PREFERENCE FOR FLEXIBILITY AND STOCHASTIC PREFERENCE

Shasikanta Nandeibam[1]
Department of Economics
University of Bath
Bath BA2 7AY
United Kingdom
E-mail: s.nandeibam@bath.ac.uk

ABSTRACT

In this note, using stochastic preference, we provide an alternative characterization of the *indirect stochastic dominance* axiom of Nehring (1999), which captures some notion of "preference for flexibility" in the context of preference over opportunity acts. At the same time, we also provide an alternative characterization of rationalizability of probabilistic choice functions by using the indirect stochastic dominance axiom. These characterizations show that there is some link between the underlying structures of the two problems.

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1. INTRODUCTION

An important issue that has received attention in the literature on preference over *opportunity sets* is the *preference for flexibility.* The classic paper on this topic is Kreps (1979), which considered an axiomatic treatment of the class of preferences over opportunity sets that can be represented in terms of the *expected indirect utilities* of the opportunity sets.[2] Nehring (1999) extented the analysis of Kreps (1979) to a stochastic setting by considering *opportunity acts* that map states to opportunity sets. He studied preferences over opportunity acts that satisfy the Savage axioms and a new preference for flexibility axiom called *indirect stochastic dominance.* Indirect stochastic dominance is an axiom which captures the notion that having "more opportunities in expectation" is preferable. The main result of Nehring (1999) shows that a preference relation over opportunity acts that satisfies the Savage axioms and indirect stochastic dominance has an additive representation in terms of an expected indirect utility function (with respect to an implicit state space) over the opportunity sets. This result contains a characterization of the indirect stochastic dominance axiom in the following sense: An additively representable preference relation over opportunity acts (i.e. the Savage axioms hold) satisfies indirect stochastic dominance if and only if the representation is in terms of an expected indirect utility function over the opportunity sets. A purpose of this note is to provide an alternative characterization of the indirect stochastic dominance axiom by using measures defined on the class of *linear orderings* over the set of alternatives augmented by a dummy alternative.

Another purpose of this note is to present an alternative characterization of the problem of rationalizability of stochastic choice behaviour, which was posed by Block and Marshak (1960). In this literature, the choice behaviour of an agent is represented by a function which maps each feasible set (which is a nonempty subset of the universal set of alternatives) to a lottery over the feasible set. The rationalizability problem is, when can this choice behaviour be induced by a probability measure over the set of linear orderings on the universal set of alternatives? An answer to this problem was provided by Falmagne (1978) (see also Barbera and Pattanaik (1986)). We present an alternative necessary and sufficient condition for this problem by requiring indirect stochastic dominance on a class of preference relations over opportunity acts which are derived from the choice behaviour of the agent. Our characterizations show that there is some underlying structural link between the notion of preference for flexibility as captured in indirect stochastic dominance and the problem of rationalizability of stochastic choice behaviour.

2. PREFERENCE FOR FLEXIBILITY

In this section we will first outline the setting developed by Nehring (1999) to consider preference for flexibility.

[2]Two other closely related papers are Koopmans (1965) and Jones and Ostroy (1984).

Some notation and definitions:

X: the finite set of alternatives.

$\mathcal{A}$: the set of nonempty subsets of X (opportunity sets)

Θ: the space of explicit states θ

$\mathcal{F}$: the class of opportunity acts $f : \Theta \to \mathcal{A}$

$[f, E; g, E^c]$: the act h such that, for $\theta \in \Theta$,

$$h(\theta) = \begin{cases} f(\theta) & \text{if } \theta \in E \\ g(\theta) & \text{if } \theta \in E^c \end{cases}$$

$\mathcal{U}$: the class of utility functions $u : \mathcal{A} \to \Re$

$\succeq$: a preference relation on $\mathcal{F}$

Given a finitely additive measure μ on 2^{Θ}, for each finitely ranged function $t : \Theta \to \Re$, define

$$\int t(\theta)d\mu = \sum_{r \in t(\Theta)} r\mu(\{\theta \in \Theta : t(\theta) = r\}).$$

Definition 1: A preference relation $\succeq$ on $\mathcal{F}$ is representable by a nonconstant utility function $u \in \mathcal{U}$ if there exists a finitely additive, convex-ranged[3] probability measure μ on 2^{Θ} such that, for all $f, g \in \mathcal{F}$,

$$f \succeq g \text{ if and only if } \int u(f(\theta))d\mu \geq \int u(g(\theta))d\mu.$$

Definition 2: Given opportunity acts $f, g \in \mathcal{F}$, f *indirectly stochastically dominates* g with respect to an ordering R on X if and only if, for all $x \in X$,

$$\{\theta \in \Theta : f(\theta) \cap \{y \in X | yRx\} \neq \emptyset\} \supseteq \{\theta \in \Theta : g(\theta) \cap \{y \in X | yRx\} \neq \emptyset\}.$$

Indirect Stochastic Dominance (ISD): A preference relation $\succeq$ over $\mathcal{F}$ satisfies *indirect stochastic dominance* if and only if, for all $f, g \in \mathcal{F}$, $f \succeq g$ whenever f indirectly stochastically dominates g for every ordering R on X.[4]

Definition 3: A nonconstant utility function $u \in \mathcal{U}$ is an *expected indirect utility* (EIU) function if there exist a finite collection of utility functions $\{v_\xi : \xi \in \Xi\} \subset \mathcal{U}$ and nonnegative weights $\{\lambda_\xi : \xi \in \Xi\}$ with $\sum_{\xi \in \Xi} \lambda_\xi = 1$ such that,[5] for all $A \in \mathcal{A}$,

$$u(A) = \sum_{\xi \in \Xi} \lambda_\xi [\max_{x \in A} v_\xi(\{x\})].$$

We can now state a weaker version of the representation theorem of Nehring (1999) as a chracterization of ISD in terms of EIU.

[3] μ is convex-ranged if, for all $E \subseteq \Theta$ and all $\rho \in [0, 1]$, there exists $F \subseteq E$ such that $\mu(F) = \rho\mu(E)$.

[4] For a discussion of how ISD embodies preference for flexibility in the sense of having "more opportunities in expectation" is better, see Nehring (1999).

[5] Nehring (1999) treats the finite index set Ξ as some preference determining implicit state space.

Theorem 1: *Suppose $\succeq$ is a preference relation over $\mathcal{F}$ that is representable by a nonconstant utility function $u \in \mathcal{U}$. Then $\succeq$ satisfies* ISD *if and only if u is an* EIU *function.*

In order to present our alternative characterization of ISD, we will introduce a dummy alternative denoted by d (i.e. d does not belong to X). Denote by $\mathcal{R}$ the set of all *linear orderings* (i.e. reflexive, connected, transitive and anti-symmetric binary relations) on $X \cup \{d\}$.

For each $A \in 2^X$, let

$$\mathcal{R}(d; A) = \{R \in \mathcal{R} : \ dRx \text{ for all } x \in A^c\}$$

where we use the notation A^c to denote the complement of A in X.

Given a utility function $u \in \mathcal{U}$, let $\mathcal{U}(u)$ be the class of utility functions in $\mathcal{U}$ that are positive affine transformations of u. Then it is clear from Definitions 1 and 3 that: (i) whenever a preference relation $\succeq$ over $\mathcal{F}$ is representable by a nonconstant utility function $u \in \mathcal{U}$, $\succeq$ is also representable by every utility function in $\mathcal{U}(u)$; (ii) whenever a nonconstant utility function $u \in \mathcal{U}$ is an EIU function, each utility function in $\mathcal{U}(u)$ is also an EIU function.

Theorem 2: *Suppose $\succeq$ is a preference relation over $\mathcal{F}$ that is representable by a nonconstant utility function $u \in \mathcal{U}$. Then $\succeq$ satisfies* ISD *if and only if there exist a probability measure m on $2^{\mathcal{R}}$ and a utility function $\hat{u} \in \mathcal{U}(u)$ such that $[1 - \hat{u}(A^c)] = m(\mathcal{R}(d; A))$ for all $A \in 2^X \setminus \{X\}$.*

The proof of Theorem 2, which is our characterization of ISD, is given in the appendix. The proof relies on an intermediate result of Nehring (1999) which provides a characterization of EIU in terms of the notion of monotonicity and total submodularity of a utility function.

For each $A \in 2^X$, $m(\mathcal{R}(d; A))$ could be interpreted as the probability that the dummy d is ranked best in the set $A^c \cup \{d\}$ by a decision maker who has a randomized preference over $X \cup \{d\}$ represented by the probability measure m. Thus, from a choice theoretic framework, Theorem 2 could be interpreted as saying that a representable preference relation satisfies ISD if and only if there is a utility function $u \in \mathcal{U}$ that represents it such that, when $1 - u(A^c)$ is viewed as the probability of choosing d by a decision maker from $A^c \cup \{d\}$ for each $A \in 2^X$, these choice probabilities could be induced by some randomized preference over $X \cup \{d\}$. This suggests that there is some link between the question addressed in this section and the problem of rationalizability in the literature on probabilistic revealed preference theory (e.g. see Block and Marshak (1960), Falmagne (1978), Barbera and Pattanaik (1986)). We will confirm this link in the next section by using the results of this section to provide a characterization of rationalizability of resolute probabilistic choice.

3. PROBABILISTIC RATIONALIZABILITY

Let X be a finite set of alternatives with $|X| = N$. Let $\mathcal{X} = 2^X \setminus \{\emptyset\}$ be the set of possible feasible sets. In this section, for each $x \in X$ and each $A \in 2^{X \setminus \{x\}}$, we will use the notation A^c for the complement of A in X, and A^c_x for the complement of A in $X \setminus \{x\}$. Also, to avoid introducing additional notation, in this section we will let $\mathcal{R}$ denote the set of all linear orderings on X, and for each $x \in X$ and each $A \in 2^{X\setminus\{x\}}$, we let

$$\begin{aligned}\mathcal{R}(x;A) &= \{R \in \mathcal{R}: \ xRy \ \forall \ y \in A_x^c\};\\ \mathcal{Q}(x;A) &= \{R \in \mathcal{R}: \ yRx \ \forall \ y \in A; \, xRy \ \forall \ y \in A_x^c\}.\end{aligned}$$

Clearly, for each $x \in X$ and each $A \in 2^{X\setminus\{x\}}$, $\{\mathcal{Q}(x;B) : B \subseteq A\}$ is a partition of $\mathcal{R}(x;A)$.

Resolute probabilistic choice function: A *resolute probabilistic choice function* (RPCF) is a function $c : X \times \mathcal{X} \to [0,1]$ such that $\sum_{x\in X} c(x;A) = \sum_{x\in A} c(x;A) = 1 \quad \forall \ A \in \mathcal{X}$.

A RPCF is the probabilistic counterpart of a single-valued choice function in the deterministic framework. We interpret $c(x;A)$ as the probability that x will be chosen from A when A is the feasible set.

Rationalizability: A RPCF c is rationalizable if there exists a probability measure m on $2^{\mathcal{R}}$ such that, for each $x \in X$, $c(x;A^c) = m(\mathcal{R}(x;A))$ for all $A \in 2^{X\setminus\{x\}}$.[6]

Rationalizability simply requires a RPCF to be induced by some randomized preference. Using simple examples (e.g. see Barbera and Pattanaik (1986)), it can be shown that not every RPCF is rationalizable. The purpose of the rest of this section is to provide a necessary and sufficient condition for rationalizability using the results from the previous section.

Given a RPCF c, for each $x \in X$, define the set function $F_x : 2^{X\setminus\{x\}} \to [0,1]$ by

$$F_x(A) = c(x;A^c) \quad \forall \ A \in 2^{X\setminus\{x\}}.$$

Then a RPCF c is rationalizable if and only if there exists a probability measure m on $2^{\mathcal{R}}$ such that, for each $x \in X$, $F_x(A) = m(\mathcal{R}(x;A))$ for every $A \in 2^{X\setminus\{x\}}$.

For each $x \in X$, let $\mathcal{A}_x$ be the set of all nonempty subsets of $X \setminus \{x\}$. Also, let Θ be the explicit state space as in the previous section. For each $x \in X$, define $\mathcal{F}_x$ to be the class of opportunity acts $f_x : \Theta \to \mathcal{A}_x$ (i.e. $\mathcal{F}_x$ is the class of opportunity acts with the range restricted to the opportunity sets without x) and $\mathcal{U}_x$ to be the class of utility functions $u : \mathcal{A}_x \to \Re$.

For each $x \in X$, let F_x^c be the conjugate of F_x, i.e. $F_x^c(A) = 1 - F_x(A_x^c)$ for all $A \in 2^{X\setminus\{x\}}$. Also, let $F_{\mathcal{A}_x}^c$ be the restriction of F_x^c to $\mathcal{A}_x$. So $F_{\mathcal{A}_x}^c$ is a utility function in $\mathcal{U}_x$. Thus, an obvious implication of Theorem 2 is that, if a RPCF c is rationalizable, then, for each $x \in X$, any preference relation over the opportunity acts in $\mathcal{F}_x$ that can be represented by the utility function $F_{\mathcal{A}_x}^c$ must satisfy ISD.

For each $R \in \mathcal{R}$ and each $i \in \{1,...,N\}$, let $t_i(R)$ be the alternative ranked in the ith position by R and let $A_i(R) = \{t_1(R),...,t_i(R)\}$. Also, let $A_0(R) = \emptyset$ for all $R \in \mathcal{R}$.

Given a RPCF c, for each $x \in X$, let

$$\pi(\mathcal{Q}(x;A)) = \sum_{A' \subseteq A} (-1)^{|A'|} F_x(A \setminus A') \qquad \forall \ A \in 2^{X\setminus\{x\}}. \tag{1}$$

[6] Although our definition of rationalizability looks somewhat different, it can be verified that it is equivalent to the definition given in Barbera and Pattanaik (1986).

Also, let

$$
\begin{aligned}
P_i(R) &= \pi\left(\mathcal{Q}(t_i(R); A_{i-1}(R))\right) \quad \forall\ R \in \mathcal{R} \text{ and } \forall\ i \in \{1, ..., N\}; && (2)\\
\Lambda_i(R) &= \sum_{x \in A_i(R)} \pi\left(\mathcal{Q}(x; A_i(R) \setminus \{x\})\right) \quad \forall\ R \in \mathcal{R} \text{ and } \forall\ i \in \{1, ..., N\}; && (3)\\
\mathcal{R}^+ &= \{R \in \mathcal{R} : P_i(R) \neq 0\ \forall\ i \in \{1, ..., N\}\}; && (4)\\
p(R) &= P_1(R)\left[\prod_{i=2}^{N}\left(\frac{P_i(R)}{\Lambda_{i-1}(R)}\right)\right] \quad \forall\ R \in \mathcal{R}^+; && (5)\\
p(R) &= 0 \quad \forall\ R \in \mathcal{R} \setminus \mathcal{R}^+ && (6)\\
m(\mathcal{Q}) &= \sum_{R \in \mathcal{Q}} p(R) \quad \text{for each } \mathcal{Q} \in 2^{\mathcal{R}} \setminus \{\emptyset\}; && (7)\\
m(\emptyset) &= 0. && (8)
\end{aligned}
$$

Lemma: *If a RPCF c is such that, for each $x \in X$, the preference relation over $\mathcal{F}_x$ that is representable by the utility function $F^c_{\mathcal{A}_x}$ satisfies ISD, then m is a probability measure on $2^{\mathcal{R}}$.*

Using the above Lemma, we can now provide our characterization result which shows that, given a RPCF c, satisfying ISD by the preference relation over $\mathcal{F}_x$ that is representable by the utility function $F^c_{\mathcal{A}_x}$ is not only necessary but also sufficient for rationalizability.

Theorem 3: *A RPCF c is rationalizable if and only if, for each $x \in X$, the preference relation over $\mathcal{F}_x$ that is representable by the utility function $F^c_{\mathcal{A}_x}$ satisfies ISD.*

Thus, Theorems 2 and 3 show that there is some link between the notion of preference for flexibility captured by the ISD axiom and the concept of rationalizability of probabilistic choice functions.

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APPENDIX

Given a nonempty and finite set S, a set function $\mu : 2^S \to [0,1]$ is:

(1) *Alternating of order* ∞ if it satisfies the following two conditions:

(a) for every $A_1, A_2 \in 2^S$, $\mu(A_1) \le \mu(A_2)$ if $A_1 \subseteq A_2$;[7]

(b) for every integer $K \ge 2$ and every $A_1, ..., A_K \in 2^S$,

$$\mu\left(\cap_{k=1}^{K} A_k\right) \le \sum_{\emptyset \ne I \subseteq \{1,...,K\}} (-1)^{|I|+1} \mu\left(\cup_{k\in I} A_k\right).$$

(2) *Monotone of order* ∞ if it satisfies the following two conditions:

(a) for every $A_1, A_2 \in 2^S$, $\mu(A_1) \le \mu(A_2)$ if $A_1 \subseteq A_2$;

(b) for every integer $K \ge 2$ and every $A_1, ..., A_K \in 2^S$,

$$\mu\left(\cup_{k=1}^{K} A_k\right) \ge \sum_{\emptyset \ne I \subseteq \{1,...,K\}} (-1)^{|I|+1} \mu\left(\cap_{k\in I} A_k\right).$$

It is well known that a set function μ is alternating of order ∞ (monotone of order ∞) if and only if its conjugate $\mu^c(A) = 1 - \mu(A^c)$ for all $A \in 2^S$ is monotone of order ∞ (alternating of order ∞).[8]

In the above, if we restrict the domain of μ to $2^S \setminus \{\emptyset\}$, then in the definition of alternating of order ∞: (i) (a) is the *monotonicity* condition of Nehring (1999); (ii) (b) is the *total submodularity* condition of Nehring (1999) if we also require $\cap_{k=1}^{K} A_k \ne \emptyset$.

For each utility function $u \in \mathcal{U}$, let u^E be the extension of u to 2^X which is given by $u^E(A) = u(A)$ for all $A \in \mathcal{A}$ and

$$u^E(\emptyset) = \inf \sum_{\emptyset \ne I \subseteq \{1,...,K\}} (-1)^{|I|+1} u\left(\cup_{k\in I} A_k\right)$$

where the infimum is taken over all $K \ge 1$ and all $A_1, ..., A_K \in \mathcal{A}$. Since we can take a positive affine transformation of u such that the range of the transformed utility function is a subset of $[0,1]$, without loss of generality we can treat the range of u^E also to be a subset of $[0,1]$. Then it can be verified that u is monotone and totally submodular if and only if u^E is alternating of order ∞. Thus, u is monotone and totally submodular if and only if the conjugate of u^E, which is given by $u^{Ec}(A) = 1 - u^E(A^c)$ for all $A \in 2^X$, is monotone of order ∞.

In the proof of Theorem 2 we will use the following intermediate result of Nehring (1999).

Lemma A1: *A utility function u is an* EIU *function if and only if it is monotone and totally submodular.*

[7]Given two sets A and B, we use $A \subseteq B$ to denote "A is a subset of B", and $A \subset B$ to denote "A is a proper subset of B".

[8]For example, see Shafer (1979).

From Lemma A1, it follows that a utility function u is an EIU function if and only if u^{Ec} is monotone of order ∞.

Lemma A2: *Let $\mu : 2^X \to [0,1]$, with $\mu(X) = 1$, be monotone of order ∞. Then there exists a probability measure m on $2^{\mathcal{R}}$ such that $m(\mathcal{R}(d;A)) = \mu(A)$ for all $A \in 2^X$.*

Proof: For each $A \in 2^X$, let

$$\mathcal{Q}(d;A) = \{R \in \mathcal{R} : \; xRd \;\forall\; x \in A; \; dRx \;\forall\; x \in A^c\}.$$

Clearly, for each $A \in 2^X$, $\{\mathcal{Q}(d;A') : A' \subseteq A\}$ is a partition of $\mathcal{R}(d;A)$. So $\{\mathcal{Q}(d;A) : A \in 2^X\}$ is a partition of $\mathcal{R}$.

For each $A \in 2^X$, let

$$m^\mu(\mathcal{Q}(d;A)) = \sum_{A' \subseteq A} (-1)^{|A'|} \mu(A \setminus A').$$

Then $m^\mu(\mathcal{Q}(d;\emptyset)) = \mu(\emptyset) \geq 0$. Now, consider any nonempty $A \in 2^X$. Without loss of generality, let $A = \{s_1, ..., s_K\}$ and also let $A_k = A \setminus \{s_k\}$ for each $k = 1, ..., K$. If $K = 1$, then condition (a) in the definition of monotone of order ∞ implies $m^\mu(\mathcal{Q}(d;A)) = \mu(A) - \mu(\emptyset) \geq 0$. Suppose $K \geq 2$. By condition (b) in the definition of monotone of order ∞,

$$0 \leq \mu(A) + \sum_{\emptyset \neq I \subseteq \{1,...,K\}} (-1)^{|I|} \mu\left(\cap_{k \in I} A_k\right).$$

Also, for each nonempty $I \subseteq \{1, ..., K\}$, $\cap_{k \in I} A_K = A \setminus \{s_k : k \in I\}$. So

$$\begin{aligned}
0 &\leq \mu(A) + \sum_{\emptyset \neq I \subseteq \{1,...,K\}} (-1)^{|I|} \mu\left(A \setminus \{s_k : k \in I\}\right) \\
&= \mu(A) + \sum_{\emptyset \neq A' \subseteq A} (-1)^{|A'|} \mu\left(A \setminus A'\right) \\
&= \sum_{A' \subseteq A} (-1)^{|A'|} \mu\left(A \setminus A'\right) \\
&= m^\mu(\mathcal{Q}(d;A)).
\end{aligned}$$

Hence, $m^\mu(\mathcal{Q}(d;A)) \geq 0$ for all $A \in 2^X$. Thus, since $\{\mathcal{Q}(d;A) : A \in 2^X\}$ is a partition of $\mathcal{R}$, there exists a measure m on $2^{\mathcal{R}}$ such that

$$m(\mathcal{Q}(d;A)) = m^\mu(\mathcal{Q}(d;A)) \qquad \forall\; A \in 2^X.$$

For each $A \in 2^X$, since $\{\mathcal{Q}(d;A') : A' \subseteq A\}$ is a partition of $\mathcal{R}(d;A)$,

$$\begin{aligned}
m(\mathcal{R}(d;A)) &= \sum_{A' \subseteq A} m(\mathcal{Q}(d;A')) \\
&= \sum_{A' \subseteq A} m^\mu(\mathcal{Q}(d;A')) \\
&= \sum_{A' \subseteq A} \sum_{A'' \subseteq A'} (-1)^{|A''|} \mu(A' \setminus A'') \\
&= \sum_{A'' \subseteq A} \left[\mu(A'') \sum_{A'' \subseteq A' \subseteq A} (-1)^{|A' \setminus A''|} \right].
\end{aligned}$$

It can be verified that, for all $A'' \subseteq A \subseteq X$,

$$\sum_{A'' \subseteq A' \subseteq A} (-1)^{|A' \setminus A''|} = \begin{cases} 0 & \text{if } A'' \neq A \\ 1 & \text{if } A'' = A \end{cases}$$

Hence, for each $A \in 2^X$, $m(\mathcal{R}(d; A)) = \mu(A)$. Also, because $\mathcal{R}(d; X) = \mathcal{R}$, we have $m(\mathcal{R}) = \mu(X) = 1$. Therefore, m is a probability measure on $2^{\mathcal{R}}$. $\|$

Proof of **Theorem 2**: Suppose a preference relation over $\mathcal{F}$ is representable by a nonconstant utility function u and satisfies ISD. Without loss of generality, let the range of u^{Ec} be in $[0, 1]$ and $u^{Ec}(X) = 1$. Then, because of Theorem 1 and Lemma A1, u^{Ec} is monotone of order ∞. Thus, Lemma A2 implies that there exists a probability measure m on $2^{\mathcal{R}}$ such that $[1 - u(A^c)] = u^{Ec}(A) = m(\mathcal{R}(d; A))$ for all $A \in 2^X \setminus \{X\}$. This completes the necessity part of the proof.

To prove the sufficiency, let m be a probability measure on $2^{\mathcal{R}}$ such that $[1 - u(A^c)] = u^{Ec}(A) = m(\mathcal{R}(d; A))$ for all $A \in 2^X \setminus \{X\}$. Because of Theorem 1 and Lemma A1, it is enough to show that u^{Ec} is monotone of order ∞. For all $A \subseteq B \subseteq X$, since $\mathcal{R}(d; A) \subseteq \mathcal{R}(d; B)$, $u^{Ec}(A) \leq u^{Ec}(B)$. Consider an integer $K \geq 2$ and $A_1, ..., A_K \in 2^X$. Then it is sufficient to show that

$$m\left(\mathcal{R}\left(d; \cup_{k=1}^K A_k\right)\right) \geq \sum_{\emptyset \neq I \subseteq \{1,...,K\}} (-1)^{|I|+1} m\left(\mathcal{R}\left(d; \cap_{k \in I} A_k\right)\right).$$

For each $B \subseteq \cup_{k=1}^K A_k$, let $I(B) = \{k \in \{1, ..., K\} : B \subseteq A_k\}$. Also, let $\mathcal{H} = \{B \subseteq \cup_{k=1}^K A_k : I(B) \neq \emptyset\}$. It can be verified that $\sum_{\emptyset \neq I \subseteq I(B)} (-1)^{|I|+1} = 1$ for all $B \in \mathcal{H}$. Thus, since $\{\mathcal{Q}(d; B) : B \subseteq A\}$ is a partition of $\mathcal{R}(d; A)$ for each $A \in 2^X$ and $\{\mathcal{Q}(d; B) : B \in \mathcal{H}\} \subseteq \{\mathcal{Q}(d; B) : B \subseteq \cup_{k=1}^K A_k\}$,

$$\begin{aligned} m\left(\mathcal{R}\left(d; \cup_{k=1}^K A_k\right)\right) &\geq \sum_{B \in \mathcal{H}} m\left(\mathcal{Q}(d; B)\right) \\ &= \sum_{B \in \mathcal{H}} \left[m\left(\mathcal{Q}(d; B)\right) \sum_{\emptyset \neq I \subseteq I(B)} (-1)^{|I|+1} \right] \\ &= \sum_{\emptyset \neq I \subseteq \{1,...,K\}} \left[(-1)^{|I|+1} \sum_{B \subseteq \cap_{k \in I} A_k} m\left(\mathcal{Q}(d; B)\right) \right] \\ &= \sum_{\emptyset \neq I \subseteq \{1,...,K\}} (-1)^{|I|+1} m\left(\mathcal{R}\left(d; \cap_{k \in I} A_k\right)\right). \quad \| \end{aligned}$$

The proof of the Lemma in section 3 follows some preliminary lemmas.

Lemma A3: *If c is a RPCF, then*

$$\sum_{x \in A^c} \pi\left(\mathcal{Q}(x; A)\right) = \sum_{x \in A} \pi\left(\mathcal{Q}(x; A \setminus \{x\})\right) \quad \forall\ A \in 2^X \setminus \{\emptyset, X\}.$$

Proof: Let $A \in 2^X \setminus \{\emptyset, X\}$. By (1),

$$\begin{aligned}
\sum_{x\in A^c} \pi\left(\mathcal{Q}(x;A)\right) &= \sum_{x\in A^c}\left[\sum_{A'\subseteq A}(-1)^{|A'|}F_x(A\setminus A')\right] \\
&= \sum_{A'\subseteq A}(-1)^{|A'|}\left[\sum_{x\in A^c}F_x(A\setminus A')\right] \\
&= \sum_{A'\subseteq A}(-1)^{|A'|}\left[\sum_{x\in A^c}c\left(x;(A\setminus A')^c\right)\right] \\
&= \sum_{A'\subseteq A}(-1)^{|A'|}\left[1-\sum_{x\in A'}c\left(x;(A\setminus A')^c\right)\right]
\end{aligned}$$

Since it can be verified that $\sum_{A'\subseteq A}(-1)^{|A'|}=0$, we have

$$\begin{aligned}
\sum_{x\in A^c} \pi\left(\mathcal{Q}(x;A)\right) &= \sum_{A'\subseteq A}(-1)^{|A'|-1}\left[\sum_{x\in A'}c\left(x;(A\setminus A')^c\right)\right] \\
&= \sum_{x\in A}\sum_{A'\subseteq A}(-1)^{|A'|-1}c\left(x;(A\setminus A')^c\right) \\
&= \sum_{x\in A}\sum_{A'\subseteq A\setminus\{x\}}(-1)^{|A'|}c\left(x;(A\setminus(A'\cup\{x\}))^c\right) \\
&= \sum_{x\in A}\sum_{A'\subseteq A\setminus\{x\}}(-1)^{|A'|}F_x\left(A\setminus(A'\cup\{x\})\right) \\
&= \sum_{x\in A}\sum_{A'\subseteq A\setminus\{x\}}(-1)^{|A'|}F_x\left((A\setminus\{x\})\setminus A'\right) \\
&= \sum_{x\in A}\pi\left(\mathcal{Q}(x;A\setminus\{x\})\right). \quad \|
\end{aligned}$$

Lemma A4: *Let c be a RPCF. Then, for each $x \in X$ and each $A \in 2^{X\setminus\{x\}}$, $\mathcal{R}^+ \cap \mathcal{Q}(x;A) \neq \emptyset$ if $\pi(\mathcal{Q}(x;A)) > 0$.*

Proof: Let $x \in X$ and $A \in 2^{X\setminus\{x\}}$ be such that $\pi(\mathcal{Q}(x;A)) > 0$. Without loss of generality, let $|A| = l-1$, $x = x_l$, $A_{l-1} = A$ and $A_l = A \cup \{x\}$.

Suppose $l \geq 2$. By Lemma A3, $\sum_{y\in A_{l-1}}\pi(\mathcal{Q}(y;A_{l-1}\setminus\{y\})) = \sum_{y\in A^c_{l-1}}\pi(\mathcal{Q}(y;A_{l-1})) > 0$. So there exists $x_{l-1} \in A_{l-1}$ with $\pi(\mathcal{Q}(x_{l-1};A_{l-1}\setminus\{x_{l-1}\})) > 0$. Let $A_{l-2} = A_{l-1}\setminus\{x_{l-1}\}$. If $l \geq 3$, by Lemma A3, $\sum_{y\in A_{l-2}}\pi(\mathcal{Q}(y;A_{l-2}\setminus\{y\})) = \sum_{y\in A^c_{l-2}}\pi(\mathcal{Q}(y;A_{l-2})) > 0$. So there exists $x_{l-2} \in A_{l-2}$ with $\pi(\mathcal{Q}(x_{l-2};A_{l-2}\setminus\{x_{l-2}\})) > 0$. Continuing in this manner, we can arrange the alternatives in A_l in a sequence $x_1,...,x_l$ such that $\pi(\mathcal{Q}(x_i;A_i\setminus\{x_i\})) > 0 \ \forall\ i = 1,...,l$, where $A_i = \{x_1,...,x_i\} \ \forall\ i = 1,...,l$.

Suppose $l \leq N-1$. By Lemma A3, $\sum_{y\in A^c_l}\pi(\mathcal{Q}(y;A_l)) = \sum_{y\in A_l}\pi(\mathcal{Q}(y;A_l\setminus\{y\})) > 0$. So there exists $x_{l+1} \in A^c_l$ such that $\pi(\mathcal{Q}(x_{l+1};A_l)) > 0$. Let $A_{l+1} = A_l \cup \{x_{l+1}\}$. Then $\pi(\mathcal{Q}(x_{l+1};A_{l+1}\setminus\{x_{l+1}\})) > 0$. If $l \leq N-2$, by Lemma A3, $\sum_{y\in A^c_{l+1}}\pi(\mathcal{Q}(y;A_{l+1})) = \sum_{y\in A_{l+1}}\pi(\mathcal{Q}(y;A_{l+1}\setminus\{y\})) > 0$. So there exists $x_{l+2} \in A^c_{l+1}$ such that $\pi(\mathcal{Q}(x_{l+2};A_{l+1})) > 0$. Let $A_{l+2} = A_{l+1}\cup\{x_{l+2}\}$. Then $\pi(\mathcal{Q}(x_{l+2};A_{l+2}\setminus\{x_{l+2}\})) > 0$. Continuing in this manner, we can arrange the alternatives in A_l^c in a sequence $x_{l+1}, ..., x_N$ such that $\pi(\mathcal{Q}(x_i; A_i \setminus \{x_i\})) > 0$ $\forall\ i = l+1, ..., N$, where $A_i = A_l \cup \{x_{l+1}, ..., x_i\}$ $\forall\ i = l+1, ..., N$.

Now, let $R \in \mathcal{R}$ be such that $t_i(R) = x_i$ for each $i = 1, ..., N$. Then it is clear that $R \in \mathcal{Q}(x; A)$ and $P_i(R) > 0$ $\forall\ i = 1, ..., N$. Hence, $\mathcal{R}^+ \cap \mathcal{Q}(x; A) \neq \emptyset$. ||

Lemma A5: *Let c be a RPCF. Then, for each $x \in X$ and each $A \subset X \setminus \{x\}$ such that $\mathcal{R}^+ \cap \mathcal{Q}(x; A) \neq \emptyset$,*

$$\sum_{R \in \mathcal{R}^+ \cap \mathcal{Q}(x;A)} \left[\prod_{i=|A|+2}^{N} \left(\frac{P_i(R)}{\Lambda_{i-1}(R)} \right) \right] = 1.$$

Proof: Let $x \in X$ and $A \subset X \setminus \{x\}$ be such that $\mathcal{R}^+ \cap \mathcal{Q}(x; A) \neq \emptyset$. Also, let

$$\mathcal{B} = \left\{ B \in 2^X : \exists\ R \in \mathcal{R}^+ \cap \mathcal{Q}(x; A) \text{ and } i \in \{|A|+1, ..., N-1\} \text{ such that } B = A_i(R) \right\}.$$

Then it can be verified that

$$\prod_{B \in \mathcal{B}} \left[\frac{\sum_{y \in B^c} \pi(\mathcal{Q}(y; B))}{\sum_{y \in B} \pi(\mathcal{Q}(y; B \setminus \{y\}))} \right] = \sum_{R \in \mathcal{R}^+ \cap \mathcal{Q}(x;A)} \left[\prod_{i=|A|+2}^{N} \left(\frac{P_i(R)}{\Lambda_{i-1}(R)} \right) \right] \neq 0.$$

Thus, by Lemma A3, we also know that

$$\left[\frac{\sum_{y \in B^c} \pi(\mathcal{Q}(y; B))}{\sum_{y \in B} \pi(\mathcal{Q}(y; B \setminus \{y\}))} \right] = 1 \qquad \forall\ B \in \mathcal{B}. \quad ||$$

Proof of **Lemma**: Let c be a RPCF such that, for each $x \in X$, the preference relation over $\mathcal{F}_x$ that is representable by the utility function $F^c_{\mathcal{A}_x}$ satisfies ISD. Then Theorem 1 and Lemma A1 imply that F_x is monotone of order ∞ for each $x \in X$. From the proof of Theorem 2 and (1), we know that, for each $x \in X$,

$$\pi(\mathcal{Q}(x; A)) = \sum_{A' \subseteq A} (-1)^{|A'|} F_x(A \setminus A') \geq 0 \qquad \forall\ A \in 2^{X \setminus \{x\}}. \tag{9}$$

So, using (2)-(5), it can be checked that $p(R) > 0$ for all $R \in \mathcal{R}^+$. Thus, given (5)-(8), we only need to show that $\sum_{R \in \mathcal{R}^+} p(R) = 1$. Let $X_+ = \{x \in X : \pi(\mathcal{Q}(x; \emptyset)) > 0\}$. By Lemma A4, $\mathcal{R}^+ \cap \mathcal{Q}(x; \emptyset) \neq \emptyset$ for all $x \in X_+$. Also, for each $x \in X_+$, (2) implies $P_1(R) = \pi(\mathcal{Q}(x; \emptyset))$ for all $R \in \mathcal{R}^+ \cap \mathcal{Q}(x; \emptyset)$. So (5) and Lemma A5 imply

$$\sum_{R \in \mathcal{R}^+ \cap \mathcal{Q}(x;\emptyset)} p(R) = \pi\left(\mathcal{Q}(x; \emptyset)\right) \qquad \forall\ x \in X_+.$$

Clearly, $\{\mathcal{R}^+ \cap \mathcal{Q}(x; \emptyset) : x \in X_+\}$ is a partition of $\mathcal{R}^+$. Thus, (9) implies

$$\sum_{R \in \mathcal{R}^+} p(R) = \sum_{x \in X_+} \left[\sum_{R \in \mathcal{R}^+ \cap \mathcal{Q}(x;\emptyset)} p(R) \right] = \sum_{x \in X_+} \pi(\mathcal{Q}(x; \emptyset)) = \sum_{x \in X} \pi(\mathcal{Q}(x; \emptyset)).$$

It can be verified from (1) that $\sum_{x \in X} \pi(\mathcal{Q}(x; \emptyset)) = \sum_{x \in X} F_x(\emptyset) = \sum_{x \in X} c(x; X) = 1$. ||

Proof of **Theorem 3**: The necessity part of the proof follows from Theorem 2. So we only need to prove sufficiency.

Let c be a RPCF such that, for each $x \in X$, the preference relation over $\mathcal{F}_x$ that is representable by the utility function $F^c_{\mathcal{A}_x}$ satisfies ISD. Then Theorem 1 and Lemma A1 imply that F_x is monotone of order ∞ for each $x \in X$. By the Lemma in section 3, the set function m defined by (7) and (8) is a probability measure on $2^{\mathcal{R}}$. We will first show that, for each $x \in X$, $m(\mathcal{Q}(x;A)) = \pi(\mathcal{Q}(x;A))$ for all $A \in 2^{X\setminus\{x\}}$.

Let $x \in X$. If $\pi(\mathcal{Q}(x;\emptyset)) = 0$, then (2), (4), (6) and (7) imply $m(\mathcal{Q}(x;\emptyset)) = 0 = \pi(\mathcal{Q}(x;\emptyset))$. Also, by (5)-(7) and Lemmas A4 and A5, if $\pi(\mathcal{Q}(x;\emptyset)) > 0$, then

$$m(\mathcal{Q}(x;\emptyset)) = \sum_{R\in\mathcal{R}^+\cap\mathcal{Q}(x;\emptyset)} p(R) = \pi(\mathcal{Q}(x;\emptyset)).$$

Let $A \in \mathcal{A}_x$. If $\mathcal{R}^+ \cap \mathcal{Q}(x;A) = \emptyset$, then (9) and Lemma A4, imply $\pi(\mathcal{Q}(x;A)) = 0$, and hence, (6) and (7) imply $m(\mathcal{Q}(x;A)) = 0 = \pi(\mathcal{Q}(x;A))$. So suppose $\mathcal{R}^+ \cap \mathcal{Q}(x;A) \neq \emptyset$. Then it can be verified that

$$\sum_{R\in\mathcal{R}^+\cap\mathcal{Q}(x;A)} \left[\prod_{i=1}^{|A|}\left(\frac{P_i(R)}{\Lambda_i(R)}\right)\right] = 1.$$

Using (5)-(7), it can also be verified that

$$m(\mathcal{Q}(x;A)) = \sum_{R\in\mathcal{R}^+\cap\mathcal{Q}(x;A)} p(R) =$$

$$\left[\sum_{R\in\mathcal{R}^+\cap\mathcal{Q}(x;A)} \left(\prod_{i=1}^{|A|}\left(\frac{P_i(R)}{\Lambda_i(R)}\right)\right)\right] \pi(\mathcal{Q}(x;A)) \left[\sum_{R\in\mathcal{R}^+\cap\mathcal{Q}(x;A)} \left(\prod_{i=|A|+2}^{N}\left(\frac{P_i(R)}{\Lambda_{i-1}(R)}\right)\right)\right],$$

where we let $\sum_{R\in\mathcal{R}^+\cap\mathcal{Q}(x;A)} \left[\prod_{i=|A|+2}^{N}\left(\frac{P_i(R)}{\Lambda_{i-1}(R)}\right)\right] = 1$ if $A = X \setminus \{x\}$. Thus, Lemma A5 implies $m(\mathcal{Q}(x;A)) = \pi(\mathcal{Q}(x;A))$.

Let $x \in X$ and let $A \in 2^{X\setminus\{x\}}$. To complete the proof it is sufficient to show that $F_x(A) = m(\mathcal{R}(x;A))$. Since $\{\mathcal{Q}(x;A') : A' \subseteq A\}$ is a partition of $\mathcal{R}(x;A)$, (1) implies

$$\begin{aligned} m(\mathcal{R}(x;A)) &= \sum_{A'\subseteq A} m(\mathcal{Q}(x;A')) \\ &= \sum_{A'\subseteq A} \pi(\mathcal{Q}(x;A')) \\ &= \sum_{A'\subseteq A}\sum_{A''\subseteq A'} (-1)^{|A''|} F_x(A' \setminus A'') \\ &= \sum_{A''\subseteq A}\left[F_x(A'') \sum_{A''\subseteq A'\subseteq A} (-1)^{|A'\setminus A''|}\right]. \end{aligned}$$

It can be verified that, for all $A'' \subseteq A$,

$$\sum_{A''\subseteq A'\subseteq A} (-1)^{|A'\setminus A''|} = \begin{cases} 0 & \text{if } A'' \neq A \\ 1 & \text{if } A'' = A \end{cases}$$

Therefore, $m(\mathcal{R}(x;A)) = F_x(A)$. $\|$