On the measurement of two-party competitiveness *

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September 15, 2025

Abstract

Duverger's Law is one of the most celebrated predictions in political science and it states that the simple majority with single ballot system favors the two-party system. However, choosing an appropriate measure to test Duverger's Law has been subject to much discussion. In the literature, many measures have been proposed to operationalize two-party competitiveness and most of them lack the required theoretical foundations. In this paper, we develop a set of axioms that an ideal two-party competitiveness measure should satisfy. We further introduce three quasi orders that order two vote share distributions following a set of meaningful axioms.

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1 Introduction

The nature and evolution of party systems are viewed as the necessary conditions for the effective function of modern democracies. However, over time what kind of party system will emerge in a democracy is largely dependent on locally adopted voting rules to find a winner. In this regard, the French political scientist Duverger (1954) made one of the most celebrated predictions in political science, dubbed later as '*Duverge's Law*', that the simple majority with single ballot system favors the two-party system. A complementary hypothesis to Duverger's original prediction is stated as "both the simple majority with double ballot system and proportional representation favor multi-partism" (Riker, 1982; Benoit, 2006). Because of the winner-take-all nature of electoral outcome in the simple majority with single ballot system (also known as the First-Past-The-Post, FPTP, rule), it creates strong incentives for strategic individual voters to coordinate and avoid voting for hopeless candidates, and this, in turn, makes the electoral outcome biased towards a two-party system (Cox, 1997). Duverger's predictions have been subject to extensive theoretical and empirical research (see Cox (1997) and Taagepera (2007) for a detailed discussion on this). Though Duverger's Law is empirically tested at various levels of electoral outcomes, it is reasonable to believe that the law's validity is meant to be tested at the constituency level as two major parties at the constituency level may differ across regions in a country and result in more than two major parties at the national level (see for examples Gaines and Taagepera, 2013; Chhibber and Kollman, 2009; Grofman, Bowler, and Blais, 2009).

Not many papers have studied the theoretical foundation of two-party competitiveness measure, exceptions being Gaines and Taagepera (2013) and Golosov (2025). The authors discuss many properties that an ideal measure of two-party competitiveness should satisfy. For example, if the vote share distribution is such that it is perfectly bipolar (i.e., $v = (0.5, 0.5)$), then a two-party competitiveness should be maximum. On the contrary, if a single party has all vote share then the vote share distribution is minimum.

Our first contribution is to develop a set of synthetic transfer axioms that would increase two-party competitiveness. We first argue that any vote transfer that increases vote share of the party that ranks second, would increase two-party competitiveness. The second axiom requires any vote transfer to the party with highest votes, where the transfer is executed from any party other the second party, should also increase two-party competitiveness. The third one essentially says that the two-party competitiveness should increase as the residual vote share (vote share of 3rd party onward) is more equally distributed. What it essentially means is whether the same residual vote share is concentrated among fewer parties or dispersed among many parties. This axiom is developed following Gaines and Taagepera, where the authors argue that, the more concentrated it is, in a relative sense it becomes a closer credible alternative to the top two parties and hence two-party competitiveness diminishes. On the contrary, if the same residual vote share is widely dispersed among more parties, then the credibility of small parties as alternatives to the top two parties diminishes and in turn to some extent it increases two-party competitiveness.

We need a final axiom of vote transfer mainly to order vote multi-party systems with equal vote shares. We again borrow an idea from Gaines and Taagepera (2013) where the authors argue that for a multi-party system where all votes are shared equally, two-party competitiveness decreases as the number of parties increases. Thus $v' = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is more two-party competitive then $v = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Note that we can write $v' = (\frac{1}{4} + \frac{1}{12}, \frac{1}{4} + \frac{1}{12}, \frac{1}{4} + \frac{1}{12})$.[1] Thus one may argue that v' is obtained by a sequence of vote transfers where the fourth party becomes obsolete by transferring votes 1/12 to the rest of parties. The transfer to party three should decrease two-party competitiveness whereas the transfer to first and second party would increase two-party

[1]See Figure 1 for a graphical illustration on how two-party competitiveness using Tagepara's Euclidean distance based index (see D(v) in the section 5) with vote share distributions where the number of parties such that all parties get equal vote shares.

competitiveness. Our final axiom that captures a kind of transfer sensitiveness which argues that a equal transfer of votes from any party ranked greater than 3 towards first/second party is valued more than that to transfers within the residual parties.

As the main contribution of this paper we develop three quasi orders that ranks two vote share distributions. A particular measure of two-party competitiveness will always rank two distributions in a specific measure. Nevertheless, it is possible that the same ranking might change when a different measure is considered. The proposed quasi orders can rank two vote share distributions in an identical manner in terms of all two-party competitiveness measures satisfying certain important axioms. To be more specific we associate the quasi orders with transfer axioms discussed in the previous two paragraphs, along with some other axioms with minimal requirements.

The rest of the paper is organized as follows. In section 2 we introduce the notation. Section 3 discusses the axioms. The next section discusses the quasi orders along with the results. Finally, in the last section we highlight the future research plans that we plan to incorporate in this paper.

2 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_1 = \mathbb{N}\backslash\{1\}$. The set of all real numbers is denoted by $\mathbb{R}$. Let $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$, and $\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$. For any $c \in \mathbb{R}$ and $k \in \mathbb{N}$, $c^{[k]}$ denotes the k-dimensional vector with c for each of its component. A permutation matrix is a square matrix where all elements are either 0 or 1, and each row and column has exactly one entry as 1. We denote the set of all permutation matrices of order n by Π_n.

Let $\mathcal{P}$ be the universal set of all parties competing in some elections. Typical elements of $\mathcal{P}$ are denoted by $p, q, \ldots$ Let v be a subset of $\mathcal{P} \times [0, 1]$ such that each element of $\mathcal{P}$ appears at most once in v. Formally, for each $p \in \mathcal{P}$, $|\{(q, x) \in v \mid q = p\}| \leq 1$. We can then define $P_v = \{p \in \mathcal{P} \mid (p, x) \in v$

for some $x \in [0,1]\}$ and, for each $p \in P_v$, we define $v(p)$ by $(p,x) \in v \implies v(p) = x$. If, in addition, $\sum_{p \in P_v} v(p) = 1$, we then say that v is a vote distribution and $v(p)$ represents the vote share obtained by party p, while P_v is the set of parties competing in v. The set $\mathbb{V}$ represents all possible vote distributions given $\mathcal{P}$. By two-party competitiveness, we mean a function $I : \mathbb{V} \to \mathbb{R}$, measuring to what extent the vote distribution v is bi-party competitive.

We will often need to arrange the vote shares in a descending vector, without reference to the parties. To do so, given a vote distribution v, we collect the votes shares in a vector $\overline{v} = (\overline{v}_1, \ldots, \overline{v}_{|v|})$ of real numbers such that $\overline{v}_1 \geq \ldots \geq \overline{v}_{|v|}$ and $v(p) = \overline{v}_{\sigma(p)}$ for some bijection $\sigma : P_v \to [|v|]$ and for every $p \in P_v$. The party p such that $\sigma(p) = i$ is called the i^{th} party.

3 Axiomatic Foundations

We introduce a set of axioms for a two-party competitiveness measure. The first axiom is Anonymity. It requires any character other than vote share vector is irrelevant in measuring two-party competitiveness.

A 1 Anonymity. *For all $v, v' \in \mathbb{V}$, $\overline{v} = \overline{v}' \implies I(v) = I(v')$.*

Our next axiom is Zero-Votes Independence which states that inclusion or exclusion of parties with zero votes, does not impact the measurement of two-party competitiveness. Formally:

A 2 Zero-Votes Independence. *If $p \notin P_v$, then $I(v) = I(v \cup \{(p, 0)\})$.*

The above axiom will be useful in the context of comparing party system with different number of competing parties. To highlight this consider two party systems $v = (.8, .2)$ and $v' = (.6, .2, .2)$. The number of parties in v and v' are different. We can formulate $W = (.8, .2, 0)$, and by Zero-Votes

Independence we have $I(v) = I(w)$. Thus the ordering between W and v' is same as v and v'.

Two-party competitiveness is maximum whenever the underlying party system is perfectly biploar, i.e., the distribution is $v = (0.5, 0.5)$. A sufficient condition to characterize this distribution is $\overline{v}_2 = 0.5$, since $\overline{v}_1 \geq \overline{v}_2$ and $\overline{v}_1 + \overline{v}_2 = 1$. If the entire vote share is enjoyed by a single party, then two-party competitiveness is minimum, we have $\overline{v}_2 = 0$. The next two axioms restricts of all two-party competitiveness that is bounded in $[0, 1]$.

A 3 Upper Bound. *For all* $v \in \mathbb{V}$, $I(v) = 1 \iff \overline{v}_2 = .5$.

A 4 Lower Bound. *For all* $v \in \mathbb{V}$, $I(v) = 0 \iff \overline{v}_2 = 0$.

We now discuss some axioms based on the notion of vote transfer. Since we observe the electoral outcomes in a static framework, the notion of transfer has no meaning literally.[2] Thus vote transfer is synthetic in nature, required only to compare two distinct party systems. In this context, we call the vote transfer as Rank Preserving Progressive (Regressive) Transfer, whenever the donor has higher (lower) vote share than the recipient both before and after the vote transfer. Formally:

Definition 1 *For* $v = (v_1, v_2, .., v_n), v' = (v'_1, v'_2, .., v'_n) \in \mathbb{V}^n$, *and for some* $i, j \in P_v$ *where* $\overline{v}_i > \overline{v}_j$, *we use the notation* $v' = \text{RPPT}_i^j(v)$ *to denote that* v' *is obtained from* v *by a Rank Preserving Progressive Transfer (RPPT) from party* i *to* j, *whenever for any* $0 < \epsilon \leq \frac{\overline{v}_i - \overline{v}_j}{2}$, *we have* $\overline{v}'_i = \overline{v}_i - \epsilon$, $\overline{v}'_j = \overline{v}_j + \epsilon$, *and* $\forall k \in P_v \setminus \{i, j\} : \overline{v}'_k = \overline{v}_k$.

Definition 2 *For* $v = (v_1, v_2, .., v_n), v' = (v'_1, v'_2, .., v'_n) \in \mathbb{V}^n$, *if* $v' = \text{RPPT}_i^j(v)$, *then* v *is said to be obtained from* v' *by a rank preserving regressive transfer (RPRT) from party* j *to* i, *which is denoted by* $v = \text{RPRT}_i^j(v')$.

[2]Such transfer makes sense when we are interested in the measurement of electoral volatility (see Sarkar and Dash, 2023, for further details).

Notice that in both the operations RPPT_i^j and RPRT_i^j, the subscript will always denote the donor whereas the superscript will denote the recipient. In some cases we are also interested in specifying the volume of transfer between parties in the notations of RPPT and RPRT. In this context, we reserve the notation $v' = \text{RPPT}_i^j(v, \epsilon)$, which implies $v' = \text{RPPT}_i^j(v)$ and the volume of transfer is ϵ (obviously $0 < \epsilon \leq \frac{\overline{v}_i - \overline{v}_j}{2}$). Note that $v' = \text{RPPT}_i^j(v, \epsilon) \iff v = \text{RPRT}_j^i(v', \epsilon)$.

We will now introduce the next axiom, which states that any transfer (either RPPT or RPRT) of votes to a second party would enhance two-party competitiveness.

A 5 Increased two-Party Competetion-I. *For all $v, v' \in \mathbb{V}^n$, such that if either $v' = \text{RPPT}_1^2(v)$, or $v' = \text{RPRT}_j^2(v)$ where $j > 2$, then $I(v') > I(v)$.*

We are now focusing on vote transfers where the first party is the recipient. It is important to note that if the second party is the donor, then, according to Increased two-Party Competetion-II two-party competitiveness will decrease.[3] To avoid conflicts with the remaining axioms, we do not allow situations where the second party acts as the donor. Our next axiom requires that any transfer of votes, in which the first party is the recipient, but the donor is any party other than the second party. This type of transfer will ensure an increase in two-party competitiveness.

A 6 Increased two-Party Competetion-II. *For all $v, v' \in \mathbb{V}^n$, if $v' = \text{RPRT}_j^1(v)$ where $j \neq \{1, 2\}$, then $I(v') > I(v)$.*

Our concern is now with the residual vote shares, i.e., all vote share concentrated not among the top two parties. We introduce the the Transfer Insenstiveness axiom, which essentially ignores any redistribution of the residual vote transfers. Formally:

[3] This follows since $v' = \text{RPRT}_j^2(v) \iff v' = \text{RPPT}_2^j(v)$.

A 7 Transfer Insenstiveness. *For all* $v, v' \in \mathbb{V}^n$*, if for any* $j > i > 2$*, we have, either* $v' = \mathrm{RPPT}_j^i(v)$ *or* $v = \mathrm{RPPT}_j^i(v')$*, then* $I(v') = I(v)$*.*

Some people may be concerned with the distribution of residual vote transfer. Gaines and Taagepera (2013) argue that, the more concentrated the residual vote shares is, it becomes a closer credible alternative to the top two parties and hence by-party competitiveness diminishes. On the contrary, if the same residual vote share is widely dispersed among more parties, then the credibility of small parties as alternatives to the top two parties diminishes and in turn to some extent it increases the two-party competitiveness. Therefore, any RPPT from the third party onward will increase by-party competitiveness. Formally:

A 8 Increased two-Party Competetion-III. *For all* $v, v' \in \mathbb{V}^n$*, if* $v' = \mathrm{RPPT}_j^i(v)$ *where* $i > j > 2$*, then* $I(v') > I(v)$*.*

Note that while defining Increased two-Party Competetion-I we use the notion of both RPPT and RPRT. However, in Increased two-Party Competetion-II we use only RPRT. Contrarily in Increased two-Party Competetion-III we use only RPPT.

A 9 Increased two-Party Competetion-IV. *For all* $v, w, v' \in \mathbb{V}^n$*, such that for some* $k > j > 2$ *and* $i \leq 2$*, we have* $w = \mathrm{RPRT}_j^i(v, \epsilon)$ *and* $v' = \mathrm{RPRT}_k^j(w, \epsilon)$*, then* $I(v') > I(v)$*.*

The transfer sensitivity axiom in the inequality measurement literature, states that transfer is valued more if it happens in the bottom of the distribution. In the present scenario, the requirement is transfer towards party 1 or 2 is valued more than any transfers among the residual parties. This axiom will enable us to rank party systems where all parties enjoys equal vote share, we highlight this further in the next two sections.

4 Quasi Orders

Any ordering that is transitive but incomplete as well as irreflexive is said to be a quasi order. In this section we develop three quasi orders which would allow us to rank two party systems identically following a class of two-party competitiveness measures satisfying some essential axioms. We first define majorization of vectors, which we use for defining the quasi orders.

Definition 3 *For all distinct $x = (x_1, x_2, .., x_n), y = (y_1, y_2, .., y_n) \in \mathbb{R}^n$, where $\forall i \in \{1, 2, .., n-1\}$: $x_{i+1} \geq x_i$ and $y_{i+1} \geq y_i$, y weakly majorises x (denoted by $y \succeq_m x$) whenever $\forall k \in \{1, 2, .., n\}$ we have $\sum_{i=1}^{k} y_i \geq \sum_{i=1}^{k} x_i$. A strong majorization of y on x is denoted by $y \succ_m x$, which implies $y \succeq_m x$ and there exist at least one $p \in \{1, 2, .., n\}$ such that $\sum_{i=1}^{p} y_i > \sum_{i=1}^{p} x_i$.*

We now introduce the three quasi orders.

Definition 4 *For all $v, v' \in \mathbb{V}$, the notation $v' \succ_1 v$ implies $(\overline{v}'_2, \overline{v}'_1) \succ_m (\overline{v}_2, \overline{v}_1)$.*

Definition 5 *For all $v, v' \in \mathbb{V}$, the notation $v' \succ_2 v$ implies $(\overline{v}'_2, \overline{v}'_1, 1 + \overline{v}'_n, 1 + \overline{v}'_{n-1}, .., 1 + \overline{v}'_3) \succ_m (\overline{v}_2, \overline{v}_1, 1 + \overline{v}_n, 1 + \overline{v}_{n-1}, .., 1 + \overline{v}_3)$.*

Definition 6 *For all $v, v' \in \mathbb{V}$, the notation $v' \succ_3 v$ implies $(2\overline{v}'_2, 2\overline{v}'_1, 2 + \overline{v}'_n, 2 + \overline{v}'_{n-1}, .., 2 + \overline{v}'_3) \succ_m (2\overline{v}_2, 2\overline{v}_1, 2 + \overline{v}_n, 2 + \overline{v}_{n-1}, .., 2 + \overline{v}_3)$.*

It is easy to check that all the quasi orders are transitive, i.e., for all $v, w, v' \in \mathbb{V}^n$: and $\forall i\{1, 2, 3\}$: $v' \succ_i w$ and $w \succ_i v \implies v' \succ_i v$. The ordering is irreflexive because it is impossible to have $v \succ_i v$. Finally, the ordering is incomplete, if the vectors required for defining $\succ_m$ exhibits $>$ for some values and $<$ for some other values. We represent such cases by $v' \not\succ_i v$.

Notice that the first quasi order is not concerned with the residual parties, whereas $\succ_2$ and $\succ_3$ is concerned with all the parties. We first show that $\succ_1$ is the weakest condition followed by $\succ_3$ and $\succ_2$. Formally:

Proposition 1 *For all* $v, v' \in \mathbb{V}$*,* $v' \succ_2 v \implies v' \succ_3 v \implies v' \succ_1 v$*.*

Proof. The proof is straightforward and is thus omitted. □

The implication of this proposition is that the chances of getting incomplete results is maximum for $\succ_2$, whereas minimum for $\succ_3$. We further stretch this point in the last two paragraphs of this section.

Before we formally introduce the association between $\succ_1$ and the set of two-party competitiveness measures, we introduce a lemma that essentially shows that following any index that satisfies Anonymity, Zero-Votes Independence, and Transfer Insenstiveness, would exhibit same degree of vote share for any $v \in \mathbb{V}^n$ to that of another vector where the residual votes in v is clubbed with the third party. Formally:

Lemma 1 *For all* $v \in \mathbb{V}^n$*, and for any* $I : \mathbb{V} \mapsto \mathbb{R}$ *that satisfies Anonymity, Zero-Votes Independence, and Transfer Insenstiveness, we have* $I(v) = I(v')$*, where* v' *is any permutation of the vector* $(\overline{v}_1, \overline{v}_2, 1 - \overline{v}_1 - \overline{v}_2) \in \mathbb{V}^3$*.*

Proof. We prove this lemma separately for $n = 2$ and $n > 2$. If $n = 2$, then $\overline{v}_1 + \overline{v}_2 = 1$. By Anonymity we have $I(v) = I(\overline{v}_1, \overline{v}_2)$, then $v = (\overline{v}_1, \overline{v}_2)$ and $v' = (\overline{v}'_1, \overline{v}'_2, 0)$. By Zero-Votes Independence we have $I(v) = I(v')$. If $n > 2$, then by Anonymity we have $I(v') = I(\overline{v}_1, \overline{v}_2, ..., \overline{v}_n)$. Define $\tilde{v} = (\overline{v}_1, \overline{v}_2, 1 - \overline{v}_1 - \overline{v}_2, 0^{[n-3]})$. By Transfer Insenstiveness $I(v') = I(\tilde{v})$. By Zero-Votes Independence $I(\tilde{v}) = I(v)$. Therefore $I(v) = I(v')$. □

We now introduce a quasi order and establish the equivalence with a set of axioms including Transfer Insenstiveness.

Theorem 1 *For all* $v, v' \in \mathbb{V}$*, and for any* $I : \mathbb{V} \mapsto \mathbb{R}$ *that satisfies Anonymity, Zero-Votes Independence, Increased two-Party Competetion-I, Increased two-Party Competetion-II, and Transfer Insenstiveness, then following conditions are equivalent* $v' \succ_1 v \iff I(v') \geq I(v)$*.*

Proof. Following Lemma 1, if I satisfies Anonymity, Transfer Insenstiveness, and Zero-Votes Independence, then $I(v) = I(\overline{v}_1, \overline{v}_2, 1-\overline{v}_1-\overline{v}_2)$ and $I(v') = I(\overline{v}'_1, \overline{v}'_2, 1-\overline{v}_1-\overline{v}'_2)$. Hence in the rest of the proof we restrict our attention only to the party systems $v = (\overline{v}_1, \overline{v}_2, 1-\overline{v}_1-\overline{v}_2), v' = (\overline{v}'_1, \overline{v}'_2, 1-\overline{v}_1-\overline{v}'_2) \in \mathbb{V}^3$.

if: Given $(\overline{v}'_2, \overline{v}'_1) \succ_m (\overline{v}_2, \overline{v}_1) \implies \overline{v}'_2 \geq \overline{v}_2$, and $\overline{v}'_1 + \overline{v}'_2 \geq \overline{v}_1 + \overline{v}_2$, with at least one strict inequality. We can write for some $\epsilon_1, \epsilon_2 \in \mathbb{R}_+$ and $\max(\epsilon_1, \epsilon_2) > 0$: $\overline{v}'_2 = \overline{v}_2 + \epsilon_2$, and $\overline{v}'_1 + \overline{v}'_2 = \overline{v}_1 + \overline{v}_2 + \epsilon_1 \implies \overline{v}'_1 = \overline{v}_1 + \epsilon_1 - \epsilon_2$. Hence, $v' = (\overline{v}_1 + \epsilon_1 - \epsilon_2, \overline{v}_2 + \epsilon_2, 1 - \overline{v}_1 - \overline{v}_2 - \epsilon_1)$. Define $w = (\overline{v}_1 - \epsilon_2, \overline{v}_2 + \epsilon_2, 1 - \overline{v}_1 - \overline{v}_2) \in \mathbb{V}^3$. If $\epsilon_1 > 0$ and $\epsilon_2 > 0$, then $w = \text{RPRT}_1^2(v, \epsilon_2)$ and $v' = \text{RPPT}_3^1(w, \epsilon_1)$. Therefore, by Increased two-Party Competetion-I, we have $I(w) > I(v)$. By Increased two-Party Competetion-II, we have $I(v') > I(w)$. Hence, by transitivity, we have $I(v') > I(v)$. If $\epsilon_1 = 0$, we use only Increased two-Party Competetion-I to prove this part. Similarly, we require only Increased two-Party Competetion-II to establish $I(v') > I(v)$ whenever, $\epsilon_2 = 0$.

Only if: We prove this by contradiction. We begin with the assumption that either a) $\overline{v}'_2 < \overline{v}_2$ or b) $\overline{v}'_1 + \overline{v}'_2 < \overline{v}_1 + \overline{v}_2$. Define the function $D : \mathbb{V} \mapsto \mathbb{R}$ such that $D(v) = \beta_1 \overline{v}_1 + (\beta_1 + \beta_2)\overline{v}_2 = \beta_1(\overline{v}_1 + \overline{v}_2) + \beta_2 \overline{v}_2$, where $\beta_1 > 0$ and $\beta_2 > 0$. It is easy to check that D satisfies all the axioms required for this Theorem. If $\overline{v}'_2 < \overline{v}_2$ then we choose β_2 high enough to get a contradiction. Contrariwise, if $\overline{v}'_1 + \overline{v}'_2 < \overline{v}_1 + \overline{v}_2$, we need to choose β_1 high enough to get $D(v) > D(v')$. □

We now introduce our next theorem, which essentially replace the Transfer Insenstiveness axiom by Increased two-Party Competetion-III.

Theorem 2 *For all distinct $v, v' \in \mathbb{V}$, and for any $I : \mathbb{V} \mapsto \mathbb{R}$ that satisfies Anonymity, Zero-Votes Independence, Increased two-Party Competetion-I, Increased two-Party Competetion-II, and Increased two-Party Competetion-III, $v' \succ_2 v \iff I(v') \geq I(v)$.*

Proof.

Let $v \in \mathbb{V}^{n_1}$ and $v' \in \mathbb{V}^{n_2}$. We can make the number of parties equal to $n = \max(n_1, n_2)$, by incorporating $n - \min(n_1, n_2)$ zeros to the party system with lower number of competing parties. Such inclusion will not impact $I(\cdot)$, whenever I satisfies Zero-Votes Independence. Anonymity ensures: $I(v) = I(\overline{v}_1, \overline{v}_2, .., \overline{v}_n)$ and $I(v') = I(\overline{v}'_1, \overline{v}'_2, .., \overline{v}'_n)$. In the rest of this proof we restrict our attention only to those indices that satisfies both Zero-Votes Independence and Anonymity. Therefore we restrict our attention only on $v = (\overline{v}_1, \overline{v}_2, .., \overline{v}_n), v' = (\overline{v}'_1, \overline{v}'_2, .., \overline{v}'_n) \in \mathbb{V}^n$.

Let $x = (x_1, .., x_n) = (\overline{v}_2, \overline{v}_1, 1 + \overline{v}_n, 1 + \overline{v}_{n-1}, .., 1 + \overline{v}_3)$. Thus $x_1 = \overline{v}_2$, $x_2 = \overline{v}_1$, and $\forall i \geq 3$: $x_i = 1 + \overline{v}_{n+3-i}$. We define the cumulative sum of the vectors $X = (X_1, X_2, .., X_n)$, where $X_i = \sum_{j=1}^{i} x_i$.

if

$v' \succ_2 v \implies x' \succ_m x$. Following Lemma 2 we can write there exists a sequence vectors K: $x = x^1, x^2, .., x^K = x'$, where $\forall a \in \{2, 3, .., K\}$, where for some $0 < \epsilon \leq \frac{x_{a+1} - x_a}{2}$, we have $x_j^{a+1} = x_j^a - \epsilon$, $x_i^{a+1} = \overline{v}_i + \epsilon$, and $\forall k \in P_v \setminus \{i, j\} : x_k^{a+1} = x_k^a$. Let the vote share vectors corresponding to $x = x^1, x^2, .., x^K = x'$ be $v = v^1, v^2, .., v^K = v'$ and $\forall i \in \{1, 2, .., K\}$: $v^i \in \mathbb{V}^n$, respectively. When we consider the vote share vectors v^a and v^{a+1}, $\forall s \in 1, 2$ and $\forall t_1, t_2 \in \{3, 4, .., n\}$ where $t_2 > t_1$, we have one of the following three possibilities: 1) $v^{a+1} = \text{RPPT}_1^2(v^a)$, 2) $v^{a+1} = \text{RPRT}_{t_1}^s(v^a)$, $v^{a+1} = \text{RPPT}_{t_1}^{t_2}(v^a)$. Thus, $I(v^{a+1}) > I(v^a)$ for (1), (2), and (3) following the axioms Increased two-Party Competetion-I, Increased two-Party Competetion-II, and Increased two-Party Competetion-III, respectively. Hence, $I(v) < I(v^1) < I(v^2) < .. < I(v^K) = I(v')$. Therefore by transitivity, we have $I(v') > I(v)$ where I satisfies Increased two-Party Competetion-I, Increased two-Party Competetion-II, and Increased two-Party Competetion-III.

Only if

We prove this by contradiction, thus we assume either $x \succ_m x'$ or $x' \not\succ_m x$. In the first case we get a contradiction $I(v) > I(v')$, since all I that satisfies the axioms will exhibit this inequality (directly follows from the *if* part). We now focus on the non-trivial part, i.e., $x' \not\succ_m x$. Given v and v' are distinct, $x' \not\succ_m x$ implies we must have for some $p \in \{1, 2, .., n\}$: $X_p > X'_p$. To get a contradiction we define the following function $D : \mathbb{V} \mapsto \mathbb{R}$, where $D(v) = \sum_{i=1}^{n} x_i w_i$, such that $\forall i \in \{1, 2, .., n\}$: $w_i = \sum_{j=i}^{n} \beta_j$, where $\beta_i > 0$. It is easy to check that $D(v)$ satisfies all the axioms required for this theorem. We can also write $D(v) = \sum_{i=1}^{n} \beta_i X_i$. To get a contradiction, i.e., $D(v) > D(v')$, we need to choose β_p high enough. □

Our next result establishes an equivalence between the third quasi order ($\succ_3$) to any two-party competitiveness measures that satisfies all the axioms underlined in Theorem 2 along with Increased two-Party Competetion-IV.

Theorem 3 *For all distinct $v, v' \in \mathbb{V}$, and for any $I : \mathbb{V} \mapsto \mathbb{R}$ that satisfies Anonymity, Zero-Votes Independence, Increased two-Party Competetion-I, Increased two-Party Competetion-II, Increased two-Party Competetion-III, and Increased two-Party Competetion-IV we have $v' \succ_3 v \iff I(v') \geq I(v)$.*

Proof. We restrict the case where $v' = (\overline{v}'_1, \overline{v}'_2, .., \overline{v}'_n), v = (\overline{v}_1, \overline{v}_2, .., \overline{v}_n) \in \mathbb{V}^n$. The generalization is same as Theorem 2. Define $\delta = \overline{v}_1 + \overline{v}_2$, $\delta' = \overline{v}'_1 + \overline{v}'_2$, and $\theta = \delta' - \delta$. We frequently use the vectors $X = (X_1, X_2, .., X_n)$ and $X = (X'_1, X'_2, .., X'_n)$, introduced in Theorem 2. For any $a, b \in P_v$, where $a < b$, we define $\Gamma_a^b = \sum_{i=a}^{b} (\overline{v}'_i - \overline{v}_i)$.

if

Following Proposition 1, we have $v' \succ_3 v \implies v' \succ_2 v$. If $v' \succ_2 v$, then this part is a simple extension of Theorem 2. Hence we restrict our attention to $v' \succ_3 v$ but $v' \not\succ_2 v$. Therefore, there exists some $i \in P_v$ such that $X'_i < X_i$. We first consider i as the only point where $X'_i < X_i$, i.e., $\forall j \in P \setminus \{i\}$, we have $X'_j \geq X_j$. Notice that $v' \succ_3 v \implies \forall i \in P_v : X'_i + \theta \geq X_i$. Let $X'_i - X_i = -\delta_i$ where $\delta_i > 0$. Therefore $\delta_i \leq \theta$. We rule out the possibility $i \leq 3$, because

$i \leq 2$ would imply $v' \not\succ_2 v$ and $i = 3 \implies \delta' + \sum_{j=i|i\geq 3}^{n} \overline{v}'_j = \delta + \sum_{j=i|i\geq 3}^{n} \overline{v}_j = n-2 \implies X'_i = X_i$. Also note that, $v' \succ_3 v$ and $v' \not\succ_2 v \implies \theta > 0$, because if $\theta = 0$, then $v' \succ_3 v \implies v' \succ_2 v$, and $\theta < 0 \implies v' \not\succ_3 v$.

It is easy to check that $X'_i - X_i = -\delta_i \implies \overline{v}'_i - \overline{v}_i = -\delta_i - \theta - \Gamma^n_{i+1}$. Thus we may argue there is a transfer of θ from i to parties 1 and/or 2, a transfer of Γ^n_{i+1} from i to any party(parties) $i+1$ to n, and a transfer of δ_i from i to any party j with rank $2 < j \leq i-1$. We call the last transfer as Anti-IB transfer. Let A_i denotes Anti-IB transfer from party i, here $A_i = \max(0, \delta_i) = \delta_i$. It is easy to check for other parties $j \in P_v \setminus \{i\}$: $A_j = 0$. To satisfy Increased two-Party Competetion-IV we must ensure $A_i \leq \theta$, which is ensured since $\theta \geq \delta_i$.

We now assume that $X'_j \geq X_j$, and $X'_t < X_t$, and $\forall j \in P_v \setminus \{i, t\}$, and $X'_i < X_i$. Let $X'_t - X_t = -\delta_t$. Thus $\overline{v}'_t - \overline{v}_t = -\delta_t - \theta - \Gamma^n_{t+1} = (-\delta_i - \Gamma^n_{i+1} - \theta) - (\delta_t - \delta_i) - (\Gamma^n_{t+1} - \Gamma^n_{i+1}) = (\overline{v}'_t - \overline{v}_t) - (\delta_t - \delta_i) - (\Gamma^n_{t+1} - \Gamma^n_{i+1})$. Now $\Gamma^n_{t+1} - \Gamma^n_{i+1} = \Gamma^{i-1}_{t+1} + (\overline{v}'_i - \overline{v}_i) \implies \overline{v}'_t - \overline{v}_t = -(\delta_t - \delta_i) - \Gamma^{i-1}_{t+1}$. Thus $A_t = \max(0, \delta_t - \delta_i)$, which implies if $\delta_t > \delta_i$, we have anti-IB transfer of $\delta_t - \delta_i$, else there is no anti-IB transfer from party t. Now $A_i + A_t = \delta_i + \max(\delta_t - \delta_i) = \max(\delta_i, \delta_t)$. Given $v' \succ_m v$ implies $\theta \geq \max(\delta_i, \delta_t)$, which ensures $I(v') > I(v)$ by successive applications of all the axioms.

To complete this part of the proof we consider the general case where there exists the following set of parties $P = \{i_1, i_2, ..., i_r\} \subset P_v$ such that for all $p \in P$: $X'_p < X_p$ and $\forall j \in P_v \setminus \{P\} : X'_j \geq X_j$. Let $\forall i_a, i_b \in P$ such that $b > a \implies \overline{v}_{i_a} > \overline{v}_{i_b}$. Following the previous cases we can write:

$$\overline{v}'_{i_1} - \overline{v}_{i_1} = -\Gamma^n_{i_1+1} - \theta - \delta_{i_1}$$

$$\overline{v}'_{i_2} - \overline{v}_{i_2} = -\Gamma^{i_1-1}_{i_2+1} - (\delta_{i_2} - \delta_{i_1})$$

Therefore we can also write $\overline{v}'_{i_3} - \overline{v}_{i_3} = -\Gamma^n_{i_3+1} - \theta - \delta_{i_3} = (-\Gamma^n_{i_1+1} - \theta - \delta_{i_1}) - \Gamma^{i_1-1}_{i_3+1} - (\overline{v}'_{i_1} - \overline{v}_{i_1}) = -\Gamma^{i_1-1}_{i_3+1} - (\delta_{i_3} - \delta_{i_1})$. We can express $\Gamma^{i_1-1}_{i_3+1} =$

$\Gamma^{i_1-1}_{i_2+1} + \Gamma^{i_2-1}_{i_3+1} + (\overline{v}'_{i_2} - \overline{v}_{i_2}) = -\Gamma^{i_2-1}_{i_3+1} - (\delta_{i_2} - \delta_{i_1})$. Hence

$$\overline{v}'_{i_3} - \overline{v}_{i_3} = -\Gamma^{i_2-1}_{i_3+1} - (\delta_{i_2} - \delta_{i_3}).$$

Hence, we can generalize this expression for any $k \in \{1, 2, .., r\}$ as follows:

$$\overline{v}'_{i_k} - \overline{v}_{i_k} = -\Gamma^{i_k-1}_{i_k+1} - (\delta_{i_k} - \delta_{i_{k-1}}).$$

Therefore, $A_{i_1} = \delta_{i_1}$ and $\forall j \in P \setminus \{i_1\} : A_{i_j} = \max(0, \delta_{i_j} - \delta_{i_{j-1}})$. It is easy to check that

$$\sum_{q=1}^{r} A_{i_q} = \max\left(\delta_{i_1}, \delta_{i_2}, .., \delta_{i_r}\right)$$

.

Given $v' \succ_3 v \implies \sum_{q=1}^{r} A_{i_q} \leq \theta$. Thus for any I that satisfies the underlying axioms implies $I(v') > I(v)$.

Only if

We prove this by contradiction, we consider the non-trivial case and assume $v' \not\succ_3 v$. Given v and v' are distinct, it implies we must have for some $p \in P_v$ such that $X'_p < X_p$. To get a contradiction we define the following function $D : \mathbb{V} \mapsto \mathbb{R}$, where $D(v) = 2w_1\overline{v}_2 + 2w_2\overline{v}_1 + \sum_{j=i|i\geq 3}^{n} \overline{v}_j w_j$, such that $\forall i \in \{1, 2, .., n\}$: $w_i = \sum_{j=i}^{n} \beta_j$, where $\beta_i > 0$. It is easy to check that $D(v)$ satisfies all the axioms required for this theorem. We can also write $D(v) = \sum_{i=1}^{n} \beta_i X_i$. To get a contradiction, i.e., $D(v) > D(v')$, we need to choose β_p high enough. □

One aspect of quasi orders is that while comparing two vote share distributions we may end up getting incomplete results. Given $\succ_1$ is the weakest of the three orders, therefore, the chances of getting conclusive results are also maximum. On the contrary, the chances of getting conclusive results are minimum for $\succ_2$. To cite an example, consider $v = (1/n, 1/n, ..., 1/n) \in \mathbb{V}^{(n)}$

and $v' = (\frac{1}{n-1}, \frac{1}{n-1}, .., \frac{1}{n-1}) \in \mathbb{V}^{n-1}$. Following Gaines and Taagepera (2013) since the later is closer to a perfectly bipolar distribution, it must exhibit higher two-party competitiveness. Unfortunately, these two vote share distributions cannot be ordered using $\succ_2$, but can be ordered using $\succ_3$. More specifically, we need Increased two-Party Competetion-IV in addition to all axioms outlined in Theorem 2 to ensure $I(v') > I(v)$. Thus the weaker orders restrict the domain of the two-party competitiveness measures.

We would like to highlight here that incomplete cases should not be viewed as a limitation of quasi-orders; rather, they reflect an inherent characteristic of the underlying vote share distributions. A similar perspective was presented by Sen (1973) when he discussed the inconclusive cases of Lorenz dominance in relation to the inequality and welfare ordering of income distributions.

5 Discussion

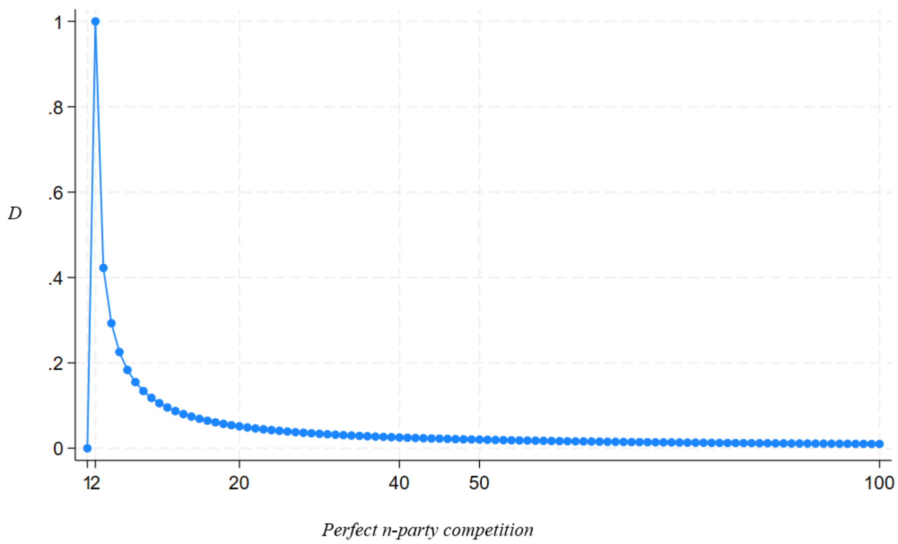
We limited our discussion only to quasi orders. As a future research plan we are interested in characterization the following index developed by Gaines and Taagepera (2013)

$$D(v) = 1 - \sqrt{2} \times \sqrt{(.5 - \overline{v}_1)^2 + (.5 - \overline{v}_2)^2 + \sum_{k=3}^{n} \overline{v}_k^2}$$

.

It is easy to check that D satisfies all the axioms discussed in section 3. For any $v \in \mathbb{V}^n$ we define the vote share vector a Perfect n-party competitive whenever all the n parties gets same vote share i.e., $v = (1/n, .., 1/n) \in \mathbb{V}^n$. D takes value zero if the number of parties is 1. This is inline with Axiom 4. Similarly, for a perfect bi-polar distribution we have the maximum value of D (D=1) in accordance to Axiom 3. Notice that as n increases two-party competitiveness decreases. A point that we already have highlighted in the introductory section of the paper. We further illustrate this in the following diagram which plots D for different perfect n-party competition.

Figure 1: Multiparty Competition using D_1



Following the tradition of axiomatically understanding indices, in this paper we make an attempt to study the properties of D that are both necessary and sufficient to characterize it. Additionally, we also plan to demonstrate that the underlying axioms are logically independent, ensuring that they are neither redundant nor contradictory. Finally, we evaluate the existing two-partyness measures through our proposed axiomatic structure and propose alternative indices satisfying this structure.

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6 Appendix

We introduce the following Lemma originally appeared in Dasgupta, Sen, and Starrett (1973).

Lemma 2 (HLP) *Given two distinct vectors* $x = (x_1, x_2, ..., x_n)$ *and* $y = (y_1, y_2, .., y_n)$*, where* $\forall i \in [n-1]$*, we have* $x_{i+1} \geq x_i$ *and* $y_{i+1} \geq y_i$*, and* $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$*. The the following conditions are equivalent.*

1. $x \succ_m y$
2. *y can be obtained from x by a non-empty finite sequence of transformations of the form:*

$$
\begin{aligned}
x_i^{\alpha+1} &= x_i^{\alpha} + e^{\alpha} \leq x_j^{\alpha}, \quad j > i \\
x_j^{\alpha+1} &= x_j^{\alpha} - e^{\alpha} \geq x_i^{\alpha}, \quad e^{\alpha} > 0 \\
x^{\alpha+1} &= x^{\alpha}, \quad \text{if } k \neq i, j
\end{aligned}
$$

3. *There exists a bistochastic matrix* B_n *of order n which is not a permutation matrix such that* $y = B_n x$*.*[4]
4. *For any strictly concave function* $U : \mathbb{R} \mapsto \mathbb{R}$*, we have* $\sum_{i=1}^{n} U(y_i) > \sum_{i=1}^{n} U(x_i)$*.*

[4] A bistochatsic matrix is a square matrix such that sum of all the rows and column are 1. A permutation matrix is a special case of bistochatsic matrix.