Influence and Counter-Influence in Networks*

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Abstract

We study the competition for agent votes between an influencer, who shapes interaction networks, and a counter-influencer, who counter-influences at a cost. Agents' votes depend on both the influence of the players and the opinions of their neighbors. If the influencer is stronger, the equilibrium networks become more centralized; if the counter-influencer is stronger, they become decentralized. We also study situations where random links are added to the influencer's network. Extensions include allowing multiple interaction periods before voting, and requiring only a majority for the influencer to win. In both extensions, the influencer needs fewer resources to win.

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1 Introduction

Leaders of organizations have the ability to shape interactions among the members by assigning team members to specific offices or arranging meetings between certain individuals. However, this ability varies significantly across organizations. For instance, leaders of criminal organizations often exert a very high degree of control over the interaction patterns of their members, thanks to the effectiveness of their incentive mechanisms and the strong social control they maintain. In contrast, leaders of organizations like political parties can influence member interactions through measures such as appointing individuals to special committees or providing platforms for deliberation on key issues. But they lack the ability to fully control how their members interact. Moreover, leaders of organizations have the ability to influence their members to achieve certain goals. The extent and effectiveness of this influence depends on the organization's patterns of interaction, which are often shaped, at least in part, by the leaders themselves. It is also not unusual for leaders to encounter adversaries who seek to counteract their influence and sway members in a different direction. This is particularly evident in the context of criminal organizations, where the leader seeks to secure the loyalty of members while the law enforcement authorities may seek to persuade some members to betray the organization.[1] In the context of political parties, consider the example of a parliamentary assembly, where the leader of the majority party seeks to persuade other members to support a proposed bill. The leader can organize working meetings to present the bill and emphasize its importance to key party members. These members can then leverage the relationships the leader has helped establish within the party to spread information about the bill and rally further support. However, the members of the majority party may also be pressured by opponents of the bill to vote against it.

Formally, we use network theory to model such a phenomenon using two players who play a game of influence and counter-influence with the goal of persuading a set of agents to vote for their opinion. Specifically, we consider a model involving two players: an influencer (IN) and a counter-influencer (CI). In the spirit of the phenomena described above, we allow only one of the two players (the influencer) to shape the pattern of interaction. The two players engage in a sequential competition, aiming to influence agents in a trinary choice scenario (0,

[1]This contrasts with Baccara and Bar-Issaac (2008), who focus on detecting criminals within a criminal network. We explore how the leadership of a criminal organization ensures the loyalty of its members under the threat of rivals.

1 or no opinion). Here, IN wants the agents to vote 1, while CI tries to avoid this outcome. Thus, the interaction between IN and CI takes the form of a *zero-sum game*: either IN will win the voting game or CI will win with neither voting themselves. IN makes two decisions: she sets up the interaction pattern or network by *forming links* between agents, and *influences* a subset of them. Then, CI decides which agents he *will influence*. Obviously, forming links and exerting influence are both costly activities. The influence activities of IN and CI determine the initial opinion of agents. Our model also captures another important aspect of the scenarios described above: agents are not just influenced by the two players; they are embedded in a network and can also affect each other's opinions before they vote. After the Influencer and Counter-influencer take their actions, each agent interacts with his neighbors. An agent's final opinion is a convex combination of his initial opinion and the opinions of his neighbors. Thus, the agents also play the role of secondary influencers. Finally, each agent votes on the basis of his final opinion.[2]

In our benchmark model, we make the following assumptions. First, we explore the role played by networks by giving IN complete control over the interaction patterns among the members of the organization.[3] Second, to understand how costs matter we require that IN must secure unanimous support for 1. Third, when both IN and CI exert influence on an agent, we assume that IN has a better influence technology than CI.[4] Each of these assumptions is subsequently relaxed in the paper.

Our first result concerns the Subgame Perfect Equilibrium (SPNE) of the sequential move game where IN forms the network and influences agents in the first stage, and CI influences agents in the second stage. In particular, we characterize the strategies employed by IN in the SPNE, i.e., her optimal strategies: we show that in equilibrium network architectures are

[2]This type of rule is commonly employed in models of influence and social learning, where the focus is often on studying the propagation of influence among non-strategic agents (see Jackson and Yariv, 2007, Golub and Jackson, 2010, Golub and Sadler, 2016, Galeotti and Goyal, 2009, Chatterjee and Dutta, 2016, and Banerjie, Chandrasekhar, Duflo, and Jackson, 2013).

[3]This assumption is admittedly extreme but holds some relevance in the context of criminal organizations, where unauthorized meetings between individuals can be monitored, reported, and, if necessary, punished. Moreover, it allows us to gain insights about the role of influence using the simplest possible model. As noted below, we also explore the limits of this assumption.

[4]An alternative assumption is that agents influenced by both CI and IN vote like those who have not been influenced, which does not change the results qualitatively.

of a centralized nature. These strategies depend on the cost of forming links relative to the cost of influencing agents for IN and the cost of influencing agents for CI. In particular, if the cost of forming links is high relative to the cost of influencing agents for IN, her optimal strategy is to form no links and influence each agent. The cost of influencing agents for CI determines the maximum number of agents he can influence. When this maximum number of agents is low, the optimal strategy for IN is to form a partial-star network where some agents are isolated, and some agents, the peripherals, have a single link with one central agent. IN influences the center, the isolated agents, and some of the peripheral agents. This strategy allows IN to benefit from not having to influence more agents than the maximum number of agents that CI can influence. When the number of agents CI can influence is high, the optimal strategy for IN is to form a quasi-core-periphery network where agents in the core are linked among themselves, while each agent in the periphery has a single link with an agent in the core. In addition, IN exclusively exerts influence on agents within the core. This strategy, while requiring more links, minimizes the number of agents that IN influences, since CI can influence all the neighbors of each agent that are not influenced by IN.[5]

Then, we allow for the possibility that additional links can appear randomly after IN has made her choices. This relaxes the assumption that IN has the ability to fully control the network.[6] In this context, we provide an upper bound on the probability of unplanned links that allows us to preserve our main results, also serving as a robustness check of our main result.

In the extension section, we first allow for repeated interactions among the agents. Not only does this give us the ability to explore the role of repeated interactions, but it also allows us to examine how the network structure matters since IN can use agents as secondary influencers to reduce her own costs. We show that, for sufficiently low link formation costs, the number of agents that IN must influence in an optimal strategy is lower than in our benchmark model. Indeed, there are situations where IN only needs to influence one agent to obtain a unanimous vote.

[5]It is worth noting that the strategy, where IN forms a partial-star network and influences the center plus some peripheral agents, is a special case of the strategy where she forms a quasi-core-periphery network and influences all the agents in the core. In the partial-star case, the core consists of the central agent and the subset of peripheral agents influenced by IN.

[6]This assumption is more applicable to organizations, such as political parties, that have partial control over their network.

Next, we examine the case where IN wins when a majority of the agents vote 1. We establish that the optimal strategies for IN are qualitatively similar to those of the benchmark model, but less costly, as they involve either fewer links or a smaller number of agents to influence. Specifically, we show that when IN only needs to secure a majority of votes in some situations, she can disregard the number of agents CI is able to influence.

Finally, we assume that CI has the superior influence technology. Consequently, when an agent is influenced by both IN and CI, he believes CI. This assumption considerably modifies IN's optimal strategies. When CI cannot influence too many agents, IN builds either a network where each agent has the same number of neighbors and influences all the agents, or a network where the agents are divided into two groups and each member of the first group is influenced by IN and forms k links with his group members, while members of the group not influenced by IN form less than k links with members of the first group. Thus, we find that the equilibrium networks are more decentralized compared to those in the benchmark model. We then show that when the number of agents that can be influenced by CI is sufficiently large, IN cannot obtain a unanimous vote for 1. This demonstrates that when CI has the better influence technology, even an incumbent influencer's ability to determine the interaction pattern among agents may not be sufficient to win the influence competition. This finding has important implications for public policy. For instance, in the context of the influence competition between the criminal organization and the authorities, if the latter have a greater ability to influence a sufficient number of the criminal organization's members, then even the ability to control the pattern of interactions among its members may not be sufficient to win the game.

To the best of our knowledge, there is only one other paper that introduces competition among influencers within a network-based framework, though it adopts a different approach from ours. Grabisch, Mandel, Rusinowska and Tanimura (2018)[7] develop a model in which two strategic players with opposing opinions compete to influence a population of non-strategic agents embedded in a network. These non-strategic agents form their opinions based on those of their neighbors. Each strategic player selects a single agent to influence directly by forming a link, after which the influence spreads through the network via that agent's neighbors. The authors analyze the resulting opinion dynamics in the network modified by the actions of the two players. In contrast, our model allows each strategic player to influence multiple agents.

[7]Comola, Rusinowska and Villeval (2024) conduct an experimental study based on a simplified version of the model proposed by Grabisch et al. (2018).

Moreover, one of the players, IN, has the additional ability to design the entire network in which the agents are embedded. Our paper also relates to multiple strands of the existing literature.

In the first strand, the focus is on network protection orchestrated by a defender/designer, mirroring the structure of our model. However, the main objective of this literature is to preserve network connectivity in the event of an attack, rather than to encourage agents to reach a shared opinion. As our model has a different objective, the optimal strategies for the designer also differ from those identified in this literature. In Dziubiński and Goyal (2013), a designer forms links between nodes and may protect some of them to preserve the network's connectivity (or maximize the size of connected components in the network).[8] Both links and protection are costly, and the attacker can target unprotected nodes to disconnect the network. To defend against such attacks, the designer's optimal strategy is either to build a star network and protect the center or to construct a densely connected network (called Harary network) without protecting any nodes. Our model differs in that achieving unanimous vote does not necessarily require network connectivity, but always requires influencing at least one agent. Even if a star network emerges as an optimal structure for the designer in both settings, in Dziubiński and Goyal only the central node is protected, whereas in our framework, some peripheral agents must also be influenced.

Goyal and Vigier (2014) extend the work of Dziubiński and Goyal (2013) by allowing the attacks (or threats) to spread like a contagion. In this model, a defender and an attacker allocate resources to nodes in a network. The attacker can reallocate resources from conquered nodes, but the defender cannot do so from protected ones. The authors identify two optimal defense strategies: build a star network and protect its center, or build a network where each protected node has few unprotected neighbors relative to the defender's investment. A key difference between the two frameworks lies in the situations they model. Goyal and Vigier study a type of 'dynamic wargame' in which the attacker develops a multi-stage strategy for deploying attacks across the network. By contrast, we examine a setting where two influencers modify agents' initial opinions, which then evolve through local interactions between agents.

[8]In Dziubiński and Goyal (2017), the defender cannot form links, making the model fundamentally different from ours. Moreover, Bravard, Charoin, and Touati (2016) modify Dziubiński and Goyal's (2013) model by considering a situation where the attacker targets links rather than nodes, and the defender has to protect these links. Therefore, this model is very different from ours.

At first glance, the results of the two models may seem similar, as both identify the star as one possible optimal strategy for the defender/influencer. However, as in the Dziubiński and Goyal model, in Goyal and Vigier's framework, the defender allocates all her resources on the center of the star, whereas in our model, some peripheral agents must also be influenced by IN. Moreover, the other optimal strategies differ between the two models. In our framework, agents influenced by IN mutually reinforce each other's opinions, so an optimal strategy links them together. Conversely, agents not influenced by IN remain vulnerable to CI's influence. To prevent them from reinforcing CI's opinion through mutual interaction, IN ensures they are not connected to one another.[9]

The second strand of literature involves decentralized protection of the network carried out by individual agents within the network. Baccara and Bar-Isaac (2008) explore situations where criminals can exchange information by creating links that enhance trust but also increase vulnerability to external threats. They show that the optimal architecture of the criminal organization has no information links, or the optimal structure consists of pairs of agents with information about each other but no information links with other members of the organization. The model is very different from ours since the external threats detect and remove agents, and do not try to counter-influence them in an opinion game. Because of these differences, our optimal networks are denser than those of Baccara and Bar-Isaac. Cabrales, Gottardi, and Vega-Redondo (2017) study the propagation of attacks in networks of financial firms where financial risk can spread between connected firms. In Acemoglu, Malekian, and Ozdaglar (2016), agents are connected but in a random network. Agents have to invest in their immunity which depends on their links and the probability of being infected in the random network. In Haller and Hoyer (2019), group members individually sponsor costly links forming an information network. A counter-influencer aims to disrupt the information flow within the network by deleting some of the links, and the authors study how the group as a whole responds to such a common enemy. In contrast to these papers, we introduce a distinctive perspective with a two-player game involving an influencer and a counter-influencer, both focusing on influencing agents rather than maintaining/removing connectivity.

The third strand of literature focuses on the spread of opinion/misinformation through social media. Bloch et al. (2018) study a situation where there are two types of players: those

[9] A related paper by Hoyer and De Jaegher (2016) allows for the possibility that certain nodes in the network remain intrinsically vulnerable and provides similar results.

who are biased in favor of the message 1 and those who are not biased. Each player can either transmit the message or block it. Biased players have an interest in transmitting messages that only favor 1, while unbiased players only transmit credible messages. Bloch et al. show that the social network acts as a filter, limiting the spread of untrustworthy messages unlike a situation where the message can be spread to the entire population by a single sender. Bravard et al. (2023) modify this framework by assuming that players are aware only of the biases of their immediate neighbors. Acemoglu et al. (2024) present a model of online sharing where agents observe an article (or piece of information) on a social network and decide whether to share it or not. The article may contain misinformation, and agents gain utility from their interactions on social media, but they lose utility if they spread misinformation. The agents are not biased, but have different prior beliefs. The authors analyze the policy that a regulator should adopt to limit the spread of misinformation given that social media wants to maximize the virality of messages. In contrast to this work, we model the competition between two players who want to persuade agents in a social network. Our goal is to explore the optimal strategies of a player who can both shape the interaction structure of the agents and influence them to resist competing messages with the agents themselves acting as secondary influencers.

The rest of the paper is organized as follows. In Section 2, we introduce the model setup. In Section 3, we establish results for the benchmark model and also present a robustness check taking into account the possibility of unplanned links. In Section 4, we deal with our three extensions. We conclude in Section 5. All proofs are given in the appendix or, where indicated, in the online appendix.

2 Model Setup

Let $[\![a, b]\!] = \{\ell \in \mathbb{N}, a \leq \ell \leq b\}$, and let $\lfloor x \rfloor$ and $\lceil x \rceil$ be respectively the largest integer less than x and the smallest integer greater than x. Further, for every set X, $\sharp X$ is its cardinality.

Players and Agents. We assume that there are two players or primary influencers called respectively the *Influencer*, IN/she, and the *Counter-influencer*, CI/he. Both IN and CI act strategically. In addition to these two players, there is a set of agents $\mathcal{N} = [\![1, n]\!]$, $n \geq 4$. In the following, we will describe the behavior of these n agents which is non-strategic in nature.

Networks. We assume that agents are located on an undirected network. An undirected network g is a pair $(\mathcal{N}, E(g))$, where $E(g) \subseteq \mathcal{N} \times \mathcal{N}$ is the set of links. We denote by $G[\mathcal{N}]$

the set of all networks with $\mathcal{N}$ as the set of agents. We denote by ij the link between agents i and j in g, i.e., $i\,j \in E(g)$. With a slight abuse of notation, we let g also be the (symmetric) adjacency matrix corresponding to the undirected network g, that is, for every $(i,j) \in \mathcal{N} \times \mathcal{N}$, $g_{i,j} \in \{0,1\}$, where $g_{i,j} = 1$ *if and only if* $i\,j \in E(g)$. By convention, $g_{i,i} = 0$.

Let $\mathcal{N}_i(g) = \{j \in \mathcal{N} : i\,j \in E(g)\}$ be the set of neighbors of agent $i \in \mathcal{N}$. Agent i is isolated in g when $\sharp\mathcal{N}_i(g) = 0$. A path $P_{i,j}(g)$ between agents $i = i_0$ and $j = i_m$, is a sequence of links of the type $i_0\,i_1, \ldots, i_\ell\,i_{\ell+1}, \ldots, i_{m-1}\,i_m$ where each link $i_\ell\,i_{\ell+1} \in E(g)$. A cycle is a path where i and j coincide. A network g is connected if there exists a path between every pair of agents. A subnetwork $g[\mathcal{N}'] = (\mathcal{N}', E(g[\mathcal{N}']))$ of network g is a network such that $\mathcal{N}' \subseteq \mathcal{N}$ and for $i,j \in \mathcal{N}'$, we have $i\,j \in E(g[\mathcal{N}'])$ *if and only if* $i\,j \in E(g)$.

Strategies of the Players. Player IN has the ability to influence agents, and to shape the interaction structure of the agents. Player CI can only influence agents. Let $\mathcal{I}_{\text{IN}} \subseteq \mathcal{N}$ and $\mathcal{I}_{\text{CI}} \subseteq \mathcal{N}$ denote the set of agents influenced by IN and CI, respectively. A strategy, s, for IN, assigns to $\mathcal{N}$ a pair $(g, \mathcal{I}_{\text{IN}})$, and a strategy, σ, for CI assigns to each pair $(g, \mathcal{I}_{\text{IN}})$ a set of agents $\mathcal{I}_{\text{CI}} \subseteq \mathcal{N}$.

Initial Opinion of Agents. We assume that before players IN and CI influence agent i, the latter has no opinion, given by $\emptyset$. If i is influenced neither by IN, nor by CI, his initial opinion, θ_i, continues to be $\emptyset$.[10] If only IN (or CI) influences agent i, then the initial opinion of i is 1 (or 0). Suppose agent i is influenced by both primary influencers, i.e., $i \in \mathcal{I}_{\text{IN}} \cup \mathcal{I}_{\text{CI}}$. Then, in this game of influence and counter-influence there are only two possibilities: either IN is successful or CI is successful.[11] First, if IN has a greater ability to influence than CI (for instance because of better technology), then $\theta_i = 1$ when $i \in \mathcal{I}_{\text{IN}} \cup \mathcal{I}_{\text{CI}}$. Second, if CI has a greater ability to influence an agent than IN, then $\theta_i = 0$ when $i \in \mathcal{I}_{\text{IN}} \cup \mathcal{I}_{\text{CI}}$. The former case is addressed in the next section, while the latter is examined in the extension section. Thus, in our benchmark

[10] The results obtained in the benchmark model are not qualitatively affected by the assumption that agents have no initial opinion and maintain this opinion when not influenced. However, if some agents have 1 as their initial opinion, then IN would be able to save resources by not influencing them.

[11] We ignore the case where both IN and CI cancel out each other's influence. This is akin to analyzing the model where these players belong to the set of uninfluenced agents and therefore is ignored here.

model, the initial opinion of each agent i is given by

$$\theta_i = \begin{cases} 1 & \text{if } i \in \mathcal{I}_{\text{IN}}, \\ 0 & \text{if } i \in \mathcal{I}_{\text{CI}} \setminus \mathcal{I}_{\text{IN}}, \\ \emptyset & \text{if } i \notin \mathcal{I}_{\text{IN}} \cup \mathcal{I}_{\text{CI}}. \end{cases}$$

The n-uple $\boldsymbol{\theta} = (\theta_i)_{i \in \mathcal{N}}$ summarizes the initial opinion of every agent $i \in \mathcal{N}$.

Final Opinion of Agents. Each agent i forms his final opinion by taking into account his own initial opinion, θ_i, and the average of his neighbors' initial opinion. We denote by $\mathcal{N}(k, \boldsymbol{\theta})$ the set of agents with *initial* opinion k. The set of neighbors of agent i with $k \in \{\emptyset, 0, 1\}$ as their initial opinion is denoted by $\mathcal{N}_i^k(g) = \{j \in \mathcal{N}_i(g) \cap \mathcal{N}(k, \boldsymbol{\theta})\}$. Moreover, when $\mathcal{N}_i^0(g) \cup \mathcal{N}_i^1(g) \neq \emptyset$, let

$$\bar{\Theta}_i = \frac{1}{\sharp\mathcal{N}_i^0(g) + \sharp\mathcal{N}_i^1(g)} \sum_{j \in \mathcal{N}_i^0(g) \cup \mathcal{N}_i^1(g)} \theta_j$$

be the opinion of i's neighbors. Thus, we assume that i forms his opinion excluding agents $j \in \mathcal{N}_i(g)$ for whom $\theta_j = \emptyset$. We assume that the final opinion of agent $i \in \mathcal{N}$, θ_i^{F}, is obtained from the following rule:

$$\theta_i^{\text{F}} = \begin{cases} (1-\alpha)\theta_i + \alpha\bar{\Theta}_i & \text{if } \theta_i \neq \emptyset \text{ and } \mathcal{N}_i^0(g) \cup \mathcal{N}_i^1(g) \neq \emptyset, \\ \bar{\Theta}_i & \text{if } \theta_i = \emptyset \text{ and } \mathcal{N}_i^0(g) \cup \mathcal{N}_i^1(g) \neq \emptyset, \\ \theta_i & \text{otherwise,} \end{cases} \tag{1}$$

where $\alpha \in (\frac{1}{2}, 1]$.[12] Note that in Equation (1), if $\mathcal{N}_i^0(g) \cup \mathcal{N}_i^1(g) = \emptyset$, then agent i's final opinion depends *only* on his initial opinion, i.e., if none of the neighbors of agent i has any opinion, then agent i considers only his initial opinion. We let $\boldsymbol{\theta}^{\text{F}} = (\theta_i^{\text{F}})_{i \in \mathcal{N}}$.

Voting behavior of agents. We assume that after all possible influences are taken into account, each agent votes for an outcome in line with his final opinion. Let $v_i(\theta_i^{\text{F}})$ be the vote of agent i, we have

$$v_i(\theta_i^{\text{F}}) = \begin{cases} 1 & \text{if } \theta_i^{\text{F}} \geq 1/2, \\ 0 & \text{if } \theta_i^{\text{F}} < 1/2, \\ \emptyset & \text{if } \theta_i^{\text{F}} = \emptyset. \end{cases} \tag{2}$$

[12]Note that if $\alpha < \frac{1}{2}$, then agent i cannot change is vote (see below). Hence, we do not consider this case.

The vote of agent i can be interpreted as his realized/chosen action. The n-uple $\boldsymbol{v} = (v_1(\theta_1^{\mathrm{F}}), \ldots, v_n(\theta_n^{\mathrm{F}}))$ provides the vote of every agent $i \in \mathcal{N}$. Following (2), for $k \in \{\emptyset, 0, 1\}$, we denote the set of agents who vote k by $\mathcal{N}(k, \boldsymbol{v}) = \{j \in \mathcal{N} : v_j(\theta_j^{\mathrm{F}}) = k\}$.

Payoff of IN and CI. Forming links and influencing agents are both costly actions. The cost that IN incurs when he forms $\sharp E(g)$ links and influences $\sharp \mathcal{I}_{\mathrm{IN}}$ agents is $C(\sharp E(g), \sharp \mathcal{I}_{\mathrm{IN}})$, where $C(\cdot, \cdot)$ is strictly increasing and convex in each of its argument. We sometimes assume that it is linear in its two components: $C(\sharp E(g), \sharp \mathcal{I}_{\mathrm{IN}}) = \sharp E(g) c_L + \sharp \mathcal{I}_{\mathrm{IN}} c_{\mathrm{IN}}$, where $c_L > 0$ is the unit cost of forming links, and $c_{\mathrm{IN}} > 0$ is the unit cost of influencing an agent for IN. The cost function of CI is linear,[13] with $c_{\mathrm{CI}} > 0$ the cost incurred by CI for each agent he influences.

The benefits of players only depend on agents' votes while the costs are player specific. Clearly, $\boldsymbol{\theta}^{\mathrm{F}}$ and $\boldsymbol{v}$ are entirely determined by the strategies of IN and CI, hence we have $\boldsymbol{\theta}^{\mathrm{F}}[s, \sigma]$ and $\boldsymbol{v}[s, \sigma]$. We assume that IN *wins if and only if every agent votes* 1, an assumption we relax subsequently. The payoff of IN, for choosing strategy s when CI chooses strategy σ is

$$u(\boldsymbol{v}[s, \sigma]) = \begin{cases} 1 - C(\sharp E(g), \sharp \mathcal{I}_{\mathrm{IN}}) & \text{if } \mathcal{N}(1, \boldsymbol{v}) = \mathcal{N}, \\ -C(\sharp E(g), \sharp \mathcal{I}_{\mathrm{IN}}) & \text{otherwise.} \end{cases} \tag{3}$$

Under unanimity every agent, even those with no initial opinion, must vote 1. We assume that the maximal cost incurred by IN is always less than 1, i.e., $C\big(\frac{n(n-1)}{2}, n\big) < 1$, to let her use any possible strategy. Similarly, the payoff of CI when he chooses σ and IN chooses s is

$$U(\boldsymbol{v}[s, \sigma]) = \begin{cases} 1 - c_{\mathrm{CI}} \, \sharp \mathcal{I}_{\mathrm{CI}} & \text{if } \mathcal{N}(1, \boldsymbol{v}) \neq \mathcal{N}, \\ -c_{\mathrm{CI}} \, \sharp \mathcal{I}_{\mathrm{CI}} & \text{otherwise.} \end{cases} \tag{4}$$

Let $k_{\mathrm{CI}} = \lfloor 1/c_{\mathrm{CI}} \rfloor$ be the maximal number of agents that CI has an incentive to influence in our benchmark model. We assume that $k_{\mathrm{CI}} \geq 1$, therefore, CI always has an incentive to influence at least one agent if it guarantees that a unanimous vote for 1 can be avoided. To sum up, IN wants all agents to vote 1, while CI wants at least one agent not to vote 1.

Timing of Moves. The game has three stages after which players obtain their payoffs.

Stage 1. IN chooses her strategy: builds the network, and influences a subset of agents.

Stage 2. Player CI *observes* the strategy of player IN and influences a subset of agents.

Stage 3. Given their initial opinion, agents interact and form their final opinion and vote.

[13] Our results do not change qualitatively if we assume that the cost of influencing agents for CI is strictly increasing.

Subgame Perfect Nash Equilibrium (SPNE). An SPNE is a pair $(s_\star, \sigma_\star)$ that prescribes the following strategic choices. Let S^{IN} and S^{CI} be the set of strategies of player IN and CI, respectively. In Stage 2, given network g, player CI plays a best response[14] $\sigma_\star$ to strategy s: $\sigma_\star \in \arg\max_{\sigma \in S^{\text{CI}}}\{U(\boldsymbol{v}[s,\sigma])\}$. Player IN obtains $u(\boldsymbol{v}[s,\sigma_\star])$ when she chooses s. In Stage 1, player IN plays $s_\star$ such that $s_\star \in \arg\max_{s \in S^{\text{IN}}}\{u(\boldsymbol{v}[s,\sigma_\star])\}$.

Example 1 Criminal Organization vs Police.[15] Criminal organizations provide a reasonable approximation for the type of model we propose here. The leader of the criminal organization[16] (IN) wants its members to be loyal (action 0) while law enforcement (CI) would prefer its members to be disloyal (action 1). IN would like to convince its members that disloyalty will be punished and would like to ensure that its members can be monitored or surveilled for this purpose. Since doing this for every member can be very costly, an alternative strategy for IN is to exert surveillance and pressure only on specific agents and to develop a network among the members of the organization in order to spread the opinion that his surveillance capacity is high. Moreover, the criminal organization is under constant threat from law enforcement who have the ability to send criminals to prison. Thus, each agent is more likely to be disloyal if he believes CI has a high ability to effectively detect and punish criminal activities.[17] In terms of our model, the agents' choices can be understood by considering two possible situations: either IN has the ability to punish its members or CI has the ability to punish the criminals. Specifically, in the first situation (say S1), IN has the ability to punish and in the second situation (say S0) CI has the ability to punish. In our model, both IN and CI have the ability to influence agents' (initial) beliefs about their ability to punish. It follows that $\theta_i = 1$ means that i has an initial opinion/belief in favor of S1 and $\theta_i = 0$ means that agent i has an initial opinion/belief in favor of S0. In the absence of any influence, a member has no initial opinion. The members of the criminal organization follow the decision rule given in Equation (1) after taking into account their own opinion as well as the opinion of their neighbors. Finally, in our benchmark model IN must win the loyalty of all members of the organization, whereas the

[14]Since CI best responds against the strategy chosen by player IN, his strategy can be interpreted as the worst possibility that IN would face. Hence, IN can also be seen as an infinitely risk-averse player.

[15]This example is inspired by Baccara and Bar-Isaac, 2008.

[16]Numerous analyses suggesting that a criminal organization can be represented as a network shaped by its leader (see for example Paoli, 2014).

[17]The question of the vulnerability of criminal organizations to law enforcement is a central topic of discussion in the paper by Baccara and Bar-Isaac (2008).

deviation of one member is enough for law enforcement to achieve their objective.

3 Model Analysis

In this section we assume that IN is the better influencer. We begin by providing an example to illustrate the importance of network architectures in determining how players IN and CI influence the agents. Specifically, we assume that IN also takes the architecture of the network as given. We consider three types of network architectures. (1) The *empty network* where all agents have formed no links. (2) A *star* network where a *central agent*, i_c, has formed links with all other agents − referred to as *peripheral agents* − while no links exist between any pair of peripheral agents. (3) The *complete network* where every agent is linked to every other agent.

Example 2 Suppose that $k_{\text{CI}} = n$ and $\alpha = 1$ in Equation (1), i.e., *only the opinion of the neighbors matters for the final opinion*. The cost function is given by $C(\sharp\mathcal{I}_{\text{IN}}) = \sharp\mathcal{I}_{\text{IN}} \times c_{\text{IN}}$, with $1 - n\, c_{\text{IN}} > 0$. Hence, IN can influence everyone. We examine some specific architectures for network g, that highlight crucial points which IN must take into account when determining her optimal strategy.

1. Let g be the empty network. Then, in an SPNE, IN has to influence all agents. Given IN's strategy, CI cannot persuade anyone to vote 0. Consequently, since influencing is a costly act, CI has no incentive to influence anyone.

2. Let g be a star, with i_c the central agent of this star. In equilibrium, IN must influence the central agent, otherwise all his neighbors will vote 0 if he is influenced by CI. Since $\alpha = 1$, when IN influences i_c, she ensures that every peripheral agent votes 1. Moreover, in an SPNE, IN must influence at least as many peripheral agents as CI to obtain $\bar{\Theta}_{i_c} \geq 1/2$. If this is not the case, the central agent will end up voting 0. Thus, IN influences $\lceil \frac{n-1}{2} \rceil$ of the neighbors of i_c and CI does not influence any agent. Note that if the 'maximal' number of agents CI can influence is $k_{\text{CI}} = 2$, then IN influences only 2 peripheral agents in addition to the central agent.

3. Let g be the complete network. Then, IN has to influence half of the population plus one agent. First, consider agent i who is not influenced by IN. Since g is complete, i has $n-1$ neighbors. In equilibrium, IN must obtain $\bar{\Theta}_i \geq 1/2$. We have $\bar{\Theta}_i = \frac{\sharp\mathcal{I}_{\text{IN}}}{n-1} \geq 1/2$, that is

the number of agents influenced by IN must be at least $\lceil \frac{n-1}{2} \rceil$. Second, consider agent i influenced by IN. We have $\bar{\Theta}_i = \frac{\sharp \mathcal{I}_{\text{IN}} - 1}{n-1} \geq 1/2$, that is $\sharp \mathcal{I}_{\text{IN}} \geq \lceil \frac{n+1}{2} \rceil$. Consequently, IN must influence $\lceil \frac{n+1}{2} \rceil$ agents.

From this example, we can conclude the following about situations where only the opinions of neighbors matter.

1. Every agent influenced by IN has at least as many neighbors who are also influenced by IN as those who are not.
2. Agents not influenced by IN must be linked with agents influenced by IN. Obviously, it is better for IN if the former agents have no connections to each other.

It follows from points 1 and 2 that, in equilibrium, unless IN influences a minimum number of agents, it is impossible to secure a unanimous vote for 1. Similarly, when IN has the option to create the network, not all strategies can be part of the SPNE. In particular, from point 2, IN has no incentive to create links between agents she does not influence. Thus, the complete network will never be an SPNE. Similarly, if link formation costs are significantly lower than the cost of influencing agents, the empty network where all agents are influenced by IN cannot be an SPNE. In this case, the latter strategy of IN will be more costly than the strategy where IN builds the complete network and influences either a number of agents that is equal to one plus half of the agents or the maximum number of agents CI can possibly influence.
Finally, observe that the value of k_{CI} plays a crucial role in determining the optimal strategy for De: in a star network, if $\alpha = 1$ and $k_{\text{CI}} < \lceil \frac{n-1}{2} \rceil$, De does not influence $\lceil \frac{n-1}{2} \rceil$ peripheral agents in addition to the central agent, but rather k_{CI} peripheral agents.

We now systematically investigate the optimal strategies chosen by IN and CI when IN has the ability to form the network. First, we analyze our benchmark model in which the social network only contains links formed by IN. Second, we introduce the possibility that links that are not formed by IN appear. Besides making the model more realistic, this scenario is used to check the robustness of the results obtained in our benchmark model.

3.1 When IN Has Better Influence Technology

Since we want to find the SPNE, we start with the optimal strategy of CI. First, given that $C(\frac{n(n-1)}{2}, n) < 1$, for any strategy adopted by CI, IN has a strategy that allows her to obtain a strictly positive payoff, with a strictly negative payoff for CI since $c_{\text{CI}} > 0$. In particular, IN can build the empty network and influence all agents, and CI cannot prevent a unanimous vote for 1, since for any agent i influenced by both players, $\theta_i = 1$ holds. Hence, CI will influence no agents and obtain a zero payoff in equilibrium as stated below.

Lemma 1 *Suppose the payoff functions of players* IN *and* CI *are given by Equations* (3) *and* (4) *respectively. In an SPNE,* CI *always chooses to influence no agents, and the strategy of* IN *is such that for every agent $i \in \mathcal{N}$, $\theta_i^F = 1$.*

Let $\mathcal{W}$ be the set of pairs $(g, \mathcal{I}_{\text{IN}})$ for which CI's best response is to influence no agents ; we refer to such pairs as *winning strategies*. A *minimal winning strategy* $(g, \hat{\mathcal{I}}_{\text{IN}})$ is a winning strategy for which $\sharp E(g)$ is minimal, i.e., $\sharp E(g) \leq \sharp E(g')$ for any winning strategy $(g', \hat{\mathcal{I}}_{\text{IN}})$. Finally, an *optimal strategy* is a winning strategy that minimizes IN's cost, i.e., a *strategy of* IN *that is an SPNE.*

In the next result, we provide a minimizing program whose solution is the optimal strategy for IN. Recall that for any agent i influenced by both players, we have $\theta_i = 1$, and CI has no incentive to influence more than k_{CI} agents. From this it follows that the maximum number of neighbors of i that CI can influence is $k_{B1}(i, g) = \min\{k_{\text{CI}}, \sharp\mathcal{N}_i(g) \setminus \mathcal{I}_{\text{IN}}\}$. Moreover, let $\kappa = 2\alpha - 1$. Thus, $\kappa \in (0, 1]$ since $\alpha \in (1/2, 1]$. Clearly, for every agent $i \in \mathcal{I}_{\text{IN}}$, the number of neighbors of i influenced by IN has to be greater than κ times the number of neighbors of i that IN does not influence. Finally, recall that any agent not influenced by IN must be connected to an agent influenced by IN.

Proposition 1 *Suppose that payoff functions of players* IN *and* CI *are given by Equations* (3) *and* (4) *respectively. Then, in an equilibrium, we have $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}}) \leq \lceil \frac{n}{2} \rceil$ for every $i \in \mathcal{I}_{\text{IN}}$. Moreover, strategy, $(g, \mathcal{I}_{\text{IN}})$, is an optimal strategy for* IN *if and only if it is a solution of the following minimizing program:*

$$\arg \min_{(g,\mathcal{I}_{\text{IN}}) \in G[\mathcal{N}] \times 2^{\mathcal{N}}} \{C(\sharp E(g), \sharp \mathcal{I}_{\text{IN}})\} \qquad (\textit{Prg})$$

$$\underline{\textit{s.t.}} \qquad \forall i \in \mathcal{I}_{\text{IN}}, \quad \sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}}) \geq \kappa k_{B1}(i, g), \qquad (\textit{Cons. 1})$$

$$\forall i \in \mathcal{N} \setminus \mathcal{I}_{\text{IN}}, \quad \exists j \in \mathcal{I}_{\text{IN}}, \mathcal{N}_i(g) = \{j\}. \qquad (\textit{Cons. 2})$$

For characterizing optimal strategies, we begin by examining the numbers of agents influenced by IN that appear in an optimal strategy. Then, for each possible value of this number $\sharp\mathcal{I}_{\text{IN}}$, we provide the minimal number of links required in any optimal strategy. Finally, we use this to establish that there exist only *three* possible optimal candidate strategies.

In the next proposition, we show that the minimal number of agents that IN has to influence in an optimal strategy, $\sharp\mathcal{I}_{\text{IN}}^{\min}$, is possibly equal to $\bar{x}$ with

$$\bar{x} = \operatorname*{arg\,min}_{x \in [\![1,n]\!]} \left\{ x \left\lfloor \frac{x-1}{\kappa} \right\rfloor \geq n - x \right\}. \tag{5}$$

In Inequality (5), x will be identified to the number of agents influenced by IN, $\sharp\mathcal{I}_{\text{IN}}$. The left-hand side corresponds to the maximum number of agents that CI can influence–each connected to an agent influenced by IN–without causing any of the latter to change their opinion, assuming they are all linked together. The right-hand side corresponds to the maximum number of agents that CI can influence, assuming k_{CI} is sufficiently large, given that IN is a better influencer.[18]

Proposition 2 *In any optimal strategy, the minimal number of agents* IN *must influence is:*

$$\sharp\mathcal{I}_{\text{IN}}^{\min} = \min\{\lceil \kappa k_{\text{CI}} \rceil + 1, \bar{x}\}.$$

We provide intuition for this result when $\alpha = 1$, i.e., $\kappa = 1$. The value of $\sharp\mathcal{I}_{\text{IN}}^{\min}$ takes into account two types of strategies of IN: (a) the case where only one agent influenced by IN, say i_c, is linked to other agents influenced by IN, and (b) the case where agents influenced by IN are all linked together. In both cases, by (Cons. 1), each agent influenced by IN has a neighborhood containing at least as many agents from $\mathcal{I}_{\text{IN}}$ as from the number of agents in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$ that CI can influence. In case (a), by (Cons. 1), the agents in $\mathcal{I}_{\text{IN}} \setminus \{i_c\}$ must each have at most one neighbor not influenced by IN, and all other agents not influenced by IN must be connected to i_c. If the latter are more than k_{CI}, it is sufficient for i_c to be connected to k_{CI} agents influenced by IN to ensure that he votes 1. Hence, IN must influence at least $k_{\text{CI}} + 1$ agents. In case (b), IN forms links between the agents she influences to reinforce the influence she exerts on them. Each agent influenced by IN can be linked with at most $\sharp\mathcal{I}_{\text{IN}} - 1$ other agents influenced by IN, and so $\sharp\mathcal{I}_{\text{IN}} - 1$ agents are not influenced by IN. Consequently, $\sharp\mathcal{I}_{\text{IN}}$ must satisfy $\sharp\mathcal{I}_{\text{IN}} (\sharp\mathcal{I}_{\text{IN}} - 1) \geq n - \sharp\mathcal{I}_{\text{IN}}$ in order to obtain a unanimous vote for 1, that is Equation (5) when $\kappa = 1$.

[18] In Lemma 4 (see Appendix A.2), we establish that either $\bar{x} = \lceil \sqrt{\kappa n} \rceil$ or $\bar{x} = \lceil \sqrt{\kappa n} \rceil + 1$. Moreover, when $\alpha = 1$, i.e., $\kappa = 1$, we have $\bar{x} = \sqrt{n}$.

We now present the minimal winning strategies, i.e., the minimal number of links that IN has to form in a winning strategy given the number of agents she influences, $\sharp\mathcal{I}_{\text{IN}}$.

Proposition 3 *For $\sharp\mathcal{I}_{\text{IN}} \geq \sharp\mathcal{I}_{\text{IN}}^{\min}$, any minimal winning strategy has at least $L^{\min}(\sharp\mathcal{I}_{\text{IN}})$ links with*

$$L^{\min}(\sharp\mathcal{I}_{\text{IN}}) = \min\left(\left\lceil\frac{\kappa(n-\sharp\mathcal{I}_{\text{IN}})}{2}\right\rceil,\ \lceil\kappa k_{\text{CI}}\rceil\right) + n - \sharp\mathcal{I}_{\text{IN}}. \tag{6}$$

Again, we provide intuition for this result when $\alpha = 1$, i.e., $\kappa = 1$. The minimal number of links that IN must form depends on both the number of agents she influences and on the type of strategy she adopts, specifically, whether it corresponds to case (a) or case (b) discussed above. In case (a), IN must form k_{CI} links between the central agent i_c and agents influenced by IN, and $n - \sharp\mathcal{I}_{\text{IN}}$ links between i_c and agents not influenced by IN. In case (b), by (Cons. 1), IN must ensure that the number of 'internal' links between agents influenced by IN is sufficient to support their 'external' links with agents not influenced by IN. Since each 'internal' link between agents influenced by IN allows them to support two 'external' links with agents not influenced by IN, the minimal number of internal links required to sustain the $n - \sharp\mathcal{I}_{\text{IN}}$ external links is $\left\lceil\frac{n-\sharp\mathcal{I}_{\text{IN}}}{2}\right\rceil$. The number of 'external' links is the same as in case (a).
We now provide an example which establishes that there are situations where it is not possible to reach the bound $L^{\min}(\sharp\mathcal{I}_{\text{IN}})$.

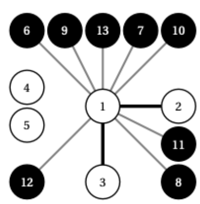
Example 3 Let $\mathcal{N} = [\![1, 27]\!]$, $k_{\text{CI}} = 27$, and $\kappa = 3/7$. By Proposition 2, we have $\sharp\mathcal{I}_{\text{IN}}^{\min} = \left\lceil\sqrt{3/7 \times 27}\right\rceil = 4$. Similarly, by Proposition 3, $L^{\min}(\sharp\mathcal{I}_{\text{IN}}^{\min}) = 23 + \lceil 3/14 \times 23\rceil = 28$. Obviously, by (Cons. 2), 23 links are required between agents in $\mathcal{I}_{\text{IN}}^{\min}$ and agents in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}^{\min}$. Suppose now that there are only 5 links between agents in $\mathcal{I}_{\text{IN}}^{\min}$. Then, two agents in $\mathcal{I}_{\text{IN}}^{\min}$ have formed links with 3 agents in $\mathcal{I}_{\text{IN}}^{\min}$, and two agents in $\mathcal{I}_{\text{IN}}^{\min}$ have formed links with 2 agents in $\mathcal{I}_{\text{IN}}^{\min}$. The former may form links with at most 14 agents in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}^{\min}$ and the latter may form links with at most $2 \times \lfloor 2 \times 7/3\rfloor = 8$ agents in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}^{\min}$. Consequently, one agent in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}^{\min}$ does not satisfy (Cons. 2), and $L^{\min}(\sharp\mathcal{I}_{\text{IN}}^{\min})$ is not a sufficient number of links for obtaining a winning strategy.

From Propositions 2 and 3, we establish the main result of this section.[19] To do so, we present specific architectures. A *partial-star* g is a network where $\mathcal{N}$ admits a partition into two subsets

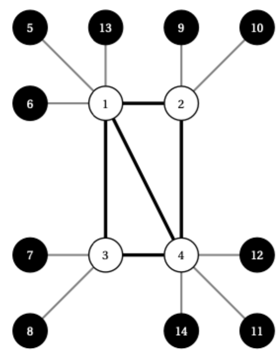
[19]In the working paper, we provide an example showing that each type of strategy presented in the theorem is optimal. We also present the exact optimal strategies for IN when the cost function is linear.

$\mathcal{X}$ and $\mathcal{N} \setminus \mathcal{X}$ such that $g[\mathcal{X}]$ is a star and agents in $\mathcal{N} \setminus \mathcal{X}$ are isolated in g. We illustrate a partial-star in Figure 1 (a) where $\mathcal{X} = [\![1, 12]\!] \setminus \{4, 5\}$ and $\mathcal{N} \setminus \mathcal{X} = \{4, 5\}$. Partial-stars with special properties play a crucial role in our next result. In these particular partial-stars, the center, i_c, belongs to a special set of agents labeled $\mathcal{Y}$ that contains all isolated agents and p agents that are peripheral. Moreover, all agents in $\mathcal{N} \setminus \mathcal{Y}$ are peripheral agents. Thus, there are two types of peripheral agents those in $\mathcal{Y}$ and those in $\mathcal{N} \setminus \mathcal{Y}$. In the following definition, we need the value of p and the set $\mathcal{Y}$.

Definition 1 *For $p \in [\![1, n-2]\!]$, and $\mathcal{Y} \subset \mathcal{N}$, g is a $(p, \mathcal{Y})$-partial-star when it is a partial-star and $\mathcal{Y}$ contains the central agent, i_c, p peripheral agents, and all the isolated agents of g.*



(a) (2, $[\![1,5]\!]$)-partial-star, g^1



(b) (1, $[\![1,4]\!]$)-MQC network, g^2

Figure 1: Illustrations of $(p, \mathcal{Y})$-partial-star and $(q, \mathcal{Y})$-MQC

Let us illustrate this definition with network g^1, drawn in Figure 1 (a). Agents colored white belong to $\mathcal{Y} = [\![1, 5]\!]$, and agents colored black belong to $\mathcal{N} \setminus \mathcal{Y} = [\![6, 13]\!]$. We observe that g^1 is a partial-star. The number of peripheral agents in $g^1[\mathcal{Y}]$, p, is $\sharp[\![2, 3]\!] = 2$. Finally, each agent in $\mathcal{N} \setminus \mathcal{Y}$ is linked to i_c.

We now present specific networks called quasi-core-periphery networks that are close to the core-periphery networks widely used in the literature. In a core-periphery network, the set of agents is divided into two disjoint subsets: the core and the periphery. Agents in the core have at least as many neighbors within the core as in the periphery, and agents in the periphery are only connected to nodes in the core. In the next definition, we present specific quasi-core-periphery networks that minimize the number of links, given that each member of the core, denoted by $\mathcal{Y}$, has q times more neighbors that belong to $\mathcal{Y}$ than neighbors that belong to $\mathcal{N} \setminus \mathcal{Y}$. Moreover, each agent in $\mathcal{N} \setminus \mathcal{Y}$ is connected to only one agent belonging to $\mathcal{Y}$.

Definition 2 *For $q \in (0,1]$ and $\mathcal{Y} \subset \mathcal{N}$, g is a $(q,\mathcal{Y})$-quasi-core-periphery network if g admits a partition of $\mathcal{N}$ into two subsets, $\mathcal{Y}$ and $\mathcal{N} \setminus \mathcal{Y}$, called the core and the periphery respectively. The following properties hold: (Q1) for every $j \in \mathcal{Y}$, $\sum_{\ell \in \mathcal{Y}} g_{j,\ell} \geq q \sum_{\ell \in \mathcal{N} \setminus \mathcal{Y}} g_{j,\ell}$, and (Q2) for every $j \in \mathcal{N} \setminus \mathcal{Y}$, $\sum_{j \in \mathcal{Y}} g_{i,j} = 1$ and $\sum_{j \in \mathcal{N} \setminus \mathcal{Y}} g_{i,j} = 0$.*
A network g is a $(q,\mathcal{Y})$-minimal-QC, denoted by $(q,\mathcal{Y})$-MQC, if it is a $(q,\mathcal{Y})$-quasi-core-periphery network and there does not exist a $(q,\mathcal{Y})$-quasi-core-periphery network with a smaller number of links.[20]

Note an important property of $(q,\mathcal{Y})$-MQC is that there are $n - \sharp\mathcal{Y}$ links between agents in $\mathcal{Y}$ and agents in $\mathcal{N} \setminus \mathcal{Y}$, and at least $\left\lceil \frac{q(n-\sharp\mathcal{Y})}{2} \right\rceil$ links between agents in $\mathcal{Y}$. We claim that g^2 in Figure 1 (b) is a $(1, [\![1,4]\!])$-MQC network. In g^2, agents in $[\![1,4]\!]$, colored white belong to $\mathcal{Y}$, and agents in $[\![5,14]\!]$, colored black belong to $\mathcal{N} \setminus \mathcal{Y}$. We have $q = 1$. Hence, (Q1) holds since each agent in $[\![1,4]\!]$ is linked with a number of agents in $[\![1,4]\!]$ that is at least equal to the number of agents in $[\![5,14]\!]$ with whom he is linked. Moreover, (Q2) is satisfied since each agent in $[\![5,14]\!]$ is linked with exactly one agent in $[\![1,4]\!]$. Finally, there is no $(1, [\![1,4]\!])$-quasi-core-periphery network with a smaller number of links than g^2.
We now present different strategies that play an important role in our analysis. In the *complete influence-empty network strategy*, IN forms the empty network and influences all agents. In a $(q,\mathcal{Y})$*-influence-MQC network strategy*, IN forms a $(q,\mathcal{Y})$-MQC network and influences agents in the core, $\mathcal{Y}$. In a $(p,\mathcal{Y})$*-influence-partial-star strategy*, IN forms a $(p,\mathcal{Y})$-partial-star and influences agents in $\mathcal{Y}$. Note that the influence-partial-star strategy can be viewed as a special case of the influence-MQC strategy, with a core composed of the central agent and the peripheral agents influenced by IN. Let us now present the main result of this section.

Theorem 1 *For any given parameters n, α and k_{CI}, one of the following strategies is optimal:*

1. *the complete influence-empty network strategy, or*
2. *a $(p, \mathcal{I}_{\text{IN}})$ influence-partial-star strategy, with $\sharp\mathcal{I}_{\text{IN}} \geq \sharp\mathcal{I}_{\text{IN}}^{\min}$ and $p \geq \lceil \kappa k_{\text{CI}} \rceil$, or*
3. *a $(\kappa, \mathcal{I}_{\text{IN}})$ influence-minimal-quasi-core periphery network strategy, with $\sharp\mathcal{I}_{\text{IN}} \geq \sharp\mathcal{I}_{\text{IN}}^{\min}$.*

For the sake of intuition, assume that $\alpha = 1$, i.e., $\kappa = 1$, and IN's cost function is linear. Clearly, when the relative cost c_L/c_{IN} is very high, the complete-influence-empty network

[20] This definition does not imply anything about the existence of $(q,\mathcal{X})$-MQC networks. In Appendix A.1, we define a class of networks that are $(q,\mathcal{X})$-MQC and provide a constructive algorithm that ensures their existence.

strategy is an optimal strategy, as IN prefers influencing more agents over establishing links. We now explain how IN chooses between the two strategies described in points 2 and 3. Assume that $n = 14$ and that CI has an incentive to influence at most three agents. Set the cost parameters to $c_{\text{IN}} = 1/1000$ and $c_L = 1/800$. Under the strategy based on star networks, IN builds a star network and protects the central agent and three peripheral agents. We call this strategy s_2. The total cost of s_2 is: $4\,c_{\text{IN}} + 13\,c_L = 0.02025$. In contrast, under strategy based on the core periphery, IN builds network g^2 shown in Figure 1(b), protecting the agents colored white.[21] We call this strategy s_3. The total cost of s_3 is: $4\,c_{\text{IN}} + 15\,c_L = 0.02275$. Thus, when CI can influence only a small number of agents, the star-based strategy s_2 is more cost-effective for IN. Next, consider a situation in which CI has an incentive to influence all 14 agents. In this case, strategy s_3 allows IN to secure a unanimous vote, whereas s_2 no longer allows IN to obtain such a unanimous vote. To achieve unanimity using the star-based strategy, IN would need to protect the central agent and six peripheral agents, incurring a total cost of: $7\,c_{\text{IN}} + 13\,c_L = 0.02325$. This example highlights the fundamental trade-off faced by IN. The optimal strategy depends on both the cost ratio, c_L/c_{IN}, and the number of agents CI has an incentive to influence, k_{CI}. Strategies based on core-periphery (e.g., s_3) are advantageous when CI has an incentive to influence many agents, as they allow to obtain unanimous vote with relatively few agents influenced by IN at the cost of a larger number of links. However, when the number of agents that CI can influence is limited, as in our previous example, star-based strategies are strictly more cost-effective, since IN needs to influence only as many neighbors of i_c as CI can influence agents. To summarize, strategies of IN based on star networks are optimal only when k_{CI} is sufficiently low and the ratio c_L/c_{IN} is sufficiently high.[22]

[21] Note that the cost incurred by IN under strategy s_2 is the same as that incurred under an alternative strategy in which she influences the same four agents. In this alternative, each of three peripheral agents in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$ is connected to a distinct peripheral agent in $\mathcal{I}_{\text{IN}}$, such that every peripheral agent in $\mathcal{I}_{\text{IN}}$ has exactly one neighbor in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$, and all other links remain unchanged.

[22] We have assumed that CI has an incentive to influence at least one agent. When CI has no incentive to influence agents ($k_{\text{CI}} = 0$), IN's optimal strategy changes if c_L/c_{IN} is sufficiently low: she builds a star and influences only the center. If c_L/c_{IN} is sufficiently high, her strategy remains to influence all agents without links.

3.2 Allowing for Random Links

In this section we want to explore the possibility that player IN does not fully control the network formation. In fact, in any social situation, agents may have a chance to meet each other and form a connection even if the influencer did not establish a connection between them. Therefore, we assume that unlinked agents have a non-negative probability of forming links $\omega \in [0,1]$, altering the neighborhood created by IN for each agent. The probability ω is *i.i.d.*, i.e., the random meetings between agents leading to new links are independent events. Moreover, ω is *common knowledge*, in particular, it is known by IN. Finally, we assume that the cost function of IN is linear, $k_{\text{CI}} = n$, and IN is risk-neutral. In this section, we consider the following timing of the game:

Stage 1. IN chooses her strategy $(g, \mathcal{I}_{\text{IN}})$, knowing that any link not formed by her in g occurs with probability ω;

Stage 2. Nature randomly forms a link between every pair of agents (i,j) who are not linked in g with probability ω;

Stage 3. CI observes the agents influenced by IN, the network she has formed, as well as the random links added by Nature before choosing his strategy.[23]

Observe that the timing of the game and $k_{\text{CI}} = n$ together ensure that IN does not obtain a strictly positive payoff when the network (and the set of influenced agents) obtained after Nature's move is no longer a winning strategy.

First, for computing the expected payoffs of IN and CI, we need to define a realization g^{ω} of g, where network g is a subnetwork of g^{ω}. Let $\lambda(g^{\omega} \mid g, \omega)$ be the probability that g^{ω} is realized given probability ω and that IN has built network g. We have:

$$\begin{aligned}\lambda(g^{\omega} \mid g, \omega) &= \prod_{ij \in E(g^{\omega}) \setminus E(g)} \omega \prod_{i'j' \notin E(g^{\omega})} (1-\omega) \\ &= \omega^{\sharp E(g^{\omega}) \setminus E(g)} (1-\omega)^{\frac{n(n-1)}{2} - \sharp E(g^{\omega})}.\end{aligned} \tag{7}$$

A winning realization is a pair $(g^{\omega}, \mathcal{I}_{\text{IN}})$ where CI has no strategy that allows him to ensure that at least one agent in the realized network g^{ω} will vote 0 when IN has influenced agents

[23] It should be noted that since Nature forms additional links just after IN plays his strategy, CI has the option to adapt his strategy to the new architecture. Hence, IN's optimal strategy needs to take account not only the fact that new links can arise with probability ω, but also the fact that agents involved in this link can be influenced by CI.

in $\mathcal{I}_{\text{IN}}$.[24] Let $WR(g;\mathcal{I}_{\text{IN}})$ be the set of winning realizations of g given $\mathcal{I}_{\text{IN}}$. Assuming that the cost function of IN is linear, her expected payoff is given by:

$$\mathbb{E}u(\boldsymbol{\theta}^{\text{F}}[s,\sigma]) = \sum_{g^{\omega}\in WR(g,\mathcal{I}_{\text{IN}})} \lambda(g^{\omega} \mid g,\omega) - c_{\text{IN}}\sharp\mathcal{I}_{\text{IN}} - c_L\sharp E(g). \tag{8}$$

Since CI observes the realized network before choosing his strategy, his expected payoff is given by Equation (4).

At first glance it might seem that IN will view the possibility of the occurrence of such unwanted links as always being harmful. Indeed, when the number of agents IN has influenced is fewer than the number of agents she has not influenced, the probability that "bad" links (which involve agents not influenced by IN) is higher than the probability that "good" links (which only involve agents influenced by IN) occur. However, the possibility that unwanted links can occur is not always harmful for IN. For instance, when $\alpha = 1$ and for $\omega = 0$, the optimal strategy of IN is the complete influence-empty network strategy. However, if $\omega^{\frac{n(n-1)}{2}}$ is sufficiently close to 1, then IN has an incentive to influence only $\lfloor n/2 \rfloor + 1$ agents instead of influencing all the agents to obtain that a unanimous vote of 1. In this case, the probability that the complete network occurs is sufficiently high and IN is able to take advantage of the fact that agents can influence each other. Hence, she obtains a higher payoff with this strategy than with the complete influence-empty network strategy. We now provide a lower bound for ω such that the strategies that are candidates for being optimal are the same as those given in Theorem 1. This statement highlights the continuity in the results obtained in our benchmark model when ω is sufficiently small. The proof is given in the online appendix (OA.2).

Proposition 4 *Suppose that $\omega \leq \frac{c_L}{4n}$. Then, the candidate strategies for being optimal are the same as those given in Theorem 1.*

It is worth noting that in Proposition 4, ω depends on n. Let us illustrate this point when $\alpha = 1$. Consider, for example, a $(1,\mathcal{I}_{\text{IN}})$-influence-MQC strategy where $\sharp\mathcal{I}_{\text{IN}} = \lceil \sqrt{n}\, \rceil$. When this strategy is played by IN, and a new link occurs, the probability that this link is formed between any agent $i \in \mathcal{N}$ and an agent $j \in \mathcal{N} \setminus \mathcal{I}_{\text{IN}}$ is at least $1 - \frac{\binom{\lceil \sqrt{n} \rceil}{2}}{\binom{n}{2}} \geq 1 - \frac{\sqrt{n}+1}{n}$. Notice that $\lim_{n\to+\infty} 1 - \frac{\sqrt{n}+1}{n} = 1$. In other words, when the number of agents is very large and a new link occurs, the probability that it involves an agent in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$ becomes very large when

[24] $(g^{\omega},\mathcal{I}_{\text{IN}})$ can be seen as a winning strategy in the benchmark model where IN forms network g^{ω} and influences agents in $\mathcal{I}_{\text{IN}}$.

IN uses a $(1, \mathcal{I}_{\text{IN}})$-influence-MQC strategy with $\sharp\mathcal{I}_{\text{IN}} = \lceil\sqrt{n}\,\rceil$. Hence, the new link makes such a strategy inefficient.

It is worth noting that when the probability of random links is very high, then the optimal strategy for IN is to form no links and influence enough agents to ensure a unanimous vote for 1 in the complete network. In contrast, when the probability of random links is moderate, intermediate strategies become optimal. In these cases, IN influences a number of agents within the range $[\![\mathcal{I}_{\text{IN}}^{\min} + 1, n-1]\!]$.[25]

4 Extensions

In this section, we systematically relax, one by one, the three key assumptions made in the benchmark model. First, we assume that agents do not have to vote following their initial interactions, but rather after a finite number of interaction periods. Second, we relax the unanimity assumption by allowing IN to win if a majority of the agents vote for 1. Third, we assume that an agent, influenced by both IN and CI, follows the opinion of CI, i.e., IN is no longer a better influencer than CI.

4.1 When Agents Interact for Multiple Periods

Our benchmark model raises a natural question: What is IN's optimal strategy if the final vote of the agents does not have to take place following their first interaction, but after several rounds of interactions? Specifically, IN needs to obtain a unanimous vote at a specific time period T. Thus, we assume a situation with $T+1$ periods, where the opinion of agent i in period $t \in [\![0, T]\!]$ is denoted by θ_i^t, with $\theta_i^0 = \theta$ and $\theta_i^T = \theta_i^{\text{F}}$. In each period $t \in [\![0, T]\!]$, the agents vote or choose an action, but only the vote cast in period T determines the payoffs of IN and CI.[26] Votes in periods before T are *non-binding*, while votes at T are *binding*. Each agent observes the votes or actions of her neighbors at time $t-1$ and adjusts her opinion at time t in response. This section allows us to show how IN can strategically use certain agents

[25] An example illustrating these cases is provided in the online appendix (OA.2).

[26] During the first $T-1$ periods, the agents' vote do not affect the players' payoffs. This type of informative vote without consequences, called a straw poll, occurs in certain decision-making processes, like an initial round of voting to seek opinions, followed by more discussions before a final vote is cast. For instance, such a process is often followed in universities while making tenure decisions.

to influence others to his advantage.[27] To simplify the presentation, in this section, we restrict attention to $\alpha = 1$ and $k_{\text{CI}} = n$. In line with our benchmark model, we have for $t \in [\![0, T]\!]$,

$$v_i^t = v_i^t(\theta_i^t) = \begin{cases} 1 & \text{if } \theta_i^t \geq 1/2, \\ 0 & \text{if } \theta_i^t < 1/2, \\ \emptyset & \text{if } \theta_i^t = \emptyset, \end{cases} \tag{9}$$

and $v_i^0 = \theta_i^0 = \theta_i$. In contrast to our benchmark model, we allow for a non-binding vote by agents after the initial stage of opinion formation.

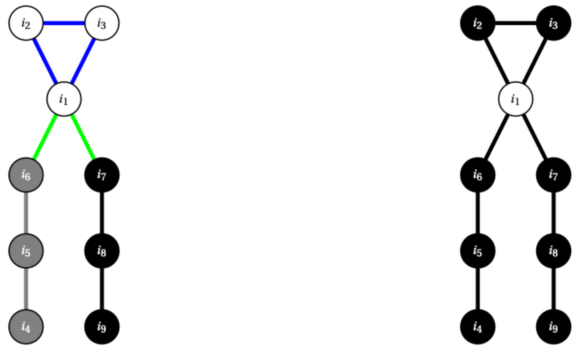
Let us denote the vector that summarizes the non-binding vote of agents at period $t \in [\![0, T-1]\!]$ by $\boldsymbol{v}^t = (v_1^t, \ldots, v_n^t)$, and the vector of binding votes is denoted by $\boldsymbol{v} = \boldsymbol{v}^T = (v_1^T, \ldots, v_n^T)$. Moreover, let $\mathcal{V}_i^k(g;t) = \{j \in \mathcal{N}_i(g) : v_j^t = k\}$ be the set of neighbors of agent i who vote $k \in \{0, 1, \emptyset\}$ in period t. Recalling that $\alpha = 1$, following the logic of the one period voting model, we have for $t \in [\![0, T]\!]$,

$$\theta_i^{t+1} = \begin{cases} \dfrac{1}{\sharp\mathcal{V}_i^0(g;t) + \sharp\mathcal{V}_i^1(g;t)} \displaystyle\sum_{j \in \mathcal{V}_i^0(g;t) \cup \mathcal{V}_i^1(g;t)} v_j^t & \text{if } \mathcal{V}_i^0(g;t) \cup \mathcal{V}_i^1(g;t) \neq \emptyset, \\ \theta_i^t & \text{otherwise.} \end{cases} \tag{10}$$

Finally, we preserve the payoff function given in Equations (3) and (4). We now present a specific network architecture, up to a relabeling of agents, that allows us to define a useful strategy for the rest of this section. Recall that $P_{i_1,i_m} = i_1\, i_2, i_2\, i_3, i_{m-1}\, i_m$ is a path between agents i_1 and i_m. Moreover, the length of a path is the number of links it contains. The (geodesic) distance between agents i and j in g, $d(i,j;g)$, is the length of any shortest path joining them.

Definition 3 *Let $\gamma = 3 + \lceil \frac{n-4}{2} \rceil$. A i_1-triangle-network g*

[27] This aspect of our paper is also related to the literature on targeting in networks where peers are used to enhance the effectiveness of targeting. Our paper is closest to Belhaj and Deorian (2019) and Belhaj, Deorian and Safi (2023) where a principal offers bilateral contracts to agents in the network to achieve their target at the lowest cost. Galeotti, Golub and Goyal (2020) considers interventions in a network aimed at increasing the marginal returns to individual agents when the planner has a given budget constraint. Zhou and Chen (2015) and Li, Zhou and Chen (2022) also exploit network synergies but instead of minimizing cost, they study a two-stage game where some players act before others to achieve the desired outcome. One significant difference with all these papers is that agents play a linear quadratic game on a given network.



(*a*) Network *g* (*b*) Vertex i_1 is influenced by *De*

Figure 2: i_1-triangle-networks and Strategy

(O1) *contains the subnetworks* $g(\{i_4, \ldots, i_\gamma\}) = P_{i_4, i_\gamma}$, *and* $g(\{i_{\gamma+1}, \ldots, i_n\}) = P_{i_{\gamma+1}, i_n}$. *Also,* $g(\{i_1, i_2, i_3\})$ *which is a triangle, i.e., a cycle that contains links* $i_1\, i_2$, $i_1\, i_3$, *and* $i_2\, i_3$*;*

(O2) *in addition network* g *contains links* $i_1\, i_\gamma$ *and* $i_1\, i_{\gamma+1}$.

By construction, $d\,(i_1, i_\ell; g) \leq \lceil \frac{n-3}{2} \rceil$ for all $\ell \in [\![4, n]\!]$. We illustrate these types of networks through the i_1-triangle-network g drawn in Figure 2 (a). We have $N = [\![i_1, i_9]\!]$, hence $\gamma = 3 + \lceil \frac{9-4}{2} \rceil = 6$. Subnetwork $g(\{i_4, i_5, i_6\}) = P_{i_4, i_6}$ is colored gray, subnetwork $g(\{i_7, i_8, i_9\}) = P_{i_7, i_9}$ is colored black, and $g(\{i_1, i_2, i_3\})$ colored white is a cycle. Links $i_1\, i_6$ and $i_1\, i_7$ are colored green.

The strategy where IN influences a unique agent, i_1, is called an *influence-i_1-triangle strategy*. Figure 2 (b) presents network g in which IN influences agent i_1.

Note that IN must influence at least one agent to ensure that all agents vote 1 in a finite number of periods. In the next result, we show that there are optimal strategies where IN influences only one agent.

Proposition 5 *Suppose that at period* t *agents vote according to Equation* (9) *and form their beliefs according to Equation* (10). *Let* IN *influence only one agent.*

1. *Any network* g *created by* IN *that leads to* $\mathcal{N}(1, \boldsymbol{v}) = \mathcal{N}$ *in a finite number of periods must be connected and contains at least* n *links.*

2. *Moreover, there exists a network* g*, with* $\sharp E(g) = n$*, that leads to* $\mathcal{N}(1, \boldsymbol{v}) = \mathcal{N}$ *in a finite number of periods.*

By inspecting the proof of this proposition, an influence-i_1-triangle strategy leads all agents to vote 1 in $2+(\max_{i_\ell \in \mathcal{N}} d(i_1, i_\ell; g))$ periods. We illustrate this convergence process for voting in the next example.

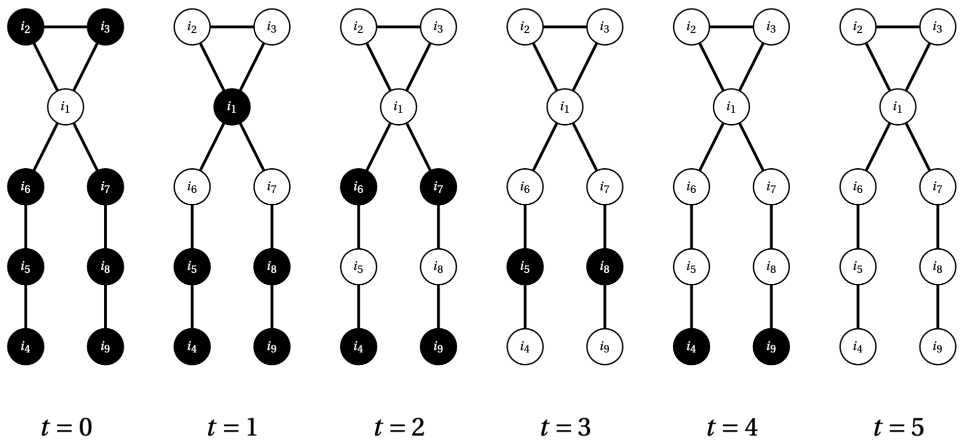


Figure 3: Vote Process with an Influence-i_1-triangle Strategy

Example 4 Let $\mathcal{N} = [\![1, 9]\!]$, $\alpha = 1$, and $k_{\text{CI}} = n$. Suppose that IN uses an influence-i_1-triangle strategy and CI influences all agents. We represent the situation at $t = 0$ in Figure 3 where black colored agents have their initial opinion equal to 0, while the white colored agent has his initial opinion equal to 1. Figure 3 illustrates the evolution of the change in agents' opinions over time for $t = 0, \ldots, 5$, and shows the various stages until a unanimous vote for 1 is obtained. For example, players i_2, i_3, i_6, and i_7 have two neighbors, including i_1. Since i_1 votes 1 at $t = 0$, each of them has an opinion equal to 1 at $t = 1$. The process continues until period 5 using the same logic – note that $\max_{i_\ell \in \mathcal{N}} d(i_1, i_\ell; g) = 3$. Thus, in equilibrium, CI has no incentive to influence agents since it is costly and does not prevent agents from voting unanimously for 1. Clearly, it is possible to increase the speed of convergence of the process by increasing the number of agents influenced by IN or the number of links she forms. Thus, if IN adds a link between i_4 and i_9, the process converges to a unanimous vote for 1 at $t = 4$. Similarly, if IN influences agents i_6 and i_7, then the process converges to a unanimous vote for 1 at $t = 2$.

Like in the benchmark model, the substitution between the number of links and the number of agents influenced continues to hold. In the next example, we show that if the influencer builds a tree network, i.e., an acyclic connected network, then she needs to influence at least two agents.[28]

Example 5 Let $\mathcal{N} = [\![1, 10]\!]$, $\alpha = 1$, and $k_{\text{CI}} = n$. Suppose that IN builds a network which contains a path and influences agents 1 and 2 as represented in Figure 4 at $t = 0$. Colored black agents have their initial opinion equal to 0, while colored white agents have their initial opinion equal to 1. Figure 4 illustrates the evolution of the change in agents' opinions over time for $t = 0, \ldots, 4$ until a unanimous vote in favor of 1 is obtained. Clearly, when IN uses this strategy, the unanimity of the agents to vote 1 is achieved in $\lceil \frac{n-2}{2} \rceil = 5$ periods.

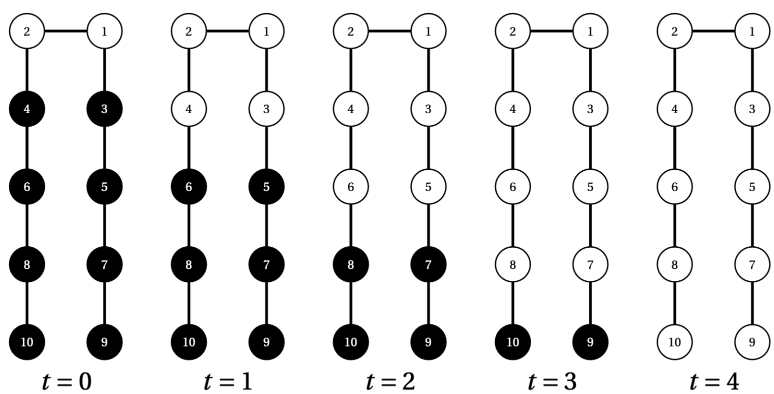


Figure 4: Vote Process When Two Agents Are Influenced by IN

In summary, if we allow the agents to interact several times instead of just once before they vote, IN can obtain the unanimous vote she needs to win using fewer resources. The network of agents acts as a powerful secondary influencer, saving the influencer resources. Thus, repeated interaction between agents favors IN.

4.2 Securing a Majority Vote

In this section, we consider a situation where IN obtains a strictly positive payoff *if and only if* at least half of the population, i.e., $\lceil n/2 \rceil$ agents, votes 1.[29] Since IN now has to influence fewer agents, she will clearly incur lower costs compared to the unanimity voting scenario. These

[28]Similarly, if the influencer save on links costs by allowing for isolated agents, these agents will all have to be individually influenced.

[29]Typically majority would require strictly greater than half, $\sharp\mathcal{N}(1, \boldsymbol{v}) \geq \lceil n/2 \rceil + 1$. But, the results presented in this section would be qualitatively the same for this case as well.

reduced costs could result from needing to establish fewer links, influence fewer agents, or both. More precisely, we will establish that strategies where IN forms a quasi-core-periphery network cannot be optimal. Such strategies are candidate for being optimal only when the required number of agents voting 1 is sufficiently large.

For simplicity, we restrict our attention to the case where the cost function of IN is linear. The majority rule is incorporated into the model by modifying IN's original unanimity based payoff function as follows

$$u_{\text{Maj}}(\boldsymbol{v}[s,\sigma]) = \begin{cases} 1 - c_L \,\sharp E(g) - c_{\text{IN}} \,\sharp \mathcal{I}_{\text{IN}} & \text{if } \sharp \mathcal{N}(1,\boldsymbol{v}) \geq \lceil \frac{n}{2} \rceil, \\ -c_L \,\sharp E(g) - c_{\text{IN}} \,\sharp \mathcal{I}_{\text{IN}} & \text{otherwise.} \end{cases} \tag{11}$$

The payoff function of CI is modified similarly:

$$U_{\text{Maj}}(\boldsymbol{v}[s,\sigma]) = \begin{cases} 1 - c_{\text{CI}} \,\sharp \mathcal{I}_{\text{CI}} & \text{if } \sharp \mathcal{N}(1,\boldsymbol{v}) < \lceil \frac{n}{2} \rceil, \\ -c_{\text{CI}} \,\sharp \mathcal{I}_{\text{CI}} & \text{otherwise.} \end{cases} \tag{12}$$

Following the arguments given in Lemma 1, in the SPNE CI does not influence any agent. Let us begin by providing some properties of the winning strategies of IN based on the size of $\mathcal{I}_{\text{IN}}$.

1. If $\mathcal{I}_{\text{IN}} = \emptyset$, then there exists no winning strategy for IN.
2. If $\mathcal{I}_{\text{IN}} = \{i_c\}$, then consider a partial-star where i_c is the central agent with $\lceil n/2 \rceil$ peripheral agents. Let the remaining agents be isolated. This constitutes a winning strategy (regardless of the value of α).
3. If $\sharp \mathcal{I}_{\text{IN}} \in [\![2, \lceil n/2 \rceil - 1]\!]$, then consider a partial-star where i_c is the central agent with $\lceil n/2 \rceil - \sharp \mathcal{I}_{\text{IN}} + 1$ peripheral agents. Let the remaining agents be isolated. IN influences $\sharp \mathcal{I}_{\text{IN}} - 1$ isolated agents and the central agent. This constitutes a winning strategy (regardless of the value of α).
4. If $\sharp \mathcal{I}_{\text{IN}} \geq \lceil n/2 \rceil$, then the empty network is a winning strategy.

In contrast to the benchmark model, there is always a winning strategy where IN builds a partial-star which is less costly than any winning strategy where IN builds a minimal-quasi-core periphery network. Indeed, in the majority case, the center of the partial-star, i_c, does not need to vote 1. Thus, property (Q1) is no longer required. Consequently, IN can influence i_c and connect him to any number of agents she does not influence to get a majority of votes – IN can also influence some isolated agents in the partial-star. Moreover, for a given $\mathcal{I}_{\text{IN}}$, the number of links IN formed in a winning strategy where she builds a partial-star strategy is at most equal to the number of links IN formed in a winning strategy where she builds a minimal quasi-core. In particular, there is no link between the agents in $\mathcal{I}_{\text{IN}}$ in a winning strategy based on a partial-star, while IN forms at least one such link in a winning strategy based on a minimal-quasi-core network. It follows that there are two possible strategies for IN: either build the empty network and influence $\lceil n/2 \rceil$ agents, or build the cheapest partial-star network that allows her to get $\lceil n/2 \rceil$ agents who vote 1. Since IN chooses the less costly strategy, her least cost strategy incorporating these two possibilities is given by

$$\min\left\{\lceil n/2\rceil c_{\text{IN}}, \min_{x\in[\![1,\lceil n/2\rceil-1]\!]}(xc_{\text{IN}} + (\lceil n/2\rceil - x + 1)c_L)\right\}.$$

The above arguments are summarized in the following proposition given without proof. In fact, it is sufficient to observe that IN *has no incentive to influence more than one agent when she forms links*, due to the linearity of the costs.

Proposition 6 *Suppose the payoff functions of players* IN *and* CI *are given by Equations* (11) *and* (12) *respectively. An optimal strategy is independent of the value of α and of c_{CI}. Further,*

1. *if $\frac{c_L}{c_{\text{IN}}} > \left(1 - \frac{1}{\lceil n/2\rceil}\right)$, then in her optimal strategy* IN *forms no links and influences exactly $\lceil n/2\rceil$ agents;*
2. *if $\frac{c_L}{c_{\text{IN}}} < \left(1 - \frac{1}{\lceil n/2\rceil}\right)$, then in her optimal strategy* IN *forms $\lceil n/2\rceil$ links and influences exactly 1 agent. The resulting network is a partial-star; and*
3. *if $\frac{c_L}{c_{\text{IN}}} = \left(1 - \frac{1}{\lceil n/2\rceil}\right)$, then the two strategies listed above are equilibria.*

The strategy where IN builds the partial-star in which she influences the center implies that peripheral agents have only one neighbor with 1 as initial opinion. Therefore, in contrast to the benchmark model, another important consequence of the majority vote requirement is that IN shapes the network according to her costs c_L and c_{IN}, without taking into account the costs for influencing agents incurred by CI, or the value α.

4.3 When CI Has Better Influence Technology

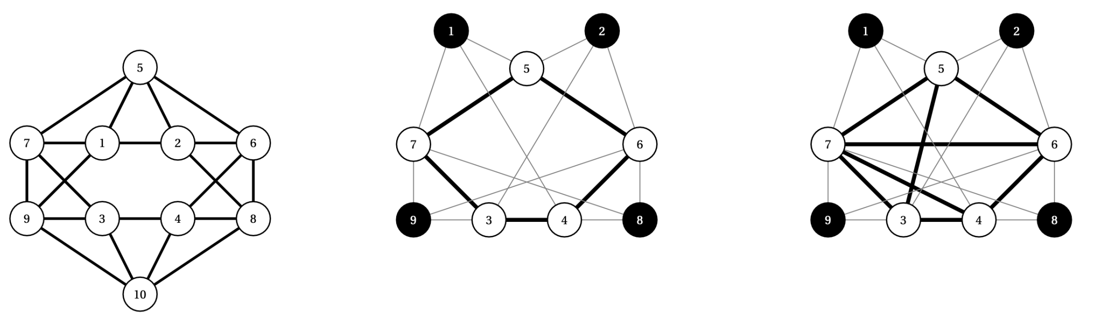
In this section, we assume that when both players IN and CI exert influence on agent i, the counter-influencer is the one with the better influencing technology, i.e., $\theta_i = 0$. In order to simplify the presentation, here we assume that n is even.[30] Formally, the initial opinion of

[30]The results obtained when n is odd are qualitatively similar, except that there exist conditions where as part of her optimal strategies player IN chooses $\sharp\mathcal{I}_{\text{IN}} = n - 1$.

each agent i is now given by

$$\theta_i = \begin{cases} 1 & \text{if } i \in \mathcal{I}_{\text{IN}} \setminus \mathcal{I}_{\text{CI}} \\ 0 & \text{if } i \in \mathcal{I}_{\text{CI}} \\ \emptyset & \text{if } i \notin \mathcal{I}_{\text{IN}} \cup \mathcal{I}_{\text{CI}} \end{cases}$$

To present the results, additional definitions of networks and strategies are required. A k-regular network g is a network where every $i \in \mathcal{N}$ has exactly k links. We illustrate a 4-regular network in Figure 5 (a).



(a) a 4-regular network (b) a $(2,3,[\![3,7]\!])$-GR network (c) a $(3,3,[\![3,7]\!])$-GR network

Figure 5: Specific Strategies of IN

Definition 4 *In a $(a,b,\mathcal{Y})$-groups-regular network, $\mathcal{N}$ is partitioned into two subsets: $\mathcal{Y}$ and $\mathcal{N} \setminus \mathcal{Y}$. Moreover, when $a \times \sharp\mathcal{Y}$ is even, we have*

(R1) *For every $i \in \mathcal{Y}$, $\sum_{j \in \mathcal{Y}} g_{i,j} = a$,*

(R2) *For every $i \in \mathcal{N} \setminus \mathcal{Y}$, $\sum_{j \in \mathcal{Y}} g_{i,j} = b$, and $\sum_{j \notin \mathcal{Y}} g_{i,j} = 0$.*

When $a \times \sharp\mathcal{Y}$ is odd, conditions provided in (R1) and (R2) hold except that there is a unique agent $i \in \mathcal{Y}$ for whom we have $\sum_{j \in \mathcal{Y}} g_{i,j} = a + 1$.

Let g be a network that satisfies (R1) and (R2) where $a \times \sharp\mathcal{Y}$ is even. (R1) implies that $g[\mathcal{Y}]$ is an a-regular network, and (R2) implies that every agent in $\mathcal{N} \setminus \mathcal{Y}$ has formed b links with agents in $\mathcal{Y}$ and no links with other agents in $\mathcal{N} \setminus \mathcal{Y}$. In Figure 5 ($b$) we illustrate a $(2,3,[\![3,7]\!])$-group-regular network where every agent in $\mathcal{Y} = [\![3,7]\!]$ is colored white. Clearly, each agent in $\mathcal{Y}$, is linked to two other agents in $\mathcal{Y}$. Moreover, each agent in $\mathcal{N} \setminus \mathcal{Y}$, colored black, has three links all with agents in $\mathcal{Y}$. In Figure 5 (c), we present the same type of strategy when

$a \times \sharp\mathcal{Y}$ is odd, where player $7 \in \mathcal{Y}$ has 4 links with other members of $\mathcal{Y}$.

In this section we focus on two specific strategies for IN. First, on an *a-regular network strategy*, IN builds an a-regular networks and influences all agents. Second, in a *$(a, b, \mathcal{Y})$-groups-regular network strategy*, IN builds a $(a, b, \mathcal{Y})$-groups-regular network and influences all agents in $\mathcal{Y}$.

Since CI is now a better influencer than IN, he chooses a (minimal) number of agents to influence so that for one agent, say i, $v_i(\theta_i^{\mathrm{F}}) = 0$. When k_{CI} is sufficiently high, there is no strategy in which IN can get all the agents to vote 1. This is the case, for example, when $k_{\mathrm{CI}} = n$. We begin our analysis by examining the conditions concerning the minimum number of agents IN must influence to achieve a unanimous vote for 1, relative to the number of agents CI has an incentive to influence. Let $\eta = \left\lceil \frac{\alpha(k_{\mathrm{CI}}-1)}{\alpha-\frac{1}{2}} \right\rceil$.

Proposition 7 *Suppose the payoff functions of players* IN *and* CI *are given by Equations* (3) *and* (4) *respectively and n is even.*

1. *Suppose that $(1-\alpha)k_{\mathrm{CI}} < 1/2$. If $n \geq \lceil 2\alpha k_{\mathrm{CI}} \rceil + 1$, then $\sharp\mathcal{I}_{\mathrm{IN}}^{\min} = \lceil 2\alpha k_{\mathrm{CI}} \rceil + 1$. Otherwise, there is no winning strategy for* IN.
2. *Suppose that $(1-\alpha)k_{\mathrm{CI}} > 1/2$. If $n \geq \eta$, then $\sharp\mathcal{I}_{\mathrm{IN}}^{\min} = \eta + 1$. Otherwise, there is no winning strategy for* IN.
3. *When $(1-\alpha)k_{\mathrm{CI}} = 1/2$, then the previous two results hold.*

We now provide an intuition for this result. First, recall that CI wants to obtain $v_i(\theta_i^{\mathrm{F}}) = 0$ for either $i \in \mathcal{I}_{\mathrm{IN}}$, or for $i \in \mathcal{N} \setminus \mathcal{I}_{\mathrm{IN}}$. Second, CI has two possible strategies: either he influences both agent i and some of his neighbors, or he only influences some neighbors of i. The threshold given in Proposition 7 follows from straightforward computations given the previous strategies that CI can use.

In the proposition below, we provide the candidate strategies can be optimal. Some additional properties of $C(\sharp E(g), \sharp\mathcal{I}_{\mathrm{IN}})$ are needed to characterize optimal strategies. Obviously, we restrict our attention to cases where IN has an incentive to influence at least some agents (and possibly form links). Clearly, the complete influence-empty network strategy cannot be optimal for IN, since CI can always influence one agent, say i, and obtain $v_i(\theta_i^{\mathrm{F}}) = 1$.

Proposition 8 *Let the cost function be convex in each of its two arguments, and let n be even.*

1. *Suppose that* $\frac{1}{2} < (1-\alpha)k_{\text{CI}} < \alpha$. *If* IN *has a winning strategy, then the strategies' candidate for being optimal are:* $(\eta, 2k_{\text{CI}}, \mathcal{I}_{\text{IN}})$*-groups-regular network strategies where* $\sharp\mathcal{I}_{\text{IN}} \in [\![\sharp\mathcal{I}_{\text{IN}}^{\min}, n-1]\!]$*, or* η*-regular network strategies.*
2. *Suppose that* $(1-\alpha)k_{\text{CI}} < \frac{1}{2}$. *If* IN *has a winning strategy, then the strategies' candidate for being optimal are:* $(\lceil 2\alpha k_{\text{CI}}\rceil, 2k_{\text{CI}}, \mathcal{I}_{\text{IN}})$*-groups-regular network strategies where* $\sharp\mathcal{I}_{\text{IN}} \in [\![\sharp\mathcal{I}_{\text{IN}}^{\min}, n-1]\!]$*, or* $\lceil 2\alpha k_{\text{CI}}\rceil$*-regular network strategies.*
3. *Suppose that* $(1-\alpha)k_{\text{CI}} > \alpha$. *If* IN *has a winning strategy, then the only optimal candidate strategies are:* $\left(\eta, \eta, \mathcal{I}_{\text{IN}}^{\min}\right)$*-groups-regular network strategies.*

Moreover, when $(1-\alpha)k_{\text{CI}} = \frac{1}{2}$*, points* 1. *and* 2. *hold, and when* $(1-\alpha)k_{\text{CI}} = \alpha$ *points* 2. *and* 3. *hold.*

In the above proposition, it is interesting to note that there are different group regular network strategies that are candidates for being optimal for IN in the first two points, depending on the value of $\sharp\mathcal{I}_{\text{IN}}$. The number of agents influenced by IN differs in these different strategies. These possibilities arise from the fact that, in these strategies, the number of neighbors of $i \in \mathcal{I}_{\text{IN}}$ that IN must influence to get $v_i(\theta_i^{\text{F}}) = 1$ is less than the number of neighbors of $j \in \mathcal{N} \setminus \mathcal{I}_{\text{IN}}$ that IN must influence to get $v_j(\theta_j^{\text{F}}) = 1$. Consequently, the fewer agents IN influences, the more links she has to create. Each of these additional links involves an agent in $\mathcal{I}_{\text{IN}}$ and an agent in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$, and allows IN to prevent agents in $N \setminus \mathcal{I}_{\text{IN}}$ from voting for 1.

5 Concluding Remarks

In this paper, we examine a situation in which two players compete to influence a set of agents, where one of the players can both establish the pattern of interactions between agents and influence them. We investigate how this player should act when faced with an opponent who can counter influence the agents. This type of competition occurs in many social and economic situations where a player, the influencer, has the ability to create links between agents by forming committees, working groups, and so on. More specifically, we study a "zero-sum" type game where the influencer wins *if and only if* she obtains the vote of all the agents. We determine the optimal strategies of the influencer for securing a unanimous vote for 1, given that creating links and influencing agents are costly activities. In the benchmark model, we assume the influencer has the superior technology for influencing the agents. We find that while the optimal strategies depend on costs, they tend to support centralized type structures which are variants of star and core periphery type architectures in addition to the empty network. We then explore the possibility that unplanned links may occur with an exogenously given probability, and provide a condition that preserves the results obtained in the benchmark model.

In the extension section of the paper, we relax the main assumptions. First, we assume that agents interact multiple times before voting. We show that the influencer's optimal strategy now costs less since the influencer can use an appropriately designed network to persuade everyone due to the repeated interactions between agents. Moreover, these networks are quite different from the optimal networks of the benchmark model. In particular, these networks allow the influencer to achieve unanimity among agents by influencing only a single agent, although the process of achieving unanimity can often be lengthy. Second, we assume that instead of unanimity, the influencer only needs the consent of a majority of agents to win the voting game. Not surprisingly, we find that again the influencer's optimal strategy requires fewer resources. Given that the influencer can afford to win by influencing only half the agents, we find that the core-periphery type architectures which are denser networks will no longer be a part of equilibrium strategies. Instead, optimal strategies are based on variants of partial stars or the empty network. As a consequence, the optimal strategies do not need to take the counter-influencer's cost into account and ultimately are not very different from the optimal strategies in the benchmark model. Third, we explore a situation where the counter-influencer is a better influencer than the influencer. We find that the influencer's optimal strategies are very different from those in the benchmark model. Instead of the asymmetric star type architectures, we show that equilibrium architectures are quite symmetric: they are either regular networks or contains regular subnetworks. This dramatic difference in the optimal strategy establishes that the outcome of a game of influence and counter-influence in networks can be largely driven by which player possesses the better influence technology. Hence the players should consider investing in the influence technology to gain additional advantage.

Finally, we believe that the influence competition framework introduced here can be used to study network based models of entities such as lobbies and alliances.

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Appendix A. Model Analysis: IN is the Better Influencer

In order to present the proof of Proposition 1, we introduce two lemmas.

Lemma 2 *Suppose that payoff functions of players* IN *and* CI *are respectively given by Equations* (3) *and* (4)*. Let* $(g, \mathcal{I}_{\text{IN}})$ *be a winning strategy. For every* $i \in \mathcal{I}_{\text{IN}}$*, we have* $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}}) \geq \kappa k_{B1}$*, with* $k_{B1} = k_{B1}(i, g)$*.*

Proof The condition is obvious for isolated agents in $\mathcal{I}_{\text{IN}}$. Consider a non-isolated agent $i \in \mathcal{I}_{\text{IN}}$. A winning strategy requires that $\theta_i^{\text{F}} = 1$. By Equation (2), this is true when $1/2 \leq (1-\alpha) + \alpha\bar{\Theta}_i = 1 - \alpha + \alpha\frac{\sharp(\mathcal{N}_i(g)\cap\mathcal{I}_{\text{IN}})}{\sharp(\mathcal{N}_i(g)\cap\mathcal{I}_{\text{IN}})+k_{B1}}$ for every $i \in \mathcal{N}$. Hence, $\left(\alpha - \frac{1}{2}\right) k_{B1} \leq \frac{1}{2}\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}})$ which leads to the conclusion. □

Lemma 3 *Suppose that payoff functions of players* IN *and* CI *are respectively given by Equations* (3) *and* (4). *Let* $(g, \mathcal{I}_{\text{IN}})$ *belong to a minimal winning strategy. If* $i \in \mathcal{N} \setminus \mathcal{I}_{\text{IN}}$*, then* $\mathcal{N}_i(g) = \{j\}$*, with* $j \in \mathcal{I}_{\text{IN}}$.

Proof Suppose that $(g, \mathcal{I}_{\text{IN}})$ is a minimal winning strategy. First, if $i \in \mathcal{N} \setminus \mathcal{I}_{\text{IN}}$, then i is not isolated. Otherwise, CI chooses to influence agent i, and the strategy is not a winning one, a contradiction. Second, we show that if $i, j \in \mathcal{N} \setminus \mathcal{I}_{\text{IN}}$, then $g_{i,j} = 1$. It is clear that links between agents $i, j \in \mathcal{N} \setminus \mathcal{I}_{\text{IN}}$ does not allow IN to save links between agents in $\mathcal{I}_{\text{IN}}$ and agents in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$. Consequently, IN has no incentive to form links between agents i and j in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$. Similarly, if $i \in \mathcal{N} \setminus \mathcal{I}_{\text{IN}}$ has a unique neighbor who is influenced by IN, then $\theta_i^{\text{F}} = 1$. Again, an additional link between i and another agent in $\mathcal{I}_{\text{IN}}$ is useless and costly and thus not formed in a minimal winning strategy. □

Proof of Proposition 1 First, we establish that $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}}) \leq \lceil \frac{n}{2} \rceil$ for every $i \in \mathcal{I}_{\text{IN}}$. To introduce a contradiction, suppose that there exists $i \in \mathcal{N}$ such that $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}}) \geq \lceil \frac{n}{2} \rceil + 1$. Then, $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}}) \geq \left\lceil \frac{\sharp\mathcal{N}_i(g)}{2} \right\rceil + 1 \geq \frac{\sharp\mathcal{N}_i(g)+2}{2}$. We have $2\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}}) \geq \sharp\mathcal{N}_i(g) + 2 \Rightarrow \sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}}) \geq \sharp\mathcal{N}_i(g) - \sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}}) + 2 \Rightarrow \sharp\mathcal{N}_i^1(g) \geq \sharp\mathcal{N}_i^0(g) + 2 \Rightarrow \sharp\mathcal{N}_i^1(g) \geq \lceil \kappa \sharp\mathcal{N}_i^0(g) \rceil + 2$. Consequently, if $\sharp(\mathcal{N}_i(g) \cap \mathcal{I}_{\text{IN}}) \geq \lceil \frac{n}{2} \rceil + 1$, then IN can decrease $C(\sharp E(g), \sharp\mathcal{I}_{\text{IN}})$ by removing a link and still obtain a winning network, a contradiction.

Second, we establish that a strategy is optimal *if and only if* it is a solution of Program given in Proposition 1. Note that an optimal strategy for IN is a minimal winning strategy. The proof is in two parts.

1. We establish that an optimal strategy $(g^\star, \mathcal{I}_{\text{IN}}^\star)$ is a solution of the program. We know that an optimal strategy $(g^\star, \mathcal{I}_{\text{IN}}^\star)$ for IN has to satisfy the two necessary conditions given in Lemmas 2 and 3, i.e., (Cons. 1) and (Cons. 2). Moreover, an optimal strategy has to minimize the cost incurred by IN. The result follows.

2. We show that a solution of the program $(g^\star, \mathcal{I}_{\text{IN}}^\star)$ is an optimal strategy. Suppose that the solution of the program is not an optimal strategy for IN. Then, there exists a winning strategy $(g, \mathcal{I}_{\text{IN}})$ less costly than $(g^\star, \mathcal{I}_{\text{IN}}^\star)$. Such a pair $(g, \mathcal{I}_{\text{IN}})$ has to violate one of the two constraints, a contradiction by Lemmas 2 and 3. □

We now introduce a lemma needed for the proof of Proposition 2.

Lemma 4 *Let*

$$\bar{x} = \underset{x \in [\![1, \sharp\mathcal{N}]\!]}{\arg\min} \left\{ x \left\lfloor \frac{x-1}{\kappa} \right\rfloor \geq n - x \right\}. \tag{13}$$

Then, either $\bar{x} = \lceil \sqrt{\kappa n} \rceil$ *or* $\bar{x} = \lceil \sqrt{\kappa n} \rceil + 1$.

Proof Consider the real valued function $f : x \mapsto x\left\lfloor \frac{x-1}{\kappa} \right\rfloor - (n - x)$. We seek $\bar{x} \in \mathbb{N}$, the minimal value such that $f(\bar{x}) \geq 0$. Let $w = \sqrt{\kappa n}$. Then, $w\left\lfloor \frac{w-1}{\kappa} \right\rfloor = \sqrt{\kappa n}\left\lfloor \frac{\sqrt{\kappa n}-1}{\kappa} \right\rfloor \leq \sqrt{\kappa n}\left(\frac{\sqrt{\kappa n}}{\kappa} - 1\right) = n - \sqrt{\kappa n} = n - w$ and thus $f(w) \leq 0$. Let $y = \sqrt{\kappa n} + 1$. Then $y\left\lfloor \frac{y-1}{\kappa} \right\rfloor = (\sqrt{\kappa n} + 1)\left\lfloor \frac{\sqrt{\kappa n}}{\kappa} \right\rfloor > (\sqrt{\kappa n} + 1)\left(\frac{\sqrt{\kappa n}}{\kappa} - 1\right) = n + \frac{\sqrt{\kappa n}}{\kappa} - \sqrt{\kappa n} - 1 = n + \frac{\sqrt{\kappa n}}{\kappa} - y > n - y$. Thus, $f(y) > 0$. Since f is strictly increasing any value $x \geq y$ satisfies $f(x) \geq 0$. Similarly, any value $x < w$ satisfies $f(x) < 0$. The conclusion follows from the fact that $\bar{x}$ is the smallest integer for which f is non-negative. □

Proof of Proposition 2 In a winning strategy, and hence in an optimal strategy, there is at least one agent in $\mathcal{I}_{\text{IN}}$. By Proposition 1, an optimal strategy satisfies (Cons. 1) and (Cons. 2). Since for every $i \in \mathcal{N}$, $k_{B1}(i, g) = \min\{k_{\text{CI}}, \sharp\mathcal{N}_i(g) \setminus \mathcal{I}_{\text{IN}}\}$ there are two possible cases.

1. Consider an optimal strategy where $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)) \leq k_{\text{CI}}$ for every $i \in \mathcal{I}_{\text{IN}}$. Then, from (Cons. 1), we have $\sharp\mathcal{N}_i^1(g) \geq \kappa\, k_{B_1}$. Thus, $\sharp\mathcal{I}_{\text{IN}} - 1 \geq \sharp\mathcal{N}_i^1(g) \geq \kappa\, \sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))$. So, $\frac{\sharp\mathcal{I}_{\text{IN}}-1}{\kappa} \geq \sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))$ and finally $\left\lfloor \frac{\sharp\mathcal{I}_{\text{IN}}-1}{\kappa} \right\rfloor \geq \sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))$ since the right-hand side is an integer. Summing over all agents in $\mathcal{I}_{\text{IN}}$, we obtain $\sharp\mathcal{I}_{\text{IN}}\left\lfloor \frac{\sharp\mathcal{I}_{\text{IN}}-1}{\kappa} \right\rfloor \geq \sum_{i \in \mathcal{I}_{\text{IN}}} \sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))$. Next, from (Cons. 2), we have $n - \sharp\mathcal{I}_{\text{IN}} = \sum_{i \in \mathcal{I}_{\text{IN}}} \sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))$ and thus $\sharp\mathcal{I}_{\text{IN}}\left\lfloor \frac{\sharp\mathcal{I}_{\text{IN}}-1}{\kappa} \right\rfloor \geq n - \sharp\mathcal{I}_{\text{IN}}$. By definition, $\bar{x}$ is the minimum integer that satisfies the above inequality. From the online appendix OA.1, a winning strategy with $\sharp\mathcal{I}_{\text{IN}} = \bar{x}$ agents exists: it is a $(\kappa, \mathcal{I}_{\text{IN}})$-influence-MQC network strategy.

2. There exists $i \in \mathcal{I}_{\text{IN}}$ such that $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)) > k_{\text{CI}}$. Since $\sharp\mathcal{N}_i^1(g) \geq \kappa k_{B1} = \kappa k_{\text{CI}}$, necessarily $\sharp\mathcal{I}_{\text{IN}} \geq \kappa k_{\text{CI}} + 1$. Consider a partial-star-network with the center being influenced by IN as well as $\lceil \kappa k_{\text{CI}} \rceil$ peripheral agents. It is a winning strategy satisfying the minimal number of influenced agents. It is possible to construct this partial-star network if and only if $n \geq \kappa \lceil k_{\text{CI}} \rceil + 1$.

□

Proof of Proposition 3 First, we deal with the number of links between agents in $\mathcal{I}_{\text{IN}}$. There are two possible cases here.

1. Consider a minimal winning strategy where $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)) \leq k_{\text{CI}}$ for every $i \in \mathcal{I}_{\text{IN}}$. From (Cons. 1), we have $\sharp\mathcal{N}_i^1(g) \geq \kappa k_{B1}$ for $i \in \mathcal{I}_{\text{IN}}$. Thus, $\sharp\mathcal{N}_i^1(g) \geq \kappa\, \sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))$. By summing over all agents in $\mathcal{I}_{\text{IN}}$, we have $\sum_{i\in\mathcal{I}_{\text{IN}}} \sharp\mathcal{N}_i^1(g) \geq \kappa \sum_{i\in\mathcal{I}_{\text{IN}}}(\sharp(\mathcal{N}_i(g)\setminus \mathcal{N}_i^1(g))$. Note that $\sum_{i\in\mathcal{I}_{\text{IN}}} \sharp(\mathcal{N}_i(g)\setminus\mathcal{N}_i^1(g))$ represents the total number of links connecting agents in $\mathcal{I}_{\text{IN}}$ to agents in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$. Thus, by (Cons. 2), $\sum_{i\in\mathcal{I}_{\text{IN}}}(\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g))) = n - \sharp\mathcal{I}_{\text{IN}}$. Therefore, the number of links in $g[\mathcal{I}_{\text{IN}}]$ is at least $\left\lceil \frac{\kappa(n-\sharp\mathcal{I}_{\text{IN}})}{2} \right\rceil$.

2. Consider a minimal winning strategy where there is $i \in \mathcal{I}_{\text{IN}}$ such that $\sharp(\mathcal{N}_i(g)\setminus\mathcal{N}_i^1(g)) > k_{\text{CI}}$. By Proposition 1, we have $\sharp\mathcal{N}_i^1(g) \leq \lceil \frac{n}{2} \rceil$. Moreover, since $\sharp\mathcal{N}_i^1(g) \geq \kappa\, k_{B1} = \kappa\, k_{\text{CI}}$, it must be that $g[\mathcal{I}_{\text{IN}}]$ contains at least $\lceil \kappa\, k_{\text{CI}} \rceil$ links involving agent i.

From (Cons. 2), the number of links joining agents in $\mathcal{I}_{\text{IN}}$ to agents in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$ is exactly $n - \sharp\mathcal{I}_{\text{IN}}$ in a minimal winning strategy. The result follows. □

Proof of Theorem 1 Recall that an optimal strategy is a winning strategy where IN cannot remove a link without forming additional links given the set of agents she influences. Consider a winning strategy such that $\sharp\mathcal{I}_{\text{IN}} = n$. Then, the unique winning strategy is the complete influence-empty network. When $\sharp\mathcal{I}_{\text{IN}} < n$, there are two possibilities:

1. Consider strategies where $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)) \leq k_{\text{CI}}$ for every $i \in \mathcal{I}_{\text{IN}}$. We establish that $(\kappa, \mathcal{I}_{\text{IN}})$-influence-MQC network strategies with $\sharp\mathcal{I}_{\text{IN}} \geq \sharp\mathcal{I}_{\text{IN}}^{\min}$ are optimal. First, they are winning strategies since they satisfy (Cons. 1), (Cons. 2), and Proposition 2. Second, the number of links in $(\kappa, \mathcal{I}_{\text{IN}})$-influence-MQC network strategies satisfies the bound given in the proof of Proposition 3 part 1.

2. Consider strategies where there is $i \in \mathcal{I}_{\text{IN}}$ such that $\sharp(\mathcal{N}_i(g) \setminus \mathcal{N}_i^1(g)) > k_{\text{CI}}$. We establish that $(p, \mathcal{I}_{\text{IN}})$-influence-partial-star strategies, with $\sharp\mathcal{I}_{\text{IN}} \geq \sharp\mathcal{I}_{\text{IN}}^{\min}$ and $p \geq \lceil \kappa k_{\text{CI}} \rceil$ are optimal. First, they are winning strategies since they satisfy (Cons. 1), (Cons. 2), and Proposition 2. Moreover, the number of links in $(p, \mathcal{I}_{\text{IN}})$-influence-partial-star strategies satisfies the bound given in the proof of Proposition 3 part 2.

□

Appendix B. Extensions

Appendix B.1. When Agents Interact for Multiple Periods

Proof of Propostion 5 We prove successively, the two parts of the proposition.

1. Suppose that IN influences one agent and g is not connected. Then, network g contains at least two distinct components. The previous reasoning can be repeated for agents within the component where no agents have been influenced by IN for obtaining a contradiction. Consequently, g is connected, i.e., it contains at least $n-1$ links. It is sufficient to show that if IN influences only one agent, say i, and builds an acyclic connected network, then for every finite T, we have $\mathcal{N}(1, \boldsymbol{v}) \neq \mathcal{N}$. Recall that $k_{\text{CI}} = n$, so it is profitable for CI to target each agent to ensure that some of them do not vote 1 at T. Therefore, IN's strategy must prevent this specific strategy of CI from leading to a situation where some agents do not vote 1 in a finite period. When CI influences all agents, we have $v_i^t \in \{0, 1\}$ for every $t \leq T$. To establish a contradiction, suppose there exists a finite period T such that $\mathcal{N}(1, \boldsymbol{v}) = \mathcal{N}$. Then, at period $t \leq T$, there are two agents j^t and k^t *linked* in g who vote 1. Since j^t and k^t are linked in an acyclic network, there is a unique path between i and j^t and a unique path between i and k^t. Moreover, $d(i, j^t; g)$ is even *if and only if* $d(i, k^t; g)$ is odd. Since g is an acyclic connected network, there are two distinct agents j^{t-1} and k^{t-1} respectively neighbors of j^t and k^t such that $v_{j^{t-1}}^{t-1} = v_{k^{t-1}}^{t-1} = 1$. Clearly, $d(i, j^{t-1}; g)$ is even *if and only if* $d(i, k^{t-1}; g)$ is odd. By reiterating this process, any period τ where there are two agents j^τ and k^τ such that $v_{j^\tau}^{t-1} = v_{k^\tau}^{t-1} = 1$, with $d(i, j^\tau; g)$ is even *if and only if* $d(i, k^\tau; g)$ is odd, requires that at period $\tau - 1$, there are two agents $j^{\tau-1}$ and $k^{\tau-1}$ such that $v_{j^{\tau-1}}^{t-1} = v_{k^{\tau-1}}^{t-1} = 1$, with $d(i, j^{\tau-1}; g)$ is even *if and only if* $d(i, k^{\tau-1}; g)$ is odd. This process stops at $t = 0$ where such agents do not exist, a contradiction.

2. Suppose that IN builds a i_1-triangle network and influences agent i_1. As in the previous point, we consider that CI influences all the agents. At period 1, we have $\mathcal{V}_i^1(g; 1) \geq \mathcal{V}_i^0(g; 1)$ for $i \in \{i_1, i_2, i_3\}$. Consequently, at period $t \geq 2$, $\mathcal{V}_i^1(g; t) \geq \mathcal{V}_i^0(g; t)$, and $v_i^t = 1$ for $i \in \{i_1, i_2, i_3\}$. Next, by construction of the triangle network, for agent at distance one of agent i_1, $\ell_1 \in \mathcal{N} \setminus \{2, 3\}$, we have $\mathcal{V}_{\ell_1}^1(g; t) \geq \mathcal{V}_{\ell_1}^0(g; t)$, for $t \geq 2$ and $v_{\ell_1}^t = 1$ for $t \geq 3$. By reiterating this argument for agent ℓ_d at distance d of agent i_1, we

have $\mathcal{V}^1_{\ell_d}(g;t) \geq \mathcal{V}^0_{\ell_d}(g;t)$, for $t \geq d+1$ and $v^t_{\ell_d} = 1$ for $t \geq d+2$. Since the population of agents is finite and the network g is connected, the distance between agent 1 and any other agent is also finite. Therefore, all agents vote for 1 after a finite number of periods.

□

B.2. Counter-influencer is the Better Influencer

We establish Propositions 7 and 8. First we begin with a lemma. Let $\mathcal{N}_i(g, \mathcal{I}_{\text{IN}})$ be the set of neighbors of agent i who are influenced by IN. Recall that $\eta = \left\lceil \frac{\alpha(k_{\text{CI}}-1)}{\alpha-\frac{1}{2}} \right\rceil$.

Lemma 5 *Suppose that payoff functions of players* IN *and* CI *are respectively given by Equations* (3) *and* (4) *and n is even. Let $\mathcal{N}_i(g, \mathcal{I}_{\text{IN}})$ be optimal for* IN.

1. *Suppose that $(1-\alpha)k_{\text{CI}} \leq \frac{1}{2}$. If $\sharp\mathcal{I}_{\text{IN}} \geq \lceil 2\alpha k_{\text{CI}} \rceil + 1$, then*

$$\sharp\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) = \begin{cases} \lceil 2\alpha k_{\text{CI}} \rceil & \text{if } i \in \mathcal{I}_{\text{IN}}, \\ 2k_{\text{CI}} & \text{otherwise}. \end{cases}$$

   If $\sharp\mathcal{I}_{\text{IN}} < \lceil 2\alpha k_{\text{CI}} \rceil + 1$, then $\sharp\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) = \emptyset$.

2. *Suppose that $\frac{1}{2} < (1-\alpha)k_{\text{CI}} \leq \alpha$. If $\sharp\mathcal{I}_{\text{IN}} \geq \eta + 1$, then*

$$\sharp\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) = \begin{cases} \eta & \text{if } i \in \mathcal{I}_{\text{IN}}, \\ 2k_{\text{CI}} & \text{otherwise}. \end{cases}$$

   If $\sharp\mathcal{I}_{\text{IN}} < \eta + 1$, then $\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) = \emptyset$.

3. *Suppose that $(1-\alpha)k_{\text{CI}} > \alpha$. If $\sharp\mathcal{I}_{\text{IN}} \geq \eta + 1$, then*

$$\sharp\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) = \eta, \text{ for every } i \in \mathcal{N}.$$

   If $\sharp\mathcal{I}_{\text{IN}} < \eta + 1$, then $\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) = \emptyset$.

Proof Note that if g is non-empty and $\mathcal{I}_{\text{IN}} \neq \emptyset$, then for every $i \in \mathcal{N}$, $\sharp\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) > k_{\text{CI}}$, otherwise IN obtains a strictly negative payoff. CI has two possibilities concerning the agents he influences when he wants to obtain $v_i(\theta^{\text{F}}_i) = 0$: either (i) he influences only the neighbors of agent i, or (ii) he influences both agent i and his neighbors. We present successively the case where CI wants to obtain $v_i(\theta^{\text{F}}_i) = 0$ for $i \in \mathcal{I}_{\text{IN}}$, and then for $i \notin \mathcal{I}_{\text{IN}}$.

(a) Suppose that CI wants to obtain $v_i(\theta_i^{\mathrm{F}}) = 0$ for $i \in \mathcal{I}_{\mathrm{IN}}$. (i) When CI influences only the neighbors of agent i, IN has to ensure that the following inequality holds: $1-\alpha+\alpha\frac{\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}})-k_{\mathrm{CI}}}{\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}})} \geq 1/2$. It follows that $\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq 2\alpha k_{\mathrm{CI}}$. (ii) When CI influences agent i and his neighbors, IN has to ensure that the following inequality holds: $\frac{\alpha}{\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}})}(\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) - k_{\mathrm{CI}}+1) \geq 1/2$, that is $\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq \frac{\alpha(k_{\mathrm{CI}}-1)}{\alpha-1/2}$.
Consequently, $v_i(\theta_i^{\mathrm{F}}) = 1 \Leftrightarrow \sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq \max\{\eta, 2\alpha k_{\mathrm{CI}}\}$. We have $\eta \geq 2\alpha k_{\mathrm{CI}} \Leftrightarrow (1-\alpha)k_{\mathrm{CI}} \geq \frac{1}{2}$. It follows that

$$v_i(\theta_i^{\mathrm{F}}) = 1 \Leftrightarrow \left[(1-\alpha)k_{\mathrm{CI}} \leq \frac{1}{2} \text{ and } \sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq 2\alpha k_{\mathrm{CI}}\right].$$

(b) Suppose that CI wants to obtain $v_i(\theta_i^{\mathrm{F}}) = 0$ for $i \notin \mathcal{I}_{\mathrm{IN}}$. (i) When CI influences only the neighbors of agent i, IN has to ensure that the following inequality holds: $\frac{\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}})-k_{\mathrm{CI}}}{\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}})} \geq 1/2$, that is $\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq 2k_{\mathrm{CI}}$. (ii) When CI influences both i and his neighbors inequality $\alpha\frac{\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}})-k_{\mathrm{CI}}+1}{\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}})} \geq 1/2$ holds, that is $\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq \eta$. We have $\eta \geq 2k_{\mathrm{CI}} \Leftrightarrow (1-\alpha)k_{\mathrm{CI}} \geq \alpha$. It follows that

$$v_i(\theta_i^{\mathrm{F}}) = 1 \Leftrightarrow \left[(1-\alpha)k_{\mathrm{CI}} \geq \alpha \text{ and } \mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq 2k_{\mathrm{CI}}\right].$$

Because $\alpha \in (1/2, 1]$, we have to examine three intervals for completing the analysis.

1. Suppose $(1-\alpha)k_{\mathrm{CI}} \leq \frac{1}{2}$. Then necessarily inequality $\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq 2\alpha k_{\mathrm{CI}}$ holds when CI influences $i \in \mathcal{I}_{\mathrm{IN}}$ and inequality $\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq 2k_{\mathrm{CI}}$ holds when CI influences $i \notin \mathcal{I}_{\mathrm{IN}}$. IN has to choose the lowest number of neighbors of i which satisfies the previous inequalities:

$$\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) = \begin{cases} \lceil 2\alpha k_{\mathrm{CI}} \rceil & \text{if } i \in \mathcal{I}_{\mathrm{IN}}, \\ 2k_{\mathrm{CI}} & \text{otherwise.} \end{cases} \tag{14}$$

Note that when $(1-\alpha)k_{\mathrm{CI}} \leq \frac{1}{2}$, $\lceil 2\alpha k_{\mathrm{CI}} \rceil + 1 \geq 2k_{\mathrm{CI}}$. If $\sharp\mathcal{I}_{\mathrm{IN}} \geq \lceil 2\alpha k_{\mathrm{CI}} \rceil + 1$, then (14) holds. Otherwise, IN cannot satisfy the necessary condition: $\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq \lceil 2\alpha k_{\mathrm{CI}} \rceil$ for $i \in \mathcal{I}_{\mathrm{IN}}$.

2. Suppose $\frac{1}{2} \leq (1-\alpha)k_{\mathrm{CI}} \leq \alpha$. Then necessarily inequality $\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq \eta$ holds when CI influences $i \in \mathcal{I}_{\mathrm{IN}}$ and $\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) \geq 2k_{\mathrm{CI}}$ holds when CI influences $i \notin \mathcal{I}_{\mathrm{IN}}$. IN has to choose the lowest number of neighbors of i which satisfies the previous inequalities:

$$\sharp\mathcal{N}_i(g,\mathcal{I}_{\mathrm{IN}}) = \begin{cases} \eta & \text{if } i \in \mathcal{I}_{\mathrm{IN}}, \\ 2k_{\mathrm{CI}} & \text{otherwise.} \end{cases} \tag{15}$$

Note that when $(1-\alpha)k_{\text{CI}} \geq \frac{1}{2}$, $\eta + 1 \geq 2k_{\text{CI}}$. If $\sharp\mathcal{I}_{\text{IN}} \geq \eta + 1$, then (15) holds. Otherwise, IN cannot satisfy the following necessary condition: $\sharp\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) \geq \eta$ for $i \in \mathcal{I}_{\text{IN}}$.

3. Suppose $(1-\alpha)k_{\text{CI}} > \alpha$. Then necessarily inequality $\sharp\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) \geq \eta$ holds when CI influences $i \in \mathcal{I}_{\text{IN}}$ or when CI influences $i \notin \mathcal{I}_{\text{IN}}$. IN has to choose the lowest number of neighbors which satisfies the previous inequality:

$$\sharp\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) = \eta. \tag{16}$$

If $\sharp\mathcal{I}_{\text{IN}} \geq \eta + 1$, then (16) holds. Otherwise, IN cannot satisfy $\sharp\mathcal{N}_i(g, \mathcal{I}_{\text{IN}}) = \eta$ for $i \in \mathcal{I}_{\text{IN}}$.

□

Proof of Proposition 7 The proof is straightforward from Lemma 5. Indeed for every value of k_{CI}, Lemma 5 provides a necessary condition for the value of $\sharp\mathcal{I}_{\text{IN}}^{\min}$:

1. If $(1-\alpha)k_{\text{CI}} \leq \frac{1}{2}$, then there is a winning strategy if $\sharp\mathcal{I}_{\text{IN}}^{\min} = \lceil 2\alpha k_{\text{CI}} \rceil + 1$;
2. if $\frac{1}{2} < (1-\alpha)k_{\text{CI}} \leq \alpha$, then $\sharp\mathcal{I}_{\text{IN}}^{\min} = \eta + 1$;
3. if $(1-\alpha)k_{\text{CI}} > \alpha$, then $\sharp\mathcal{I}_{\text{IN}}^{\min} = \eta + 1$.

□

Proof of Proposition 8 Due to Proposition 7, we know the minimal size of $\sharp\mathcal{I}_{\text{IN}}$ for a winning strategy, $\sharp\mathcal{I}_{\text{IN}}^{\min}$. It follows that because of the convexity in each of its two arguments of the cost function, in a winning strategy, $\sharp\mathcal{I}_{\text{IN}} \in [\![\sharp\mathcal{I}_{\text{IN}}^{\min}, n]\!]$. Moreover, by Lemma 5, we know conditions that a winning strategy has to satisfy. Clearly, when $(1-\alpha)k_{\text{CI}} \leq \frac{1}{2}$, $(\lceil 2\alpha k_{\text{CI}} \rceil, 2k_{\text{CI}}, \sharp\mathcal{I}_{\text{IN}})$-groups-regular network strategies, with $\sharp\mathcal{I}_{\text{IN}} \in [\![\sharp\ \mathcal{I}_{\text{IN}}^{\min}, n-1]\!]$, and $\lceil 2\alpha k_{\text{CI}} \rceil$-regular network strategies allow to satisfy conditions given in Lemma 5 and minimize the number of links given the size of $\mathcal{I}_{\text{IN}}$. Similarly, by using Lemma 5, we obtain the two other parts of the proposition. Note that the only two types of strategies candidate for being optimal are $(\eta, \eta, \mathcal{I}_{\text{IN}})$-groups-regular network strategy where $\sharp\mathcal{I}_{\text{IN}} \in [\![\sharp\mathcal{I}_{\text{IN}}^{\min}, n-1]\!]$, and η-regular network strategies. Obviously, the number of links in all these networks is the same, since the degree of each agent in each of these networks is η. Consequently, only $\left(\eta, \eta, \mathcal{I}_{\text{IN}}^{\min}\right)$-groups-regular network strategies are optimal for IN since the number of agents influenced by IN is minimal. □

6 Online Appendix

OA.1. Existence of $(q, \mathcal{Y})$-MQC Network

To show the existence of a $(q, \mathcal{Y})$-MQC network for every $q \leq 1$, it is sufficient to establish that the set of $(q, \mathcal{Y})$-quasi-core-periphery networks is non-empty. Indeed, since this set is finite, it admits at least one minimal element with regard to the number of links. Since $q \leq 1$, a $(1, \mathcal{Y})$-MQC is a $(q, \mathcal{Y})$-quasi-core-periphery network. The existence of $(1, \mathcal{Y})$-MQC implies that the set of $(q, \mathcal{Y})$-MQC networks is non-empty. Let us construct a process that leads to a $(1, \mathcal{Y})$-MQC network g.

1. Start with the empty network.
2. While $\sharp E(g[\mathcal{Y}]) = \frac{1}{2}\sum_{i,j \in \mathcal{Y}} g_{i,j}$ is such that $\sharp E(g[\mathcal{Y}]) < \left\lceil \frac{n - \sharp\mathcal{Y}}{2} \right\rceil$, take two unlinked agents $i, j \in \mathcal{Y}$ such that $\sharp\mathcal{N}_i(g), \sharp\mathcal{N}_j(g) \in \min_{\ell \in \mathcal{Y}}\{\sharp\mathcal{N}_\ell(g)\}$, do $i\,j \in E(g)$. When $\sharp E(g[\mathcal{Y}]) = \left\lceil \frac{n - \sharp\mathcal{Y}}{2} \right\rceil$ go to 3.
3. While there exists $j \in \mathcal{N}\backslash\mathcal{Y}$, with $\mathcal{N}_j(g) = \emptyset$, take $i \in \mathcal{Y}$ with $\sum_{\ell \in \mathcal{Y}} g_{i,\ell} \geq \sum_{\ell \in \mathcal{N}\backslash\mathcal{Y}} g_{i,\ell} + 1$, do $i\,j \in E(g)$. Stop.

OA.2. Allowing for Random Links

OA.2.1. Allowing for Random Links: Proposition 4

Proof of Proposition 4 Let $\mathbb{P}^W(g)$ be the probability to obtain a winning network from g, and $\mathbb{P}_i^W(g)$ be the probability that agent i satisfies (Cons. 1 & 2) after some links have been formed by Nature. Note that $\mathbb{P}^W(g) = \prod_{i \in \mathcal{N}} \mathbb{P}_i^W(g)$ since every agent has to satisfy (Cons. 1 & 2) in a winning network.

First, we provide a lower bound for the expected payoff associated with a network, say g^{mw}, which is a minimal winning network before Nature forms links. We have $\mathbb{P}^W(g^{mw}) = \prod_{i \in \mathcal{N}} \mathbb{P}_i^W(g^{mw}) \geq \prod_{i \in \mathcal{N}}(1-\omega)^n = n(1-\omega)^n \geq (1-\omega)^{n^2} = ((1-\omega)^n)^2 \geq (1-n\omega)^2 \geq (1 - \frac{c_L}{4})^2$. The first inequality follows the fact that if Nature does not form any links, then the realization of g^{mw} is a winning network. The third and the last inequalities follow the assumption that $\omega \leq \frac{c_L}{4n}$ and the fact that $\frac{c_L}{4n} \leq \frac{1}{n}$. We conclude that $\mathbb{P}^W(g^{mw}) \geq 1 - 2\frac{c_L}{4} = 1 - \frac{c_L}{2}$.

We now establish that IN has no incentive to build a non-minimal winning network, g^w, instead of g^{mw}. Recall that the set $\mathcal{I}_{\text{IN}}$ is given. The difference between the expected payoff of g^w and

g^{mw} is bounded by: $1-\sharp E(g^w)c_L-(1-\frac{c_L}{2}-c_L\sharp E(g^{mw}))=(\frac{1}{2}-(\sharp E(g^w)-\sharp E(g^{mw})))c_L<0$ since $\sharp E(g^w)-\sharp E(g^{mw})\geq 1$.

Finally, we establish that In has no incentive to build a network, say g^ℓ, which is non-winning before Nature forms links. For every agent $i\in\mathcal{N}$ for which (Cons. 1 & 2) do not hold in g^ℓ, Nature has to form at least b_i links in order to obtain agent i satisfies (Cons. 1 & 2) in a winning network g^w. Let $M(\ell)$ be the minimal set of links that allows g^ℓ to be a winning network, i.e., there is no set of links with lower cardinality that allows to obtain a winning network. Clearly, $m(\ell)=\sharp M(\ell)\geq 1$. Similarly, let $\mathcal{S}(\ell)$ be the set of agents that are involved in links in $M(\ell)$. We have $\mathbb{P}^W(g^\ell)\leq\sum_{k=b_i}^{B}\binom{B}{k}\omega^k$, with $i\in\mathcal{S}_\ell$ and where $B=\sharp\mathcal{I}_{\text{IN}}-\sharp\mathcal{N}_i^1(g^\ell)$. Note that $\binom{B}{k}\omega^k=B\omega\frac{(B-1)\omega}{2}\dots\frac{(B-k)\omega}{k}\leq(B\omega)^k$. Moreover, $\sum_{k=b_i}^{B}(B\omega)^k=(B\omega)^{b_i}\frac{(1-(B\omega)^{B+1-b_i})}{1-B\omega}\leq\frac{(B\omega)^{b_i}}{1-B\omega}\leq\frac{B\omega}{1-B\omega}$ since $b_i\geq 1$ and $B\omega<1$. Since $\omega<\frac{c_L}{4n}$ and $B<n$, $\frac{B\omega}{1-B\omega}<\frac{B\frac{c_L}{4n}}{1-B\frac{c_L}{4n}}\leq\frac{nc_L}{4n-nc_L}<\frac{c_L}{2}$. We conclude that $\mathbb{P}^W(g^\ell)\leq\frac{c_L}{2}$. We now compute the difference between the minimal expected payoff of In with g^{mw} and the maximal one with g^ℓ: $1-\frac{c_L}{2}-\sharp E(g^{mw})c_L-(\frac{c_L}{2}-(\sharp E(g^{mw})-m(\ell))c_L)>1-c_L-\left(\frac{n(n-1)}{2}-1\right)c_L=1-\frac{n(n-1)}{2}c_L\geq 0$. □

OA.2.2. Allowing for Random Links: Example

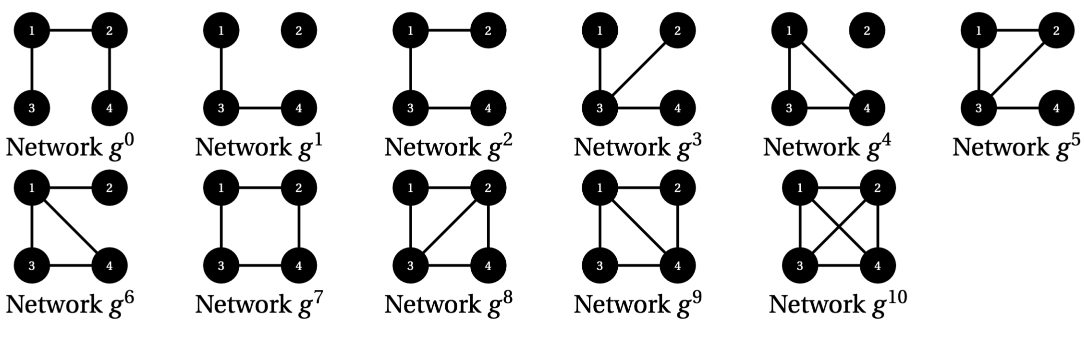

Figure 6: Networks for the Probabilistic Model when $\mathcal{N}=[\![1,4]\!]$

We now illustrate the probabilistic case in a specific situation where $\alpha=1$, $\mathcal{N}=[\![1,4]\!]$, thus $\sharp\mathcal{I}_{\text{IN}}^{\min}=2$. Clearly, $\sharp\mathcal{I}_{\text{IN}}\in[\![2,4]\!]$ and In always obtains $1-4c_{\text{IN}}$ when she influences 4 agents. Note that In does not form any links between agents who are not influenced by her. Let us explore in turn the cases where $\sharp\mathcal{I}_{\text{IN}}=2$ and $\sharp\mathcal{I}_{\text{IN}}=3$ and provide the probability of achieving a unanimous vote for 1.

1. $\sharp\mathcal{I}_{\text{IN}} = 2$, say $\mathcal{I}_{\text{IN}} = \{1, 2\}$. Note that there are only two networks where all agents vote 1 – up to a relabeling of agents: g^0 and g^7 in Figure 6. Suppose IN has formed no links. Then, the probability that g^0 or g^7 occur is $2\omega^3(1-\omega)^2$. When IN forms one link, she has two possibilities: either she forms a link between two agents in $\mathcal{I}_{\text{IN}}$, or she forms a link between an agent in $\mathcal{I}_{\text{IN}}$ and an agent in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$. The first case leads to a probability of obtaining g^0 or g^7 equal to $2\omega^2(1-\omega)^2$, and the second case leads to a probability of obtaining g^0 or g^7 equal to $\omega^2(1-\omega)^2$. Consequently, IN always has an incentive to form a link between two agents she influences. When IN forms two links, she has two possibilities: either she forms a link between two agents in $\mathcal{I}_{\text{IN}}$ and one link between an agent in $\mathcal{I}_{\text{IN}}$ and an agent in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$, or both links are between an agent in $\mathcal{I}_{\text{IN}}$ and an agent in $\mathcal{N} \setminus \mathcal{I}_{\text{IN}}$. In both cases the probability of obtaining g^0 or g^7 is equal to $\omega(1-\omega)^2$. Next, when IN forms three links, the probability that g^0 or g^7 occurs is $(1-\omega)^2$. When IN forms g^7, network g^0 cannot occur and the probability that the realized network is g^7 is $(1-\omega)^2$. Thus, IN faces the same probability of success forming g^0 as forming g^7, while g^0 allows her to save a costly link. Moreover, if IN forms more than 4 links, then she cannot obtain a winning realization given $\sharp\mathcal{I}_{\text{IN}} = 2$. Consequently, it is suboptimal for IN to form more than 3 links.

2. $\sharp\mathcal{I}_{\text{IN}} = 3$, say $\mathcal{I}_{\text{IN}} = \{1, 2, 3\}$. We draw in Figure 6 the different networks where all agents vote 1 when $\mathcal{I}_{\text{IN}} = \{1, 2, 3\}$ up to a relabeling of agents. By using similar arguments as in the previous point and the list of networks g^1 to g^{10} we obtain the following results.[31] When IN forms no links the probability that all agents vote 1 is $\omega^6 + 6\omega^2(1-\omega)^2 + 6\omega^4(1-\omega)$. When IN forms 1 link the probability that all agents vote 1 is $\omega + \omega(1-\omega)$. When IN forms 2 links the probability that all agents vote 1 is: $\max\{1-\omega(1-\omega)^2, 1-(1-\omega)^3\}$. When IN forms 3 links, she can ensure to obtain that all agents vote 1 with network g^2. It is also possible when she forms 4, 5 or 6 links.

Let us now provide the optimal strategies of IN for some specific sets of parameters. More precisely, we assume that $N = [\![1, 4]\!]$. We define the following strategies for IN: $S_1 : \mathcal{I}_{\text{IN}} = \{1, 2\}, E(g) = \{12, 13, 24\}$, $S_2 : \mathcal{I}_{\text{IN}} = \{1, 2, 3\}, E(g) = \{12, 13, 34\}$, and $S_3 : \mathcal{I}_{\text{IN}} = \{1, 2, 3\}, E(g) = \{12, 13\}$, $S_4 : \mathcal{I}_{\text{IN}} = \{1, 2, 3\}, E(g) = \emptyset$, and $S_5 : \mathcal{I}_{\text{IN}} =$

[31] Here we indicate the probability associated with the strategy of IN that maximizes the probability of getting all agents to vote 1.

$\{1, 2, 3, 4\}, E(g) = \emptyset$. In the following table, we provide an optimal strategy for IN up to a relabeling of agents for several value of ω and c_L given that $c_{\text{IN}} = 0.07$.

ω \ c_L	1/1000	1/100	11/100
1/1000	S_1	S_1	S_5
87/100	S_2	S_3	S_4
99/100	S_4	S_4	S_4

Let us provide some observations based on these examples.

1. When the probability of random links is very low as described in Proposition 4, the result of Theorem 1 is preserved. Therefore, the optimal strategy of IN depends on the relative cost of c_L and c_{IN}.

2. When the probability of random links is very high, then the optimal strategy is to play S_4 where IN forms no links, and influences enough agents to ensure a unanimous vote for 1 in the complete network. In this case, IN will incur lower costs than in the benchmark model if c_{IN} is sufficiently low relative to c_L.

3. When the probability of random links is moderate, some intermediate strategies, where IN influences a number of agents in $[\![\mathcal{I}_{\text{IN}}^{\min} + 1, n - 1]\!]$ become optimal. In particular, in S_3, IN influences 3 agents, $3 > \sharp\mathcal{I}_{\text{IN}}^{\min}$. Moreover, the number of links and the number of agents IN influences depend on the relative cost of c_L and c_{IN}.