Rich by Accident: the Second Welfare Theorem with a Redundant Asset under Imperfect Foresight*

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Abstract

In a T-period model with perfect foresight and no uncertainty, markets are complete with short-term bonds. The no-arbitrage principle renders any additional asset redundant, with no implications for allocation. Relaxing the perfect foresight assumption, we consider a reasonable version of temporary equilibria that accommodates sensible forecasts while maintaining no-arbitrage. With only short-term bonds, even allowing for forecasting errors, only a T-dimensional subset of efficient allocations can arise as Walrasian equilibria. However, if long-term bonds are traded in addition, essentially all efficient allocations can be achieved as equilibria, whereas forecasting errors may be arbitrarily small. We argue that minute errors in price forecasts can generate any feasible wealth transfer within the model, and that the beneficiaries of such transfers are determined by chance, not by superior forecasting ability. (JEL classification numbers: D51, D53, D61)

Keywords: General equilibrium; Efficient temporary equilibrium; Endogenous price forecasts; Redundant Assets

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1 Introduction

What allocational role might a redundant financial asset play in an intertemporal frictionless Walrasian setting? Traditional wisdom would suggest none, since by definition, a redundant financial asset can be completely replicated by trading other assets dynamically at market prices so that any trader is indifferent between holding it and ignoring it. Thus its presence in no way alters the possibilities of income transfers across periods/states. This conclusion appears not to depend on the accuracy of price forecasts; if the market prices are not correctly anticipated but the redundant asset creates some additional market value, positive or negative, one would find some arbitrage opportunity through trading the asset, which cannot arise in frictionless markets.

We shall argue however that this conclusion relies entirely on the axiom of perfect foresight: we dispense with perfect foresight and show that essentially *all* intertemporally efficient allocations can arise as Walrasian equilibria when a redundant asset is traded, while the prices are arbitrarily close to perfect foresight prices ex post. Therefore, there are profound welfare effects arising from Walrasian trade which are completely missed out in rational expectations models.

We formulate our question in a one good economy which lasts for T periods with no uncertainty, and which has two types of nominal assets. The first of these is a discount bond traded in periods $t = 0, 1, ..., T - 1$, which matures in one period, i.e., the bond traded in period t pays out \$1 in period $t + 1$. The second is a discount bond with a longer maturity, which will be called the L-bond: the L-bond can be traded in every period, but pays \$1 in period T, and nothing in other periods. With no uncertainty about the fundamentals of the economy at all, a textbook conclusion is that by the no arbitrage principle its yield must be given by the compound one year interest rates, since the payout of the L-bond can be replicated by an iterative one year saving. As is well known, *if perfect foresight (rational expectation) is assumed*, the markets are already complete with bonds with one period maturity, and any additional asset markets such as the L-bond markets have no additional implication on the allocation of the good except for indeterminacy of asset trade, which arises since the L-bond and the dynamic plan are perfect substitutes at any time.

However, notice that this argument implicitly assumes that the one year rates are known, which in particular means that the forecasts are perfectly aligned and thus homogenous across agents.[1] Even though there is no uncertainty, the one year rates that will prevail in the future years are not realized yet, leaving some room for heterogenous forecasts. One might still think that if the degree of heterogeneity of forecasts is small enough, the market outcome will be close to the one predicted in a perfect foresight model, and the presence of the L-bond like above will not add much qualitatively. To assess this claim theoretically, we need to consider competitive markets allowing for heterogeneous forecasts, which naturally leads us to inspect temporary equilibria (TE) of these markets. By definition, a TE occurs when all markets clear under some forecasts which need not be common.

There are numerous TE in a model like this and one cannot bring out the allocative implications of the asset markets in a sharp manner. To avoid the myriad of TE allocations that emerge when allowing for heterogenous forecasts, one needs to put some more discipline into the analysis. One way of doing so would be to put restrictions directly on the sort of forecasts agents are allowed to hold.[2] Instead, we will focus on, with no restrictions on forecasts a priori, those TE which induce Pareto efficient allocations, and let the efficiency requirement indirectly put restrictions on forecasts.[3] By definition, differences in efficient allocations can be seen as a result of wealth transfers across agents.

[1] It goes without saying that homogeneity of forecasts is an excessively stringent assumption (see Radner [1982] and Grandmont [1988] for instance). Various kinds of evidence against alignment of forecasts can be supplied even in more restrictive contexts of asset pricing (see e.g., Bossaerts [2002]).

[2] Recent literature in macroeconomics and finance explores implications of various kinds of forecasts. For instance, Woodford and Xie [2022] and Woodford, [2013]) consider bounded rationality in the design of fiscal and monetary policy and inflation targeting. Adam et al [2016] considers the CAPM model to show that small deviations from the rational expectations generate realistic amounts of stock price volatility. This literature allows some sorts of deviations of forecasts from the rational expectations hypothesis that cause inefficiencies and studies the scope of policy interventions in making welfare improvements.

[3] Allocative implications did not receive attention in the earlier literature on temporary equilibrium (Grandmont [1988]), which was focussed on existence (achieved by restricting the dependence of forecasts on current period prices). Subsequently, the literature examined learning behavior wherein forecasts were updated in a structured way (using OLS or Bayesian updating for instance (Chatterji [1995])) with a view to investigating the stability of a perfect foresight equilibrium. Our interest is in scenarios where heterogeneity of forecasts persists and we investigate its allocational implications.

These efficient TE (ETE) are then, in the very least, a natural benchmark for any sort of welfare analysis in these markets. Our question can then be formulated as enquiring how the L-bond markets affect the structure of ETE and induce wealth transfers.

Note that in any TE, irrespective of efficiency, the forecasts of the households in any period are such that they perceive no arbitrage opportunities, i.e., the forecast prices satisfy the no arbitrage principle, or else the households would find lucrative gains from additional trade of assets. Thus every household's mathematical model explains the bond prices perfectly at any time under its forecast prices, so that in our simple world, each household is no worse than a financial firm which uses very sophisticated model to find out the correct valuations of redundant assets in every trading opportunity. However, unless one imposes that forecasts are perfectly correct, ex post forecasting errors are inevitable, although they might be very small for sophisticated traders. These *ex post* forecasting errors may lead to *ex post* arbitrage opportunities.

Consider first those TE where an ex post arbitrage opportunity does not exist. The L-bond markets can be dispensed with since the payouts from the L-bond can be completely replicated by the bond trading in the TE: that is, there are allocationally equivalent TE where the L-bond is not traded at all. These therefore constitutes a class of TE where the redundant asset has no allocational role. A perfect foresight equilibrium (PFE) is an instance of such a TE, since the forecasts obey no arbitrage, while the correctness of the forecasts implies that the ex post prices inherit this no arbitrage property. We then examine the role of efficiency by characterizing ETE with no ex post arbitrage opportunities. It turns out that the dimension of ETE allocations of this kind can be as large as T, one less than the number of trading periods, while, as is well known, the set of efficient allocations is $H-1$ dimensional. More importantly, an ETE allocation of this kind is close to the PFE allocation if its arbitrage free ex post prices are close to the PFE prices, and vice versa. In this sense, if the L-bond is redundant ex post, the PFE is robust to minor noise in prices.

Next consider a TE where ex post prices admit arbitrage. Recall that the price forecasts underlying these prices cannot admit any arbitrage opportunities and the two types of the bonds must be deemed perfect substitutes at any time by any household, i.e., a household will never find it advantageous or disadvantageous to trade the L-bond in equilibrium. A plausible conjecture might then be that the L-bond markets do not have any additional allocational role and the set of ETE allocations is the same as under the case where ex post prices exhibit no arbitrage opportunity, or equivalently that such a ETE does not exist to begin with. But surprisingly, not only does such a ETE exist, but also the set of ETE allocations of this kind is of dimension $H - 1$, and includes *all* intertemporally efficient allocations in the vicinity of the PFE allocation. Thus we obtain a version of the second welfare theorem in the vicinity of the PFE allocation where the income transfers, whose existence is postulated in the second welfare theorem, are generated endogenously via the combination of forecasting errors and trade in an redundant asset. Moreover, in these ETE, while the observed prices may resemble arbitrarily closely the sequence of perfect foresight prices, the associated allocation is bounded away from the PFE allocation, and hence the aforementioned robustness property does not hold.

This result means that any direction of income transfer can arise implicitly in ETE, i.e., there will be winners and losers relative to PFE, whenever we deviate in the slightest from perfect foresight. Roughly speaking, since the bond and the L-bond are deemed perfect substitutes, a household is prepared to accept any affordable portfolio of the two types of the bonds in any period. Say that the return of the L-bond turns out just slightly larger than that of the bond ex post in a particular period. Then those who held the L-bond in the previous period will receive an unexpected gain and those who sold the bond will suffer from some unexpected losses. The size of gains and losses can be arbitrarily large in principle since an affordable portfolio might consist of a very large long/short positions. It can be verified that any income transfer of this sort is consistent with efficiency as well as individual optimization, if its volume is moderate enough. We therefore argue that the beneficiaries of such unanticipated income transfers are determined not by the superiority of forecasts but rather by accident, since there is no direct link between the accuracy of forecasts and the chosen portfolios in the description above, and hence between the winners/losers of trading.

The allocational role of the redundant asset is most pronounced in the instance of an economy composed of identical agents. The unique PFE allocation is the no-trade allocation. In the absence of perfect foresight, it is conceivable that heterogeneous forecasts might induce households to trade to a different efficient allocation. We show that this is not possible if one confines attention to equilibria where the ex post prices satisfy no arbitrage. On the other hand, our characterization of trade using a redundant asset when ex post prices violate no arbitrage, shows that all income transfers consistent with efficiency may arise as a consequence of households trading under heterogeneous forecasts. The trade thus induced may be interpreted as being purely speculative, and stands in sharp contrast to the so called no trade theorem under rational expectations.

Our finding brings into question the interpretation of the determinacy of the perfect foresight (rational expectation) allocation in more general contexts. For instance, derivative securities in financial markets are redundant assets since by construction their payouts can be replicated by some alternative trading strategy in the markets. Even if those financial markets were frictionless, introduction of derivative securities might bring about unintended winners and losers by chance if one allows for miniscule forecasting errors. Elimination of redundant assets in a model of frictionless markets is far from innocuous if one is pragmatic enough to incorporate possibilities of forecasting errors into the formalism.

The remainder of the paper is organized as follows. In Section 2, we provide a leading example to sketch our main findings. The same example will be utilized throughout this paper. Section 3 formally introduces the model, and states the definitions of TE and the no arbitrage condition. Sections 4 studies wealth transfers using the key intermediary concept of budget compatibility. Section 5 characterizes the set of efficient allocations that are obtained in the model. Section 6 briefly explores the possibility of speculative trade in this set up while Section 7 concludes.

2 Leading Example

Consider the case of $T = 2$, and four households $h = 1, ..., 4$ who have the identical utility function $\ln\left(x_h^0\right) + \ln\left(x_h^1\right) + \ln\left(x_h^2\right)$, where x_h^t is the quantity consumed in period t by household h. Each household h is endowed with e_h^t units of the good in period t, which are given as in the following table:

e_h^t	$t=0$	$t=1$	$t=2$
$h=1$	$\frac{1-\varepsilon}{3}+2\eta$	$\frac{1-\varepsilon}{3}-\eta$	$\frac{1-\varepsilon}{3}-\eta$
$h=2$	$\frac{1-\varepsilon}{3}-\eta$	$\frac{1-\varepsilon}{3}+2\eta$	$\frac{1-\varepsilon}{3}-\eta$
$h=3$	$\frac{1-\varepsilon}{3}-\eta$	$\frac{1-\varepsilon}{3}-\eta$	$\frac{1-\varepsilon}{3}+2\eta$
$h=4$	ε	ε	ε

(1)

where $0<\varepsilon<1$ and $0\leq\eta<\frac{1-\varepsilon}{3}$. That is, household h, $h=1,2,3$, has a high endowment in period $t=h-1$, and a low endowment in the other periods, whereas household $h=4$ has a constant endowment ε in every period. Note that the total endowment is one in every period, and hence a feasible allocation of goods is efficient intertemporally if and only if it assigns a time invariant consumption to every household. Then it is readily confirmed that in the Arrow- Debreu (AD) equilibrium, each household consumes the average of her endowments across the three periods (details can be found in Section 4). Notice that $e_h^0+e_h^1+e_h^2=1-\varepsilon$ for $h=1,2,3$, and hence these households consume $\frac{1-\varepsilon}{3}$ in every period whereas household 4 consumes ε in each period in the AD equilibrium. Under perfect foresight, the intertemporal equilibrium allocation coincides with the AD allocation, and the L-bond is a redundant asset with no allocational significance. We shall refer to this class of economies as the leading example throughout this paper. All the examples refer to the leading example.

When ε is "large" so that household 4 is rather well endowed at the expense of the remaining households, traditional wisdom, based on perfect foresight, suggests that the first three households are consigned to poverty since by trading in Walrasian markets they are unable to go beyond the average of their endowments across time. We will construct an efficient intertemporal equilibria where household 4 may end up consuming less than her endowment, while the first three households consume more. These equilibria are characterized by a miniscule ex post arbitrage opportunity in the price of the long term bond that was unanticipated by the markets, but which households one to three are able to exploit at the expense of household 4, as it were, by accident, owing to a fortuitous choice of a trading position.

When $\eta=0$ and $\varepsilon=\frac{1}{4}$ so that all households are identical, autarchy is the unique AD equilibrium where each household has the constant consumption $\frac{1}{4}$. We will see in

Section 6 that in this case while the PFE is the only ETE if ex post prices satisfy the no arbitrage condition, there are non autarkic ETE in which trade creates income transfers from the winners to the losers who are determined “by accident”.

3 The Model and Temporary Equilibrium

3.1 Set up

We consider a very simple market economy whose properties are well-known, but we nonetheless summarize its key properties for completeness. Let there be $T+1$ periods starting with period 0, where $T > 0$. A single non-storable good is available in every period. There is no uncertainty in the economy.

There are H households, labeled by $h = 1, 2, ..., H$. Household h is endowed with e_h^t units of the good in period t, $t = 0, 1, ..., T$, which is known to household h, and we write $e_h = \left(\cdots, e_h^t, \cdots\right) \in \left(\mathbb{R}_+^{T+1}\right)$. To avoid triviality and zero income, we assume $H > 1$ and $e_h >> 0$ for every h. We shall write $x_h^t \geq 0$ for the consumption of household h in period t, and $x_h = \left(\cdots, x_h^t, \cdots\right)$ for the sequence of consumption. An allocation of the goods, $x = (\cdots, x_h, \cdots) \in \left(\mathbb{R}_+^{T+1}\right)^H$, is feasible if $\sum_{h=1}^H \left(x_h^t - e_h^t\right) = 0$ for $t = 0, 1, ..., T$. Household h's preferences are represented by an additively time separable increasing utility function $u_h(x_h) = u_h^0\left(x_h^0\right) + u_h^1\left(x_h^1\right) + \cdots + u_h^T\left(x_h^T\right)$. The additive structure allows us to provide a clean analysis of the efficient allocations which arise as temporary equilibria. It also eliminates the conceptual issues about continuation utility which might appear as a potential hazard for the intertemporal general equilibrium analysis. However, it will become clear that the additive structure is not needed mathematically for the general point about induced income transfers.

We will consider two kinds of assets in the economy. The first of these is a discount bond traded in every period, which matures in one period. That is, the bond traded in period t pays out \$1 in period $t+1$, where $t = 0, 1, ..., T-1$. The second is a discount bond with a longer maturity, called the L-bond, which can also be traded in every period. The L-bond pays \$1 in period T, nothing in other periods. Note that the payout is fixed in units of account, not in units of good. The net supply of any of these bonds is zero. In every period, the good and the bonds are traded competitively. By assumption no default occurs.

We write p^t for the prevailing price of the good in units of account in period t, and $p = \left(\cdots, p^t, \cdots\right)$ for a sequence of prices. Similarly, write q^t and q_L^t for the respective prevailing prices of the bond and the L-bond in period t, and q and q_L for sequences of prices. Denote by b_h^t and l_h^t the respective amounts of the bond and the L-bond household h holds at the end of period t. Thus $l_h^t - l_h^{t-1}$ is the amount of the L-bond traded in period t, which costs $q_L^t \left(l_h^t - l_h^{t-1}\right)$. Unlimited short sales are allowed, so b_h^t and l_h^t are possibly any negative number, but recall that default is not allowed. We write $b^t = (..., b_h^t, ...)$ and $l^t = (..., l_h^t, ...)$ for allocations of the bond and the L-bond in period t, respectively, and write $\mathbf{b} = \left(\cdots, b^t, \cdots\right)$ and $\mathbf{l} = \left(\cdots, l^t, \cdots\right)$ for a sequence of such allocations. Since the bonds are in zero net supply, we say $\mathbf{b}$ (resp. $\mathbf{l}$) is feasible if $\sum_{h=1}^H b_h^t = 0$ (resp. $\sum_{h=1}^H l_h^t = 0$) holds in every period t.

As is known, if the bond prices are given, the payoff of L-bond is can be replicated by a plan of dynamic transactions of the bonds: buy $q^{t+1} \cdots q^{T-1}$ units of the bond in period t, which costs $q^t q^{t+1} \cdots q^{T-1}$, and use the payout of the bond to buy $q^{t+2} \cdots q^{T-1}$ units of the bond in the next period t, and so on. This synthetic trading strategy has exactly the same yield as a unit of the long term bond in period T, and hence by the no arbitrage principle or the law of one price, the market price of the L-bond in period t, q_L^t, must be the same as the cost of the plan. If they are different, an arbitrarily large amount of profits will be extracted with no cost, i.e., there is free lunch in the bond markets. It can be readily verified that the converse holds, too: the bond markets admit *no free lunch if and only if* $q_L^t = q^t q^{t+1} \cdots q^{T-1}$ holds for $t = 0, 1, ..., T-1$.[4] Thus we shall use the following convention:

Definition 1 *For a sequence of bond prices starting in period t, $q^t, q^{t+1}, ..., q^{T-1}$ and $q_L^t, q_L^{t+1}, ,, , q_L^{T-1}$, the no arbitrage condition is satisfied in period t if $q_L^s = q^s \cdots q^{T-1}$ holds for $s = t, t+1, ..., T-1$.*

[4]It is clear that the bonds have linearly independent payoffs and so any (positive) prices are consistent. Then no free lunch is equivalent to the L-bond being properly priced, which is exactly the condition states. More generally, as is well known (e.g., Lemma 19.E.1 of Mas-Collel et al [1995]), in the rational expectations model, no free lunch is equivalent to the existence of state prices, and asset prices are derived using the state prices. In our context, the (normalized) state prices are discounted values of forecast future bond prices.

In other words, for given prices, the no arbitrage condition holds in period t if and only if the L-bond and the bond are perfect substitutes under those prices in period t as well as the following periods. Notice in particular that $q_L^{T-1} = q^{T-1}$ holds since the two bonds are indistinguishable perfect substitutes in period $T-1$ by construction. Consequently, if the realized prices satisfies the no arbitrage condition, the market for the L-bond has no additional implication on the allocation of the good except for indeterminacy of asset trade, which arises since the L-bond and the dynamic plan are perfect substitutes.

As is well known, since under perfect foresight the forecast prices turn out to be correct, the prices which realize ex post satisfy the no arbitrage condition in every period, and hence the L-bond market has no allocational implications. To ask to what extent the conclusion above depends on perfect foresight, we shall first define temporary equilibrium which accommodates heterogeneous forecasts. Now a household h will trade with some forecast prices in mind, which are not necessarily correct ex post, given market prices prevailing in period t, i.e., p^t q^t and q_L^t. It means in particular that households might anticipate different rate of real returns of the bond. For t, $t = 0, 1, ..., T$, write $\hat{p}_{h|t} = \left(\hat{p}_{h|t}^{t+1}, ..., \hat{p}_{h|t}^{T}\right)$, $\hat{q}_{h|t} = \left(\hat{q}_{h|t}^{t+1}, ..., \hat{q}_{h|t}^{T-1}\right)$ and $\hat{q}_{Lh|t} = \left(\hat{q}_{Lh|t}^{t+1}, ..., \hat{q}_{Lh|t}^{T-1}\right)$ for the forecast prices of the good and respectively the prices of the short and the L-bond, where the subscript $h|t$ indicates that it is the forecast of household h made in period t. As mentioned in the introduction, in what follows we do not restrict forecasts a priori; thus in particular we do not impose any specific learning procedure at this point, though requiring markets to be in a temporary equilibrium will dictate that forecasts cannot admit arbitrage opportunities.

In every period t, $t = 0, 1, ..., T$, household h optimizes given prices p^t, q^t and q_L^t as well as the outstanding bond holdings b_h^{t-1} and l_h^{t-1} (where $l_h^{-1} = b_h^{-1} = 0$ by convention) under the following constraints,

$$
\begin{aligned}
p^t x^t + q^t b^t + q_L^t \left(l^t - l_h^{t-1}\right) &\leq p^t e_h^t + b_h^{t-1}, \qquad (2)\\
\hat{p}_{h|t}^{t+1} \hat{x}^{t+1} + \hat{q}_{h|t}^{t+1} \hat{b}^{t+1} + \hat{q}_{Lh|t}^{t+1} \left(\hat{l}^{t+1} - l^t\right) &\leq \hat{p}_{h|t}^{t+1} e_h^{t+1} + b^t,\\
&\vdots\\
\hat{p}_{h|t}^{T-1} \hat{x}^{T-1} + \hat{q}_{h|t}^{T-1} \hat{b}^{T-1} + \hat{q}_{Lh|t}^{T-1} \left(\hat{l}^{T-1} - \hat{l}^{T-2}\right) &\leq \hat{p}_{h|t}^{T-1} e_h^{T-1} + \hat{b}^{T-2},\\
\hat{p}_{h|t}^{T} \hat{x}^{T} &\leq \hat{p}_{h|t}^{T} e_h^{T} + \hat{b}^{T-1} + \hat{l}^{T-1},
\end{aligned}
$$

with variables x^t, b^t, l^t, $\hat{x}^{t+1},...,\hat{x}^T$, $\hat{b}^{t+1},...,\hat{b}^{T-1}$ and $\hat{l}^{t+1},...,\hat{l}^{T-1}$. The choice variables with hats are also forecasts (where these variables are written without the subscript $h|t$ for convenience) and yet to be realized at the time household h trades x^t, b^t and l^t. The forecasts do not necessarily coincide with the actual trades carried out in the future, and they need not constitute feasible allocations.

The inequalities in (2) exhibits some homogeneity, and one might wonder if the price of the good each period can be normalized to 1 without loss of generality. Indeed if all future prices were perfectly anticipated, it would be natural to set p_t and $\hat{p}^{t+s}_{h|t}$, $s = 1,...,T-t$ equal to one. In our setting, the price forecasts are heterogeneous and presumably updated each period. Setting the price of the good to one each period, though technically feasible, is somewhat problematic conceptually as it does not sit well with the heterogeneity of forecasts and their updating across households. We therefore choose to proceed without normalization.

Remark 2 *Suppose that a sequence of forecast bond prices in period t, $\hat{q}^{t+1}_{h|t},...,\hat{q}^{T-1}_{h|t}$ and $\hat{q}^{t+1}_{Lh|t},...,\hat{q}^{T-1}_{Lh|t}$, satisfy the no arbitrage condition with the ex post market prices q^t and q^t_L; that is, $q^t_L = q^t\hat{q}^{t+1}_{h|t}\cdots\hat{q}^{T-1}_{h|t}$ and $\hat{q}^s_{Lh|t} = \hat{q}^s_{h|t}\cdots\hat{q}^{T-1}_{h|t}$ holds for $s = t+1,...,T-1$. Then multiplying period s budget in (2) with $q^t\hat{q}^{t+1}_{h|t}\cdots\hat{q}^s_{h|t}$ and summing up, we obtain a single inequality:*

$$p^t\left(x^t - e^t_h\right) + \sum_{s=t+1}^{T} \underline{\hat{p}}^s\left(\hat{x}^s - e^s_h\right) \leq b^{t-1}_h + q^t_L l^{t-1}_h, \tag{3}$$

where $\underline{\hat{p}}^s := q^t\hat{q}^{t+1}_{h|t}\cdots\hat{q}^{s-1}_{h|t}\hat{p}^s$ is the discounted forecast price of period s good. It can be readily confirmed that if a consumption stream satisfies (3) then there are some trading plan of the bonds such that (2) holds. Of course, this can be seen as just an instance of the well known property that in a dynamically complete markets, the sequential budget is equivalent to a single budget constraint which is referred to as the Arrow Debreu (AD) budget. The constraint (3) is referred to as the perceived AD budget in period t given $b^{t-1}_h + q^t_L l^{t-1}_h$.

We say that choice variables in period t are *justifiable* if there are forecasts such that the chosen value of the variables is part of an optimal trade, given period t market prices. That is, the current choice is justifiable if they constitute household h's demand for the good and the bonds in period t for some forecasts. Thus $\left(x^t, b^t, l^t\right)$ is justifiable at prices $\left(p^t, q^t, q_L^t\right)$ (and b_h^{t-1}, l_h^{t-1}) in period t for household h if there exist forecasts $\hat{p}_{h|t}$, $\hat{q}_{h|t}$, and $\hat{q}_{Lh|t}$ about future prices such that household h's utility is maximized at x^t, b^t, l^t, $\hat{x}^{t+1}, ..., \hat{x}^T$, $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$and $\hat{l}^{t+1}, ..., \hat{l}^{T-1}$ for some $\hat{x}^{t+1}, ..., \hat{x}^T$, $\hat{b}^{t+1}, ..., \hat{b}^{T-1}$ and $\hat{l}^{t+1}, ..., \hat{l}^{T-1}$ (given b_h^{t-1}, l_h^{t-1}).

Sequential markets are in temporary equilibrium if the demand meets the supply in every market; that is, a temporary equilibrium (TE) is defined as follows:

Definition 3 *Prices p^t, q^t and q_L^t, $t = 0, 1, ..., T$, and an allocation $(x, \mathbf{b}, \mathbf{l})$ constitute a temporary equilibrium (TE) if (1) x, $\mathbf{b}$, and $\mathbf{l}$ are feasible, (2) for every household h, in every period $t = 0, 1, ..., T$, $\left(x_h^t, b_h^t, l_h^t\right)$ is* justifiable *at prices $\left(p^t, q^t, q_L^t\right)$ (and b_h^{t-1}, l_h^{t-1}). The prices p^t, q^t and q_L^t, $t = 0, 1, ..., T$ will be referred to as ex post temporary equilibrium (ex post TE) prices. A TE with an efficient consumption allocation is called a ETE. If the prices (p, q, q_L) can be used as the forecasts in the justifiability condition for every household in every period, TE is called a perfect foresight equilibrium (PFE).*

By construction, a PFE obtains if prices and forecasts are assumed to be the same up front. Hence it is readily verified that the definition above is equivalent to the standard definition of a perfect foresight equilibrium.

Two important observations follow. The first concerns the role of the no arbitrage condition in forecasts.

Observation 4 *Justifiability implies that household h's forecasts must not allow any arbitrage opportunity to itself at any time, or else the utility maximization problem has no solution since it will provide itself with a free lunch at some point whereas there is no limit on the volume of trade. Consequently, although there is no explicit intertemporal link among the prices and forecasts in the definition, each household must deem the two types of the bonds as perfect substitutes under their forecasts in every period. It also means that in any period, the dynamic choice problem of consumption and bonds for a household is reduced to the consumer's problem under the perceived AD budget constraint (3) (See Remark 2).*

The second is about the ex post budget constraint and forecasts.

Observation 5 *Since the forecast prices might not be correct, the planned behavior might turn out to be infeasible. Nevertheless, the trades carried out must respect the budget constraint. Hence as part of equilibrium requirements, extreme forecasts and trading behavior which are incompatible with the budget constraint are excluded.*

Therefore for prices p^t, q^t and q_L^t and a feasible allocation x, b, and l to arise as TE, since the total expenditure must be equal to the total income in every period by monotonicity, the following budget equations, which we shall refer to as the *ex post budget constraint*, must hold for x_h, b_h, l_h for every household h:

$$\begin{aligned} p^0 x_h^0 + q^0 b_h^0 + q_L^0 l_h^0 &= p^0 e_h^0 \\ p^1 x_h^1 + q^1 b_h^1 + q_L^1 \left(l_h^1 - l_h^0\right) &= p^1 e_h^0 + b_h^0 \\ &\vdots \\ p^t x_h^t + q^t b_h^t + q_L^t \left(l_h^t - l_h^{t-1}\right) &= p^t e_h^t + b_h^{t-1} \qquad (4) \\ &\vdots \\ p^{T-1} x_h^{T-1} + q^{T-1} b_h^{T-1} + q_L^{T-1} \left(l_h^{T-1} - l_h^{T-2}\right) &= p^{T-1} e_h^{T-1} + b_h^{T-2} \\ p^T x_h^T &= p^T e_h^T + b_h^{T-1} + l_h^{T-1} \end{aligned}$$

It turns out that the ex post budget constraint alone gives a good deal of information when we restrict our attention to efficient allocations, which we shall explore in the next section.

Remark 6 *Since we only consider TE and $q_L^{T-1} = q^{T-1}$ is a requirement of the no arbitrage condition (the two bonds are identical in period $T-1$), we shall automatically assume $q_L^{T-1} = q^{T-1}$ whenever ex post TE prices are considered.*

4 Budget Compatibility and Wealth Transfer by the Redundant Asset

4.1 Efficient allocations

Since TE can accommodate a large class of feasible allocations, the aforementioned redistribution role associated with the L-bond will be hard to identify if all TE allocations are considered. An obvious modelling choice is to focus on the efficient allocations; if two distinct efficient allocations are compared, one can be deemed as a result of an (efficient) wealth transfer operated on the other. Thus we are primarily interested in a ETE in the following analysis.

It will then be useful to work with an economy where the set of efficient allocations has a simple structure so that the purely distributional effects can be observed in a transparent manner. Therefore, in addition to the additively time separable utility function $u_h\left(x_h^0\right)+u_h\left(x_h^1\right)+\cdots+u_h\left(x_h^T\right)$, with $u_h'>0$, $u_h''<0$ and $u_h'(0)=+\infty$, we assume that the total endowment is one in every period, i.e., $\sum_{h=1}^H e_h^t=1$ for $t=0,1,...,T$. Consequently, a feasible allocation of goods is efficient intertemporally if and only if it assigns a time invariant consumption to every household.[5] An efficient allocation can therefore be parameterized by a tuple of positive numbers $\xi_1,\xi_2,\cdots,\xi_H$ with $\sum_{h=1}^H \xi_h=1$, where ξ_h is the time invariant consumption level of household h. We shall identify an efficient allocation with a tuple $(\cdots,\xi_h,\cdots)$ of H positive numbers summing up to one, which might be viewed as a wealth distribution, which will help us to identify the redistribution role.

As a first benchmark, consider an AD equilibrium $(p,x)\in\mathbb{R}^{T+1}\times\left(\mathbb{R}^{T+1}\right)^H$ of this economy, where $p=\left(\cdots,p^t,\cdots\right)$ are positive prices of the goods and $x=(\cdots,x_h,\cdots)$ is the associated allocation of the goods; that is, each household is maximizing utility at x_h given prices p and income $p\cdot e_h$, and x is feasible. As is known, and we will see later, an AD equilibrium allocation is an equilibrium allocation in a PFE, and vice versa. As far as the allocational property is concerned, we will use AD and PFE interchangeably.

The allocation x in an AD equilibrium is of course efficient by the first fundamental theorem of welfare economics. From utility maximization and the additive time separability of the utility function, for every household h, prices p must be proportional to the gradient vector $u_h'(x_h)=\left(u_h'\left(x_h^0\right),u_h'\left(x_h^1\right),\cdots,u_h'\left(x_h^T\right)\right)=\left(\cdots,u_h'(\xi_h),\cdots\right)$ which is also time invariant. Thus the AD equilibrium price system must also be time invariant.

By the homogeneity of equilibrium prices, the time invariant AD equilibrium price of

[5] Indeed, it can be readily confirmed that a time invariant feasible allocation is efficient. Conversely, for any consumption path $\left(x^0,x^1,...,x^T\right)$, every household prefers the path which provides its average consumption $\frac{1}{T+1}\sum_{t=0}^T x^t$ every period, so an efficient allocation must be time invariant.

the good can be normalized to be one. The market value of household h's endowments is $\frac{1}{T+1}\sum_{t=0}^{T} e_h^t$, and the market value of the time invariant consumption of ξ_h is $(T+1)\,\xi_h$. So from the budget constraint, we conclude that the AD equilibrium allocation is unique and household h consumes $\xi_h = \frac{\sum_{t=0}^{T} e_h^t}{T+1}$, i.e., the time average endowments, in every period.

Example 7 *In the unique AD equilibrium, the time invariant consumption is* $\bar{x}_h = \frac{1}{3}(1-\varepsilon)$ *for* $h = 1, 2, 3$, *and* $\bar{x}_4 = \varepsilon$. *In particular if* $\varepsilon = \frac{1}{4}$, *every household consumes* $\frac{1}{4}$ *in every period.*

4.2 Budget compatible Allocations

To address the issue of wealth transfer with heterogeneous forecasts in ETE, it will turn out to be convenient to first examine allocations which satisfy the ex post budget constraints with some stream of ex post market prices. A pair of a consumption allocation and ex post prices, $(\mathbf{x}, (p, q, q_L))$, will be called an *allocation - price pair* (AP). We say that a AP $(\mathbf{x}, (p, q, q_L))$ is *budget compatible* if there are allocations of the bonds $(\mathbf{b}, \mathbf{l})$ such that the ex post budget constraint (4) holds for every household. Since the ex post budget (4) must hold in any TE, a TE must induce a budget compatible AP (CAP). Conversely, for a CAP $(\mathbf{x}, (p, q, q_L))$, if the allocation is efficient and the justifiability requirement is met for all households with some forecasts, then we have a ETE.[6] We shall first study the structure of *CAP*, and the issue of justifiability will be taken up in the next section. It will become evident that keeping the justifiability requirement separate from budget compatibility in this way makes the analysis more tractable and transparent. While the concept of budget compatibility has little to do with efficiency, we will see how the presence of the L-bond might induce an additional channel of wealth transfers.

Notice that the ex post prices in the ex post budget constraint (4) need not satisfy

[6] As we have seen, the ex post budget constaint is implied by the justifiability requirement of TE, and so there is a bit of logical repetition in separating only the budget constraints. Furthermore, since there is no requirement that the forecast values of future variables that are used in justifying an allocation actually clear markets, there could be many different profiles of forecasts that may justify the same TE allocation.

the no arbitrage condition even for TE. We use this observation to separate CAP into two categories: if the no arbitrage conditions holds for the associated ex post prices in every period, it will be termed Category 1. If otherwise, i.e., the no arbitrage condition is violated for the associated ex post prices in some periods, it will be termed Category 2.

The following is an critical observation about Category 1 CAP. Since the no arbitrage condition $q_L^t = q^t q^{t+1} \cdots q^{T-1}$ holds for $t = 0, 1, ..., T-1$ by assumption, a single budget constraint analogous to (3) is obtained:

$$\sum_{t=0}^{T} \underline{p}^t \left(x_h^t - e_h^t \right) = 0 \tag{5}$$

where $\underline{p}^t ;= q^0 q^1 \cdots q^{t-1} p^t$ is the discounted period t (ex post) price. It can be readily seen that a feasible allocation x of the goods satisfies (5) for every h if and only if there exist a feasible allocation $\mathbf{b}$ and $\mathbf{l}$ of bond and the L-bond such that (4) holds for every h with equality.

Remark 8 *While in principle a household may hold a portfolio comprising bonds and the L-bond (Observation 4), each household can finance its consumption in a category 1 CAP by just trading the bond. Indeed, if the household were to act as if the L-bond market was shut, i.e., in the ex post budget constraint (4) where $l_h^t = 0$ is required throughout, (4) still reduces to the same single ex post budget constraint (5). Then a feasible allocation x of the goods satisfies (5) for every h if and only if there exist a feasible allocation* $\mathbf{b}$ *of bond such that (4) holds with $l_h^t = 0$ for every h.*

An important implication of Remark 8 above is that CAP allocations of category 1, irrespective of their efficiency status, can be generated with trade in only the bond. The L-bond is therefore a redundant asset, even without perfect foresight, for CAP allocations of category 1.

The analysis above shows that $(\mathbf{x}, (p, q, q_L))$ is a CAP of category 1 where allocation x is feasible if and only if there are positive weights $p^0, p^1, ...p^T$, satisfying the following equations:

$$\begin{array}{rcll} \sum_{t=0}^{T} p^t \left(x_h^t - e_h^t \right) & = & 0 & \text{for } h = 1, 2, .., H \\ \left(\sum_{h=1}^{H} x_h^t \right) - 1 & = & 0 & \text{for } t = 0, 1, .., T \end{array} \tag{6}$$

The first set of H equations implies budget consistency for all households and the second set of T equations implies feasibility.

When restricted to efficient allocations, i.e., time invariant allocations, the system of equations (6) can be significantly simplified. Consider first the allocation in a PFE. Then constraint (5) holds ex ante with forecast prices, i.e., household h plans to choose a utility maximizing x_h given constraint (5), and trades the bonds to finance, i.e., to satisfy (4). This is of course the well known reason why a PFE is equivalent to an AD equilibrium in this model. The ex post prices must then satisfy the no arbitrage condition, and the allocation is efficient by the first fundamental theorem of welfare economics. Consequently (6) is satisfied at a PFE: that is, with correctly forecasted prices, the corresponding CAP belongs to category 1.

But (6) has many other solutions, even when restricted to efficient allocations. Letting $e^t = \left(\cdots, e_h^t, \cdots\right)$ be the row vector of initially endowed goods among households in period t, the set of efficient allocations in CAP of category 1 under the maintained assumptions has the following clean structure. We refer the reader to the Appendix for its derivation.

Proposition 9 *The set of efficient allocations arising in some CAP of category 1 is given by the set* $\left\{\sum_{t=0}^{T} p^t e^t : \sum_{t=0}^{T} p^t = 1, p^t > 0, \text{ for } t = 0, 1, ..., T\right\}$, *i.e., the set is the relative interior of the convex hull of* $T+1$ *vectors* $e^0, e^1, ..., e^T$ *in* $\mathbb{R}^H$.

The set of efficient allocations found in Proposition 9 is a convex set of dimension at most $\min(T, H-1)$.[7] In comparison with the set of PFE, which is a singleton set, we see that heterogeneity of forecasts alone might create a great deal of wealth transfers among the households. If $T < H - 1$, which is the case we focus on, there are efficient allocations which do not arise in any CAP of category 1.

Example 10 *In any CAP of category 1, household 4 consumes* ε *in every period by Proposition 9.*

[7] If any choice of $\min(T+1, H)$ vectors among $e^0, e^1, e^2, \cdots, e^T$ are affine independent, then the dimension is exactly $\min(T, H-1)$. Since such affine independence is a generic property, we conclude that generically in endowments (with total resource equal to one in every period),the dimension of category 1 CAP allocations is $\min(T, H-1)$.

Now we turn to the examination of category 2 CAP. It turns out that any feasible allocation can arise in a category 2 CAP. Moreover, the associated ex post prices in the CAP can be arbitrarily set as long as they violate the no arbitrage condition. Thus the failure of the no arbitrage condition for ex post prices, no matter how minor it might be, has significant implications on income transfers.

To see this, fix any ex post prices $p^0, p^1 ..., p^T$, $q^0, q^1, ..., q^{T-1}$, and $q_L^0, q_L^1, ..., q_L^{T-1}$ such that $q^{T-1} = q_L^{T-1}$. The discounted prices are written as $\underline{p}^t := q^0 q^1 \cdots q^{t-1} p^t$, $\underline{q}^t := q^0 q^1 \cdots q^{t-1} q^t$, and $\underline{q}_L^t := q^0 q^1 \cdots q^{t-1} q_L^t$. Assume that the no arbitrage condition fails in some period, which then means that $\underline{q}_L^{T-1} - \underline{q}_L^{T^*} \neq 0$ for some $t = T^*$, $0 \leq T^* < T-1$, since the no arbitrage condition would hold at every t if the discounted L-bond price $\underline{q}_L^t$ were constant over time. Fix any feasible allocation x. We shall demonstrate that there are feasible allocations of the bond and the L-bond such that the consumption stream x_h meats the ex post budget constraint (4) for every household h.

Consider the following dynamic trading plan, where the L-bond is traded only in period T^*: $l_h^t = 0$ for $t < T^*$, and then keep household h from any additional transaction of the L-bond after period T^*, so that $\left(l_h^t - l_h^{t-1}\right) = 0$ for $t = T^* + 1, ..., T^* - 1$. For later reference we shall refer to such a canonical trading plan as a CT(T^*), or simply a CT. Hence by construction, $q_L^t \left(l_h^t - l_h^{t-1}\right) = 0$ holds for $t = 1, ...T - 1$, except $t = T^*$ when $l_h^{T^*}$ units of the L-bond is bought. The amount $l_h^{T^*}$ such that $\sum_h l_h^{T^*} = 0$ will be specified later. For $t = 0, 1, ..., T^* - 1$, set b_h^t iteratively to meet period t budget, $p^t x_h^t + q^t b_h^t = p^t e_t^t + b_h^{t-1}$, where $b_h^{-1} = 0$. Let $b_h^{T^*}$ solve the period T^* budget, $p^{T^*} x_h^{T^*} + q^{T^*} b_h^{T^*} + q_L^{T^*} l_h^{T^*} = p^{T^*} e_h^{T^*} + b_h^{T^*-1}$, where $b_h^{T^*-1}$ and $l_h^{T^*}$ have already been determined as above. For $t = T^* + 1, ..., T - 1$, set b_h^t iteratively to meet period t budget, $p^t x_h^t + q^t b_h^t = p^t e_t^0 + b_h^{t-1}$.

Then by construction the budget in each period $t = 0, 1, ..., T - 1$ is satisfied, i.e., the plan finances the given stream of consumption at the given prices up to period $T - 1$, regardless of the choice of $l_h^{T^*}$:

$$\begin{aligned} p^0x_h^0 + q^0b_h^0 &= p^0e_h^0 \\ p^1x_h^1 + q^1b_h^1 &= p^1e_h^0 + b_h^0 \\ &\vdots \\ p^{T^*}x_h^{T^*} + q^{T^*}b_h^{T^*} + q_L^{T^*}l_h^{T^*} &= p^{T^*}e_h^{T^*} + b_h^{T^*-1} \\ p^{T^*+1}x_h^{T^*+1} + q^{T^*+1}b_h^{T^*+1} &= p^te_h^t + b_h^{T^*} \\ &\vdots \\ p^{T-1}x_h^{T-1} + q^{T-1}b_h^{T-1} &= p^{T-1}e_h^{T-1} + b_h^{T-2} \end{aligned} \tag{7}$$

Consequently, the constructed trading plan $b_h^0, b_h^1, ..., b_h^{T-1}$ and $l_h^0, l_h^1, ..., l_h^{T-1}$ is budget feasible if the period T budget equation is satisfied in addition, which is

$$p^Tx_h^T = p^Te_h^T + b_h^{T-1} + l_h^{T^*}, \tag{8}$$

since $l_h^t = l_h^{T^*}$ for $t > T^*$. It can then be readily confirmed that the constructed allocation $\mathbf{b}$ and $\mathbf{l}$ are feasible from these budget equations and the feasibility of x. Thus we obtain the desired allocations if we find suitable $l_h^{T^*}$ which satisfies (8) for all h.

If (8) holds, multiply the period t budget in (7) with $q^0q^1\cdots q^{t-1}$ for $t = 1, 2, ..., T$, summing them from period 0 to T, we have

$$\sum_{t=0}^{T} \underline{p}^t \left(x_h^t - e_h^t\right) = (\underline{q}_L^{T-1} - \underline{q}_L^{T^*})l_h^{T^*}. \tag{9}$$

Thus it can be readily shown that (8) is satisfied as well if and only if (9) holds. Since $\underline{q}_L^{T-1} - \underline{q}_L^{T^*} \neq 0$ by assumption, (9) holds for $l_h^{T^*}$ given by the rule:

$$l_h^{T^*} = \frac{\sum_{t=0}^{T} \underline{p}^t \left(x_h^t - e_h^t\right)}{\underline{q}_L^{T-1} - \underline{q}_L^{T^*}} \tag{10}$$

By the feasibility of x, $\sum_h l_h^{T^*} = 0$ holds.

To sum up, we have established the following result:

Proposition 11 *For any ex post prices (p, q, q_L) where the no arbitrage condition does not hold, for any feasible consumption allocation x, $(x, (p, q, q_L))$ is a CAP of category 2. Moreover, the supporting bond trading can be set to be a CT.*

To appreciate Proposition 11, first recall that the set of feasible allocations arising from category 1 CAP can be sustained without the L-bond (recall Proposition 9). On the other hand, the particular CT constructed in the proof of Proposition 11 uses trade in the L-bond in a central way. Therefore, the L-bond, which is redundant for category 1 CAP, can have a significant distributional role.

Remark 12 *The role of the L-bond becomes clearer if one recalls the AD budget constraint: notice that equation (10) says that consumption x_h is AD budget feasible if household h is provided with an extra income of $(\underline{q}_L^{T-1} - \underline{q}_L^{T^*}) l_h^{T^*}$, which might be negative of course. If allocation $(\cdots, x_h, \cdots)$ is feasible, $\sum_{h=1}^{H} \sum_{t=0}^{T} \underline{p}^t \left(x_h^t - e_h^t \right) = 0$, and so $\left(\underline{q}_L^{T^*} - \underline{q}_L^{T-1} \right) l_h^{T^*}$, $h = 1, 2, ..., H$, constitute income transfers among households.*

From this viewpoint, the essence of Proposition 11 is that the failure of no arbitrage condition in any one period ex post, no matter how minor it might be, is consistent with any amount of income transfer while maintaining budget compatibility. The economic intuition is simple. Notice that in our construction, required income transfers effectively occur through trade of the L-bond in only one period when the no arbitrage condition fails ex post. In that period, the law of one price is broken ex post: if the same object has two prices simultaneously, no matter how small the difference is, any kind of transfers can be established in competitive markets with no limit on the volume of trade.

Finally we note that the trading strategy proposed here is not the unique one that would work for establishing the Proposition; we chose it as it will be convenient to work with it for the subsequent justifiability argument.

5 Justifiability of CAP

5.1 General Argument

Recall that a CAP is a TE if the allocation is justifiable, i.e., in each period, there are suitable forecasts about future prices with which the prescribed amount of consumption is the quantity demanded at the prevailing prices in that period. We shall argue that justifiability is expected to hold, at least when the allocation is close enough to a PFE allocation. We maintain the assumption of history-free updating of forecasts, and hence there is no ad hoc learning procedure.[8]

Let $(x,(p,q,q_L))$ be a CAP, and fix a portfolio of the bond and the L-bond associated with the consumption, i.e., (4) is satisfied. We shall first ask if $\left(x^t,b^t,l^t\right)$ is justifiable in period t. Since the forecast prices must satisfy the no arbitrage condition (Observation 4), household h maximizes utility under the perceived AD budget constraint (3) given b_h^{t-1} and l_h^{t-1} (Remark 2).[9] The utility maximization can be solved by first finding a consumption bundle which maximizes utility $u_h\left(x^t\right)+\sum_{s=t+1}^{T}u_h\left(\hat{x}^s\right)$ under (3), and then determining bond transactions which finances the consumption $\left(x^t,\hat{x}^{t+1},...,\hat{x}^T\right)$. Then $\left(x_h^t,b_h^t,l_h^t\right)$ of household h is justified in period t if and only if x_h^t is the quantity demanded of the good in period t under the perceived AD budget constraint (3) for some forecasts about the discounted prices, $\underline{\tilde{p}}^{t+1},...,\underline{\tilde{p}}^{T}$, and the associated $\left(b_h^t,l_h^t\right)$ is found to satisfy the dynamic budget (2) for period t.

In principle we need to describe the forecast prices of the bond and the L-bond, and the dynamic portfolio of the two types of assets which appear to be a complicated task. But here is a key: as is seen in Observation 4, the forecasts must be set in such a way that the two bonds must be perfect substitutes in this construction, which means that when consumption x^t is demanded, any combination $\left(b_h^t,l_h^t\right)$ which satisfies the period t budget can be demanded. Thus if we want to induce the household to choose a particular $\left(b_h^t,l_h^t\right)$, the period t budget is a necessary and sufficient condition, and it is not necessary to fix the details of the forecast bond prices or the planned portfolio of the assets. To sum up, all we need to show is x^t is the quantity demanded under the perceived AD budget constraint (3) for some forecasts for some forecast $\underline{\hat{p}}^s$, $s=t+1,...,T$.

Suppose that the demand function under consideration is responsive to forecasts in the sense that as a function of price forecasts, the demand changes in any direction. This property will be generically true under some mild and plausible conditions on utility

[8] We refer to Chatterji-Kajii [2023] for general techniques and issues therein. In particular, stochastic forecasts would make justifiability easier to satisfy, and hence the set of ETE might be larger than what we shall report. But we restrict attention to point forecasts since it suffices to make our point, besides expositional simplicity.

[9] Since the saving decision in period $t-1$ might be done with very incorrect forecasts, it is possible that the household is practically bankrupt. However, as long as endowments are positive, there always exist (very optimisitic) forecasts with which household can repay the debt in future.

functions and endowments, at least if x_h^t is the AD (thus PFE) consumption in period t. We therefore contend that any CAP with its feasible allocation close enough to the AD equilibrium allocation with prices close to the PFE prices is justifiable in any period.

For a specific case of additive log function, i.e., $u_h(z) = \ln(z)$, we can give a formal and more constructive argument. In this case, in period t, $t < T$, household h spends its AD income evenly in the remaining periods including period t, and the quantity demand in period t is found by dividing the intended expenditure by the prevailing market price of the good. From (3) we see that the AD income consists of the market value of the endowment and the outstanding L-bond with the payout from the bond, which is $\underline{p}^t e_h^t + \underline{q}^{t-1} b_h^{t-1} + \underline{q}_L^t l_h^{t-1}$, plus the forecast (discounted) income $\hat{m} := \sum_{s=t+1}^{T} \hat{\underline{p}}^s e_h^s$, which depends on price forecasts, but not time t market variables. Thus the problem of justifiability is reduced to finding a forecast income $\hat{m}$ such that

$$\underline{p}^t x_h^t = \frac{\underline{p}^t e_h^t + \underline{q}^{t-1} b_h^{t-1} + \underline{q}_L^t l_h^{t-1} + \hat{m}}{(T - t + 1)}. \tag{11}$$

By varying price forecasts, the forecast income $\hat{m}$ can take any positive value. Thus we find a necessary and sufficient condition for justifiability as follows:

$$(T - t + 1)\,\underline{p}^t x_h^t > \underline{p}^t e_h^t + \underline{q}^{t-1} b^{t-1} + \underline{q}_L^t l_h^{t-1}, \tag{12}$$

In summary, we have

Lemma 13 *When $u_h(z) = \ln(z)$, a consumption stream x_h^0, x_h^1, ..., x_h^T which arises in a CAP $(x, (p, q, q_L))$ is justifiable for household h if and only if inequality (12) is satisfied for every $t = 0, 1, ..., T - 1$.*

5.2 Justifiability of category 1 CAP

Consider a consumption stream $x_h^0, x_h^1, ..., x_h^T$ which arises in a category 1 CAP $(x, (p, q, q_L))$. Recall that the consumption stream can be sustained by a trading plan where the L-bond is not traded (Remark 8). Thus as far as the consumption allocation is concerned, we can focus on the case $l_h^{t-1} = 0$ in the perceived AD budget constraint (3). The variable b^{t-1} can then be expressed in terms of consumption before period t, since for a category 1 CAP, the ex post budget (4) can be reduced to a single budget. From the budget equations till $t - 1$, we have $\sum_{s=0}^{t-1} \underline{p}^s (x_h^s - e_h^s) + \underline{q}^{t-1} b_h^{t-1} = 0$; in words, since transactions before t are already completed and they meet the budget equation (4), b_h^{t-1} is equal to the total net saving accumulated before t. So (3) can be written as

$$\underline{p}^t x^t = \underline{p}^t e_h^t - \sum_{s=0}^{t-1} \underline{p}^s \left(x_h^s - e_h^s\right) + \sum_{s=t+1}^{T} \underline{\hat{p}}^s \left(\hat{x}^s - e_h^s\right), \tag{13}$$

and the justifiability can be judged from the consumption and the discounted ex post prices for a category 1 CAP.

Now we consider the case of the log utility. Using $\sum_{s=0}^{t-1} \underline{p}^s \left(x_h^s - e_h^s\right) + \underline{q}^{t-1} b_h^{t-1} = 0$, condition (12) can be re-written as follows:

$$\sum_{s=0}^{t} \underline{p}^s \left(x_h^s - e_h^s\right) + (T-t)\, \underline{p}^t x_h^t > 0. \tag{14}$$

Therefore Lemma 13 can be written as follows:

Lemma 14 *When $u_h(z) = \ln(z)$, a consumption stream x_h^0, x_h^1, ..., x_h^T which arises in a category 1 CAP $(x, (p, q, q_L))$ is justifiable for household h if and only if inequality (14) is satisfied for every $t = 0, 1, ...T$.*

Note that this result is not restricted to being in a neighborhood of the AD equilibrium consumption. Condition (14) is trivially satisfied if $\sum_{s=0}^{t} \underline{p}^s \left(x_h^s - e_h^s\right) \geq 0$, i.e., the discounted value of household h's consumption is non-negative, which means household h is a borrower at the end of period t. In general the inequality will hold if household h's lending is not excessively large.

5.3 Justifiability of category 2 CAP

For a category 2 CAP $(x, (p, q, q_L))$, since the ex post prices might allow arbitrage between the two types of the bonds, the ex post budget (4) is not reduced to a single equation. Thus the justifiability issue in period t depends on the portfolio of the two types of the bonds. A complete characterization of justifiability is complex, since it will depend on gains and losses owing to recurrent unexpected arbitrage opportunities before period t. We shall therefore examine category 2 CAP of special interest, which satisfy the following: (1) the no arbitrage condition for ex post prices fails only in one period, T^*; (2) household h adopts a CT(T^*); (3) the consumption stream x_h^0, x_h^1, ..., x_h^T is constant over time and (4) ex post prices p and q are the same as the PFE prices. Conditions (1) and (2) mean that the unexpected income transfer occurs solely due to trade in period T^*. Condition (3) means that the consumption stream arises in a efficient allocation, which is our primary interest. Condition (4) with condition (1) means that as far as the ex post prices are concerned, they are almost indistinguishable from the PFE prices.

We shall omit the reference to household h for simplicity. Write $\bar{x}$ for the constant consumption. Since the discounted prices of the good is constant over time in the PFE, we might as well assume $p^t = 1$ for $t = 0, 1, ..., T$ and $q^t = 1$ for $t = 0, 1, ..., T-1$, $q_L^t = 1$ for $t \neq T^*$ and $q_L^{T^*} \neq 1$ for $t \leq T^*$.

From the general argument, to see the quantity demanded in period τ, we only need to specify the forecast discounted prices $\underline{\hat{p}}^t$ of period t good in period τ for $t = \tau+1, \cdots, T$ and the forecast consumption good price as well as bond and L-bond prices can be implicitly determined, although we will take a detour to give them explicitly to facilitate an economic interpretation. Specifically we shall set the consumption and the discount forecast prices by the rule $\hat{x}^t = \bar{x}$ and $\underline{\hat{p}}^t = 1$ for $t \neq T^*+1$, and the household will hold these values as forecasts except in period T^*. Then the marginal utility of consumption (per forecast price) equals to $u'(\bar{x})$ except for T^*+1. For $t = T^*+1$, we shall set $\underline{\hat{p}}^{T^*+1}$ and $\hat{x}^{T^*+1}$ such that

$$u'(\bar{x}) = \frac{u'\left(\hat{x}^{T^*+1}\right)}{\underline{\hat{p}}^{T^*+1}}. \tag{15}$$

The household forecast for date t consumption and discounted price is $\left(\hat{x}^t, \underline{\hat{p}}^t\right)$. So for instance in period $s < T^*$, the household expects that the consumption and the forecasts of discounted price will remain $\hat{x}^t = \bar{x}$ and $\underline{\hat{p}}^t = 1$ for $t \leq T^*$while for $t > T^*+1$, these revert to $\bar{x}$ and 1 respectively.

We now provide a forecast specification for the prices which sustains the discount forecast prices described above. The forecasted values for a period $t \neq T^*$ are $\hat{p}^t = 1$ and $\hat{q}^t = 1$, and $\hat{q}_L^t = 1$ follows by no arbitrage. Then $\underline{\hat{p}}^t = 1$ for $t \neq T^*$ by construction. So it remains to show that $\underline{\hat{p}}^{T^*+1}$ can arise as the discount forecast price of period T^*+1 in any period $t \leq T^*$. In period $t < T^*$ set the forecast nominal price $\hat{p}^{T^*+1} = \underline{\hat{p}}^{T^*+1}$, and then by construction $\underline{\hat{p}}^{T^*+1}$ is the discounted forecast price since the bond price forecast is 1 throughout these periods. In period $t = T^*$, the household forecasts the bond prices in such a way that the observed L-bond price $q_L^{T^*} \neq 1$ $(= q^{T^*})$ satisfies the no arbitrage condition. We do not specify the details of these prices but note that for any such bond prices the nominal price forecast of the good $\hat{p}^{T^*+1}$ can be set in a way that ensures that the discounted forecast remains constant at $\underline{\hat{p}}^{T^*+1}$, as is desired.

The specification of the bond prices above is not unique, but it exhibits a natural learning property as follows: when $t < T^*$, the forecasts of the bond and L-bond are held fixed at 1 and these forecasts are self-fulfilling. At T^*, for the first time the household observes a price which is different from the forecast price since $q_L^{T^*} \neq 1$. This necessitates a revision of forecast bond prices from $T^* + 1$ onwards so as to satisfy the no-arbitrage condition. At $T^* + 1$, the ex post bond and L-bond prices revert to 1, which conforms with the prices which have been forecast in any period before T^*, and the forecasts of the bond and L-bond prices revert to the self-fulfilling value of 1 from $T^* + 1$ onwards, as do the price forecasts of the good so that the discounted price forecasts of the good revert to the self-fulfilling value of 1.

With these specifications of the discount forecast prices, the planned consumption stream which allocates $\bar{x}$ for $t = 0, ..., T^*$, allocates $\hat{x}^{T^*+1}$ for $T^* + 1$ followed by $\bar{x}$ for $t = T^* + 2, ...T$ would satisfy the period 0 perceived AD budget constraint if the following holds:

$$\sum_{t=0}^{T^*} \left(\bar{x} - e^t\right) + \underline{\hat{p}}^{T^*+1} \left(\hat{x}^{T^*+1} - e^{T^*+1}\right) + \sum_{t=T^*+1}^{T} \left(\bar{x} - e^t\right) = 0 \tag{16}$$

Proposition 15 *If (15) and (16) have a non negative solution $\underline{\hat{p}}^{T^*+1}$ and $\hat{x}^{T^*+1}$, the constant stream of $\bar{x}$ is justifiable in any period t with our specifications of the forecasts.*

Proof. Find a CT(T^*) plan $\left(b^t, l^t\right)$, $t = 0, ..., T - 1$, which sustains a constant consumption stream $\bar{x}$ given the ex post prices. Since p^t and q^t are equal to one in any period t, from the budget equations from $t = 0$ to T^* in (7) and (9), we have

$$\sum_{t=0}^{T^*} \left(\bar{x} - e^t\right) + b^{T^*} + q_L^{T^*} l^{T^*} = 0 \tag{17}$$

and

$$\sum_{t=0}^{T} \left(\bar{x} - e^t\right) = (1 - \underline{q}_L^{T^*}) l^{T^*}. \tag{18}$$

We shall show that in any period t, consuming $\bar{x}$ and choosing $\left(b^t, l^t\right)$ is utility maximizing given some forecasts.

First consider period $t = 0$. Since (16) corresponds to the perceived AD budget for the household under the forecasts, (15) and (16) imply that the aforementioned planned consumption stream turns out to be utility maximizing on the perceived AD budget set and the household chooses $\bar{x}$ in period 0, and hence $b^0 = e^0 - \bar{x}$, as in the CT(T^*). For each $t < T^*$, since the ex post prices of the good, the bond and the L-bond prices are all one, the planned consumption is feasible under the perceived AD budget in period t with the same forecasts as required. It means that the planned consumption is still utility maximizing, the household continues to choose $\bar{x}$ and adjusting the bond according to the CT(T^*).

In period $t = T^*$, we have $q_L^{T^*} \neq 1$, whereas the forecast price was 1. By construction, the forecast discounted prices remain the same: $\underline{\hat{p}}^{T^*+1}$ for T^*+1 and 1 after T^*+1. By our assumption that households follow the CT(T^*) plan, the household does not trade the L-bond except at T^*. It means that the household does not receive any unexpected income in period T^*, and hence the equation (16) continues to assure that the consumption of $\bar{x}$ in period T^* with the planned consumption satisfies the perceived AD budget equation. Hence from (15), consuming $\bar{x}$ and saving the rest of income, which is $\sum_{t=0}^{T^*} \left(e^t - \bar{x}\right)$ is still utility maximizing. Since the bond and the L-bond are equivalent as saving devices under the forecasts, choosing the portfolio $\left(b^{T^*}, l^{T^*}\right)$ as in (17) is optimal.

In period $T^* + 1$, the price of the good is 1 and the bond prices are 1, contrary to the forecasts. The income from the bonds is $b^{T^*} + l^{T^*}$, which is equal to $-\sum_{t=0}^{T^*} \left(\bar{x} - e^t\right) + \left(1 - q_L^{T^*}\right) l^{T^*}$ by (17). Hence under the forecast prices the household' perceived AD budget constraint is

$$\sum_{t=0}^{T^*} \left(\bar{x} - e^t\right) + \left(x^{T^*+1} - e^{T^*+1}\right) + \sum_{t > T^*+1} \left(x^t - e^t\right) = \left(1 - q_L^{T^*}\right) l^{T^*} \tag{19}$$

From (18), we see that the constant stream of $x^t = \bar{x}$ for $t = T^* + 1, ..., T$ satisfies (19). Equating the marginal utility of consumption throughout, the constant stream is utility maximizing under (19). Hence it is optimal to set $x^{T^*+1} = \bar{x}$ and follow the CT(T^*) to select b^{T^*+1} and $l^{T^*+1} = l^{T^*}$.

For $t > T^* + 1$, the argument is analogous to $t = T^* + 1$, and a constant consumption of $\bar{x}$ remains budget feasible under the forecasts and optimal, and hence consumption of $\bar{x}$ is justifiable with the the CT(T^*). This completes the proof. ■

Remark 16 *In the construction above, the household maintains the same forecasts up to* T^**, in which only* $\underline{\hat{p}}^{T^*+1}$ *and* $\hat{x}^{T^*+1}$ *might turn out to be incorrect. The forecasts held in period* T^* *might be totally incorrect, but still they do not provide any arbitrage opportunity. After* T^*+1*, the household's forecasts are perfect. Conditional on this specification one can relate the nature of forecast error to the consumption level. If the household is to consume more than the PFE consumption level, the perceived income must be higher than that in the PFE. Thus, other things being equal,* $\underline{\hat{p}}^{T^*+1}$ *must be higher than the PFE price.*

We next note that such a solution to (15) and (16) exists if $\bar{x}$ is close enough to the AD equilibrium consumption level, which is $\frac{1}{1+T}\sum_{t=0}^{T} e^t$, provided a mild regularity condition at the AD equilibrium is satisfied.

Corollary 17 *Let* $x^* := \frac{1}{1+T}\sum_{t=0}^{T} e^t$*. Assume that* $\Delta \neq 0$*, where* $\Delta := u'(x^*) + u''(x^*)\left(x^* - e^{T^*+1}\right)$*. Then every* $\bar{x}$ *close enough to* x^* *can be justified using our specification of forecasts.*

Proof. At $\bar{x} = x^*$, (15) and (16) have a non-negative solution. The result then follows by the implicit function theorem. ■

Note that for the case of $u(z) = \ln(z)$, the regularity condition $\Delta \neq 0$ identified in Corollary (17) holds under our maintained assumption that $e^{T^*+1} > 0$.

The sufficient condition in Proposition 15 is general, but since we required that the discounted forecast prices coincide with those in PFE except about period T^*+1 price, it is far from necessary. In the appendix we consider the log utility case with $T = 2$ and with $T^* = 0$, to show that period 1 justifiability is warranted if and only if

$$\underline{p}^1 \bar{x} > \underline{p}^2\left(\bar{x} - e^2\right). \tag{20}$$

and period 0 justifiability holds if and only if

$$3\bar{x} > e^0 \tag{21}$$

The period 2 justifiability holds automatically from the budget constraint. In summary, for the log utility case, constant consumption $\bar{x}$ arising in a category 2 CAP is justifiable in every period if and only if conditions (20) and (21) are satisfied.

Notice that condition (20) holds with $\underline{p}^1 = \underline{p}^2$, and it explains why the justifiability is warranted with the forecasts identical to the PFE prices for periods after $T^* + 1$ in the general construction of Proposition 15. On the other hand condition (21) is weaker than its counterpart in Proposition 15.

5.4 Structure of ETE

Following the analysis of the previous section, a ETE is said to be of category 1 if the ex post prices satisfy the no arbitrage condition and of category 2 if they do not.

We first examine the structure of category 1 ETE. Following the analysis of section 5.2, we take it for granted that the consumption path is justifiable, generically, for every household when its allocation and prices are close enough to those of a PFE. As far as the allocation is concerned, we may assume that each household trades only the bond, and the market clearing condition for the bond follows from the feasibility of consumption allocation. Then Proposition 9 effectively characterizes the structure of category 1 ETE around the PFE, and thus we obtain the following:

Proposition 18 *The set of category 1 ETE allocations contains a set of at most* $\min(T, H-1)$ *dimension around the PFE allocation. If a sequence of category 1 ETE prices,* (p_n, q_n, q_{Ln}) *with allocation* $(x_n, \mathbf{b}_n, \mathbf{l}_n)$*,* $n = 1, 2, ...,$ *converges to the PFE prices as* $n \to \infty$*, the consumption allocation* x_n *converges to the PFE allocation as* $n \to \infty$*.*

Proof. The first statement follows directly from Proposition 9, which shows that category 1 CAP is expressed as a system of linear equations of at most rank $\min(T, H-1)$.

For the second statement, note that since the set of feasible allocation is compact, sequence x_n has an accumulation point, $\bar{x}$. It can be readily shown that a limit point of efficient allocations is efficient under our assumptions, and hence $\bar{x}$ is an efficient allocation. Since the ex post budget depends continuously on prices, $\bar{x}$ is also budget compatible for all households under the limit prices. Therefore by the second fundamental theorem of welfare economics, $\bar{x}$ is the unique AD equilibrium allocation. It means that any accumulation point of sequence x_n is the PFE allocation, and hence x_n must converge to the PFE allocation. ■

For the category 2 ETE, we have the following:

Proposition 19 *Assume that there is* $T^* < T-1$ *such that* $u'(x_h^*)+u''(x_h^*)\left(x_h^* - e_h^{T^*+1}\right) \neq 0$ *for every* h*, where* $x_h^* := \frac{1}{1+T}\sum_{t=0}^{T} e_h^t$*. Then there is a set of category 2 ETE of the following property: (1) the allocations contains a* $H-1$ *dimensional set around the PFE: (2) in each ETE, ex post prices are the same as the PFE prices except for the L-bond price* $q_L^{T^*}$ *in period* T^**. Moreover,* $q_L^{T^*}$ *can be arbitrarily close to the PFE price.*

Proof. Recall that $x_h^* = \frac{1}{1+T}\sum_{t=0}^{T} e_h^t$ is the unique constant consumption level at the PFE. Fix $q_L^{T^*} \neq 1$ arbitrarily close to 1. Consider the set of category 2 CAP with efficient allocations where the ex post prices of the good and the bonds are all 1, except $q_L^{T^*} \neq 1$. By Proposition 15 and Corollary 17, the consumption stream close enough to x_h^* is justifiable with a CT(T^*) for household h, and hence if restricted to allocations in a $H-1$ dimensional set around the PFE, any of such a CAP is justifiable for all households.

Thus it remains to show that if every household adopts the particular CT in the construction used in Proposition 15, all the bond markets clear in every period. For $t \neq T^*$, the L-bond market trivially clears since there is no trade. The bond market clearing follows from the market clearance for the good and the period t budget constraint. For $t = T^*$, the L-bond market clearing holds by construction (see (10)). Then the bond market must clearing follows from the market clearance for the good and the period T^* budget constraint. This completes the proof. ∎

Proposition 19 shows that any direction of transfer around PFE can occur even when the ex post prices as well as the underlying price forecasts are almost indistinguishable from PFE since the existence of the accidental income transfers seen above does not depend on the magnitude of ex post arbitrage opportunity. Consequently, when the observed market prices are almost the same as PFE prices, an analyst might be tempted to conclude that the economy was very close to the PFE, but the allocation might be drastically different from the PFE allocation. This contrasts sharply with Category 1 ETE (Proposition 18), where price proximity ensures allocation proximity.

From now on, we focus on the log utility case where the structure of ETE for both category 1 and 2 can be seen explicitly. Write $\bar{x}_h$ for the time invariant consumption of household h. Efficient allocations arising in some CAP of category 1 can be expressed as a convex combination of weights, $p^0, p^1, ..., p^T$ (Proposition 9), which we shall interpret as (normalized) discounted prices. Apply condition (14) to the case where household h's consumption is a constant stream of $\bar{x}_h = \sum_{t=0}^{T} p^t e_h^t$, so that it reads:

$$\left(\sum_{s=0}^{t} p^s + (T-t)\, p^t\right) \sum_{t=0}^{T} p^t e_h^t - \sum_{s=0}^{t} p^s e_h^s > 0 \tag{22}$$

where the first term is the value of the whole consumption stream assuming that the discounted price stays at p^t after period t. Therefore, we have

Proposition 20 *In the log utility case, an efficient allocation $(\cdots, \bar{x}_h, \cdots)$ is a category 1 ETE allocation if and only if there are $\sum_{t=0}^{T} p^t = 1$, $p^t > 0$, for $t = 0, 1, ..., T$, such that for every h, $\bar{x}_h = \sum_{t=0}^{T} p^t e_h^t$ and (22) are satisfied for every $t = 0, 1, ...T$.*

Example 21 *Fix $\sum_{t=0}^{T} p^t = 1$, $p^t > 0$, for $t = 0, 1, ..., T$, and then $\bar{x}_h = \sum_{t=0}^{2} p^t e_h^t = \frac{1-\varepsilon}{3} + \left(3p^h - 1\right)\eta$, for $h = 1, 2, 3$, and $\bar{x}_4 = \varepsilon$ (recall Example 10). For household 1, conditions (22) for $t = 0$ and 1 are*

$$3p^0 \left(\frac{1-\varepsilon}{3} + \left(3p^0 - 1\right)\eta\right) - p^0 \left(\frac{1-\varepsilon}{3} + 2\eta\right) = p^0 \left(2\frac{1-\varepsilon}{3} - 5\eta + 9p^0\eta\right) > 0$$

$$\left(p^0 + 2p^1\right)\left(\frac{1-\varepsilon}{3} + \left(3p^0 - 1\right)\eta\right) - \left(p^0\left(\frac{1-\varepsilon}{3} + 2\eta\right) + p^1\left(\frac{1}{4} - \eta\right)\right) > 0$$

and household 1's consumption is justifiable if the simultaneous (quadratic) inequalities above are met, and the solution range can be computed. Notice that when η is small enough, both inequalities are satisfied for any prices. A similar computations show that the other households' consumption streams are justifiable if η is small enough. Thus when η is small enough, any CAP of category 1 with an efficient allocation constitutes a ETE.

For category 2 ETE allocations, the justifiability condition utilizing a CT for the log case follows from Lemma 13. Therefore we have

Proposition 22 *In the log utility case, an efficient allocation $(\cdots, \bar{x}_h, \cdots)$ is a category 2 ETE allocation with the CT(T^*) if and only if inequality (12) is satisfied for every $t = 0, ..., T-1$. The set of category 2 ETE allocations is a $H-1$ dimensional set containing the PFE allocation.*

Proof. The first statement is a direct consequence of Lemma 13. The second can be established by inspection of (12), which holds if the discount forecast prices are close enough to the PFE prices. Alternatively, it follows from Corollary (17) since the regularity condition $\Delta \neq 0$ is satisfied in the log case. ■

By the definition of efficiency, in any ETE which is not a PFE, some are better off than in the PFE and some are worse off. For the case of trading $CT(0)$ we have constructed, wealth transfers in effect are generated by the specific amount traded in period T^*. The bond and the L-bond are perfect substitutes under the forecasts held in period T^*, so from the view point of each household the choice is done by accident. That is, the ones who are "rich by accident", are subsidized by the ones who are "poor by accident", and there is no particular link to the quality of their forecasts. Indeed since the construction of a CAP and its justifiability issue can be established separately, it can be readily inferred that those who are benefitted from the implicit transfers do not necessarily have forecasts which are accurate ex post; the former is determined by the construction of a CAP, whereas the latter is related to the issue of justifiability. This observation is indeed valid, even in the common log utility model where the quality of a forecast might appear to be the only source for an advantageous trade.

Example 23 *When $\eta < \frac{1-\varepsilon}{3}$ there is a category 2 ETE where household 4 consumes $\bar{x}_h^4 \in \left(\frac{2}{3}\varepsilon, \varepsilon\right)$ and households $h = 1, 2, 3,$ consume $\bar{x}_h = \frac{1}{3}(1-\bar{x}_4)$. This can be confirmed by applying justifiability conditions (20) and (21) where all ex post prices with the exception of q_L^0 as well as forecasts about period 2 prices are set to coincide with the perfect foresight prices: $q^0 = p^0 = p^1 = p^2 = 1 = q^1 = q_L^1$, $q_L^0 \neq 1$, and $\hat{p}_h^2 = \underline{\hat{p}}_h^2 = 1$. The details can be found in Appendix. Recall that household 4 consumes ε at the PFE, so it consumes less in these ETE. When ε is large, i.e., household 4 is "richly endowed" relative to the others, households 1, 2 and 3, who are rather poorly endowed end up richer in the ETE at the expense of household 4. There is of course a lower bound on how much household 4 may end up forsaking in such an ETE since $\bar{x}_4 > \frac{2}{3}\varepsilon$ has to hold.*

6 On Speculative Trade

The so called no-trade theorem asks if a purely speculative trade based on private information is possible in a rational expectation equilibrium. Since it is hard to distinguish speculative motive from other genuine motives based on perceived gains from trade, work in this literature typically start with an ex ante efficient allocation and ask if there is an equilibrium where trade takes place based on private information. If there is one, it can be regarded as a result of pure speculation. A general conclusion in this literature is that there tends not to be any purely speculative trade, which is referred to as the no (speculative) trade theorem.[10]

We can carry out the following exercise with a similar motivation in spirit in our framework. Suppose that there are many, identical households. The initial allocation is efficient by construction, and a unique perfect foresight equilibrium occurs with no trade. The question is whether or not there is a non-trivial ETE where households trade in this economy. If there is, one might interpret that the trade is driven by heterogeneous (and incorrect) forecasts, i.e., lack of rational expectation.

Of course, there are many inefficient TE, i.e., households might choose trades that distort intertemporal efficiency. One might think that there might also be trades based on heterogeneity of forecasts that preserve efficiency: households whose price forecasts disagree seem to find (incorrectly) that they have mutually beneficial trading opportunities. Even if the initial endowments are efficient, a household which thinks the price will be very low is willing to sell the good today to another household which thinks the price will be very high. This process might induce effective income transfers among households from the ones with good forecasts to the ones with bad forecasts, without distorting efficiency. But in general this is not straightforward even for category 1 ETE since good forecasts are not necessarily recipient of transfers, as is seen in Example 23.

It turns out that there is no category 1 ETE other than PFE. To see this recall the characterization result Proposition 9: The set of category 1 CAP allocation is $\left\{\sum_{t=0}^{T} p^t e^t : \sum_{t=0}^{T} p^t = 1, p^t > 0, \text{ for } t = 0, 1, ..., T\right\}$, where $e^t = \left(\cdots, e^t, \cdots\right)$. Since the

[10]The literature was initiated by Milgrom and Stokey [1982], and a clean, efficiency based formulation was given in Morris [1995]. See Kajii and Ui [2009] for its relation with the fundamental theorems of welfare economics.

efficiency of initial allocation implies that e^t is equal to a constant vector $\bar{e}$ for $t = 0, 1, ..., T$, it can be readily seen that $\sum_{t=0}^{T} p^t e^t = \left(\sum_{t=0}^{T} p^t\right) \bar{e} = \bar{e}$ for any element of this set; that is, it is a singleton set consisting of the initial, no trade allocation, and it is exactly the set of category 1 ETE allocations consisting of the PFE allocation. In conclusion, lack of rational expectation does not necessarily invoke trade that leads to another efficient allocation when the ex post prices satisfy the no arbitrage condition.

The conclusion, however, is not correct for category 2 ETE: note that the regularity condition for Proposition 19 is satisfied if there is no trade, the set of category 2 ETE allocations contains an $H - 1$ dimensional set around the PFE allocation. That is, lack of rational expectation might invoke income transfers among the households if the no arbitrage condition is violated ex post by an arbitrarily small amount. Therefore, the presence of an asset which is redundant under perfect foresight creates the opportunities for "speculative trade".

Example 24 *Set $\varepsilon = \frac{1}{4}$ and $\eta = 0$. The set of category 1 ETE coincides with PFE, but application of Proposition 22 reveals that any efficient allocation which gives more than $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ to every household can arise as a category 2 ETE. When $q_L^0 < 1$, i.e., where the long term bond is inexpensive according to the ex post prices, those who enjoy consumption higher than $\frac{1}{4}$ every period, i.e., the winners, buy the L-bond in period 0 as in (10), whereas those who sell the L-bond consume less than $\frac{1}{4}$ to be the losers. Since the bonds are perfect substitutes under their forecasts, the particular position on the L-bond is chosen by accident.*

7 Final Remarks

We comment on the robustness of PFE in the context of our findings. In the absence of a redundant asset, PFE is in a sense robust: allowing only for small deviations from PFE prices, the set of ETE allocations is a small set of dimension T around PFE which in a model with large H, can be interpreted as saying that efficient temporary equilibria "approximate" PFE. On the other hand,in the presence of a redundant asset, this robustness property is entirely lost. The predictive power of a perfect foresight model with no redundant assets is exaggerated, since even small deviations are consistent with large welfare effects through trading a redundant asset. While one interpretation of this feature might be that temporary equilibrium approach is flawed,[11] we are inclined to see the efficient income transfers we characterize as being relevant since these arise endogenously from market equilibria, much in the spirit of the second welfare theorem (albeit without a planners intervention). These welfare effects of redundant assets rather point to a worrying lack of robustness of the rational expectation paradigm to small departures from "rationality".

One might then wonder if our findings are contrived because of the restrictive set up. The key observation about the distributional role of redundant assets in a model without uncertainty is very general, as it does not rely on anything but budget equations. It therefore holds in a model with more goods, a longer period, or with general utility functions. For instance, in an infinite-horizon setup, the dynamic budget constraint for a category 1 CAP, with a suitable requirement, can still be reduced to a one-shot AD budget constraint, and hence our observation can be readily applied. Although we no longer obtain a clear-cut result about the dimension of CAP allocations, it can be verified that not all efficient allocations around the AD equilibrium arises as a category 1 CAP. Similarly, any kind of income transfers can be generated for a category 2 CAP, utilizing suitably modified canonical trading strategies.

The justifiability problem on the other hand gets more complex in a more general environment. However, one can readily accommodate stochastic forecasts, which enlarge the set of justifiable allocations while the structure of CAP allocations is unaltered, and hence our main points about the wealth transfers remain unchanged. Our justifiability result takes advantage of additive time separability, and we expect that the result can be extended for general time-separable utility functions. On the other hand, time non-separable utility functions are difficult to analyze, and we shall leave it for future research.

The message from the speculative trade also seems very general. The works in this literature typically asks possibility of speculative trade under rational expectations, and thus in particular, the way individual forecasts might be related to private information is common knowledge. One can view the forecasts in our model being related to some

[11] This interpretation would probably underlie the reason why the temporary equilibrium approach fell out of favour as the rational expectations approach became ubiquitous.

unmodeled private information, and from this perspective our finding can be understood as the possibility of speculation without common knowledge of forecasts: trade occurs not only by lack of common knowledge about forecasts but also the structure of (redundant) assets. The latter issue has been overlooked in our view, although our setup needs to be more elaborated to facilitate a proper comparison with the existing results.

A natural and important extension is to accommodate uncertainty in the model. It is especially important in the context of our interest in studying the role of redundant assets in financial markets. In particular, derivative securities, which constitute a rich class of financial assets, can only be studied in models that explicitly incorporate uncertainty.[12] It would be interesting to investigate whether our observation regarding the indeterminacy of wealth transfers under efficiency can be made by studying a model with uncertainty which accommodates a wider variety of financial assets. We expect that under our formulation, the presence of assets which are equivalent under rational expectations would provide some channels of income transfers, and thus would expand the set of attainable (ex ante) efficient allocations beyond the set of rational expectations allocations.

We make some further comments on the quality of forecasts in our framework. For a household who consumes more than in the PFE, the perceived AD budget must be larger than the perfect foresight one. Thus as a rule of thumb, the forecast discount future income tends to be higher for such a household, and importantly the accuracy of forecast has little logical connection to prosperity.[13] This (in)dependence of the quality of forecasts to the quality of life appears to be very general, but more research is needed to articulate this phenomenon. For the category 2 ETE we constructed, winners and losers will emerge by accident, since the bonds are deemed perfect substitutes for both

[12] The classical Black-Scholes option pricing formula finds the theoretical price of an option contract as a derivative asset assuming the relevant price processes are rationally expected. It has also been argued (first by Ross [1976] and subsequently elaborated by Polemarchakis and Ku [1990], Krasa and Werner [1991], Kajii [1997], among others) that under rational expectations, the presence of options might complete the markets, and consequently a rational expectations equilibrium with options is efficient and determinate.

[13] We have observed in earlier work (Chatterji *et al* [2018]) that households whose forecasts turn out more accurate need not be the beneficiaries in the ETE induced by heterogeneous forecasts. The same remains true here.

the winners and the losers. This is a very general point.

One might argue that the lack of implications on forecast accuracy rather reveals that the behavioral rule in our model is too primitive, and is the root cause of the robustness failures. In this regard we observe that choosing to require specific learning and forecasting rules as part of TE would change the set of justifiable allocations, but would not alter the set of CAP allocations. This suggests that an alternative path for a robustness check might involve endogenous and subjective frictions of trade: for instance, if one anticipates potentially unexpected gains or losses from trading the L-bond, it might be preferred to keep the volume of trade as small as possible. This reduced trading activity would limit endogenous income transfers in CAP, thereby enhancing the robustness of PFE. Further investigation into behavioral assumptions of this kind constitutes a promising avenue for future research in our view.

Finally, we point out that the model we use can be seen as a multiperiod extension of an earlier model (Chatterji and Kajii [2023]) where we have studied the structure of ETE allocations in a two period set up with multiple non-storable goods with one nominal asset (bond), and shown that there is a one dimensional set of ETE allocations around each perfect foresight equilibrium (PFE) allocation, generically in endowments. In this one good set up, Proposition 20 generalizes the indeterminacy finding of that paper to a multi-period set up and shows the dimension of the set of category 1 ETE allocations is generically one less than the number of periods. As we wrote above, we expect this finding to hold more generally with multiple goods and general utilities. But for such a generalization, as is pointed out in the earlier model, there are additional complications about justifiability even in the two period case when utilities are non time-separable.

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Appendix

Proof of Proposition 9

Recall that allocation x arises as part of a category 1 CAP if and only if the following system of equations for some positive prices:

$$\begin{array}{rcl} \sum_{t=0}^{T} p^t \left(x_h^t - e_h^t\right) & = & 0 \quad \text{for } h = 1,2,..,H \\ \left(\sum_{h=1}^{H} x_h^t\right) - 1 & = & 0 \quad \text{for } t = 0,1,..,T \end{array} \tag{23}$$

Notice that the last feasibility equation is redundant in (23) since for time invariant allocations the first H equations imply $\sum_{t=0}^{T} p^t \left(x_h - e_h^t\right) = \left(\sum_{t=0}^{T} p^t\right) x_h - \sum_{t=0}^{T} p^t e_h^t = 0$ for every h, and hence

$$\begin{aligned} \sum_{h=1}^{H} x_h &= \sum_{h=1}^{H} \frac{\sum_{t=0}^{T} p^t e_h^t}{\left(\sum_{t=0}^{T} p^t\right)} \\ &= \frac{1}{\sum_{t=0}^{T} p^t} \sum_{t=0}^{T} p^t \left(\sum_{h=1}^{H} e_h^t\right) \\ &= \frac{1}{\sum_{t=0}^{T} p^t} \sum_{t=0}^{T} p^t \\ &= 1. \end{aligned}$$

Since $\sum_{h=1}^{H} x_h = 1$, (23) also implies that $1 = \sum_{h=1}^{H} \left(\sum_{t=0}^{T} p^t e_h^t\right) = \sum_{t=0}^{T} p^t \left(\sum_{h=1}^{H} e_h^t\right) = \sum_{t=0}^{T} p^t$. Therefore, the system of equations (23) is equivalent to

$$\sum_{t=0}^{T} p^t e_h^t \quad = \quad x_h \quad \text{for } h = 1,2,..,H \ , \tag{24}$$

where $\sum_{t=0}^{T} p^t = 1$ and $p^t > 0$ for every t. It means that the set of efficient allocations arising in some CAP of category 1 is given by the set $\left\{\sum_{t=0}^{T} p^t e^t : \sum_{t=0}^{T} p^t = 1, p^t > 0, \text{ for } t = 0,1,...,T\right\}$, as is desired.

Justifiability for the log case

Derivation of inequalities (20) and (21)

We set $T = 2$ where the household consumes a constant amount $\bar{x}$ in every period, and $q^0 q^1 - q_L^0 \neq 0$. Denote by $\underline{p}^t$ the discounted price. We set b^0, b^1 in accordance with the iterative procedure[14] of a CT(T^*), and specify l^0 $(= l^1)$ according to (10); that is,

$$l^0 = \frac{\left(p^0 + \underline{p}^1 + \underline{p}^2\right)\bar{x} - \left(p^0 e^0 + \underline{p}^1 e^1 + \underline{p}^2 e^2\right)}{\left(q^0 q^1 - q_L^0\right)},$$
$$b^0 = \frac{1}{q^0}\left(p^0 e^0 - p^0 \bar{x} - q_L^0 l^0\right),$$
$$b^1 = \frac{1}{q^1}\left(p^1 e^1 + b^0 - p^1 \bar{x}\right).$$

Now we shall examine the justifiability for each period.

Period 2: This holds automatically from the period 2 budget feasibility.

Period 1: Applying condition (12) with $t = 1$ and 2, we find that the necessary and sufficient condition for justifiability is $2\underline{p}^1\bar{x} > \underline{p}^1 e^1 + \underline{q}^0 b^0 + \underline{q}_L^1 l^0$. Substituting b^0 found above (recall $\underline{q}^0 = q^0$), it is equivalent to $\left(p^0 + 2\underline{p}^1\right)\bar{x} > p^0 e^0 + \underline{p}^1 e^1 + \left(\underline{q}_L^1 - q_L^0\right) l^0$. Substituting l^0 found above (recall $\tilde{q}_L^1 = q^0 q_L^1$ by definition, and $q^1 = q_L^1$ since the two bonds are identical in period 1), it is equivalent to $\left(p^0 + 2\underline{p}^1\right)\bar{x} > p^0 e^0 + \underline{p}^1 e^1 + \left(\left(p^0 + \underline{p}^1 + \underline{p}^2\right)\bar{x} - \left(p^0 e^0 + \underline{p}^1 e^1 + \underline{p}^2 e^2\right)\right)$, which is reduced to $\underline{p}^1\bar{x} > \underline{p}^2\left(\bar{x} - e^2\right)$ (inequality (20))

Period 0: Applying condition (12) with $T = 2$ and $t = T^* = 0$, we find that the necessary and sufficient condition for justifiability is $\underline{p}^0\left(\bar{x} - e^0\right) + 2\underline{p}^0\bar{x} + l^0\left(\underline{q}_L^0 - \underline{q}_L^0\right) > 0$, which is readily simplified as $3\bar{x} > e^0$ (inequality (21))

In summary, for the log utility case, constant consumption $\bar{x}$ arising in a category 2 CAP is justifiable in every period if and only if conditions (20) and (21) are satisfied.

Details on Example 23.

We shall identify efficient allocations that satisfy the justifiability where all ex post prices with the exception of q_L^0 as well as forecasts about period 2 prices are set to coincide with the perfect foresight prices: $q^0 = p^0 = p^1 = p^2 = 1 = q^1 = q_L^1$, $q_L^0 \neq 1$, and $\hat{p}_h^2 = \hat{\underline{p}}_h^2 = 1$. Note that condition ((20) holds with $\underline{p}^1 = \underline{p}^2$, and so justifiability is warranted with the forecasts identical to the PFE prices for periods for $t = 1, 2$.

It remains to verify justifiability at $t = 0$ with (12). For purpose of illustration, it is convenient to compute $\hat{\underline{p}}_h^1$ for each h that satisfy (12) and thereby justify an efficient allocation at period 0. Recall that period 0 justifiability then requires that

$$3\bar{x}_h - e_h^0 = \hat{\underline{p}}_h^1 e_h^1 + e_h^2 \tag{25}$$

[14] Recall we had set b^t iteratively to meet period t budget, $p^t x^t + q^t b^t = p^t e_t^t + b^{t-1}$, where $b^{-1} = 0$.

holds for each h. The table below solves for $\underline{\hat{p}}_h^1$ and substitutes values of e_h^0, e_h^2 respectively from (1).

h	$\underline{\hat{p}}_h^1$
1	$\frac{3}{(1-\varepsilon)-3\eta}\left(3\bar{x}_1 - 2\left(\frac{1-\varepsilon}{3}\right) - \eta\right)$
2	$\frac{3}{(1-\varepsilon)+6\eta}\left(3\bar{x}_2 - 2\left(\frac{1-\varepsilon}{3}\right) + 2\eta\right)$
3	$\frac{3}{(1-\varepsilon)-3\eta}\left(3\bar{x}_3 - 2\left(\frac{1-\varepsilon}{3}\right) - \eta\right)$
4	$\frac{1}{\varepsilon}\left(3\bar{x}_4 - 2\varepsilon\right)$

(26)

An efficient allocation $\bar{x}$ is justifiable if and only if the values of $\underline{\hat{p}}_h^1$ are strictly positive for each h. The positivity requirement for $h = 4$ says that $3\bar{x}_4 > 2\varepsilon$.

One configuration we investigate is where household 4 consumes less than the AD equilibrium and the others benefit equally from this decline in consumption. Set $\bar{x}_4 \in (\frac{2\varepsilon}{3}, \varepsilon)$ so that positivity of discounted forecast holds for $h = 4$. For $h = 1, 2, 3$, $\bar{x}_h = \frac{1}{3}(1-\bar{x}_4)$ so that $\bar{x}_h \in \left(\frac{1-\varepsilon}{3}, \frac{3-2\varepsilon}{9}\right)$. Then $3\bar{x}_h - 2\left(\frac{1-\varepsilon}{3}\right) > \frac{1-\varepsilon}{3}$ and so $\underline{\hat{p}}_h^1 > 0$ for $h = 1, 2, 3$ if $\eta < \frac{1-\varepsilon}{3}$. For instance, these inequalities hold for $\varepsilon = 0.9$, $\bar{x}_4 \in (0.6, 0.9)$, $\eta < \frac{1}{30}$. This completes the details on Example 23.

There is an ETE where $h = 2$ consumes less than the AD allocation and the beneficiary is $h = 4$, and the others consume their AD allocation. Inspection of (26) shows that such an ETE exists when $\frac{1(1-\varepsilon)}{3} > \eta$. In particular, when ε is "small", $h = 4$ is the poorest household who receives a transfer from the well endowed household 2.

Note that the calculations above are based on the presumption of $CT(0)$ and $\hat{p}_h^2 = p^2 = 1$ for all h. If these are not insisted on, one has greater flexibility to justify ETE by allowing price forecasts of both periods 1 and 2 to be adjusted appropriately.