, the findings emphasize the need for carefully calibrated entry policies that strike a balance between fostering innovation and mitigating the adverse effects of rent-seeking and inequality.

This paper delves into the critical role of policymakers in balancing these opposing forces. On the one hand, policymakers are influenced by bribes offered by rent-seeking incumbents, which can lead to restrictive entry policies. On the other hand, they are tasked with promoting welfare objectives, such as reducing income inequality and fostering economic growth. The analysis highlights how these conflicting incentives shape policy decisions and, consequently, the broader economic environment.

The theoretical framework employed here integrates insights from endogenous growth theory, particularly the Schumpeterian model of innovation, with a focus on rent-seeking and lobbying as key determinants of market outcomes. By formalizing the relationship between firm entry policies, innovation incentives, and income distribution, this paper aims to contribute to the literature on the political economy of growth. It underscores the importance of designing balanced policies that mitigate rent-seeking while encouraging innovation, thereby promoting sustainable economic growth and equitable income distribution.

The remainder of this paper is organized as follows. Section 2 provides a detailed description of the consumer and production structures within the economy. Section 3 examines the role of intermediate goods and sectoral output dynamics. Section 4 defines the gross domestic product and inequality in the model, examining how they are affected by innovation and market entry. Section 5 explores the innovation and entry behavior of incumbent firms, distinguishing between advanced and backward firms. Section 6 characterizes the steady-state distribution of advanced firms and its implications for aggregate productivity. Finally, Section 6 introduces the policymaker's optimization problem and discusses the implications of entry policies for growth and inequality. Section 7 introduces the policymaker and examines their role in optimizing entry policies to balance welfare objectives and economic efficiency. Section 8 discusses the process of determining the firm entry policy, considering the trade-offs between fostering innovation and mitigating rent-seeking. Section 9 presents the main results of our paper. Section 10 concludes the paper.

2 Consumer Behavior

The economy consists of $L_{u,t}$ unskilled workers and $L_{s,t}$ skilled workers, who supply labor inelastically and have identical preferences over the unique consumption good, Y. Their utility is given by

$$U_{k,t} = \int_t^\infty e^{-rt} c_{k,t} dt, \quad k \in \{u, s\} \tag{1}$$

where $c_{k,t}$ is the consumption of worker k at time t and r is the discount rate, which due to linear utility is also the interest rate.

The consumption good is a costless assembly of two distinct final goods, $Y_{u,t}$ and $Y_{s,t}$, which are competitively produced. This means that the consumption good is a bundle of and unskilled good, $Y_{u,t}$, which is produced using unskilled labor, and a skilled good, $Y_{s,t}$, which is produced using skilled labor and a continuum of intermediate inputs. The consumption bundle is given by

$$Y = [Y_{u,t}^{\rho} + \eta Y_{s,t}^{\rho}]^{1/\rho}, \quad 0 < \rho < 1; \quad \eta > 1, \tag{2}$$

where the elasticity of substitution between $Y_{u,t}$ and $Y_{s,t}$ is $\frac{1}{1-\rho}$. Equation (2) conveys that while households consume one unit of the unskilled sector good, they consume $\eta > 1$ units of the skilled sector good. The unskilled-sector good could be considered akin to a basic homogenous good consumed for subsistence. The skilled-sector good is a sophisticated variety of goods. The prices of the two final goods are considered to be P_u and P_s respectively. Therefore, the price of the consumption bundle is given by

$$P_Y = \left[P_u^\rho + \eta P_s^\rho\right]^{\frac{1}{\rho}}$$

Laborers earn wages for the labor provided, which is the only source of income for households. Therefore, the budget equation is given by

$$\int_0^\infty P_Y \cdot c_{k,t} dt \leq \int_0^\infty w_{k,t} dt, \tag{3}$$

where w_u and w_s are the wages of the unskilled and skilled laborers respectively. $P_Y \cdot c_{k,t}$ represents the expenditure on consumption by household k at time t, and $w_{k,t}$ represents the wage income of household k at time t. The budget constraint ensures that the total present value of consumption does not exceed the total present value of income. Households maximize utility subject to their respective budget constraints. The Lagrangian for this maximization exercise is given by

$$\mathcal{L} = \int_0^\infty e^{-rt} c_{k,t} dt + \lambda \left[\int_0^\infty w_{k,t} dt - \int_0^\infty P_Y \cdot c_{k,t} dt\right].$$

Taking the derivative of $\mathcal{L}$ with respect to $c_{k,t}$ and setting it to zero, we obtain the first-order condition

$$\frac{\partial \mathcal{L}}{\partial c_{k,t}} = e^{-rt} - \lambda P_Y = 0,$$

which on solving for λ we obtain

$$\lambda = \frac{e^{-rt}}{P_Y}.$$

This indicates that the marginal utility of consumption, discounted over time, equals the marginal cost of consumption.

Since $c_{k,t} = Y$, the household's problem is now to choose $Y_{u,t}$ and $Y_{s,t}$ to maximise their utility. Given that the prices of $Y_{u,t}$ and $Y_{s,t}$ are P_u and P_s respectively, the cost minimization problem can be formulated as

$$min_{Y_{u,t},Y_{s,t}} \quad P_u Y_{u,t} + P_s \eta Y_{s,t},$$

subject to

$$c_{k,t} = \left[Y_{u,t}^\rho + \eta Y_{s,t}^\rho\right]^{1/\rho}.$$

We set-up the Lagrangian as

$$\mathcal{L} = P_u Y_{u,t} + P_s \eta Y_{s,t} + \mu\Big(c_{k,t} - \Big[Y_{u,t}^{\rho} + \eta Y_{s,t}^{\rho}\Big]^{1/\rho}\Big),$$

and the first-order conditions are given by

$$\frac{\partial \mathcal{L}}{\partial Y_{u,t}} = P_u - \mu \cdot \frac{1}{\rho}\Big[Y_{u,t}^{\rho} + \eta Y_{s,t}^{\rho}\Big]^{\frac{1}{\rho}-1} \cdot \rho Y_{u,t}^{\rho-1} = 0,$$

and

$$\frac{\partial \mathcal{L}}{\partial Y_{s,t}} = P_s - \mu \cdot \frac{1}{\rho}\Big[Y_{u,t}^{\rho} + \eta Y_{s,t}^{\rho}\Big]^{\frac{1}{\rho}-1} \cdot \rho\eta Y_{s,t}^{\rho-1} = 0.$$

The above equations can be simplified as

$$P_u = \mu \cdot \Big[Y_{u,t}^{\rho} + \eta Y_{s,t}^{\rho}\Big]^{\frac{1}{\rho}-1} \cdot Y_{u,t}^{\rho-1}, \tag{4}$$

and

$$P_s = \mu \cdot \Big[Y_{u,t}^{\rho} + \eta Y_{s,t}^{\rho}\Big]^{\frac{1}{\rho}-1} \cdot \eta Y_{s,t}^{\rho-1}. \tag{5}$$

respectively. On dividing equations (4) and (5), we obtain the relative demand as

$$\frac{P_u}{P_s} = \frac{\mu \cdot \Big[Y_{u,t}^{\rho} + \eta Y_{s,t}^{\rho}\Big]^{\frac{1}{\rho}-1} \cdot Y_{u,t}^{\rho-1}}{\mu \cdot \Big[Y_{u,t}^{\rho} + \eta Y_{s,t}^{\rho}\Big]^{\frac{1}{\rho}-1} \cdot \eta Y_{s,t}^{\rho-1}},$$

which can be simplified to

$$\frac{P_u}{P_s} = \frac{Y_{u,t}^{\rho-1}}{\eta Y_{s,t}^{\rho-1}},$$

which can be re-written as

$$\frac{P_s}{P_u} = \frac{1}{\eta}\Big[\frac{Y_{u,t}}{Y_{s,t}}\Big]^{1-\rho}.$$

Equation (6) gives us the standard relative demand equation for the two final goods. This relationship helps in understanding how the quantities of unskilled and skilled sector goods, Y_u and Y_s, are chosen given their relative prices, P_u and P_s.

In our subsequent analysis, we have normalized the price of the final good in the skilled sector, P_s, to be 1. In doing so, we simplify the algebra and make it easier to interpret the model's results without a loss of generality. This normalization implies that the price of the unskilled sector good, P_u, and the price of the consumption bundle, P_Y, are expressed relative to the price of the skilled sector good, P_s. Consequently, the relative demand equation derived earlier now simplifies to

$$\frac{1}{P_u} = \frac{1}{\eta}\left[\frac{Y_{u,t}}{Y_{s,t}}\right]^{1-\rho}. \tag{6}$$

And the price of the consumption bundle, P_Y, will now be

$$P_Y = \left[P_u^{\rho} + \eta\right]^{\frac{1}{\rho}}.$$

This completes the formulation of the consumption side of our model.

3 Sectoral Production and Output Dynamics

As stated earlier, the two final goods, $Y_{u,t}$, and $Y_{s,t}$, are competitively produced. The productivity in the unskilled sector is less than the productivity of the skilled sector. This is captured by the condition

$$A_u < A_{is,t} \leq \bar{A}_{s,t},$$

where A_u is the productivity of the unskilled sector, $A_{is,t}$ is the productivity of i^{th} intermediate good in the skilled sector, at time t. $\bar{A}_{s,t}$ is the frontier productivity of the skilled sector at time t. An individual firm, i, in the intermediate sector may be at the frontier of technology, or below it, as will be seen in the section on firm entry and incumbent innovation.

We now state the production technologies for each of these goods.

3.1 The Unskilled Sector

Output in the unskilled sector is produced using only unskilled labor. The production technology is given by

$$Y_{u,t} = A_u L_{u,t} \tag{7}$$

where $L_{u,t}$ is the labor used in the production of the unskilled good, and A_u is the productivity parameter associated with this sector. Note that

$$\frac{\partial Y_{u,t}}{\partial L_{u,t}} = A_u. \tag{8}$$

Equation (8) shows that the production technology in the unskilled sector exhibits constant returns to scale. The profits of the producer in this sector are given by

$$\Pi_u = P_u Y_{u,t} - w_u L_{u,t}, \tag{9}$$

where P_u is the price of the unskilled good and w_u is the wage rate of the unskilled labor. On substituting for $Y_{u,t}$ from equation (7) we obtain

$$\Pi_u = P_u A_u L_{u,t} - w_u L_{u,t}.$$

The profit maximization exercise of the producer yields the first-order condition

$$\frac{\partial \Pi_u}{\partial L_{u,t}} = P_u A_u - w_u = 0,$$

which can also be written as

$$w_u = P_u A_u. \tag{10}$$

Equation (10) gives us the optimal wage rate of the unskilled labor engaged in the production of the unskilled good, $Y_{u,t}$, as a function of the price of the final good of this sector, P_u and the productivity parameter of this sector, A_u.

3.2 The Skilled Sector

Output in the skilled sector is produced using both skilled labor and a continuum of intermediary goods. These intermediary goods can be thought of either as inputs used in the manufacture of the final good of this sector or as machines employed in the production of the final good of this sector.

The final good of this sector can be used either for consumption, or as an input in the process of production of intermediate goods, or as investments in the R&D activity. The production technology in this sector is given by

$$Y_{s,t} = L_{s,t}^{1-\alpha} \int_0^1 A_{is,t}^{1-\alpha}\, x_{i,t}^{\alpha}\, di; \quad 0 < \alpha < 1, \tag{11}$$

where $L_{s,t}$ is the labor employed in this sector. $x_{i,t}$ is the quantity of intermediate good i used in the production of $Y_{s,t}$ and $A_{is,t}$ is the productivity associated with intermediate good i, at time t. This term determines the overall efficiency with which skilled labor and intermediate goods are combined to produce the final output $Y_{s,t}$ of the skilled sector. In any period, the productivity will vary across the intermediate goods, depending on whether it is at or behind the frontier and whether the firm decides to innovate in the time period in consideration. The parameter α signifies the elasticity of substitution between skilled labor and intermediate goods in the production process, which reflects the degree to which skilled labor can be substituted for intermediate goods and vice versa, influencing the sector's production dynamics and cost structure. Specifically, a lower value of α indicates a lower elasticity of substitution, implying that skilled labor and intermediate goods are less substitutable. Conversely, a higher value of α signifies a higher elasticity of substitution, suggesting that these inputs can be more easily interchanged in the production process.

Note that

$$\frac{\partial Y_{s,t}}{\partial L_{s,t}} = (1-\alpha)L_{s,t}^{-\alpha} \int_0^1 A_{is,t}^{1-\alpha}\ x_{i,t}^{\alpha}\ di, \tag{12}$$

and

$$\frac{\partial Y_{s,t}}{\partial x_{i,t}} = \alpha L_{s,t}^{1-\alpha} A_{is,t}^{1-\alpha} x_{i,t}^{\alpha-1}. \tag{13}$$

The profits of the producer in this sector are given by

$$\Pi_s = P_s Y_{s,t} - w_s L_{s,t} - \int_0^1 p_{i,t} x_{i,t} di, \tag{14}$$

where P_s is the price of the skilled good (which we have normalized to 1) and w_s is the wage rate of the skilled labor. $p_{i,t}$ is the price of the intermediate good $x_{i,t}$. On substituting for $Y_{s,t}$ from equation (11), we obtain

$$\Pi_s = L_{s,t}^{1-\alpha} \int_0^1 A_{is,t}^{1-\alpha}\ x_{i,t}^{\alpha}\ di - w_s L_{s,t} - \int_0^1 p_{i,t} x_{i,t} di \tag{15}$$

The profit maximization exercise of the producer yields the first-order conditions

$$\frac{\partial \Pi_s}{\partial L_{s,t}} = (1-\alpha)L_{s,t}^{-\alpha} \int_0^1 A_{is,t}^{1-\alpha}\ x_{i,t}^{\alpha}\ di - w_s = 0,$$

and

$$\frac{\partial \Pi_s}{\partial x_{i,t}} = \alpha \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha} x_{i,t}^{\alpha-1} - p_{i,t} = 0,$$

which can be re-written as

$$w_s = (1-\alpha) L_{s,t}^{-\alpha} \int_0^1 A_{is,t}^{1-\alpha}\ x_{i,t}^{\alpha}\ di, \tag{16}$$

and

$$p_{i,t} = \alpha \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha} x_{i,t}^{\alpha-1}, \tag{17}$$

respectively. Equations (16) and (17) give us the optimal wage rate of the skilled labor, w_s, and optimal price for the intermediate good, $p_{i,t}$. We will re-visit these equations after characterizing and solving the intermediate goods sector.

3.3 The Intermediate Goods Sector

Each intermediate good is uniquely produced by a monopolist in each period, using the final good as the only input. Each unit of the intermediate good uses one unit of the final good of the skilled sector as input. The final output of the skilled sector that is not used for intermediate goods production is available for consumption and research, and it constitutes the economy's gross domestic product (GDP). We will revisit this aspect when we discuss growth in this economy.

There are two types of firms in the intermediate goods sector: advanced and backward. The type of each firm is determined by its proximity to the technological frontier. We will present this discussion in the section on firm entry and incumbent innovation. Irrespective of their proximity to the technology frontier, all intermediate firms have the same production and profit structure. The profits of the intermediate good monopolist are given by

$$\pi_{i,t} = p_{i,t} x_{i,t} - x_{i,t}, \tag{18}$$

where as defined earlier, $p_{i,t}$ is the price of the intermediate good $x_{i,t}$. On substituting for $p_{i,t}$ from the first-order condition of the skilled final good manufacturer given in equation (17), we obtain

$$\pi_{i,t} = \alpha \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha} x_{i,t}^{\alpha-1} \cdot x_{i,t} - x_{i,t},$$

which simplifies to

$$\pi_{i,t} = \alpha \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha} x_{i,t}^{\alpha} - x_{i,t}. \tag{19}$$

The profit maximization exercise by the intermediate monopolist yields

$$\frac{\partial \pi_{i,t}}{\partial x_{i,t}} = \alpha^2 \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha} x_{i,t}^{\alpha-1} - 1 = 0, \tag{20}$$

which can also be expressed as

$$\alpha^2 \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha} x_{i,t}^{\alpha-1} = 1,$$

which can be further expressed as

$$x_{i,t}^{\alpha-1} = \alpha^{-2} \cdot L_{s,t}^{\alpha-1} \cdot A_{is,t}^{\alpha-1},$$

which can be simplified to

$$x_{i,t} = \alpha^{\frac{-2}{\alpha-1}} \cdot L_{s,t} \cdot A_{is,t},$$

which can also be written as

$$x_{i,t} = \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t}. \tag{21}$$

We substitute $x_{i,t}$ obtained from the above equation (21) back in the profit function of the intermediate monopolist, given by equation (19) and obtain

$$\pi_{i,t} = \alpha \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha} \Big[\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t} \Big]^{\alpha} - \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t}.$$

The above equation can be re-expressed as

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t} \Bigg[\alpha \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha} \Big[\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t} \Big]^{\alpha-1} - 1 \Bigg],$$

which can further be written

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t} \Bigg[\alpha \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha} \Big[\alpha^{\frac{2(\alpha-1)}{1-\alpha}} \cdot L_{s,t}^{\alpha-1} \cdot A_{is,t}^{\alpha-1} \Big] - 1 \Bigg],$$

which can be simplified to

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t} \Bigg[\alpha \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha} \Big[\alpha^{-2} \cdot L_{s,t}^{\alpha-1} \cdot A_{is,t}^{\alpha-1} \Big] - 1 \Bigg],$$

which can be further simplified to

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t}\left(\frac{1}{\alpha} - 1\right). \tag{22}$$

Equation (22) gives the profits of the intermediate monopolist as a function of the labor employed in the skilled sector, $L_{s,t}$, the elasticity of substitution between skilled labor and the intermediate goods, α, and the productivity associated with the i^{th} intermediary good, $A_{is,t}$.

We now substitute $x_{i,t}$ obtained in equation (21) in the optimal wage of the skilled labor, given in equation (16).

$$w_s = (1-\alpha)L_{s,t}^{-\alpha} \int_0^1 A_{is,t}^{1-\alpha} \left[\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t}\right]^{\alpha} di,$$

which can be simplified as

$$w_s = (1-\alpha)L_{s,t}^{-\alpha} \int_0^1 A_{is,t}^{1-\alpha} \alpha^{\frac{2\alpha}{1-\alpha}} \cdot L_{s,t}^{\alpha} \cdot A_{is,t}^{\alpha} di,$$

which can further be simplified as

$$w_s = (1-\alpha) \cdot \alpha^{\frac{2\alpha}{1-\alpha}} \cdot L_{s,t}^{-\alpha} \cdot L_{s,t}^{\alpha} \int_0^1 A_{is,t}^{1-\alpha} A_{is,t}^{\alpha} di,$$

which can also be written as

$$w_s = (1-\alpha) \cdot \alpha^{\frac{2\alpha}{1-\alpha}} \int_0^1 A_{is,t} di. \tag{23}$$

We express the above equation (23) as

$$w_s = (1-\alpha) \cdot \alpha^{\frac{2\alpha}{1-\alpha}} A_{s,t}, \tag{24}$$

where

$$A_{s,t} = \int_0^1 A_{is,t} di \tag{25}$$

is the weighted numerical aggregate of all individual productivity parameters in the skilled-good sector.

Substituting $x_{i,t}$ obtained in equation (21) in the optimal price of the intermediate good obtained in equation (17), yields

$$p_{i,t} = \alpha \cdot L_{s,t}^{1-\alpha} \cdot A_{is,t}^{1-\alpha}\left[\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t}\right]^{\alpha-1},$$

which can be simplified to

$$p_{i,t} = \alpha \cdot \alpha^{-2} = \frac{1}{\alpha}. \tag{26}$$

Equation (26) indicates that the price charged by the monopolist producer of the intermediate good is a markup over his marginal cost.

Finally, we substitute $x_{i,t}$ obtained in equation (21) in the final output of the skilled good sector, as given in equation(11), and obtain

$$Y_{s,t} = L_{s,t}^{1-\alpha} \int_0^1 A_{is,t}^{1-\alpha} \left[\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t}\right]^{\alpha} di, \tag{27}$$

which can also be written as

$$Y_{s,t} = L_{s,t}^{1-\alpha} \int_0^1 A_{is,t}^{1-\alpha} \alpha^{\frac{2\alpha}{1-\alpha}} \cdot L_{s,t}^{\alpha} \cdot A_{is,t}^{\alpha} \; di,$$

which can be simplified to

$$Y_{s,t} = \alpha^{\frac{2\alpha}{1-\alpha}} L_{s,t}^{1-\alpha} \cdot L_{s,t}^{\alpha} \int_0^1 A_{is,t}^{1-\alpha} \cdot A_{is,t}^{\alpha} \; di,$$

which can be further simplified as

$$Y_{s,t} = \alpha^{\frac{2\alpha}{1-\alpha}} L_{s,t} \int_0^1 A_{is,t} \; di.$$

We then substitute equation (25), which defines aggregate productivity in the economy, into the above equation and obtain

$$Y_{s,t} = \alpha^{\frac{2\alpha}{1-\alpha}} L_{s,t} A_{s,t} \tag{28}$$

Equation (28) expresses the final output of the skilled-good sector as a function of the labor employed in the skilled sector, $L_{s,t}$, the elasticity of substitution between skilled labor and the intermediate goods, α, and the aggregate productivity of the skilled sector, $A_{s,t}$. This last term determines the overall efficiency with which skilled labor and intermediate goods are combined to produce the final output $Y_{s,t}$ of the skilled sector. Higher values of $A_{s,t}$ indicate greater technological advancement or efficiency gains in production, allowing the sector to produce more output $Y_{s,t}$ for a given input combination of $L_{s,t}$ and $x_{i,t}$. Thus $A_{s,t}$ plays a crucial role in shaping the growth and competitiveness of the skilled sector within the broader economy.

4 Gross Domestic Product and Inequality

We now discuss the gross domestic product (GDP) and inequality in our model. The ensuing analysis of GDP and income equality leads us to recognize the one route through which the planner can address concerns regarding both growth and income inequality, which is the aggregate productivity of the skilled sector, $A_{s,t}$. In this section, we will show how both the GDP and the skill premium, ω_t, are functions of $A_{s,t}$. In the section where we model the politician's maximization exercise, we will present an analysis of how this aggregate productivity can be affected by the firm entry policy that is set by the policymaker.

4.1 Gross Domestic Product

Recall that in each period, the intermediate product is produced by a monopolist using the final good of the skilled good sector as input, one-for-one. The final output of the skilled-good sector, to the extent not used for intermediate production, is available for consumption and research. Therefore, the GDP in this economy is defined as

$$GDP_t = Y_{u,t} + Y_{s,t} - \int_0^1 x_{i,t} di. \tag{29}$$

In the above equation, we substitute the final output of the unskilled sector, Y_u, from equation (7), the final output of the skilled sector, Y_s, from equation (28), and output of the intermediate good sector, $X_{i,t}$, from equation (21) and obtain

$$GDP_t = A_u L_{u,t} + \alpha^{\frac{2\alpha}{1-\alpha}} L_{s,t} A_{s,t} - \int_0^1 \alpha^{\frac{2}{1-\alpha}} L_{s,t} A_{is,t}\ di. \tag{30}$$

Substituting for aggregate productivity from equation (25) and further simplification yields,

$$GDP_t = A_u L_{u,t} + \left[\alpha^{\frac{2}{1-\alpha}}\right]^{\alpha} L_{s,t} A_{s,t} - \alpha^{\frac{2}{1-\alpha}} L_{s,t} A_{s,t},$$

which can also be written as

$$GDP_t = A_u L_{u,t} + \alpha^{\frac{2}{1-\alpha}} L_{s,t} A_{s,t} \left[\left[\alpha^{\frac{2}{1-\alpha}}\right]^{\alpha-1} - 1\right],$$

which can be simplified as

$$GDP_t = G_t = A_u L_{u,t} + \alpha^{\frac{2}{1-\alpha}} L_{s,t} A_{s,t} \Big(\frac{1}{\alpha^2} - 1 \Big). \tag{31}$$

Equation (31) conveys that the GDP is a function of the unskilled labor, $L_{u,t}$, the skilled labor, L_s, the productivity in the unskilled sector, A_u, the aggregate productivity in the skilled sector, $A_{s,t}$, and the share of intermediate good in the production of the skilled good, α.

Note that the GDP is a linear function of the aggregate productivity in the skilled sector, $A_{s,t}$. This becomes of topical interest to us when we model the planner, who will be determining the entry probability, θ, which will affect the aggregate productivity, $A_{s,t}$. This, in turn, affects the GDP of the economy. We will return to his discussion in the relevant section while modeling the planner's maximization problem. Also note that the productivity in the unskilled sector, A_u, is parametrically given and not considered to be evolving over time in our analysis. Therefore, when we differentiate equation (31) with respect to time, we obtain

$$\frac{dG_t}{dt} = A_u \frac{dL_{u,t}}{dt} + \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha^2} - 1 \right) \left[\frac{dL_{s,t}}{dt} A_{s,t} + \frac{dA_{s,t}}{dt} L_{s,t} \right]. \tag{32}$$

Modeling skill acquisition and consequent changes in labor composition would make for an interesting extension of our work. However, in our present analysis, we do not consider changes in the labor composition over time. This means that $\frac{dL_{u,t}}{dt} = 0$ and $\frac{dL_{s,t}}{dt} = 0$. Henceforth, we will also drop the time subscript on labor. Therefore, we will rewrite equation (32) as

$$\frac{dG_t}{dt} = \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha^2} - 1 \right) \frac{dA_{s,t}}{dt} L_s. \tag{33}$$

From equation (33), we see that the only driver of growth in our model would be the growth in the aggregate productivity, $A_{s,t}$ in our model.

4.2 Inequality

In this section, we model inequality. We define skill premium ω_t as

$$\omega_t = \frac{w_s}{w_u}, \tag{34}$$

which means ω_t indicates how much more skilled labor earns compared to unskilled labor. A higher ω_t signifies a greater wage differential, implying that skilled labor is relatively more valuable or scarce, compared to unskilled labor. Substituting from equation (10) for w_u and from equation (24) for w_s, we obtain

$$\omega_t = \frac{(1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} A_{s,t}}{P_u A_u},$$

which can be re-written as

$$\omega_t = \frac{(1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} A_{s,t}}{A_u} \cdot \frac{1}{P_u}. \tag{35}$$

Recall from equation (6) that $\frac{1}{P_u} = \frac{1}{\eta}\left[\frac{Y_{u,t}}{Y_{s,t}}\right]^{1-\rho}$. In this equation (6), we substitute for $Y_{u,t}$ and $Y_{s,t}$ from equations (7) and (28) respectively and obtain

$$\frac{1}{P_u} = \frac{1}{\eta}\left[\frac{A_u L_u}{\alpha^{\frac{2\alpha}{1-\alpha}} L_s A_{s,t}}\right]^{1-\rho}. \tag{36}$$

Note from equation (35) that as the productivity of the unskilled sector, A_u, increases, the relative prices, $\frac{1}{P_u}$ increases, which means P_u decreases relative to $P_s = 1$. Similarly, with an increase in the productivity of the skilled sector, $A_{s,t}$, the relative prices, $\frac{1}{P_u}$ decreases, which means that P_u increases relative to $P_s = 1$. These results align with the intuition that an increase in the productivity of unskilled labor, A_u, makes the unskilled sector good cheaper relative to the skilled sector good, thus increasing $\frac{P_s}{P_u} = \frac{1}{P_u}$. Conversely, an increase in the productivity of skilled labor, $A_{s,t}$, makes the skilled sector good relatively cheaper, thus decreasing $\frac{P_s}{P_u} = \frac{1}{P_u}$. Substituting equation (36) in equation (35) above, we obtain

$$\omega_t = (1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} \cdot \frac{A_{s,t}}{A_u} \cdot \frac{1}{\eta}\left[\frac{A_u L_u}{\alpha^{\frac{2\alpha}{1-\alpha}} L_s A_{s,t}}\right]^{1-\rho}, \tag{37}$$

which can be re-written as

$$\omega_t = (1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} \cdot \left(\alpha^{\frac{-2\alpha}{1-\alpha}}\right)^{1-\rho} \cdot \frac{1}{\eta} \cdot \frac{A_{s,t}}{A_u} \cdot \left[\frac{A_{s,t}}{A_u}\right]^{\rho-1} \cdot \left[\frac{L_u}{L_s}\right]^{1-\rho},$$

which can be simplified to

$$\omega_t = (1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha} - \frac{2\alpha(1-\rho)}{1-\alpha}} \cdot \frac{1}{\eta} \cdot \left[\frac{A_{s,t}}{A_u}\right]^{1+\rho-1} \cdot \left[\frac{L_u}{L_s}\right]^{1-\rho},$$

which can further be simplified to

$$\omega_t = (1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}} \cdot \frac{1}{\eta} \cdot \left[\frac{A_{s,t}}{A_u}\right]^{\rho} \cdot \left[\frac{L_u}{L_s}\right]^{1-\rho}. \tag{38}$$

Equation (38) expresses the skill premium, ω_t, as a function of the elasticity of substitution between skilled labor and the intermediate goods, α, the elasticity of substitution between the unskilled and skilled goods in the consumption bundle, ρ, the consumer's preference for the skilled good, η, the unskilled labor force, L_u, the skilled labor force, L_s, the productivity in the unskilled sector, A_u, and the productivity in the skilled sector, $A_{s,t}$.

When α is high, skilled labor and intermediate goods are highly substitutable. This means that firms can adjust their production processes more easily by either using more skilled labor or more intermediate goods, depending on relative prices and productivity. Therefore, firms may substitute skilled labor with intermediate goods more readily when the cost of skilled labor rises relative to intermediate goods. This could potentially lower the demand for skilled labor and thus reduce ω_t. Conversely, if the cost of intermediate goods rises relative to skilled labor, firms may substitute intermediate goods with skilled labor, potentially increasing ω_t. On the other hand, when α is low, skilled labor and intermediate goods are less substitutable. This implies that skilled labor is more specialized or unique in its contribution to production, making it more valuable relative to intermediate goods. This leads to higher wages for the skilled labor and thus a higher value for ω_t, the skill premium.

A higher elasticity of substitution between the unskilled and skilled goods in the consumption bundle, ρ, implies that consumers are more likely to switch their consumption towards the cheaper (in terms of relative productivity) option, thereby affecting the wage premium. A higher value of ρ also increases the sensitivity of ω_t to changes in the productivity ratio, $\frac{A_{s,t}}{A_u}$. The converse is the case with a lower value of ρ. A higher value of η, which is the consumers' preference for the skilled good relative to the unskilled good, indicates a stronger demand for skilled labor, thereby influencing wages.

Recall that $0 < \rho < 1$. Therefore, the term $(1-\rho)$ is positive. When there is a higher supply of skilled labor force relative to unskilled labor force increases, $\frac{L_u}{L_s}$ will be higher. From equation (38) this means that it leads to lower wages for unskilled workers compared to skilled workers, decreasing ω_t. Similarly, if the supply of unskilled labor force relative to the skilled labor force increases, the skill premium also increases.

Finally, if the aggregate productivity of the skilled sector, $A_{s,t}$ increases relative to the productivity of the unskilled sector, A_u, that is to say, for a higher $\frac{A_{s,t}}{A_u}$, skilled workers are more productive compared to unskilled workers. This productivity advantage can lead to higher wages for skilled workers relative to unskilled workers, thereby increasing the skill premium, ω_t. The converse will be true if A_u increases relative to $A_{s,t}$.

5 Firm Entry and Incumbent Innovation

In this section, we discuss how firms in the intermediate goods sector make their innovation decisions in the light of a threat posed by the entry of a new firm. Let θ be the probability that a potential entrant shows up in the intermediate goods sector. In a subsequent section, this parameter, θ, will be chosen by the policymaker. In doing so, the policymaker not only exhibits a concern for the overall GDP and income inequality in the economy but is also subject to influence by incumbent firms that want to prevent entry.

As stated earlier, within the intermediate sector, there are two types of firms, advanced firms and backward firms, depending on their proximity to the technological frontier, $\bar{A}_{s,t}$, which represents the highest productivity that firms in the intermediate sector can achieve.

Advanced firms are defined as those operating at the frontier level of productivity. In other words, the productivity level of an advanced firm is equal to the highest possible productivity level in the skilled sector at time t. This relationship can be expressed as

$$A_{is,t} = \bar{A}_{s,t}, \tag{39}$$

where $\bar{A}_{s,t}$ represents the frontier productivity level at time t.

On the other hand, backward firms are defined as those operating below the frontier level of productivity. In other words, the productivity level of a backward firm lags behind the highest possible productivity level in the skilled sector at time t, by a factor $\frac{1}{\gamma}$. This relationship is expressed by

$$A_{is,t} = \frac{1}{\gamma}\bar{A}_{s,t}; \quad \gamma > 1 \tag{40}$$

where $\gamma > 1$ is the constant rate at which the frontier productivity level grows over time. Therefore,

$$\bar{A}_{s,t} = \gamma \bar{A}_{s,t-1}. \tag{41}$$

We assume that a potential entrant is always at the technological frontier. When, at time t, an incumbent firm that was at the frontier in time $t-1$ faces the threat of a potential entrant arriving with leading-edge technology, it uses its first mover advantage to block entry and consequently retain its monopoly power. However, if entry occurs and the incumbent firm fails to reach the new technological frontier at time t, the technologically superior new entrant will replace the incumbent in the ensuing Bertrand competition.

For an incumbent firm, we define the cost of investing in research and development (R&D) activity at time t, as

$$c_{i,t} \cdot A_{is,t-1}, \tag{42}$$

where $A_{is,t-1}$ is the incumbent's pre-innovation productivity. We assume $c_{i,t}$ to be random and independently and identically distributed across intermediate sectors. $c_{i,t}$ can take two values

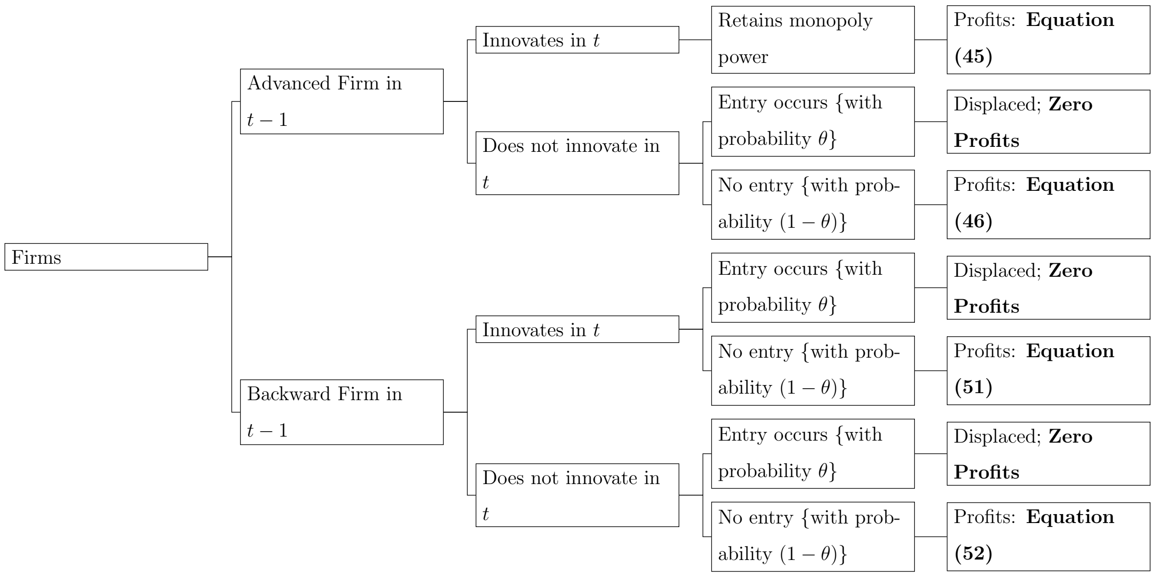
$$c_{i,t} \in \{0, \bar{c}\}, \tag{43}$$

and

$$Pr(c_{i,t} = 0) = Pr(c_{i,t} = \bar{c}). = \frac{1}{2} \tag{44}$$

How incumbent firms react to an entry threat, θ, depends on the marginal benefit that they expect to receive from an innovation, given $c_{i,t}$ and θ. This also varies

Figure 1: Payoffs of Incumbent Firms



depending on whether the incumbent is an advanced or a backward firm. In our subsequent sub-sections, we analyze how firms make these decisions.

The payoffs of incumbent firms, both advanced and backward, contingent upon whether they innovate and whether entry occurs, are summarised in Figure 1.

5.1 Innovation by Advanced Incumbent Firms

We first consider firms that were at the frontier level of technology in the previous time period $t-1$. Accordingly, their productivity level will be

$$A_{is,t-1} = \bar{A}_{s,t-1}.$$

If, in the current time period, t, this firm innovates, it will remain at the frontier in this period, too. This will make it immune to potential entry by an advanced firm. Upon successful innovation, an advanced firm can earn gross profits (i.e., before deducting R&D costs) equal to

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot \bar{A}_{s,t} \left(\frac{1}{\alpha} - 1 \right). \tag{45}$$

Note that this is the same as equation (22) except that the productivity of this firm will be at the frontier, $A_{is,t} = \bar{A}_{s,t}$.

On the other hand, if this firm does not innovate in time period t, then with probability θ, it will be eliminated by a potential entrant and consequently make zero profits. However, with probability $(1-\theta)$, it will survive the entry threat and thereby make a profit of

$$\pi_{i,t} = (1-\theta)\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot \bar{A}_{s,t-1}\left(\frac{1}{\alpha} - 1\right). \tag{46}$$

Given these two possibilities, an advanced firm facing a cost of innovation $c_{i,t}\bar{A}_{s,t-1}$ will innovate only when the incremental benefit from the innovation is greater than the cost of innovation itself. This condition is given by

$$\underbrace{\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot \bar{A}_{s,t}\left(\frac{1}{\alpha} - 1\right)}_{(a)} - \underbrace{(1-\theta)\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot \bar{A}_{s,t-1}\left(\frac{1}{\alpha} - 1\right)}_{b} > \underbrace{c_{i,t}\bar{A}_{s,t-1}}_{(c)}. \tag{47}$$

In equation (47) above, term (a) is the benefit obtained by the firm by innovating. Term (b) is the benefit obtained by the firm by not innovating. Term (c) is the cost of innovation. By substituting from equation (39), the above equation can be re-written as

$$\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot \gamma\bar{A}_{s,t-1}\left(\frac{1}{\alpha} - 1\right) - (1-\theta)\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot \bar{A}_{s,t-1}\left(\frac{1}{\alpha} - 1\right) > c_{i,t}\bar{A}_{s,t-1},$$

and by collecting like terms, it can be simplified to

$$\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot \bar{A}_{s,t-1}\left(\frac{1}{\alpha} - 1\right)\Big[\gamma - (1-\theta)\Big] > c_{i,t}\bar{A}_{s,t-1},$$

which on cancelling out the term $\bar{A}_{s,t-1}$ can further be simplified to

$$\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha} - 1\right) \cdot L_{s,t}\Big[\gamma - 1 + \theta\Big] > c_{i,t}. \tag{48}$$

Equation (48) captures the condition for an advanced firm to innovate, given the entry probability and cost of innovation. The left-hand side of the inequality represents the marginal benefit of innovation for the firm. The term $\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha} - 1\right) \cdot L_{s,t}$ captures the combined effect of the elasticity of substitution, the amount of skilled labor, and the frontier productivity level. The factor $\big[\gamma - 1 + \theta\big]$ adjusts this benefit based on the growth rate of the technological frontier and the entry probability. The right-hand side of the inequality represents the cost of innovation for the firm. For the firm to decide to innovate, the marginal benefit of innovation must exceed the cost of innovation, that is, equation (48) should hold true.

From equation (48) it can be seen that with an increase in the entry threat, θ, the term $\left[\gamma - 1 + \theta\right]$ increases, which means that the marginal benefit of innovation increases as the threat of new entrants becomes more significant. This encourages the advanced incumbent firm to innovate. On the other hand, when θ reduces, the term $\left[\gamma - 1 + \theta\right]$ reduces, which means that the marginal benefit of innovation decreases as the threat of new entrants becomes less significant. This reduces the incentive for firms to innovate since the risk of displacement by new entrants is lower. Consequently, fewer firms will find it beneficial to innovate, especially if the cost, $c_{i,t}$, is high.

Intuitively, a firm that is at the technological frontier in the previous time period $t-1$, responds to an entry threat in time period t, by innovating and thereby escaping the threat of displacement due to entry. When the probability of new entrants, θ, is high, the competitive pressure motivates incumbent firms to invest in innovation to maintain their leading position. By advancing their technology, these firms can push the frontier further, securing their market dominance and mitigating the risk of being outcompeted by new entrants. Conversely, when θ is low, the likelihood of new entrants is minimal, reducing the immediate threat to the incumbent firms' market position. In such a scenario, the urgency to innovate diminishes, as the lower expected gains may not justify the costs of innovation. This potentially leads to slower technological advancement in the long run. Understanding this dynamic highlights the critical role of entry threats in driving innovation and shaping the competitive landscape.

If this firm does innovate, its profits, net of R&D cost will be

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot \bar{A}_{s,t} - c_{i,t} \bar{A}_{s,t-1}. \tag{49}$$

And substituting equation (39) in the equation (49) above, we obtain

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot \gamma \bar{A}_{s,t-1} - c_{i,t} \bar{A}_{s,t-1},$$

which on collecting like terms, can be simplified to

$$\pi_{i,t} = \left[\alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot \gamma \cdot L_{s,t} - c_{i,t} \right] \bar{A}_{s,t-1}. \tag{50}$$

Equation (50) captures the profits made by an advanced incumbent firm, given θ and $c_{i,t}$.

5.2 Innovation by Backward Incumbent Firms

In this subsection, we consider an incumbent firm that was a backward firm in the previous time period, $t-1$. Accordingly, their productivity will be

$$A_{is,t-1} = \frac{1}{\gamma} \bar{A}_{s,t-1}.$$

Such a firm will remain backward even if it innovates in the current time period, t, since the technological frontier would also have advanced by γ in the current time period. Therefore, this firm will make zero profits if entry occurs with probability θ in time period t, irrespective of whether it innovates. Entry, which is always of a technologically advanced firm, will displace this firm.

On the other hand, if an entry does not occur, the probability for which is given by $(1-\theta)$, the firm does survive and make profits. Note that this is the only case where the firm can make any profits.

If the firm innovates, and entry does not occur, its gross profits (i.e., before deducting R&D costs) will be

$$\pi_{i,t} = \underbrace{\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t} \left(\frac{1}{\alpha} - 1 \right)}_{(a)} \underbrace{\left[1 - \theta \right]}_{(b)}. \tag{51}$$

Note that term (a) in equation (51) is the same as equation (22). The productivity parameter $A_{is,t}$ indicates that this firm has innovated in the current time period but is still not at the technological frontier. Term (b) is the probability that an entry does not occur in the intermediate good sector.

On the other hand, if this firm does not innovate and entry does not occur, its profits will be

$$\pi_{i,t} = \underbrace{\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t-1}\left(\frac{1}{\alpha} - 1\right)}_{(a)}\underbrace{\Big[1-\theta\Big]}_{(b)}. \tag{52}$$

As earlier, term (a) in equation (51) is the same as equation (22). The difference is that the productivity parameter $A_{is,t-1}$ indicates that this firm has not innovated in the current time period but is still not at the technological frontier. Term (b) is, as earlier, the probability that an entry does not occur in the intermediate good sector.

A backward firm, with innovation cost $c_{i,t}A_{is,t-1}$, will innovate only when the incremental benefit from innovation exceeds the innovation cost. This is captured by the condition

$$\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t}\left(\frac{1}{\alpha}-1\right)\Big[1-\theta\Big] - \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t-1}\left(\frac{1}{\alpha}-1\right)\Big[1-\theta\Big] > c_{i,t}A_{is,t-1}. \tag{53}$$

Recall that $A_{is,t} = \gamma A_{is,t-1}$, which we substitute for in the above equation and obtain

$$\alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot \gamma A_{is,t-1}\left(\frac{1}{\alpha}-1\right)\Big[1-\theta\Big] - \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t-1}\left(\frac{1}{\alpha}-1\right)\Big[1-\theta\Big] > c_{i,t}A_{is,t-1},$$

which on the collection of like terms can be written as

$$\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right) \cdot L_{s,t} \cdot A_{is,t-1}\Big[1-\theta\Big]\Big[\gamma-1\Big] > c_{i,t}A_{is,t-1},$$

which on canceling out like terms, can be simplified as

$$\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)L_{s,t}\Big[1-\theta\Big]\Big[\gamma-1\Big] > c_{i,t}. \tag{54}$$

Equation (54) captures the condition under which a backward firm innovates, given θ and $c_{i,t}$. It can be noticed that the backward firm's incentive to innovate has a negative relationship with the entry probability θ. Intuitively, this means that since the firm is far below the technological frontier, it is not going to survive an entry threat, irrespective of whether it innovates. Therefore, it is discouraged from innovating if the entry threat increases because it cannot escape the displacement caused by the entry of a technologically advanced entrant.

If the firm does innovate and survive the entry threat, its profit net of R&D investment will be

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot A_{is,t} \left(\frac{1}{\alpha} - 1 \right) \Big[1 - \theta \Big] - c_{i,t} A_{is,t-1}.$$

In the above equation, we substitute for $A_{is,t} = \gamma A_{is,t-1}$, and obtain

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \cdot L_{s,t} \cdot \gamma A_{is,t-1} \left(\frac{1}{\alpha} - 1 \right) \Big[1 - \theta \Big] - c_{i,t} A_{is,t-1},$$

which on collecting like terms, can also be written as

$$\pi_{i,t} = \left[\alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) L_{s,t} \cdot \gamma \Big[1 - \theta \Big] - c_{i,t} \right] A_{is,t-1}. \tag{55}$$

Equation (55) captures the profits made by a backward firm, net of R&D investment, if it innovates, given θ and $c_{i,t}$.

6 The Steady-State Share of Advanced Firms

In this section, we characterize the steady-state fraction of advanced firms in our model. We make the following two assumptions before we proceed to characterize the steady state.

Assumption 1. We assume that initially, the entry threat, θ, is zero.

Assumption 1 allows for the analysis of firms' natural inclination towards innovation in an environment free from the pressure of potential new entrants. It helps to identify the innovation behavior of firms solely driven by their internal cost structures and productivity levels without external competitive pressure. Without the entry threat, firms' decisions to innovate are influenced only by their innovation costs, denoted by $c_{i,t}$, and the existing technological frontier.

We also make the following additional assumption.

Assumption 2. Absent any entry threat, that is to say, when $\theta = 0$, no firm with innovation cost equal to $\bar{c}$, ever innovates. This assumption takes the form

$$(\gamma - 1)\alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} < \bar{c}$$

Assumption 2 mathematically expresses that the expected benefit of innovation, when there is no threat of new entrants, is less than the cost of innovation for firms with the highest cost, $\bar{c}$. This assumption effectively underscores that, without the pressure of potential new entrants, the expected returns from innovation are insufficient to justify the expenditure for firms facing the highest cost of innovation.

Assumption 2 serves as a critical baseline for analyzing the effects of policy-induced entry threats on firm innovation behavior. By stating that no firm with the highest innovation cost, $\bar{c}$, will innovate in the absence of an entry threat($\theta = 0$), we delineate the conditions under which firms are inert in terms of innovation. This baseline helps in contrasting the innovation behavior when entry threats, ($\theta > 0$), are introduced.

Assumption 2 also sets the stage for exploring the impact of policy interventions on innovation. If policymakers increase the probability of entry, θ, they alter the cost-benefit analysis for incumbent firms. This increases threat, $\theta > 0$, creates an additional incentive for firms to innovate, as failure to do so could result in loss of market position and profits. We thereby demonstrate how policy-induced competition can stimulate innovation, especially among firms that would otherwise remain stagnant due to high innovation costs.

We make use of these two assumptions to determine the steady-state fraction of advanced firms in the economy, conditional upon $\theta = 0$. Let β_t denote the fraction of advanced firms at time period t. Suppose that an advanced firm that successfully innovates at date t starts out in time period $t + 1$ as an advanced firm, and all other firms start out as backward firms. Additionally, with exogenous probability ε, a backward firm at the end of time period t is replaced by a new, advanced firm at time period $t + 1$. The dynamic equation governing the fraction of advanced firms can be written as

$$\underbrace{\beta_{t+1}}_{(a)} = \underbrace{z_A \beta_t}_{(b)} + \underbrace{\varepsilon(1 - z_A \beta_t)}_{c}, \tag{56}$$

where $z_A = Pr(c = 0) = \frac{1}{2}$, is the probability that an advanced firm innovates if $\theta = 0$. Recall that as per assumption 2, firms do not innovate if $\theta = 0$ and $c \neq 0$. In equation (56), term (a) is the fraction of advanced firms in time period $t + 1$. The term (b) is the fraction of advanced firms that started out as advanced firms and have successfully innovated and continued as advanced firms in time period $t+1$. Term (c) is the fraction of backward firms that have been replaced by new advanced firms. By substituting for $z_A = \frac{1}{2}$, and the fact that in the steady-state, $\beta_{t+1} = \beta_t = \beta^*$, the steady-state fraction of advanced firms is given by

$$\beta = \frac{1}{2}\beta + \varepsilon\Big(1 - \frac{1}{2}\beta\Big),$$

which can be simplified to

$$\beta - \frac{1}{2}\beta + \frac{\varepsilon}{2}\beta = \varepsilon,$$

which can further be simplified to

$$\beta\Big(1 - \frac{1}{2} + \frac{\varepsilon}{2}\Big) = \beta\Big(1 - \frac{1}{2}(1-\varepsilon)\Big) = \beta\Big(\frac{2-1+\varepsilon}{2}\Big) = \beta\Big(\frac{1+\varepsilon}{2}\Big) = \varepsilon.$$

Therefore, the steady-state fraction of advanced firms in the economy is given by

$$\beta^* = \Big(\frac{2\varepsilon}{1+\varepsilon}\Big) \tag{57}$$

Note from equation (57) that the steady-state fraction of advanced firms, β^*, is determined based on the exogenous parameter ε, representing the probability of a backward firm being replaced by an advanced firm. This abstraction allows us to focus our analysis on understanding the nuanced dynamics of firm behavior under varying conditions of entry threat, $\theta > 0$. By abstracting the determination of β^* as exogenous, we establish a framework to explore the direct implications of policy interventions on firm behavior. This approach enables us to isolate and analyze the causal relationships between policy decisions, entry threats, and innovation incentives. The main focus of our study is to isolate and study the impact of firm entry policy set by the policymaker (determining θ) on firm behavior. The firm entry policy, θ, is set by the policymaker who not only pays attention to increasing the output (GDP) in the economy and reducing income inequality (ω_t) but also succumbs to bribes offered by incumbent firms to restrict entry.

With the steady-state share of advanced firms being given by equation (57), the aggregate productivity of the skill sector, when there is no entry threat faced by incumbent firms (i.e. $\theta = 0$), is given by

$$A_{s,t} = \beta^* \bar{A}_{s,t} + (1 - \beta^*) \cdot \frac{1}{\gamma} \bar{A}_{s,t}$$

This equation illustrates how the aggregate productivity, $A_{s,t}$, in the skilled sector is determined by the productivity levels of both advanced and backward firms in the steady state. Specifically, $\bar{A}_{s,t}$ represents the frontier productivity level, and γ is the factor by which backward firms lag behind the frontier. The term $\beta^* \bar{A}_{s,t}$ represents the contribution to aggregate productivity from the advanced firms, which operate at the frontier productivity level. The term $(1 - \beta^*) \cdot \frac{1}{\gamma} \bar{A}_{s,t}$ represents the contribution from the backward firms, which operate at a productivity level of $\frac{1}{\gamma} \bar{A}_{s,t}$.

In the absence of an entry threat ($\theta = 0$), the steady-state share of advanced firms, β^*, determines the proportion of firms at the frontier. The aggregate productivity is thus a weighted average of the productivity levels of the advanced and backward firms. The weight for the advanced firms is β^*, while the weight for the backward firms is $1 - \beta^*$, adjusted by their relative productivity level, $\frac{1}{\gamma}$. This equation highlights the impact of the distribution of firms' productivity levels on the overall productivity of the skilled sector.

7 Introducing the Policymaker

We now introduce a policymaker who sets the firm entry policy, θ in each period. On the one hand, the policymaker responds to bribes offered by incumbent firms to restrict entry. On the other hand, he is also concerned about reducing the wage gap between the skilled and unskilled labor force, ω_t[1]. In this section, we now proceed to show how each of these variables - the bribes offered by incumbent firms, the GDP, and the skill premium, ω_t, can be expressed as functions of the firm entry policy, θ.

[1] At the end of this section, it will be shown that both bribes and GDP are linear functions of θ. Therefore, we abstract away from considering an increase in output as an additional concern for the policymaker since a policymaker that sets an entry policy that maximizes his bribes is automatically also maximizing the output. However, while comparing the impact of the entry policy, we consider both the GDP and the skill premium, which is a measure of inequality in the economy.

We will then proceed to set the policymaker's objective function where he sets the entry policy, θ.

7.1 Bribes Offered by Incumbent Firms

In this section, we compute the total bribes that incumbent firms will be willing to pay to the policymaker to prevent him from moving from an initial entry probability of $\theta = 0$ to $\theta > 0$. To compute these bribes, we first need to compute the total payoffs of each type of firm, and for each cost realization - $c_{i,t} = 0$ or $c_{i,t} = \bar{c}$. We show that these payoffs are functions of the firm entry policy, θ.

The bribes offered by firms are summarized in Figure 2.

7.1.1 Bribes by Advanced Incumbent Firms

Consider first an incumbent intermediary firm that was advanced in the previous time period. Recall that we have specified two possibilities for the cost of innovation: $c_{i,t} = 0$ or $C_{i,t} = \bar{c}$.

Case 1: Cost of innovation, $c_{i,t}$, is zero.

If the innovation cost in time period t is $c_{i,t} = 0$, an incumbent intermediary firm will always innovate, irrespective of the entry threat, since, by doing so, it will make itself immune to the entry threat. Therefore, for an advanced incumbent firm facing $c_{i,t} = 0$, the post-innovation profit will be independent of the entry probability. From equation (50), we know that the profits of an advanced incumbent firm are given by

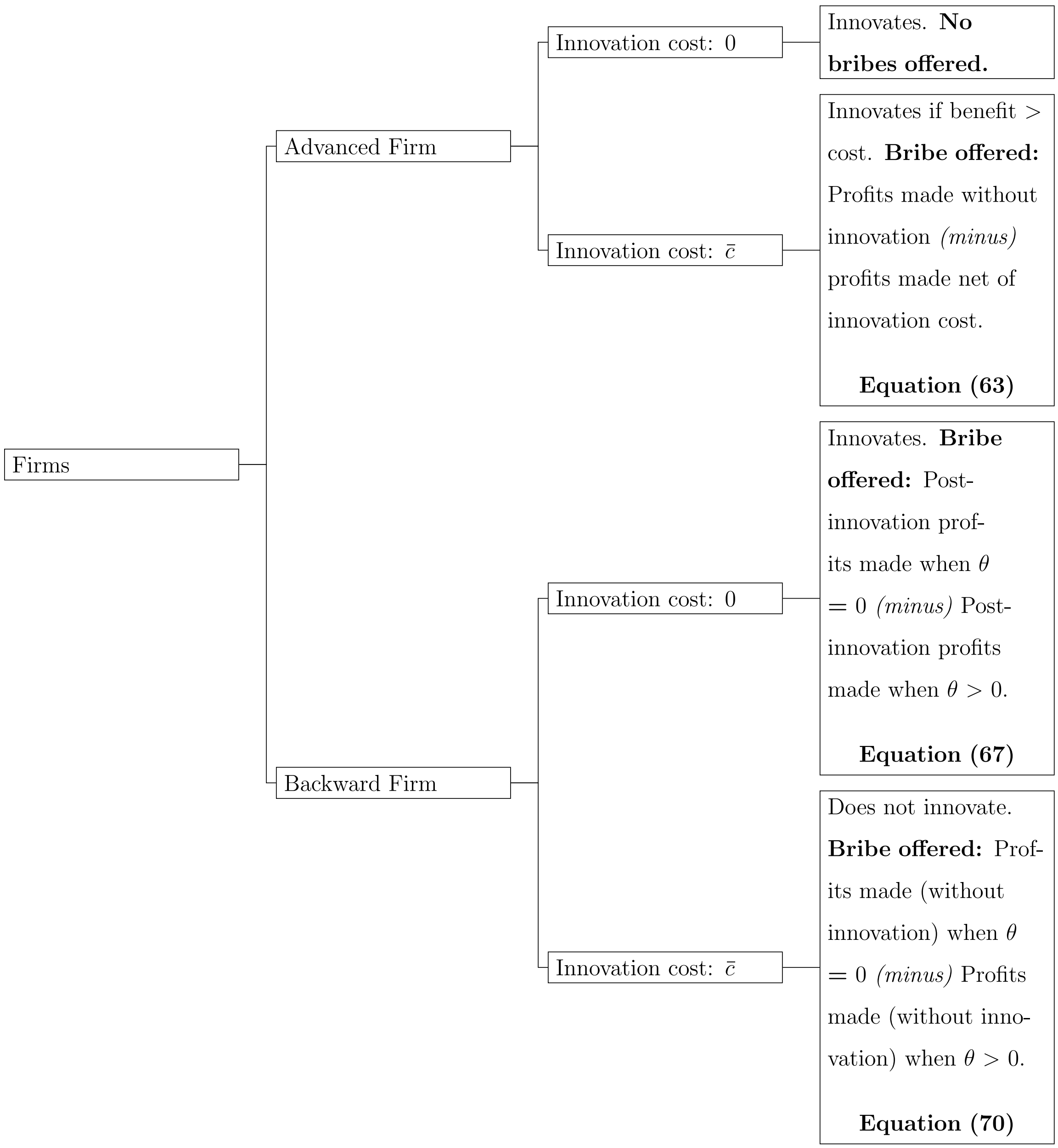
$$\pi_{i,t} = \left[\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha} - 1\right) \cdot \gamma \cdot L_{s,t} - c_{i,t}\right]\bar{A}_{s,t-1}.$$

Since $c_{i,t} = 0$, the profit for this type of firm would reduce to

$$\pi_{i,t} = \left[\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha} - 1\right) \cdot \gamma \cdot L_{s,t}\right]\bar{A}_{s,t-1}. \tag{58}$$

Essentially, if a firm is advanced in the previous period, and if innovation is costless in the current period, it has no incentive to bribe the policymaker. This is because

Figure 2: Bribes Offered by Incumbent Firms



the firm can secure its position and maintain its profits simply by innovating. There is no additional benefit to be gained from bribing the policymaker because the firm's profits are already maximized through innovation.

Case 2: Cost of innovation, $c_{i,t}$, is $\bar{c}$.

An incumbent intermediary firm facing cost, $c_{i,t} = \bar{c}$, will innovate only if the entry threat, θ, becomes sufficiently high that equation (48) holds, which means that the post-innovation profits, given the entry threat, θ, exceeds the cost of innovation. If it does innovate, it loses a chunk of its profits compared to its pre-innovation profits. Therefore, in time period t, the maximum bribe that an incumbent intermediary firm that was advanced in time period $t-1$, facing cost, $c_{i,t} = \bar{c}$, would be willing to pay to the policymaker will be the difference between the profits it would have made without innovation and the profits (net of innovation cost) that it would be making if it innovates.

The pre-innovation profits of an incumbent intermediary firm are given by equation (28). Note that for an incumbent firm that was advanced in the previous period, $t-1$, and has chosen not to innovate in the current time period, t, the productivity parameter in time period t would be $A_{is,t} = \bar{A}_{s,t-1}$. Therefore, we substitute for this fact in equation (28) to obtain the current period pre-innovation profit of a firm that was advanced in the previous period and has not chosen to innovate in the current period. This is given by

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot \bar{A}_{s,t-1}. \tag{59}$$

On the other hand, if the incumbent intermediary firm chooses to innovate, its profits will be given by equation (50). Substituting for $c_{i,t} = \bar{c}$ in equation (50), we obtain

$$\pi_{i,t} = \left[\alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot \gamma \cdot L_{s,t} - \bar{c} \right] \bar{A}_{s,t-1}. \tag{60}$$

The maximum bribe that an incumbent firm that was advanced in time period $t-1$, which faces an entry threat $\bar{c}$, would be willing to pay to the policymaker would therefore be the difference between equations (59) and (60). We express it as

$$\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)\cdot L_{s,t}\cdot\bar{A}_{s,t-1}-\left[\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)\cdot\gamma\cdot L_{s,t}-\bar{c}\right]\bar{A}_{s,t-1}, \tag{61}$$

which can be simplified as

$$\bar{A}_{s,t-1}\left[\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)\cdot L_{s,t}\big(1-\gamma\big)+\bar{c}\right],$$

which can also be written as

$$\bar{A}_{s,t-1}\underbrace{\left[\bar{c}-\big(\gamma-1\big)\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)\cdot L_{s,t}\right]}_{(a)}. \tag{62}$$

Note that by assumption 2, the term (a) in equation (62) is positive. Equation (62) gives us the maximum bribe that each incumbent firm that was advanced in the previous period will be willing to pay to the policymaker to prevent entry in time period t. Since the fraction of advanced firms in the steady state is given by β^*, and the probability that the cost of innovation is $\bar{c}$, is $Pr(c_{i,t}=\bar{c})=\frac{1}{2}$, the total bribes offered by all incumbent firms that were advanced in the previous period and face an innovation cost of $\bar{c}$, that want the policymaker to restrict entry of new technologically advanced firms in time period t, will be

$$B_{a,t}=\bar{A}_{s,t-1}\cdot\beta^*\cdot\frac{1}{2}\left[\bar{c}-\big(\gamma-1\big)\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)\cdot L_{s,t}\right]\cdot 1_{\Phi},$$

which on substituting for the fact that $\bar{A}_{s,t-1}=\frac{1}{\gamma}\bar{A}_{s,t}$, can also be written as

$$B_{a,t}(\theta)=\frac{1}{\gamma}\bar{A}_{s,t}\cdot\beta^*\cdot\frac{1}{2}\left[\bar{c}-\big(\gamma-1\big)\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)\cdot L_{s,t}\right]\cdot 1_{\Phi}. \tag{63}$$

where the term 1_{Φ} is equal to 1 if equation (48) holds, that is the condition for an advanced firm to innovate, given the entry probability and cost of innovation, holds. If this condition does not hold, the term 1_{Φ} would take the value zero, and the total bribes given by these firms would also be zero since they would not be engaging in innovation.

Note from equation (63) that the total bribes that advanced incumbent firms as a group would pay to the policymaker when faced with an innovation cost of $\bar{c}$ in the current time period is not entirely independent of the entry threat, θ. Essentially, the entry threat, θ, impacts the firm's threshold of innovation decision, as given in equation (48), but not the resultant profit differential, which is used to determine the bribes. Once the advanced firms decide to innovate so as to stay at the frontier and avoid displacement, their post-innovation profit becomes a constant factor, not influenced by the entry threat θ. It has already been seen in equation (48) that a higher entry threat, θ, encourages advanced incumbent firms to innovate.

7.1.2 Bribes by Backward Incumbent Firms

We now consider incumbent intermediary firms that were backward in the previous time period, $t-1$. Such firms will innovate in time period t if and only if their cost of innovation is zero, that is, $c_{i,t} = 0$, irrespective of their entry threat. This is because even if a firm that was backward in time period $t-1$ innovates in time period t, it will still remain backward in time period t and will be displaced when entry occurs in time period t.

Case 1: Cost of innovation, $c_{i,t}$, is zero.

Consider, first, the scenario that it faces an innovation cost of $c_{i,t} = 0$. In such a scenario, the incumbent firm that was backward in the previous period would choose to innovate since innovation is costless. However, it would survive in time period t only if entry threat $\theta = 0$. Therefore, such a firm has an incentive to bribe the policymaker to restrict entry. The backward incumbent firm would be willing to pay a maximum bribe of the post-innovation profits that it would be foregoing when moving from $\theta = 0$ to $\theta > 0$. This means that the maximum bribe that this firm would be willing to pay to the policymaker would be the difference between the profits that it makes when it innovates and $\theta = 0$, and the profits that it would make when it innovates and $\theta > 0$. The post-innovation profit of an incumbent backward firm is given by equation (55). When not faced by an entry threat, that is $\theta = 0$, and when the cost of innovation, $c_{i,t} = 0$, this would be suitably modified as written as

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) L_{s,t} \cdot \gamma A_{is,t-1}. \tag{64}$$

On the other hand, when the entry threat is $\theta > 0$, if a backward firm innovates, its post-innovation profits are given by equation (55). When considering the fact that in the present case $c_{i,t} = 0$, equation (55) is suitably modified and written as

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) L_{s,t} \cdot \gamma A_{is,t-1} \Big[1 - \theta \Big]. \tag{65}$$

As mentioned earlier, the maximum bribe that a firm facing $c_{i,t} = 0$ would be willing to pay would be the difference in post-innovation profits when $\theta = 0$ and when $\theta > 0$. This is given by the difference between equations (64) and (65).

$$\alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) L_{s,t} \cdot \gamma A_{is,t-1} - \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) L_{s,t} \cdot \gamma A_{is,t-1} \Big[1 - \theta \Big], \tag{66}$$

which can be simplified as

$$\alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) L_{s,t} \cdot \theta \gamma A_{is,t-1}. \tag{67}$$

Case 2: Cost of innovation, $c_{i,t}$, is $\bar{c}$. If a firm that was backward in the previous time period faces a cost of innovation $c_{i,t} = \bar{c}$ in the current time period, it will not innovate if entry threat $\theta > 0$. This is because, even if it innovates and takes a cut on its profits, it would not be able to survive an entry threat. In such a scenario, the maximum bribe that such a firm would be willing to pay the policymaker would be the difference between profits made when it does not innovate and $\theta = 0$, and the profits made when it does not innovate and $\theta > 0$. The profits of a backward firm, when it does not innovate, is given by equation (52). When we consider $\theta = 0$, equation (52) is suitably modified as

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot A_{is,t-1}. \tag{68}$$

The profits of a backward firm that does not innovate and faces an entry threat of $\theta > 0$, are given by equation (52)

$$\pi_{i,t} = \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot A_{is,t-1} \Big[1 - \theta \Big]$$

Therefore, the maximum bribe that a backward firm facing $c_{i,t} = \bar{c}$, would be willing to pay to the policymaker is given by the difference between the above two equations, which is given by

$$\alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot A_{is,t-1} - \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot A_{is,t-1} \Big[1 - \theta \Big], \tag{69}$$

which can be simplified as

$$\alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot \theta \cdot A_{is,t-1}. \tag{70}$$

Equation (67) gives the maximum bribe that a backward firm will be ready to pay when $C_{i,t} = 0$ and equation (70) gives the maximum bribe that a backward firm will be ready to pay when $C_{i,t} = \bar{c}$. Since ex ante, the firm has equal probabilities of facing either of the two scenarios, the maximum bribe that the backward firms as a group would be willing to pay the policymaker is given by

$$B_{b,t} = (1-\beta^*) \left[\frac{1}{2} \cdot \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) L_{s,t} \cdot \theta \gamma \cdot A_{is,t-1} + \frac{1}{2} \cdot \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot \theta \cdot A_{is,t-1} \right], \tag{71}$$

where $(1-\beta^*)$ is the steady state share of backward firms in the economy. The above equation can also be written as

$$B_{b,t} = (1-\beta^*) A_{is,t-1} \left[\frac{1}{2} \cdot \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) L_{s,t} \cdot \theta \gamma + \frac{1}{2} \cdot \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot \theta \right],$$

which, on substituting for $A_{is,t-1} = \frac{1}{\gamma} \bar{A}_{s,t-1} = \frac{1}{\gamma^2} \bar{A}_{s,t}$, can be re-written as

$$B_{b,t} = (1-\beta^*) \frac{1}{\gamma^2} \bar{A}_{s,t} \left[\frac{1}{2} \cdot \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) L_{s,t} \cdot \theta \gamma + \frac{1}{2} \cdot \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) \cdot L_{s,t} \cdot \theta \right],$$

which can be simplified as

$$B_{b,t}(\theta) = (1-\beta^*) \frac{1}{\gamma^2} \bar{A}_{s,t} \left[\frac{1}{2} \cdot \alpha^{\frac{2}{1-\alpha}} \left(\frac{1}{\alpha} - 1 \right) L_{s,t} \cdot \theta \right] \Big(\gamma + 1 \Big). \tag{72}$$

Note from equation (72) that the total bribes offered by backward firms change linearly with θ. This reflects the economic reality that backward firms are more willing to pay higher bribes to avoid being displaced when the threat of entry is significant. A higher entry threat, θ, means that these firms are increasingly vulnerable to being displaced from the economy.

7.1.3 Total Bribes Offered by Incumbent Firms

On adding equations (63) and (72), we get the total bribes that incumbent firms will be willing to pay the policymaker so as to prevent him from increasing the entry threat from $\theta = 0$ to $\theta > 0$. Thus, the total bribes will be

$$B_t(\theta) = B_{a,t}(\theta) + B_{b,t}(\theta) = \frac{1}{\gamma}\bar{A}_{s,t} \cdot \beta^* \cdot \frac{1}{2}\left[\bar{c} - \big(\gamma - 1\big)\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha} - 1\right) \cdot L_{s,t}\right] \cdot 1_\Phi$$
$$+ (1-\beta^*)\frac{1}{\gamma^2}\bar{A}_{s,t}\left[\frac{1}{2} \cdot \alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha} - 1\right)L_{s,t} \cdot \theta\right]\Big(\gamma + 1\Big). \quad (73)$$

Equation (73) captures the aggregate behavior of incumbent firms in terms of their willingness to bribe the policymaker to restrict entry. The equation combines the incentives of both advanced and backward firms, reflecting their strategic responses to the entry threat parameter, θ. Note that for advanced firms, θ indirectly affects the bribes by affecting their decision to innovate, which is conditional upon equation (48) holding true. This is captured by the term 1_Φ in the equation. For a backward firm θ directly scales the bribe amount, showing a linear relationship. This indicates that as the entry threat, θ increases, backward firms are more willing to pay higher bribes to avoid being displaced.

We analyze now the total bribe paid by incumbent firms changes with a change in the entry threat, θ. The first-order derivative with respect to θ is given by

$$\frac{\partial B_t(\theta)}{\partial \theta} = (1-\beta^*)\frac{1}{\gamma^2}\bar{A}_{s,t}\left[\frac{1}{2} \cdot \alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha} - 1\right)L_{s,t}\right]\Big(\gamma + 1\Big), \quad (74)$$

which is positive. Note that the term 1_Φ is an indicator function, which can only be 0 or 1, and is therefore constant with respect to θ. The second-order derivative is given by

$$\frac{\partial^2 B_t(\theta)}{\partial \theta^2} = 0. \tag{75}$$

Since $\frac{\partial B_t(\theta)}{\partial \theta} > 0$ and $\frac{\partial^2 B_t(\theta)}{\partial \theta^2} = 0$, it is clear that the total bribes are a linear function of the entry probability, θ. Since θ lies in the range of $[0, 1]$, the policymaker can earn the highest bribes by setting the entry policy as $\theta = 1$.

7.2 The Dynamics of Aggregate Productivity

It has been shown that the GDP and the wage premium are both functions of the aggregate productivity in the skilled goods sector, $A_{s,t}$. In this sub-section, we show how this aggregate productivity is influenced by firms' decision to innovate, which in turn is influenced by the entry threat θ and the cost realization faced by each firm, $c_{i,t}$. Note that irrespective of whether the firm is advanced or backward, if entry occurs in sector i with probability θ, the productivity in that sector will always be at the frontier since the new entrant comes with frontier technology. Therefore, in such a scenario, the productivity of the i^{th} firm will be $\bar{A}_{s,t}$. When entry does not occur, for which the probability is given by $(1 - \theta)$, the change in a firm's productivity depends on the type of the firm and the cost of innovation faced by each of them.

The changes in the productivity of firms, conditional upon the entry threat, θ, are summarized in Figure 3.

Case 1: Contributions to Aggregate Productivity by Advanced Firms

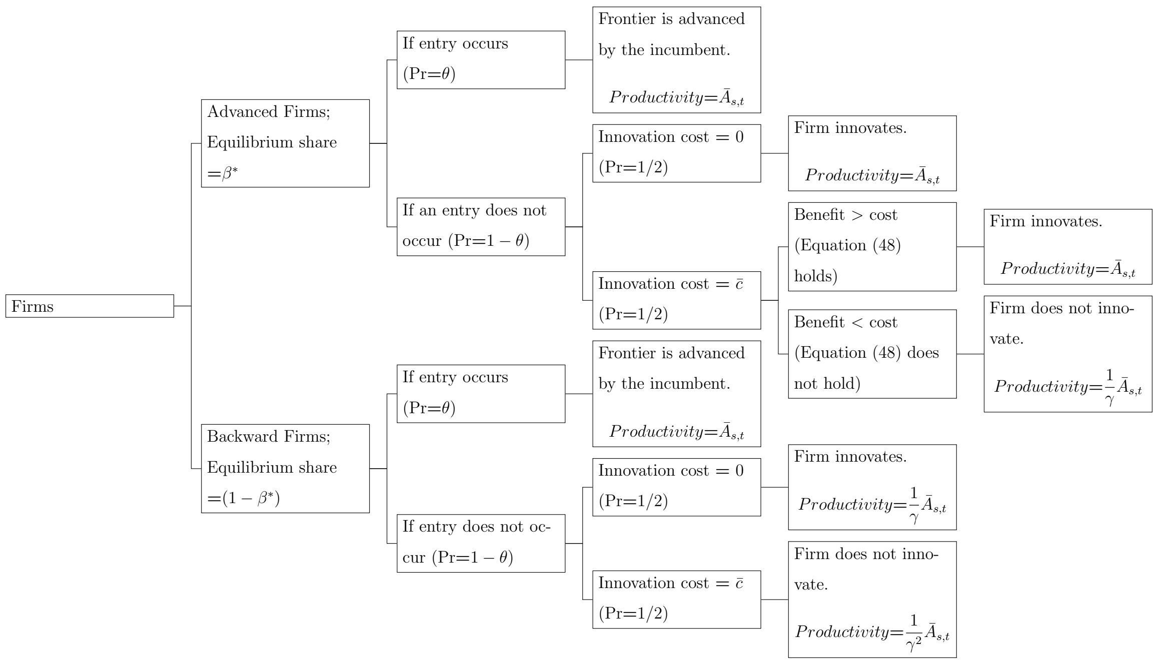
In the case of an advanced firm, if entry occurs in the i^{th} sector with probability θ, then the productivity of that sector would be at the frontier since the new entrant comes with frontier technology. Therefore, the contribution to aggregate productivity by sectors in which entry has occurred is given by

$$\beta^* \theta \bar{A}_{s,t}, \tag{76}$$

where β^* is the equilibrium share of advanced firms in the economy.

When there is no entry threat, the decision to innovate depends on the cost of innovation faced by the firm. If the innovation cost faced by the firm is $c_{i,t} = 0$, for

Figure 3: Changes in Productivity of Firms, Conditional upon the Entry Threat θ



which the probability is $\frac{1}{2}$, an advanced firm innovates. Therefore, the contribution to aggregate productivity by advanced firms that have survived an entry threat, faced an innovation cost of $c_{i,t} = 0$, and innovated will be

$$\beta^*(1-\theta)\frac{1}{2}\bar{A}_{s,t}. \tag{77}$$

On the other hand, if it faces a cost of innovation $C_{i,t} = \bar{c}$, whether the firm innovates depends on whether the benefit from innovation exceeds the cost of innovation, that is, when equation (48) holds. In such a scenario, the contribution to aggregate productivity by advanced firms that face an innovation cost of $\bar{c}$ and for whom equation (48) holds, is given by

$$\beta^*(1-\theta)\frac{1}{2}\bar{A}_{s,t} \cdot 1_{\Phi}, \tag{78}$$

where the term 1_{Φ} takes the value one if equation (48) holds, and zero otherwise. Advanced firms that survive entry threat and face an innovation cost of $\bar{c}$ do not innovate if the benefit from innovation does not exceed the cost of innovation, that is if equation (48) does not hold. Consequently, the productivity of an i^{th} firm in this scenario will be the same as the productivity in the previous period; that is, it will be $\bar{A}_{s,t-1} = \frac{1}{\gamma}\bar{A}_{s,t}$. The contribution to aggregate productivity by such advanced firms is given by

$$\beta^*(1-\theta)\frac{1}{2}\bar{A}_{s,t-1} \cdot 1_\Psi = \beta^*(1-\theta)\frac{1}{2} \cdot \frac{1}{\gamma}\bar{A}_{s,t} \cdot 1_\Psi, \tag{79}$$

where the term 1_Ψ takes the value one if equation (48) does not hold, and zero otherwise. Adding equations (76) to (79) gives us the total contribution to aggregate productivity by advanced firms. This addition yields

$$A_{s,t,a} = \beta^*\theta\bar{A}_{s,t} + \beta^*(1-\theta)\frac{1}{2}\bar{A}_{s,t} + \beta^*(1-\theta)\frac{1}{2}\bar{A}_{s,t} \cdot 1_\Phi + \beta^*(1-\theta)\frac{1}{2} \cdot \frac{1}{\gamma}\bar{A}_{s,t} \cdot 1_\Psi, \tag{80}$$

which, on collection of terms, can also be written as

$$A_{s,t,a} = \beta^*\left[\theta\bar{A}_{s,t} + (1-\theta)\frac{1}{2}\bar{A}_{s,t} + (1-\theta)\frac{1}{2}\bar{A}_{s,t}\left[1 \cdot 1_\Phi + \frac{1}{\gamma} \cdot 1_\Psi\right]\right],$$

and on factoring out the term $\bar{A}_{s,t}$ can be simplified as

$$A_{s,t,a} = \beta^*\left[\frac{1}{2}\bar{A}_{s,t} + \theta\frac{1}{2}\bar{A}_{s,t} + (1-\theta)\frac{1}{2}\bar{A}_{s,t}\left[1 \cdot 1_\Phi + \frac{1}{\gamma} \cdot 1_\Psi\right]\right],$$

and on combining like terms, this reduces to

$$A_{s,t,a}(\theta) = \frac{\beta^*\bar{A}_{s,t}}{2}\left[1 + \theta + (1-\theta)\left[1 \cdot 1_\Phi + \frac{1}{\gamma} \cdot 1_\Psi\right]\right], \tag{81}$$

Note that equation (81) denotes the contribution made by advanced firms to the aggregate productivity in the economy. This contribution not only depends on the entry threat, θ, but also the growth rate of the frontier, γ, and also on the marginal benefits and costs associated with innovation. Further, it is also a function of the steady-state share of advanced firms, β^*, the frontier productivity level, $\bar{A}_{s,t}$, the indicator function, 1_Φ, which takes value 1 if equation (48) holds and 0 otherwise, and the indicator function, 1_Ψ which takes the value 1 if equation (48) does not hold and 0 otherwise.

Case 2: Contributions to Aggregate Productivity by Backward Firms

In the case of a backward firm, once again, if entry occurs in the i^{th} sector, the productivity in that sector will be at the frontier since the entrant comes with frontier technology. The contribution to aggregate productivity by such firms will be given by

$$(1-\beta^*)\theta\bar{A}_{s,t}. \tag{82}$$

When there is no entry threat, for which the probability is given by $(1-\theta)$, firms decide whether to innovate, depending on the cost of innovation, $C_{i,t}$. If the cost of innovation is zero, for which the probability is $\frac{1}{2}$, these firms innovate. The contribution to aggregate productivity by such firms is given by

$$(1-\beta^*)(1-\theta)\frac{1}{2}A_{is,t} = (1-\beta^*)(1-\theta)\frac{1}{2}\cdot\frac{1}{\gamma}\bar{A}_{s,t}. \tag{83}$$

On the other hand, if the cost of innovation is $\bar{c}$, for which the probability is $\frac{1}{2}$, these firms do not innovate. The contribution to aggregate productivity by such firms is given by

$$(1-\beta^*)(1-\theta)\frac{1}{2}A_{s,t-1} = (1-\beta^*)(1-\theta)\frac{1}{2}\cdot\frac{1}{\gamma}\bar{A}_{s,t-1} = (1-\beta^*)(1-\theta)\frac{1}{2}\cdot\frac{1}{\gamma^2}\bar{A}_{s,t}. \tag{84}$$

The total contribution to aggregate productivity by backward firms is given by adding equations (82) to (84). This addition yields

$$A_{s,t,b} = (1-\beta^*)\theta\bar{A}_{s,t} + (1-\beta^*)(1-\theta)\frac{1}{2}\cdot\frac{1}{\gamma}\bar{A}_{s,t} + (1-\beta^*)(1-\theta)\frac{1}{2}\cdot\frac{1}{\gamma^2}\bar{A}_{s,t}, \tag{85}$$

which, on factoring out $(1-\beta^*)\bar{A}_{s,t}$ can be simplified as

$$A_{s,t,b} = (1-\beta^*)\bar{A}_{s,t}\left[\theta + (1-\theta)\frac{1}{2}\cdot\frac{1}{\gamma}\left[1+\frac{1}{\gamma}\right]\right], \tag{86}$$

which can also be written as

$$A_{s,t,b}(\theta) = (1-\beta^*)\bar{A}_{s,t}\left[\theta + (1-\theta)\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\right] \tag{87}$$

Note that equation (87) denotes the contribution made by the backward firms to the aggregate productivity in the economy. This contribution is a function of the share of backward firms, $1-\beta^*$, the frontier productivity, $\bar{A}_{s,t}$, the entry threat, θ, and the growth rate of the frontier, γ.

The total contribution to productivity made by advanced firms and backward firms put together is obtained by adding equations (81) and (87), which yields

$$A_{s,t}(\theta) = A_{s,t,a}(\theta) + A_{s,t,b}(\theta) = \frac{\beta^* \bar{A}_{s,t}}{2}\left[1 + \theta + (1-\theta)\left[1 \cdot 1_\Phi + \frac{1}{\gamma} \cdot 1_\Psi\right]\right] + (1-\beta^*)\bar{A}_{s,t}\left[\theta + (1-\theta)\frac{1}{2} \cdot \frac{1+\gamma}{\gamma^2}\right]. \quad (88)$$

Equation (88) encapsulates the complex interplay between aggregate productivity in the skilled sector and the entry probability, θ, innovation costs, $c_{i,t}$, and the rate of growth of the frontier γ. To see how the aggregate productivity of the skilled sector, $A_{s,t}$, behaves with respect to θ, we find the partial derivative of equation (85) with respect to θ, which yields

$$\frac{\partial A_{s,t}}{\partial \theta} = \underbrace{\frac{\beta^* \bar{A}_{s,t}}{2}\left[1 - \left[1 \cdot 1_\Phi + \frac{1}{\gamma} \cdot 1_\Psi\right]\right]}_{(a)} + \underbrace{(1-\beta^*)\bar{A}_{s,t}\left[1 - \frac{1}{2} \cdot \frac{1+\gamma}{\gamma^2}\right]}_{(b)} \quad (89)$$

Note from equation (89) that term (b) is always positive, given that $0 < \beta < 1$ and $\gamma > 1$. The sign of term (a) depends on whether equation (48) holds, that is, whether the benefit from innovation exceeds the cost of innovation for advanced firms. Two scenarios are possible here, depending on whether equation (48) holds.

Scenario 1: Equation (48) holds

In such a case, the term 1_Φ will be equal to 1 and the term 1_Ψ will be equal to zero. Consequently, term (a) of equation (86) will be

$$\frac{\beta^* \bar{A}_{s,t}}{2}\left[1 - \left[1 \cdot 1 + \frac{1}{\gamma} \cdot 0\right]\right] = \frac{\beta^* \bar{A}_{s,t}}{2}\left[1 - 1\right] = 0,$$

in which case equation (89) will be positive.

Scenario 2: Equation (48) does not hold

In such a case, the term 1_Φ will be equal to zero and the term 1_Ψ will be equal to 1. Consequently, term (a) of equation (86) will be

$$\frac{\beta^* \bar{A}_{s,t}}{2}\left[1 - \left[1 \cdot 0 + \frac{1}{\gamma} \cdot 1\right]\right] = \frac{\beta^* \bar{A}_{s,t}}{2}\left[1 - \frac{1}{\gamma}\right] = \frac{\beta^* \bar{A}_{s,t}(\gamma - 1)}{2\gamma},$$

in which case, again, equation (89) will be positive.

Thus, in both scenarios, equation (89) is positive. This means that the aggregate productivity of the skilled sector, $A_{s,t}$, increases with respect to θ.

$$\frac{\partial A_{s,t}}{\partial \theta} > 0. \tag{90}$$

Intuitively, a higher θ leads to a more competitive environment where firms are driven to continuously improve their productivity through innovation.

From equation (89), we further arrive at the second-order derivative, which is given by

$$\frac{\partial^2 A_{s,t}}{\partial \theta^2} = 0. \tag{91}$$

This indicates that the relationship between the aggregate productivity in the skilled sector, $A_{s,t}$, and the entry probability, θ, is linear[2]. This means that the $A_{s,t}$ attains maximum when $\theta = 1$.

7.2.1 Relationship of Aggregate Productivity in the Skilled Sector, $A_{s,t}$ with the GDP

Equation (31) expresses the GDP as a function of, inter-alia, the aggregate productivity in the skilled sector.

$$GDP_t(A_{s,t}(\theta)) = A_u L_u + \alpha^{\frac{2}{1-\alpha}} L_s A_{s,t}(\theta)\Big(\frac{1}{\alpha^2} - 1\Big).$$

Recall from equation (31) that the GDP is a linear function of the aggregate productivity of the skilled sector, $A_{s,t}$. We now analyze how GDP behaves with respect to changes in the entry probability, θ. The first-order derivative with respect to θ is given by

$$\frac{\partial G_t(A_{s,t}(\theta))}{\partial \theta} = \alpha^{\frac{2}{1-\alpha}}\Big(\frac{1}{\alpha^2} - 1\Big) L_s \frac{\partial A_{s,t}(\theta)}{\partial \theta}. \tag{92}$$

Since $\frac{\partial A_{s,t}(\theta)}{\partial \theta} > 0$, the above equation is positive. The second-order derivative with respect to θ is given by

$$\frac{\partial^2 G_t(A_{s,t}(\theta))}{\partial \theta^2} = \alpha^{\frac{2}{1-\alpha}}\Big(\frac{1}{\alpha^2} - 1\Big) L_s \frac{\partial^2 A_{s,t}(\theta)}{\partial \theta^2}. \tag{93}$$

[2]Note that the terms 1_Φ and 1_Ψ are indicator functions, which can only be 0 or 1, and therefore are constants with respect to θ. Recall that these indicators depend on whether equation (48) holds.

And since $\frac{\partial^2 A_{s,t}(\theta)}{\partial \theta^2} = 0$, the above equation becomes zero. Therefore, we now have that

$$\frac{\partial G_t(A_{s,t}(\theta))}{\partial \theta} > 0; \quad \frac{\partial^2 G_t(A_{s,t}(\theta))}{\partial \theta^2} = 0,$$

which means that the GDP is a linear function of θ. Since θ lies in the range of $[0, 1]$, the GDP attains maximum when the firm entry policy, θ, is set to 1.

7.2.2 Relationship of Aggregate Productivity in the Skilled Sector, $A_{s,t}$ with the Inverse of Skill Premium

We now proceed to analyze how the skill premium behaves with respect to the entry probability, θ. Recall from equation (38) that the skill premium is given by

$$\omega_t(A_{s,t}(\theta)) = (1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}} \cdot \frac{1}{\eta} \cdot \left[\frac{A_{s,t}(\theta)}{A_u}\right]^{\rho} \cdot \left[\frac{L_u}{L_s}\right]^{1-\rho}.$$

We differentiate the above equation with respect to θ and obtain

$$\frac{\partial \omega_t(A_{s,t}(\theta))}{\partial \theta} = (1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}} \cdot \frac{1}{\eta} \cdot \left[\frac{1}{A_u}\right]^{\rho} \cdot \left[\frac{L_u}{L_s}\right]^{1-\rho} \cdot \rho \cdot \Big[A_{s,t}(\theta)\Big]^{\rho-1} \cdot \frac{\partial A_{s,t}(\theta)}{\partial \theta}. \quad (94)$$

Since we know from equation (89) that $\frac{\partial A_{s,t}}{\partial \theta} > 0$, and since $0 < \rho < 1$, equation (94) is positive, that is, $\frac{\partial \omega_t}{\partial \theta} > 0$. The second-order derivative is given by

$$\frac{\partial^2 \omega_t}{\partial \theta^2} = (1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}} \cdot \frac{1}{\eta} \cdot \left[\frac{1}{A_u}\right]^{\rho} \cdot \left[\frac{L_u}{L_s}\right]^{1-\rho} \cdot \rho \cdot \Bigg[(\rho-1)\Big[A_{s,t}(\theta)\Big]^{\rho-2} \cdot \left[\frac{\partial A_{s,t}(\theta)}{\partial \theta}\right]^2 + \Big[A_{s,t}(\theta)\Big]^{\rho-1} \cdot \frac{\partial^2 A_{s,t}(\theta)}{\partial \theta^2}\Bigg]. \quad (95)$$

From equation (90), we know that $\frac{\partial^2 A_{s,t}(\theta)}{\partial \theta^2} = 0$. Therefore, equation (95) can be reduced to

$$\frac{\partial^2 \omega_t}{\partial \theta^2} = (1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}} \cdot \frac{1}{\eta} \cdot \left[\frac{1}{A_u}\right]^{\rho} \cdot \left[\frac{L_u}{L_s}\right]^{1-\rho} \cdot \rho \cdot \Bigg[(\rho-1)\Big[A_{s,t}(\theta)\Big]^{\rho-2} \cdot \left[\frac{\partial A_{s,t}(\theta)}{\partial \theta}\right]^2\Bigg]. \quad (96)$$

Since $0 < \rho < 1$, $(\rho - 1) < 0$. Therefore, equation (96) is negative. From equations (94) and (96) we have that

$$\frac{\partial \omega_t}{\partial \theta} > 0; \quad \frac{\partial^2 \omega_t}{\partial \theta^2} < 0.$$

This shows that the skill premium, that ω_t is a concave function of the entry probability, θ. This means that the skill premium reaches a maximum at the point where $\frac{\partial \omega_t}{\partial \theta} = 0$.

The policymaker is interested in reducing the wage differential between skilled and unskilled laborers. To this end, we consider that the inverse of the skill-premium enters his objective function, which he sets to maximize. Therefore, we now proceed to analyze how the inverse of the skill-premium behaves with respect to the entry probability, θ. Equation (38) gives the skill-premium, from which the inverse of the skill-premium is obtained as

$$\omega_t(A_{s,t}(\theta))^{-1} = \left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}} \cdot \frac{1}{\eta}\right]^{-1} \cdot \left[\frac{A_u}{A_{s,t}(\theta)}\right]^{\rho} \cdot \left[\frac{L_s}{L_u}\right]^{1-\rho}. \tag{97}$$

Note that ω_t^{-1} is inversely proportional to the aggregate productivity of the skill sector, $A_{s,t}$. This inverse relationship is an essential characteristic of how the inverse of skill-premium responds to changes in aggregate productivity in the skilled sector, which we will invoke while discussing our main results.

From equation (97), we obtain the first-order derivative of ω_t^{-1} with respect to θ as

$$\frac{\partial \omega_t(A_{s,t}(\theta))^{-1}}{\partial \theta} = \left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}} \cdot \frac{1}{\eta}\right]^{-1} \left[\frac{L_s}{L_u}\right]^{1-\rho} \left[A_u\right]^{\rho} \left[-\rho\left[A_{s,t}(\theta)\right]^{-(\rho+1)} \cdot \frac{\partial A_{s,t}(\theta)}{\partial \theta}\right]. \tag{98}$$

Since $0 < \rho < 1$, and $\frac{\partial A_{s,t}(\theta)}{\partial \theta} > 0$, equation (98) is negative. The second-order derivative is given by

$$\frac{\partial^2 \omega_t(A_{s,t}(\theta))^{-1}}{\partial \theta^2} = \left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}} \cdot \frac{1}{\eta}\right]^{-1} \left[\frac{L_s}{L_u}\right]^{1-\rho} \left[A_u\right]^{\rho}$$
$$\cdot \left[\rho(\rho+1)\left[A_{s,t}(\theta)\right]^{-(\rho+2)} \cdot \left[\frac{\partial A_{s,t}(\theta)}{\partial \theta}\right]^2 - \rho\left[A_{s,t}(\theta)\right]^{-(\rho+1)} \cdot \frac{\partial^2 A_{s,t}(\theta)}{\partial \theta^2}\right].$$

Since $\frac{\partial^2 A_{s,t}(\theta)}{\partial\theta^2} = 0$, the above equation reduces to

$$\frac{\partial^2 \omega_t(A_{s,t}(\theta))^{-1}}{\partial\theta^2} = \left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}} \cdot \frac{1}{\eta}\right]^{-1} \left[\frac{L_s}{L_u}\right]^{1-\rho} \left[A_u\right]^{\rho} \cdot \left[\rho(\rho+1)\left[A_{s,t}(\theta)\right]^{-(\rho+2)} \cdot \left[\frac{\partial A_{s,t}(\theta)}{\partial\theta}\right]^2\right], \quad (99)$$

which is positive since $0 < \rho < 1$. Therefore, from equations (98) and (99), we have

$$\frac{\partial\omega_t^{-1}}{\partial\theta} < 0; \quad \frac{\partial^2\omega_t^{-1}}{\partial\theta^2} > 0. \quad (100)$$

This means that the inverse of the skill premium is a decreasing and convex function with respect to θ. This suggests that as the entry probability θ increases, the inverse of the skill premium decreases at an increasing rate. Therefore, to maximize the inverse of the skill premium, a lower value of θ would be preferable. Since θ lies in the range of $[0, 1]$, this means that the inverse of skill-premium is maximized when $\theta = 0$.

The relationships between the aggregate productivity in the skill sector, $A_{s,t}$, the entry probability, θ, the total bribes, B, the GDP and the inverse of the skill premium, ω_t^{-1}, are summarized in Table 1.

Table 1: Relationship between $A_{s,t}$, θ, Bribes, GDP and ω_t^{-1}

	Aggregate Productivity ($A_{s,t}$)	Entry Probability (θ)
Aggregate Productivity, ($A_{s,t}$)	-	Linear
Bribe (B)	-	Linear
GDP (G_t)	Linear	Linear
Inverse of Skill-premium (ω_t^{-1})	Inversely Proportional to $A_{s,t}$	Convex and Decreasing with θ

8 Determining the Firm Entry Policy, θ

We now present the politician's objective function, based on which he determines the firm entry policy. The policymaker is not only concerned about addressing inequality, ω_t^{-1} in the economy but also influenced by rent-seeking activities by incumbent firms, who offer bribes to the politicians to restrict the entry of technologically advanced firms. Thus, the objective function of the policymaker is given by

$$\Theta(\theta) = a \cdot B_t(\theta) + (1-a) \cdot \omega_t^{-1}(\theta); \quad a \in [0,1] \tag{101}$$

where a is the weight that the policymaker assigns on the total bribes offered by incumbent firms, B_t, and $(1-a)$ is the weight assigned by him on the inverse of the skill-premium, ω^{-1}.

Proposition 1. A policymaker that does not have a welfare concern sets a highly competitive entry policy. The aggregate productivity of the skilled sector is at the frontier of technology in the skilled sector.

Proof. A policymaker that does not have a welfare concern does not aim to reduce the wage inequality between the two types of laborers in the economy. Therefore, he sets a weight $a = 1$, and as a result, the weight on the inverse of skill premium is zero. We call this regime as **"Regime 1: The No-welfare Regime"**. In this regime, the objective function of the policymaker is given by

$$\Theta_{nw}(\theta) = a \cdot B_t(\theta). \tag{102}$$

The first-order and second-order derivatives of this objective function, with respect to θ, are given by

$$\frac{\partial \Theta_{nw}(\theta)}{\partial \theta} = a \cdot \frac{\partial B_t(\theta)}{\partial \theta} > 0,$$

and

$$\frac{\partial^2 \Theta_{nw}(\theta)}{\partial \theta^2} = a \cdot \frac{\partial^2 B_t(\theta)}{\partial \theta^2} = 0,$$

respectively. Therefore, the objective function of a policymaker that does not have a welfare concern is a strictly linear increasing function of the entry policy, θ. Given that θ lies in the range $[0,1]$, the policymaker will choose $\theta = 1$ to maximize $\Theta_{nw}(\theta)$. Setting an entry policy, $\theta = 1$, means that the threat of an entrant entering the economy and displacing the incumbent is high. Consequently, the incumbent firms, recognizing the high risk of displacement, may offer substantial bribes to the policymaker to deter entry. Thus, a policymaker without a welfare concern, focusing solely on maximizing bribes, will prefer a high entry threat, $\theta = 1$. This concludes the proof.

When the entry policy is set to $\theta = 1$, the aggregate productivity in the skill sector is obtained by substituting for $\theta = 1$ in equation (87). This yields

$$A_{s,t}(\theta=1) = \frac{\beta^* \bar{A}_{s,t}}{2}\left[1+1+(1-1)\left[1\cdot 1_{\Phi}+\frac{1}{\gamma}\cdot 1_{\Psi}\right]\right]+(1-\beta^*)\bar{A}_{s,t}\left[1+(1-1)\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\right], \tag{103}$$

which can be simplified to

$$A_{s,t}(\theta=1) = \frac{\beta^* \bar{A}_{s,t}}{2}\cdot\big(2\big)+(1-\beta^*)\bar{A}_{s,t}\cdot\big(1\big) = \beta^*\bar{A}_{s,t}+(1-\beta^*)\bar{A}_{s,t} = \bar{A}_{s,t}. \tag{104}$$

From equation (104), it can be seen that the aggregate productivity of the skilled sector, when the policymaker sets $\theta = 1$, is exactly at the frontier. This means that

$$\frac{A_{s,t}(\theta=1)}{\bar{A}_{s,t}} = 1. \tag{105}$$

Equation (105) captures the fact that a high entry threat induces a highly competitive environment where incumbent firms are continuously pressured to innovate and improve their productivity to avoid being displaced by new entrants. The constant threat of entry ensures that the market remains dynamic, fostering innovation and efficiency among firms.

This completes the proof.

Proposition 2. A policymaker that is only concerned about reducing the income inequality in the economy sets the most restrictive entry policy. The the aggregate productivity of the skill sector is lower than the maximum efficiency level that it can reach.

Proof. A policymaker that is solely concerned about reducing the income inequality in the economy sets weights $a = 0$. Consequently, he assigns zero-weight to total bribes offered by firms and a weight of 1 on the inverse of skill premium. We call this as **"Regime 2: The Only-welfare Regime"**. Such a policymaker has the following objective function

$$\Theta_{ow}(\theta) = (1 - a) \cdot \omega_t^{-1}(\theta). \tag{106}$$

The first-order and second-order derivatives of this objective function, with respect to θ, are given by

$$\frac{\partial \Theta_{ow}(\theta)}{\partial \theta} = (1 - a) \cdot \frac{\partial \omega_t^{-1}(\theta)}{\partial \theta} < 0,$$

and

$$\frac{\partial^2 \Theta_{ow}(\theta)}{\partial \theta^2} = (1 - a) \cdot \frac{\partial^2 \omega_t^{-1}(\theta)}{\partial \theta^2} > 0$$

respectively. The first-order derivative indicates that the objective function $\Theta_{ow}(\theta)$ is decreasing in θ. The second-order derivative being positive confirms that $\Theta_{ow}(\theta)$ is a convex function. Therefore, to minimize $\Theta_{ow}(\theta)$, the policymaker sets θ to its lower bound, $\theta = 0$, which corresponds to the least competitive, or the most restrictive, entry policy. In this scenario, the incumbent firms do not face the pressure to innovate to avoid displacement by new entrants.

The absence of entry threat, that is, $(\theta = 0)$, allows the incumbent firms to maintain their market positions without needing continuous innovation. This leads to a more stable but less dynamic market environment, which reduces the overall productivity growth. However, the reduction in competitiveness leads to a decrease in wage inequality. Hence, by setting $\theta = 0$, the policymaker successfully reduces income inequality in the economy, achieving the desired objective.

When the entry policy is set to $\theta = 0$, the aggregate productivity in the skill sector is obtained by substituting for $\theta = 0$ in equation (87). This yields[3]

$$A_{s,t}(\theta=0) = \frac{\beta^* \bar{A}_{s,t}}{2}\left[1 + 0 + (1-0)\left[\frac{1+\gamma}{2\gamma}\right]\right] + (1-\beta^*)\bar{A}_{s,t}\left[0 + (1-0)\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\right],$$

which can be simplified to

$$\begin{aligned} A_{s,t}(\theta=0) &= \frac{\beta^* \bar{A}_{s,t}}{2}\left[1 + \frac{1+\gamma}{2\gamma}\right] + (1-\beta^*)\bar{A}_{s,t}\frac{1+\gamma}{2\gamma^2} \\ &= \frac{\bar{A}_{s,t}}{2\gamma}\left[\beta * \cdot \frac{3\gamma+1}{2} + (1-\beta^*)\frac{1+\gamma}{\gamma}\right] = \frac{\bar{A}_{s,t}}{2\gamma}\left[\frac{\beta^*(3\gamma^2+1) + 2(1-\beta^*)(1+\gamma)}{2\gamma}\right], \end{aligned}$$

which is further simplified to

$$\frac{A_{s,t}(\theta=0)}{\bar{A}_{s,t}} = \frac{1}{4\gamma^2}\left[\beta^*(3\gamma^2+1) + 2(1-\beta^*)(1+\gamma)\right]. \tag{107}$$

Equation (107) gives the distance between the aggregate productivity in the skill sector and the frontier technology in the skill sector, when $\theta = 0$. This provides a measure of how close the overall productivity is to the maximum achievable productivity. Note that the right-hand side of equation (107) is linear in terms of β^*. Further, note that,

$$\lim_{\beta^* \to 0} \frac{1}{4\gamma^2}\left[\beta^*(3\gamma^2+1) + 2(1-\beta^*)(1+\gamma)\right] = \frac{1+\gamma}{2\gamma^2} < 1$$

and

$$\lim_{\beta^* \to 1} \frac{1}{4\gamma^2}\left[\beta^*(3\gamma^2+1) + 2(1-\beta^*)(1+\gamma)\right] = \frac{3}{4} + \frac{1}{4\gamma^2} < 1.$$

Therefore, for $\gamma > 1$ and $0 < \beta^* < 1$, the right-hand side of equation (107) will always be less than one. This shows that when the policymaker sets the entry policy as $\theta = 0$, the aggregate productivity of the skill sector is lower than the maximum efficiency level that it can reach, which is captured as

$$\frac{A_{s,t}(\theta=0)}{\bar{A}_{s,t}} < 1. \tag{108}$$

[3]Note that when $\theta = 0$, if $c_{i,t} = \bar{c}$ then equation (48) does not hold by virtue of assumption 2. In such a case, $1_\Phi = 0$ and $1_\Psi = 1$. On the other hand, if $c_{i,t} = 0$, then equation (48) will hold. In such a case, $1_\Phi = 1$ and $1_\Psi = 0$. Since ex ante the probability for either scenarios is $\frac{1}{2}$, the term $\left[1 \cdot 1_\Phi + \frac{1}{\gamma} \cdot 1_\Psi\right]$ will be equal to $\left[\frac{1}{2}\left(0 + \frac{1}{\gamma}\right) + \frac{1}{2}\left(1+0\right)\right] = \frac{1+\gamma}{2\gamma}$.

This completes the proof.

Proposition 3: If the policymaker is office-motivated, the aggregate productivity is greater than the aggregate productivity in Regime 2, provided that the effective labor employed in the skilled sector is greater than or equal to the effective labor employed in the unskilled sector.

Proof. We recognize that an office-motivated policymaker would not only require votes to remain in office but also campaign contributions to contest elections. Therefore, he would assign equal weights to both total bribes offered by incumbent firms and also to the inverse of the skill premium. We call this as **"Regime 3: The Office-motivated Regime"**. The objective function of such a policymaker is given by equation (101) with $a = 0.5$, which we term as Θ_{om}. The first-order derivative of the policymaker's objective function, with respect to the firm entry policy, θ, is given by

$$\frac{\partial \Theta_{om}(\theta)}{\partial \theta} = 0.5 \cdot \underbrace{\frac{\partial B_t(\theta)}{\partial \theta}}_{>0} + 0.5 \cdot \underbrace{\frac{\partial \omega_t^{-1}(\theta)}{\partial \theta}}_{<0}. \tag{109}$$

From equation (74), we know that $\frac{\partial B_t(\theta)}{\partial \theta} > 0$, and from equation (98), we know that $\frac{\partial \omega_t^{-1}(\theta)}{\partial \theta} < 0$. Therefore, the overall sign of equation (104) is ambiguous.

The second-order derivative of the policymaker's objective function is given by

$$\frac{\partial^2 \Theta_{om}(\theta)}{\partial \theta^2} = 0.5 \cdot \underbrace{\frac{\partial^2 B_t(\theta)}{\partial \theta^2}}_{=0} + 0.5 \cdot \underbrace{\frac{\partial^2 \omega_t^{-1}(\theta)}{\partial^2 \theta}}_{>0}. \tag{110}$$

From equation (75), we know that $\frac{\partial^2 B_t(\theta)}{\partial \theta^2} = 0$, and from equation (99), we know that $\frac{\partial^2 \omega_t^{-1}(\theta)}{\partial \theta^2} > 0$. Therefore, the overall sign of equation (110) is positive. This shows that the objective function of such a policymaker is convex. The policymaker will set θ to maximize $\Theta(\theta)$, which involves solving for θ where $\frac{\partial \Theta(\theta)}{\partial \theta} = 0$. This is given by

$$0.5 \cdot \frac{\partial B_t(\theta)}{\partial \theta} + \ 0.5 \cdot \frac{\partial \omega_t^{-1}(\theta)}{\partial \theta} = 0,$$

which can also be written as

$$\frac{\partial \omega_t^{-1}(\theta)}{\partial \theta} = -\frac{\partial B_t(\theta)}{\partial \theta}.$$

Substituting for $\frac{\partial \omega_t^{-1}(\theta)}{\partial \theta}$ from equation (98), we can write the above euqation as

$$\left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}} \cdot \frac{1}{\eta}\right]^{-1} \left[\frac{L_s}{L_u}\right]^{1-\rho} \left[A_u\right]^{\rho} \left[-\rho\left[A_{s,t}(\theta)\right]^{-(\rho+1)} \cdot \frac{\partial A_{s,t}(\theta)}{\partial \theta}\right] = -\frac{\partial B_t(\theta)}{\partial \theta}, \quad (111)$$

which simplifies to

$$\frac{A_{s,t}(\theta=\theta_{om})}{\bar{A}_{s,t}} = \bar{A}_{s,t}^{-1}\left[\rho A_u\right]^{\frac{\rho}{1+\rho}} \left[\frac{2(1-\beta^*)\cdot(\gamma+1)}{\gamma\beta^*\left[\gamma+1\right]+2(1-\beta^*)\left[2\gamma^2-(1+\gamma)\right]}\right]^{\frac{-1}{1+\rho}} \cdot \left[(1-\alpha)^2\alpha^{\frac{1+\alpha(2\rho+1)}{1-\alpha}} \cdot \frac{1}{\eta}\right]^{\frac{-1}{1+\rho}} \cdot \left[\frac{\left(L_s\right)^{\rho}}{\left(L_u\right)^{1-\rho}}\right]^{\frac{1}{1+\rho}}. \quad (112)$$

[Refer to Appendix 1 for a detailed derivation.]

The policymaker who assigns non-zero weights to bribes and the inverse of skill-premium, sets an entry policy, θ, that satisfies equation (112). Thus, the optimal entry policy, θ, balances the effects of bribes, GDP, and the inverse of the skill premium based on their respective weights a, b, and c. θ is also a function of the frontier productivity, $\bar{A}_{s,t}$, elasticity of substitution between skilled and unskilled labor, α, steady-state fraction of advanced firms, β^*, growth rate of the technological frontier, γ, the skilled labor, L_s, the unskilled labor, L_u, the productivity of unskilled labor, A_u, the preference for the skilled good, η, and the substitution factor between the skilled and unskilled good, ρ.

In Proposition 1, it was shown that in Regime 1, the aggregate productivity of the skilled sector is at the frontier. In Proposition 2, it was shown that in Regime 2, the aggregate productivity of the skilled sector is below the frontier. We now compare the distance between aggregate productivity and the technological frontier in the skilled sector in Regimes 2 and 3. We calculate the ratio

$$\frac{A_{s,t}(\theta=0)}{\bar{A}_{s,t}} : \frac{A_{s,t}(\theta=\theta_{om})}{\bar{A}_{s,t}}, \quad (113)$$

where the left-hand side is given by equation (107) and the right-hand side is given by equation (112). If this ratio is less than 1, then the distance between aggregate productivity and the frontier productivity under Regime 3 is greater than the distance in Regime 2. This would mean that the aggregate productivity in Regime 3 is lower than the aggregate productivity in Regime 2, which has consequences on growth and income inequality, which we will discuss in the section on the main results of our paper.

For the given parametric restrictions, if

$$\frac{\left[L_s\right]^{\rho}}{\left[\bar{A}_{s,t}\right]^{1+\rho}} \geq \frac{\left[L_u\right]^{1-\rho}}{\left[\rho A_u\right]^{\rho}}, \tag{114}$$

then,

$$\frac{A_{s,t}(\theta = 0)}{\bar{A}_{s,t}} < \frac{A_{s,t}(\theta = \theta_{om})}{\bar{A}_{s,t}}. \tag{115}$$

[*See Appendix 2 for a detailed derivation of this condition.*]

Equations (114) gives the sufficient condition when the distance between the aggregate productivity and the technological frontier of the skilled sector can be said to be smaller in the *"only-welfare"* regime, *vis-à-vis* the *"office-motivated"* regime. Equation (114) implies that the effective labor employed in the skilled sector must be greater than or equal to the effective labor employed in the unskilled sector. This sufficient condition essentially requires a balanced relationship between the productivity and labor force in the skilled and unskilled sectors. Specifically, the productivity in the skilled sector should not vastly outstrip that in the unskilled sector, and there should be a significant allocation of labor in the skilled sector relative to the unskilled sector.

If equation (114) holds and consequently equation (115) also holds, then,

$$A_{s,t}(\theta = 0) < A_{s,t}(\theta = \theta_{om}).$$

This suggests that an office-motivated policymaker who optimizes entry policies based on broader economic incentives can significantly enhance aggregate productivity in the skilled sector, provided that the skilled sector is relatively more effective.

This completes the proof of Proposition 3.

It has been shown in Appendix 2 that if the sufficient condition given by equation (115) is not met, then it would not be analytically possible to rank the aggregate productivities of Regime 2 and Regime 3.

Table 2: Results of the Policymaker's Maximization Exercise in the Three Regimes

Regime	**Firm Entry Policy** (θ)	**Distance between Aggregate Productivity and the Frontier**
No-welfare	$\theta = 1$	$\frac{A_{s,t}(\theta=1)}{\bar{A}_{s,t}} = 1$
Only-welfare	$\theta = 0$	$\frac{A_{s,t}(\theta=0)}{\bar{A}_{s,t}} < 1$
Office-motivated	$\theta = \theta_{om}$ satisfying Eq. (112)	$\frac{A_{s,t}(\theta=0)}{\bar{A}_{s,t}} < \frac{A_{s,t}(\theta=\theta_{om})}{\bar{A}_{s,t}}$

This completes the section on the determination of firm entry policy by the policymaker. We summarize the results of this section in Table 2.

9 Main Results

We now proceed to present the main results of our paper. Recall from Table 1 that GDP and the aggregate productivity of the skilled sector, $A_{s,t}$, have a linear relationship. On the other hand, the inverse of skill-premium, ω^{-1}, and the aggregate productivity of the skilled sector, $A_{s,t}$, have an inverse relationship. We present our main results as the following two propositions.

Proposition 4. The Gross Domestic Product and the income inequality in the economy are lower in the "welfare-only" regime vis-à-vis the "office-motivated" regime if effective labor employed in the skilled sector is greater than or equal to the effective labor employed in the unskilled sector.

Proof. From equation (31), we know that the GDP is a linear function of the aggregate productivity in the skill sector. Equation (38) shows that the skill premium, which is a measure of income inequality in the economy, is directly proportional to the aggregate productivity in the skilled sector. From Proposition 3, we know that if the sufficient condition, given by equation (114), holds true, then the aggregate productivity of the skill sector in the *"only-welfare"* regime is lower than the aggregate productivity in the *"office-motivated"* regime. Given the nature of the relationships given by equations (31) and (38), this means that GDP and income inequality are lower in the *"only-welfare"* than the *"office-motivated"*.

Proposition 5. In the "No-welfare" regime, the Gross Domestic Product in the economy is the highest, and the income inequality in the economy is also the highest.

Proof. From equation (31), we know that the GDP is a linear function of the aggregate productivity in the skill sector. On the other hand, equation (38) shows that the skill-premium, which is a measure of income inequality in the economy, is directly proportional to the aggregate productivity in the skilled sector. (Alternatively, equation (97) shows that the inverse of skill-premium, which is a measure of income equality in the economy, is inversely proportional to the aggregate productivity in the skilled sector.) From Proposition 1, we know that the aggregate productivity is at the frontier, which is the highest achievable productivity, in the *"no-welfare"* regime. Consequently, the GDP is the highest, and the inequality is also highest under the *"no-welfare"* regime where the aggregate productivity is also at the frontier of technology.

10 Conclusion

We have explored the intricate dynamics between firm entry policies, rent-seeking behavior, and technological innovation within a dual-sector economy. By extending the Schumpeterian growth framework, this analysis highlights the pivotal role of entry threats in shaping the incentives of advanced and backward firms to innovate or engage in rent-seeking. The model reveals that entry threats are a double-edged sword: while they encourage innovation among advanced firms, they also incentivize backward firms to engage in rent-seeking activities, ultimately distorting competition and reducing aggregate productivity.

The findings underscore the critical role of policymakers in mediating these dynamics. Policymakers face a delicate balancing act, influenced by conflicting pressures from welfare objectives and rent-seeking incumbents. The analysis demonstrates that well-calibrated entry policies can foster innovation and long-term economic growth, while excessive protectionism may entrench inefficiencies and exacerbate income inequality.

This study also contributes to the broader literature by integrating insights from political economy and growth theory, particularly highlighting the endogenous role of policymakers. By formalizing the relationships between entry policies, innovation incentives, and income distribution, the model offers a comprehensive framework to analyze the trade-offs inherent in growth policy design.

Future research could extend this framework by incorporating dynamic elements such as evolving technology gaps between advanced and backward firms, as well as exploring the interactions between international trade policies and domestic entry dynamics. Furthermore, empirical validation of the model's predictions using industry-level data could provide valuable insights into the practical implications of entry policies and rent-seeking behavior.

In conclusion, we underscore the importance of fostering an economic environment where innovation is rewarded, and rent-seeking is curtailed. Achieving this balance requires careful policy design that aligns incentives across firms and policymakers, thereby promoting sustainable economic growth and equitable income distribution.

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Appendix 1

Consider equation (111)

$$\left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\right]^{-1}\left[\frac{L_s}{L_u}\right]^{1-\rho}\left[A_u\right]^{\rho}\left[-\rho\left[A_{s,t}(\theta)\right]^{-(\rho+1)}\cdot\frac{\partial A_{s,t}(\theta)}{\partial\theta}\right] = -\frac{\partial B_t(\theta)}{\partial\theta},$$

which upon multiplying both sides by (-1), can be re-written as

$$\left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\right]^{-1}\left[\frac{L_s}{L_u}\right]^{1-\rho}\left[A_u\right]^{\rho}\left[\rho\left[A_{s,t}(\theta)\right]^{-(\rho+1)}\cdot\frac{\partial A_{s,t}(\theta)}{\partial\theta}\right] = \frac{\partial B_t(\theta)}{\partial\theta},$$

and by isolating the term $A_{s,t}(\theta)$, this can be written as

$$\left[A_{s,t}(\theta)\right]^{-(\rho+1)} = \left[\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\rho\big(A_u\big)^{\rho}\cdot\frac{\partial A_{s,t}(\theta)}{\partial\theta}}\right]\cdot\left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\right]\left[\frac{L_u}{L_s}\right]^{1-\rho},$$

which can further be simplified to

$$A_{s,t}(\theta) = \left[\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\rho\big(A_u\big)^{\rho}\cdot\frac{\partial A_{s,t}(\theta)}{\partial\theta}}\right]^{\frac{-1}{1+\rho}}\cdot\left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\right]^{\frac{-1}{1+\rho}}\left[\frac{L_u}{L_s}\right]^{\frac{-(1-\rho)}{1+\rho}},$$

which can also be written as

$$A_{s,t}(\theta) = \left[\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\rho\big(A_u\big)^{\rho}\cdot\frac{\partial A_{s,t}(\theta)}{\partial\theta}}\right]^{\frac{-1}{1+\rho}}\cdot\left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\right]^{\frac{-1}{1+\rho}}\left[\frac{L_s}{L_u}\right]^{\frac{1-\rho}{1+\rho}}.$$

By substituting for $A_{s,t}(\theta)$ from equation (87), we get

$$\begin{aligned}A_{s,t}(\theta) = \frac{\beta^*\bar{A}_{s,t}}{2}\left[1+\theta+(1-\theta)\left[1\cdot 1_\Phi+\frac{1}{\gamma}\cdot 1_\Psi\right]\right]+(1-\beta^*)\bar{A}_{s,t}\left[\theta+(1-\theta)\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\right]\\ = \left[\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\rho\big(A_u\big)^{\rho}\cdot\frac{\partial A_{s,t}(\theta)}{\partial\theta}}\right]^{\frac{-1}{1+\rho}}\cdot\left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\right]^{\frac{-1}{1+\rho}}\left[\frac{L_s}{L_u}\right]^{\frac{1-\rho}{1+\rho}},\end{aligned}$$

which can also be written as

$$\begin{aligned}A_{s,t}(\theta) = \frac{\beta^*\bar{A}_{s,t}}{2}\left[1+\theta+(1-\theta)\left[1\cdot 1_\Phi+\frac{1}{\gamma}\cdot 1_\Psi\right]\right]+(1-\beta^*)\bar{A}_{s,t}\left[\theta+(1-\theta)\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\right]\\ = \left[\frac{\big(A_u\big)^{-\rho}}{\rho}\cdot\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\frac{\partial A_{s,t}(\theta)}{\partial\theta}}\right]^{\frac{-1}{1+\rho}}\cdot\left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\right]^{\frac{-1}{1+\rho}}\left[\frac{L_s}{L_u}\right]^{\frac{1-\rho}{1+\rho}}.\end{aligned}$$

which can further be written as

$$A_{s,t}(\theta) = \frac{\beta^* \bar{A}_{s,t}}{2}\left[1+\theta+(1-\theta)\left[1\cdot 1_\Phi + \frac{1}{\gamma}\cdot 1_\Psi\right]\right] + (1-\beta^*)\bar{A}_{s,t}\left[\theta+(1-\theta)\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\right]$$

$$= \left[A_u\right]^{\frac{\rho}{1+\rho}}\left[\rho\right]^{\frac{1}{1+\rho}}\left[\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\frac{\partial A_{s,t}(\theta)}{\partial\theta}}\right]^{\frac{-1}{1+\rho}}\cdot\left[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\right]^{\frac{-1}{1+\rho}}\left[\frac{L_s}{L_u}\right]^{\frac{1-\rho}{1+\rho}}. \quad \text{(A1)}$$

From equations (74) and (88), we calculate the ratio

$$\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\frac{\partial A_{s,t}(\theta)}{\partial\theta}} = \frac{(1-\beta^*)\frac{1}{\gamma^2}\bar{A}_{s,t}\left[\frac{1}{2}\cdot\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)L_s\right]\left(\gamma+1\right)}{\frac{\beta^*\bar{A}_{s,t}}{2}\left[1-\left[1\cdot 1_\Phi+\frac{1}{\gamma}\cdot 1_\Psi\right]\right]+(1-\beta^*)\bar{A}_{s,t}\left[1-\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\right]},$$

which can be simplified to

$$\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\frac{\partial A_{s,t}(\theta)}{\partial\theta}} = \frac{\frac{(1-\beta^*)}{2}\frac{\gamma+1}{\gamma^2}\bar{A}_{s,t}\left[\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)L_s\right]}{\frac{\bar{A}_{s,t}}{2}\left[\beta^*\left[1-\left[1\cdot 1_\Phi+\frac{1}{\gamma}\cdot 1_\Psi\right]\right]+2(1-\beta^*)\left[1-\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\right]\right]},$$

which upon cancelling out like terms in the numerator and the denominator, can be written as

$$\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\frac{\partial A_{s,t}(\theta)}{\partial\theta}} = \frac{(1-\beta^*)\cdot\frac{\gamma+1}{\gamma^2}\left[\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)L_s\right]}{\beta^*\left[1-\left[1\cdot 1_\Phi+\frac{1}{\gamma}\cdot 1_\Psi\right]\right]+2(1-\beta^*)\left[1-\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\right]}.$$

In footnote 3, it has been shown that the term $\left[1\cdot 1_\Phi+\frac{1}{\gamma}\cdot 1_\Psi\right]$ will be equal to $\left[\frac{1}{2}\left(0+\frac{1}{\gamma}\right)+\frac{1}{2}\left(1+0\right)\right]=\frac{1+\gamma}{2\gamma}$. Substituting this in the above equation, we obtain

$$\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\frac{\partial A_{s,t}(\theta)}{\partial\theta}} = \frac{(1-\beta^*)\cdot\frac{\gamma+1}{\gamma^2}\left[\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)L_s\right]}{\beta^*\left[1-\left[\frac{1+\gamma}{2\gamma}\right]\right]+2(1-\beta^*)\left[1-\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\right]},$$

which can be simplified to

$$\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\frac{\partial A_{s,t}(\theta)}{\partial\theta}} = \frac{(1-\beta^*)\cdot\frac{\gamma+1}{\gamma^2}\left[\alpha^{\frac{2}{1-\alpha}}\left(\frac{1}{\alpha}-1\right)L_s\right]}{\frac{\beta^*}{2\gamma}\left[2\gamma-(1+\gamma)\right]+\frac{(1-\beta^*)}{\gamma^2}\left[2\gamma^2-(1+\gamma)\right]},$$

which upon factoring out $\frac{1}{2\gamma^2}$ in the denominator, can be written as

$$\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\frac{\partial A_{s,t}(\theta)}{\partial\theta}} = \frac{(1-\beta^*)\cdot\frac{\gamma+1}{\gamma^2}\Big[\alpha^{\frac{2}{1-\alpha}}\Big(\frac{1}{\alpha}-1\Big)L_s\Big]}{\frac{1}{2\gamma^2}\Bigg[\gamma\beta^*\Big[\gamma+1\Big]+2(1-\beta^*)\Big[2\gamma^2-(1+\gamma)\Big]\Bigg]},$$

and on canceling out γ^2 in the numerator and denominator, the above equation can be written as

$$\frac{\frac{\partial B_t(\theta)}{\partial\theta}}{\frac{\partial A_{s,t}(\theta)}{\partial\theta}} = \frac{2(1-\beta^*)\cdot(\gamma+1)\Big[\alpha^{\frac{2}{1-\alpha}}\Big(\frac{1}{\alpha}-1\Big)L_s\Big]}{\gamma\beta^*\Big[\gamma+1\Big]+2(1-\beta^*)\Big[2\gamma^2-(1+\gamma)\Big]},$$

Substituting the above in equation $(A1)$, we obtain

$$\begin{aligned}
A_{s,t}(\theta) = \frac{\beta^*\bar{A}_{s,t}}{2}\Bigg[1+\theta+(1-\theta)\Big[1\cdot 1_\Phi+\frac{1}{\gamma}\cdot 1_\Psi\Big]\Bigg]+(1-\beta^*)\bar{A}_{s,t}\Bigg[\theta+(1-\theta)\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\Bigg] \\
= \Big[A_u\Big]^{\frac{\rho}{1+\rho}}\Big[\rho\Big]^{\frac{1}{1+\rho}}\Bigg[\frac{2(1-\beta^*)\cdot(\gamma+1)\Big[\alpha^{\frac{2}{1-\alpha}}\Big(\frac{1}{\alpha}-1\Big)L_s\Big]}{\gamma\beta^*\Big[\gamma+1\Big]+2(1-\beta^*)\Big[2\gamma^2-(1+\gamma)\Big]}\Bigg]^{\frac{-1}{1+\rho}} \\
\cdot\Bigg[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\Bigg]^{\frac{-1}{1+\rho}}\Bigg[\frac{L_s}{L_u}\Bigg]^{\frac{1-\rho}{1+\rho}},
\end{aligned}$$

which can be simplified to

$$\begin{aligned}
A_{s,t}(\theta) = \frac{\beta^*\bar{A}_{s,t}}{2}\Bigg[1+\theta+(1-\theta)\Big[1\cdot 1_\Phi+\frac{1}{\gamma}\cdot 1_\Psi\Big]\Bigg]+(1-\beta^*)\bar{A}_{s,t}\Bigg[\theta+(1-\theta)\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\Bigg] \\
= \Big[\rho A_u\Big]^{\frac{\rho}{1+\rho}}\Bigg[\frac{2(1-\beta^*)\cdot(\gamma+1)}{\gamma\beta^*\Big[\gamma+1\Big]+2(1-\beta^*)\Big[2\gamma^2-(1+\gamma)\Big]}\Bigg]^{\frac{-1}{1+\rho}} \\
\cdot\Bigg[\alpha^{\frac{2}{1-\alpha}}\Big(\frac{1}{\alpha}-1\Big)L_s\Bigg]^{\frac{-1}{1+\rho}}\cdot\Bigg[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\Bigg]^{\frac{-1}{1+\rho}}\Bigg[\frac{L_s}{L_u}\Bigg]^{\frac{1-\rho}{1+\rho}},
\end{aligned}$$

which can be re-written as

$$\begin{aligned}
A_{s,t}(\theta) = \frac{\beta^*\bar{A}_{s,t}}{2}\Bigg[1+\theta+(1-\theta)\Big[1\cdot 1_\Phi+\frac{1}{\gamma}\cdot 1_\Psi\Big]\Bigg]+(1-\beta^*)\bar{A}_{s,t}\Bigg[\theta+(1-\theta)\frac{1}{2}\cdot\frac{1+\gamma}{\gamma^2}\Bigg] \\
= \Big[\rho A_u\Big]^{\frac{\rho}{1+\rho}}\Bigg[\frac{2(1-\beta^*)\cdot(\gamma+1)}{\gamma\beta^*\Big[\gamma+1\Big]+2(1-\beta^*)\Big[2\gamma^2-(1+\gamma)\Big]}\Bigg]^{\frac{-1}{1+\rho}} \\
\cdot\Bigg[\alpha^{\frac{2}{1-\alpha}}\Big(\frac{1}{\alpha}-1\Big)\Bigg]^{\frac{-1}{1+\rho}}\cdot\Bigg[(1-\alpha)\alpha^{\frac{2\alpha\rho}{1-\alpha}}\cdot\frac{1}{\eta}\Bigg]^{\frac{-1}{1+\rho}}\cdot\Big[L_s\Big]^{\frac{-1}{1+\rho}}\cdot\Bigg[\frac{L_s}{L_u}\Bigg]^{\frac{1-\rho}{1+\rho}},
\end{aligned}$$

which can be simplified to

$$A_{s,t}(\theta) = \frac{\beta^* \bar{A}_{s,t}}{2} \Bigg[1 + \theta + (1-\theta) \Big[1 \cdot 1_\Phi + \frac{1}{\gamma} \cdot 1_\Psi \Big] \Bigg] + (1-\beta^*) \bar{A}_{s,t} \Bigg[\theta + (1-\theta) \frac{1}{2} \cdot \frac{1+\gamma}{\gamma^2} \Bigg]$$
$$= \big[\rho A_u \big]^{\frac{\rho}{1+\rho}} \Bigg[\frac{2(1-\beta^*) \cdot (\gamma+1)}{\gamma \beta^* \Big[\gamma + 1 \Big] + 2(1-\beta^*) \Big[2\gamma^2 - (1+\gamma) \Big]} \Bigg]^{\frac{-1}{1+\rho}}$$
$$\cdot \Bigg[\Big(\frac{1}{\alpha} - 1 \Big) (1-\alpha) \alpha^{\frac{2(1+\alpha\rho)}{1-\alpha}} \cdot \frac{1}{\eta} \Bigg]^{\frac{-1}{1+\rho}} \cdot \Bigg[\frac{\big(L_s \big)^\rho}{\big(L_u \big)^{1-\rho}} \Bigg]^{\frac{1}{1+\rho}},$$

which can be further simplified to

$$A_{s,t}(\theta) = \frac{\beta^* \bar{A}_{s,t}}{2} \Bigg[1 + \theta + (1-\theta) \Big[1 \cdot 1_\Phi + \frac{1}{\gamma} \cdot 1_\Psi \Big] \Bigg] + (1-\beta^*) \bar{A}_{s,t} \Bigg[\theta + (1-\theta) \frac{1}{2} \cdot \frac{1+\gamma}{\gamma^2} \Bigg]$$
$$= \big[\rho A_u \big]^{\frac{\rho}{1+\rho}} \Bigg[\frac{2(1-\beta^*) \cdot (\gamma+1)}{\gamma \beta^* \Big[\gamma + 1 \Big] + 2(1-\beta^*) \Big[2\gamma^2 - (1+\gamma) \Big]} \Bigg]^{\frac{-1}{1+\rho}}$$
$$\cdot \Bigg[\frac{(1-\alpha)^2}{\alpha} \alpha^{\frac{2(1+\alpha\rho)}{1-\alpha}} \cdot \frac{1}{\eta} \Bigg]^{\frac{-1}{1+\rho}} \cdot \Bigg[\frac{\big(L_s \big)^\rho}{\big(L_u \big)^{1-\rho}} \Bigg]^{\frac{1}{1+\rho}},$$

which can also be written as

$$A_{s,t}(\theta) = \frac{\beta^* \bar{A}_{s,t}}{2} \Bigg[1 + \theta + (1-\theta) \Big[1 \cdot 1_\Phi + \frac{1}{\gamma} \cdot 1_\Psi \Big] \Bigg] + (1-\beta^*) \bar{A}_{s,t} \Bigg[\theta + (1-\theta) \frac{1}{2} \cdot \frac{1+\gamma}{\gamma^2} \Bigg]$$
$$= \big[\rho A_u \big]^{\frac{\rho}{1+\rho}} \Bigg[\frac{2(1-\beta^*) \cdot (\gamma+1)}{\gamma \beta^* \Big[\gamma + 1 \Big] + 2(1-\beta^*) \Big[2\gamma^2 - (1+\gamma) \Big]} \Bigg]^{\frac{-1}{1+\rho}}$$
$$\cdot \Bigg[(1-\alpha)^2 \alpha^{\frac{2(1+\alpha\rho) - (1-\alpha)}{1-\alpha}} \cdot \frac{1}{\eta} \Bigg]^{\frac{-1}{1+\rho}} \cdot \Bigg[\frac{\big(L_s \big)^\rho}{\big(L_u \big)^{1-\rho}} \Bigg]^{\frac{1}{1+\rho}},$$

which, on dividing throughout by $\bar{A}_{s,t}$, yields

$$\frac{A_{s,t}(\theta)}{\bar{A}_{s,t}} = \frac{\beta^*}{2} \Bigg[1 + \theta + (1-\theta) \Big[1 \cdot 1_\Phi + \frac{1}{\gamma} \cdot 1_\Psi \Big] \Bigg] + (1-\beta^*) \Bigg[\theta + (1-\theta) \frac{1}{2} \cdot \frac{1+\gamma}{\gamma^2} \Bigg]$$
$$= \bar{A}_{s,t}^{-1} \big[\rho A_u \big]^{\frac{\rho}{1+\rho}} \Bigg[\frac{2(1-\beta^*) \cdot (\gamma+1)}{\gamma \beta^* \Big[\gamma + 1 \Big] + 2(1-\beta^*) \Big[2\gamma^2 - (1+\gamma) \Big]} \Bigg]^{\frac{-1}{1+\rho}}$$
$$\cdot \Bigg[(1-\alpha)^2 \alpha^{\frac{1+\alpha(2\rho+1)}{1-\alpha}} \cdot \frac{1}{\eta} \Bigg]^{\frac{-1}{1+\rho}} \cdot \Bigg[\frac{\big(L_s \big)^\rho}{\big(L_u \big)^{1-\rho}} \Bigg]^{\frac{1}{1+\rho}},$$

which is the equation (112) in our model.

Appendix 2

The comparison of the ratio mentioned in equation (113) is a comparison of the terms contained in

$$\frac{A_{s,t}(\theta=0)}{\bar{A}_{s,t}} = \underbrace{\frac{1}{4\gamma^2}\Bigg[\beta^*(3\gamma^2+1)+2(1-\beta^*)(1+\gamma)\Bigg]}_{Term\ 1},$$

and

$$\frac{A_{s,t}(\theta=\theta_{om})}{\bar{A}_{s,t}} = \bar{A}_{s,t}^{-1}\left[\rho A_u\right]^{\frac{\rho}{1+\rho}} \underbrace{\left[\frac{2(1-\beta^*)\cdot(\gamma+1)}{\gamma\beta^*\Big[\gamma+1\Big]+2(1-\beta^*)\Big[2\gamma^2-(1+\gamma)\Big]}\right]^{\frac{-1}{1+\rho}}}_{Term\ 2}$$
$$\cdot\underbrace{\left[(1-\alpha)^2\alpha^{\frac{1+\alpha(2\rho+1)}{1-\alpha}}\cdot\frac{1}{\eta}\right]^{\frac{-1}{1+\rho}}}_{Term\ 3}\cdot\left[\frac{\left(L_s\right)^{\rho}}{\left(L_u\right)^{1-\rho}}\right]^{\frac{1}{1+\rho}}.$$

We now analyze each of the terms in the two equations. Consider, first, term 1.

$$\frac{1}{4\gamma^2}\Bigg[\beta^*(3\gamma^2+1)+2(1-\beta^*)(1+\gamma)\Bigg].$$

Recall that $0<\beta^*<1$ and $\gamma>1$. We evaluate term 1 at the limits of β^*.
When $\beta^*=1$,

$$\frac{1}{4\gamma^2}(3\gamma^2+1)=\frac{3\gamma^2+1}{4\gamma^2}=\frac{3}{4}+\frac{1}{4\gamma^2}<1.$$

And, when $\beta^*=0$

$$\frac{1}{4\gamma^2}\cdot 2(1+\gamma)=\frac{1+\gamma}{2\gamma^2}<1$$

Term 1 is a convex combination of these two cases, which implies that term 1 will always be less than 1.

We next consider Term 2:

$$\left[\frac{2(1-\beta^*)\cdot(\gamma+1)}{\gamma\beta^*\left[\gamma+1\right]+2(1-\beta^*)\left[2\gamma^2-(1+\gamma)\right]}\right]^{\frac{-1}{1+\rho}}.$$

We first evaluate the expression within the parentheses at the limits of β^*.

When $\beta^* = 1$

$$\frac{2(1-1)(\gamma+1)}{\gamma(\gamma+1)} = \frac{0}{\gamma(\gamma+1)} = 0.$$

And, when $\beta^* = 0$,

$$\frac{2(\gamma+1)}{2\left[2\gamma^2-(1+\gamma)\right]} = \frac{2(\gamma+1)}{2\left[2\gamma^2-1-\gamma\right]} = \frac{\gamma+1}{2\gamma^2-\gamma-1} < 1$$

For intermediate values of β^* where $0 < \beta^* < 1$,

1. Consider the numerator: $2(1-\beta^*)(\gamma+1)$. Since $0 < \beta^* < 1$, $1-\beta^*$ is positive and lies between 0 and 1. Thus, $2(1-\beta^*)(\gamma+1)$ is positive and lies between 0 and $2(\gamma+1)$.

2. Consider the denominator: $\gamma\beta^*(\gamma+1) + 2(1-\beta^*)\left[2\gamma^2-(1+\gamma)\right]$. Since $0 < \beta^* < 1$, the term $\gamma\beta^*(\gamma+1)$ is positive and lies between 0 and $\gamma(\gamma+1)$, and the term $2(1-\beta^*)\left[2\gamma^2-(1+\gamma)\right]$ is positive and lies between $2[2\gamma^2-(1+\gamma)]$ and 0.

Combining these, the denominator lies between $\gamma(\gamma+1)$ and $2[2\gamma^2-(1+\gamma)]$. Therefore, since the expression is equal to 0 when $\beta^* = 1$, equal to $\frac{\gamma+1}{2\gamma^2-\gamma-1} < 1$ when $\beta^* = 0$, and for intermediate values of β^*, the expression within the parentheses lies between 0 and a positive number less than 1.

$$0 < \frac{2(1-\beta^*)(\gamma+1)}{\gamma\beta^*(\gamma+1)+2(1-\beta^*)[2\gamma^2-(1+\gamma)]} < \frac{\gamma+1}{2\gamma^2-\gamma-1} < 1$$

Since the term inside the parentheses is positive and less than 1 for $0 < \beta^* < 1$ and $\gamma > 1$, raising this fraction to the power $\frac{-1}{1+\rho}$ (where $0 < \rho < 1$) results in a value greater than 1.

Therefore, for $0 < \beta^* < 1$, $\gamma > 1$, and $0 < \rho < 1$, the term 2 is greater than 1.

We next consider term 3:

$$\left[(1-\alpha)^2\alpha^{\frac{1+\alpha(2\rho+1)}{1-\alpha}} \cdot \frac{1}{\eta}\right]^{\frac{-1}{1+\rho}}.$$

Recall that $0 < \alpha < 1$ and $\eta > 1$. With these conditions, the expression with the parentheses is positive and less than one. Therefore, being raised to a negative power makes the entire expression greater than 1. Therefore, term 3 is greater than 1 for the given parametric restrictions.

Therefore, a comparison between $\frac{A_{s,t}(\theta=0)}{\bar{A}_{s,t}}$ and $\frac{A_{s,t}(\theta=\theta_{om})}{\bar{A}_{s,t}}$ reduces to

$$Term\ 1\ :\ Term\ 2 \times Term\ 3 \times\ \bar{A}_{s,t}^{-1}\left[\rho A_u\right]^{\frac{\rho}{1+\rho}} \cdot \left[\frac{\left(L_s\right)^{\rho}}{\left(L_u\right)^{1-\rho}}\right]^{\frac{1}{1+\rho}}.$$

On dividing both sides by $Term\ 2 \times Term\ 3$, we obtain

$$\frac{\overbrace{Term\ 1}^{<1}}{\underbrace{Term\ 2}_{>0} \times \underbrace{Term\ 3}_{>1}} : \bar{A}_{s,t}^{-1}\left[\rho A_u\right]^{\frac{\rho}{1+\rho}} \cdot \left[\frac{\left(L_s\right)^{\rho}}{\left(L_u\right)^{1-\rho}}\right]^{\frac{1}{1+\rho}}.$$

The left-hand side of the above ratio is less than 1. If the right-hand side of the above ratio is greater than or equal to 1, then our original comparison results in

$$\frac{A_{s,t}(\theta=0)}{\bar{A}_{s,t}} < \frac{A_{s,t}(\theta=\theta_{om})}{\bar{A}_{s,t}}.$$

On the other hand, if the right-hand side is also less than 1, then we cannot rank between the two distances from the frontier, $\frac{A_{s,t}(\theta=0)}{\bar{A}_{s,t}}$ and $\frac{A_{s,t}(\theta=\theta_{om})}{\bar{A}_{s,t}}$.

Accordingly, for

$$\frac{A_{s,t}(\theta=0)}{\bar{A}_{s,t}} < \frac{A_{s,t}(\theta=\theta_{om})}{\bar{A}_{s,t}},$$

we need to have

$$\bar{A}_{s,t}^{-1}\left[\rho A_u\right]^{\frac{\rho}{1+\rho}} \cdot \left[\frac{\left(L_s\right)^{\rho}}{\left(L_u\right)^{1-\rho}}\right]^{\frac{1}{1+\rho}} \geq 1,$$

which can also be written as

$$\left[\rho A_u\right]^{\frac{\rho}{1+\rho}} \cdot \left[\frac{\left(L_s\right)^{\rho}}{\left(L_u\right)^{1-\rho}}\right]^{\frac{1}{1+\rho}} \geq \bar{A}_{s,t},$$

and on raising the power by $1+\rho$ on both sides, we obtain

$$\left[\rho A_u\right]^{\rho} \cdot \left[\frac{\left(L_s\right)^{\rho}}{\left(L_u\right)^{1-\rho}}\right] \geq \left[\bar{A}_{s,t}\right]^{1+\rho},$$

which can be re-written as

$$\frac{\left[L_s\right]^{\rho}}{\left[\bar{A}_{s,t}\right]^{1+\rho}} \geq \frac{\left[L_u\right]^{1-\rho}}{\left[\rho A_u\right]^{\rho}}$$