On the existence of EFX allocations for goods*

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Abstract

We study a fair division problem involving the allocation of a finite set of indivisible goods among a group of agents, where each agent has a valuation function defined over all possible bundles of goods. An allocation is said to be *envy-free up to any good* (EFX) if no agent envies another after the removal of any single good from the other's allocated bundle. A valuation function is termed *set monotonic* if it does not decrease with the addition of goods (i.e., the valuation of every strict superset of a bundle is at least as high), and *size monotonic* if it does not decrease with the increase in the cardinality of the bundle. We prove two results concerning the existence of EFX allocations under valuation domains defined by set monotonicity and size monotonicity. While the results are somewhat technical in nature, the following corollary serves as a notable and illustrative special case: An EFX allocation always exists when any two agents have arbitrary set monotonic valuation functions, and the remaining $n-2$ agents have arbitrary size monotonic valuation functions.

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1. Introduction

The problem of fairly allocating indivisible goods among multiple agents lies at the heart of fair division theory. In this work, we explore the existence of fair allocations under a compelling envy-based fairness notion, namely, *envy-freeness up to any good* (EFX). Despite significant interest, the existence of EFX allocations remains a central open question in algorithmic game theory.

1.1 Background and Motivation

We consider a finite set E of m indivisible goods to be allocated among a set N of $n \geq 2$ agents. Each agent is endowed with a valuation function over bundles of goods. A valuation function is said to be *set monotonic* if the valuation of a bundle weakly increases when more goods are added, and *size monotonic* if the valuation (weakly) increases with the number of goods in the bundle. Clearly, size monotonicity implies set monotonicity, but not vice versa.

An allocation is said to be *envy-free* (EF) if no agent prefers another agent's bundle over their own. However, EF allocations may not exist in general; for instance, with only one good and two agents, envy-freeness is unachievable. To address this, Caragiannis et al. (2019) proposed the notion of EFX allocations, where an agent does not envy another agent after the hypothetical removal of *any* single good from the other's bundle.

1.2 Related Work

The question of whether envy-free up to any item (EFX) allocations exist has been the subject of significant attention in the fair division literature. A number of positive existence results have been established under specific structural assumptions on agents' valuation functions.

Plaut and Roughgarden (2020) showed that EFX allocations are guaranteed to exist when all agents have identical, set monotonic valuation functions, as well as in the two-agent case. Extending this, Akrami et al. (2023) demonstrated the existence of EFX allocations for three agents when at least one agent has an MMS-feasible valuation function (see Definition 3.1), while the remaining agents may have arbitrary set monotonic valuations. Also, Mahara (2023) proved that EFX allocations exist for any number of agents as long as there are at most two distinct additive set monotonic valuation functions among them.

In a different vein, Amanatidis et al. (2021) proved the existence of EFX allocations when agents' valuation functions are bi-valued—i.e., each good is assigned one of two possible values. Several researchers have also focused on the existence of *partial* EFX allocations, where some goods may remain unallocated. Chaudhury et al. (2021) established that an EFX allocation always exists with at most $n-1$ goods unallocated, such that no agent envies the set of unallocated goods. This bound was improved by Berger et al. (2022) and Mahara (2024), who showed that EFX allocations can be achieved with at most $n-2$ unallocated goods. Moreover, they proved that for the special case of $n=4$, an EFX allocation with at most one unallocated good is always possible. More recently, Ghosal et al. (2025) generalized these results by showing that if agents have at most k distinct valuation functions, then an EFX allocation exists with at most $k-2$ unallocated goods.

Despite these advances, the question of whether EFX allocations exist for arbitrary set monotonic valuations remains open. This general case continues to be one of the most challenging unresolved problems in the domain of fair division.

1.3 Our Contribution

We establish a new existence result for Envy-Free up to any item (EFX) allocations under a hybrid valuation framework. Specifically, we consider settings in which a subset of agents have valuation functions satisfying a generalization of set monotonicity, while the remaining agents satisfy a generalized form of size monotonicity (see Theorem 3.2 for details). A notable corollary of our main theorem is the following:

> *An EFX allocation always exists when any two agents have arbitrary set monotonic valuation functions, and the remaining $n-2$ agents have size monotonic valuation functions.*

Our result extends the landscape of known EFX existence results by permitting arbitrary heterogeneity among two agents, without requiring any structure on their valuation functions.

1.4 Proof Overview

Our approach deviates from prior techniques. We begin by fixing an agent, say Agent 1, and construct an initial allocation $X^1 = (X_1, \ldots, X_n)$ such that X_1 is Agent 1's most preferred bundle of size one, and the remaining bundles are assigned in a specific way, which implies that either:

- X^1 is already EFX, or
- Agent 1 envies another agent.

The existence of such an allocation X^1 is guaranteed by the size monotonicity of all agents except Agents 1 and 2.

We then iteratively refine the allocation. At each step k, we obtain an allocation X^k such that (i) no agent from $\{2, \ldots, n\}$ envies any other, and (ii) the size of Agent 1's bundle strictly increases: $|X_1^{k-1}| < |X_1^k|$. This process must terminate (loosely speaking, due to boundedness), and at termination, we obtain an EFX allocation.

1.5 Comparison with Existing Work

To the best of our knowledge, this is the first work to prove EFX existence for any number of agents where up to two agents may have arbitrary set monotonic valuations. In contrast, recent work by HV et al. (2024) establishes EFX existence when each agent's valuation is drawn from a predefined set of *three* distinct additive valuation functions. Our result imposes no such restriction on the agents. Moreover, our proof technique yields a new existence result for the case $n = 2$, distinct from that of Plaut and Roughgarden (2020), and offers fresh insights that prove crucial for the general case.

1.6 Open Problems

Our results prompt several compelling open questions. Chief among them are:

- Does an EFX allocation exist for $n \geq 3$ agents with arbitrary set monotonic valuations?
- Does such an allocation exist for $n \geq 4$ agents when all have additive, set monotonic valuation functions?

These remain fundamental and challenging problems in the theory of fair division with indivisible goods.

2. Model

Let E be a set of m indivisible goods, and let $N = \{1, \ldots, n\}$ be a set of agents, where $n \geq 2$. Each agent is endowed with a valuation function, which is a mapping $v : 2^E \to \mathbb{R}$ assigning a real value to each subset of E.[1]

[1] We denote by 2^E the power set of E, i.e., the set of all subsets of E.

A valuation function v is said to be *set monotonic* if for all $X, X' \in 2^E$ with $X \subseteq X'$, it holds that $v(X) \leq v(X')$. Let $1 \leq k < l \leq m$. A valuation function v is said to be *$\langle k, l \rangle$-set monotonic* if for all $X \subseteq Y \subseteq E$ with $k \leq |X| < |Y| \leq l$, we have $v(X) \leq v(Y)$. Observe that v is set monotonic if and only if it is $\langle 1, m \rangle$-set monotonic. Similarly, v is called *$\langle k, l \rangle$-size monotonic* if for all distinct subsets $X, Y \subseteq E$ with $k \leq |X| < |Y| \leq l$, it holds that $v(X) \leq v(Y)$. We say that v is *size monotonic* if it is $\langle 1, m \rangle$-size monotonic.

Each agent $i \in N$ is assumed to have a set monotonic valuation function v_i. A collection of such functions $v_N := (v_i)_{i \in N}$ is referred to as a *set monotonic valuation profile*. A collection $X_N := (X_i)_{i \in N}$ of subsets of E is called an *allocation* if $\{X_1, \ldots, X_n\}$ forms a partition of E.[2]

Given a valuation function v and subsets $X, Y \subseteq E$, we say that X *is envy-free from Y up to any one item under v*, and write $X \, E_v \, Y$, if $v(X) \geq v(Y \setminus \{y\})$ for all $y \in Y$.

Definition 2.1. An allocation X_N is said to be *envy-free up to any good (EFX)* with respect to a valuation profile v_N if $X_i \, E_{v_i} \, X_j$ for all $i, j \in N$.

OBSERVATION 2.1. *An EFX allocation exists for every set monotonic valuation profile whenever $m \leq n$. In particular, any allocation X_N in which $|X_i| \leq 1$ for each $i \in N$ is EFX under every set monotonic valuation profile. Note that the full strength of set monotonicity is not required in this case; the allocation remains EFX as long as $v(\{x\}) \geq v(\emptyset)$ for each $x \in E$.*

In light of Observation 2.1, we assume $m > n$ for the remainder of this paper.

A valuation function $\hat{v}$ is said to be *strict* if $\hat{v}(X) \neq \hat{v}(Y)$ for all $X \neq Y$. A valuation profile $\hat{v}_N$ is called *strict* if each $\hat{v}_i$ is strict. For clarity, we refer to general (possibly non-strict) valuation functions as *weak* valuation functions.

OBSERVATION 2.2. *Let $\bar{v}_N$ be a weak valuation profile and $\hat{v}_N$ be a strict valuation profile such that for each $i \in N$ and for all distinct $X, Y \in 2^E$, it holds that $\bar{v}_i(X) > \bar{v}_i(Y)$ implies $\hat{v}_i(X) > \hat{v}_i(Y)$. Suppose X_N is an EFX allocation under $\hat{v}_N$. Then X_N is also EFX under $\bar{v}_N$.*

Proof Sketch. Assume, for contradiction, that X_N is not EFX under $\bar{v}_N$. Then there exist agents $i, j \in N$ and an item $x_j \in X_j$ such that $\bar{v}_i(X_i) < \bar{v}_i(X_j \setminus \{x_j\})$. By our assumption, this implies $\hat{v}_i(X_i) < \hat{v}_i(X_j \setminus \{x_j\})$, contradicting the EFX property of X_N under $\hat{v}_N$. ∎

[2] A collection of (possibly empty) subsets $\{X_1, \ldots, X_n\}$ forms a partition of E if $X_i \cap X_j = \emptyset$ for all $i \neq j$, and $\bigcup_{i \in N} X_i = E$.

3. Existence Results for EFX Allocations

In the study of fair division of indivisible goods, *Envy-Freeness up to any item* (EFX) stands out as a compelling and stringent fairness notion. Although the existence of EFX allocations remains unresolved in general, a number of partial results establish its existence under certain structural conditions on agents' valuations.

Plaut and Roughgarden (2020) demonstrate that EFX allocations are guaranteed to exist when (i) all agents have identical *set monotonic* valuations, or (ii) there are exactly two agents. Building upon this, recent work by Akrami et al. (2023) introduces the notion of *MMS-feasibility*, a structural constraint on valuation functions that permits the extension of existence results to more general settings.

3.1 MMS-Feasibility and Existing Results

We begin by formalizing the notion of MMS-feasibility, which intuitively captures a kind of "consistency" in the maximal and minimal values across all bipartitions of a given subset of goods.

Definition 3.1 (MMS-Feasibility)**.** A set monotonic valuation function v is *MMS-feasible* if, for every subset $X \subseteq E$ and for any two bipartitions $\{X_1, X_2\}$ and $\{X'_1, X'_2\}$ of X, the following inequality holds:

$$\max\{v(X_1), v(X_2)\} \geq \min\{v(X'_1), v(X'_2)\}.$$

This condition ensures that no two bipartitions of a subset X can yield values so unequal that the maximum of one is strictly smaller than the minimum of the other partition, preserving a form of "min-max fairness" within subsets.

Under this constraint, the following result from Akrami et al. (2023) extends the existence of EFX allocations to a broader setting:

Theorem 3.1 (Akrami et al. (2023))**.** *Let v_N be a profile of set monotonic valuation functions for $n = 3$ agents. If at least $n - 2 = 1$ of the agents have MMS-feasible valuations, then there exists an EFX allocation at v_N.*

3.2 New Results: Size Monotonicity and EFX Existence

We now introduce a notion called *size monotonicity*, which restricts how the valuation of sets evolves with their cardinality. By identifying the relationship between the total number of items m and the number of agents n, we derive existence results for EFX allocations under size monotonic valuation functions. We also consider relaxed versions of set monotonicity assumptions for our main theorems below. As previously noted, $\langle k, \ell \rangle$-size monotonicity implies $\langle k, \ell \rangle$-set monotonicity for any $1 \leq k < \ell \leq m$.

Let us first write m in terms of n using the division algorithm:

$$m = n \cdot \left\lfloor \frac{m}{n} \right\rfloor + m \bmod n,$$

where $\lfloor \cdot \rfloor$ denotes the floor function, and $m \bmod n$ is the remainder.

We now present our first main result, which establishes the existence of EFX allocations under the condition that, depending on the value of $m \bmod n$, certain subsets of agents have valuation functions that satisfy size monotonicity for specific cardinalities of item bundles.

Theorem 3.2. *Let v_N be an arbitrary valuation profile. Let $m \bmod n$ denote the remainder when m is divided by n.*

(i) Suppose $0 \leq m \bmod n \leq 1$. Then, an EFX allocation exists at v_N if:

- *$m \bmod n$ agents have $\langle \left\lfloor \frac{m}{n} \right\rfloor - 1, \left\lfloor \frac{m}{n} \right\rfloor \rangle$-size monotonic and $\langle \left\lfloor \frac{m}{n} \right\rfloor, \left\lfloor \frac{m}{n} \right\rfloor + 1 \rangle$-set monotonic valuation functions,*
- *a disjoint set of $n - 1 - (m \bmod n)$ agents have $\langle \left\lfloor \frac{m}{n} \right\rfloor - 1, \left\lfloor \frac{m}{n} \right\rfloor \rangle$-size monotonic valuation functions, and*
- *the remaining agent has a $\langle \left\lfloor \frac{m}{n} \right\rfloor - 1, \left\lfloor \frac{m}{n} \right\rfloor \rangle$-set monotonic valuation function.*

(ii) Suppose $1 < m \bmod n \leq n - 2$. Then, an EFX allocation exists at v_N if:

- *$m \bmod n$ agents have $\langle \left\lfloor \frac{m}{n} \right\rfloor, \left\lfloor \frac{m}{n} \right\rfloor + 1 \rangle$-size monotonic and $\langle \left\lfloor \frac{m}{n} \right\rfloor - 1, \left\lfloor \frac{m}{n} \right\rfloor \rangle$-set monotonic valuation functions,*
- *a disjoint set of $n - 1 - (m \bmod n)$ agents have $\langle \left\lfloor \frac{m}{n} \right\rfloor - 1, \left\lfloor \frac{m}{n} \right\rfloor \rangle$-size monotonic valuation functions, and*
- *the remaining agent has a $\langle \left\lfloor \frac{m}{n} \right\rfloor - 1, \left\lfloor \frac{m}{n} \right\rfloor \rangle$-set monotonic valuation function.*

(iii) *Suppose* $m \bmod n = n-1$. *Then, an EFX allocation exists at* v_N *if all* $n-1$ *agents have both* $\langle \left\lfloor \frac{m}{n} \right\rfloor, \left\lfloor \frac{m}{n} \right\rfloor + 1\rangle$*-size monotonic and* $\langle \left\lfloor \frac{m}{n} \right\rfloor - 1, \left\lfloor \frac{m}{n} \right\rfloor\rangle$*-set monotonic valuation functions.*

The proof is deferred to Appendix A.1.

We now present another existence result where only $n-2$ agents have structured preferences. However, now the agents have more stringent requirement for size monotonicity compared to Theorem 3.2.[3]

In this case, we reformulate $m-1$ using the $(n-1)$-division:

$$m - 1 = (n-1) \cdot \left\lfloor \frac{m-1}{n-1} \right\rfloor + (m-1) \bmod (n-1).$$

Theorem 3.3. *Let* v_N *be an arbitrary valuation profile. Let* $r := (m-1) \bmod (n-1)$.

(i) *Suppose* $0 \leq r \leq 1$. *Then, an EFX allocation exists at* v_N *if:*

- r *agents have* $\langle 1, \left\lfloor \frac{m-1}{n-1} \right\rfloor\rangle$*-size monotonic and* $\langle \left\lfloor \frac{m-1}{n-1} \right\rfloor, \left\lfloor \frac{m-1}{n-1} \right\rfloor + 1\rangle$*-set monotonic valuation functions,*
- *a disjoint set of* $n-2-r$ *agents have* $\langle 1, \left\lfloor \frac{m-1}{n-1} \right\rfloor\rangle$*-size monotonic valuation functions, and*
- *the remaining two agents have* $\langle 1, \left\lfloor \frac{m-1}{n-1} \right\rfloor\rangle$*-set monotonic valuation functions.*

(ii) *Suppose* $1 < r \leq n-2$. *Then, an EFX allocation exists at* v_N *if:*

- r *agents have* $\langle 1, \left\lfloor \frac{m-1}{n-1} \right\rfloor + 1\rangle$*-size monotonic valuation functions,*
- *a disjoint set of* $n-2-r$ *agents have* $\langle 1, \left\lfloor \frac{m-1}{n-1} \right\rfloor\rangle$*-size monotonic valuation functions, and*
- *the remaining two agents have* $\langle 1, \left\lfloor \frac{m-1}{n-1} \right\rfloor\rangle$*-set monotonic valuation functions.*

The proof of Theorem 3.3 is provided in Appendix A.2.

To aid the reader's understanding, we present a corollary to Theorem 3.3, derived under a significantly stronger yet simpler assumption.

Corollary 3.1. *There exists an EFX allocation at* v_N *if* $n-2$ *agents have size monotonic valuation functions and the remaining two agents have any arbitrary set monotonic valuation functions.*

[3] It is worth noting that neither of Theorem 3.2 and Theorem 3.3 imply each other.

3.3 Comparing Size Monotonicity and MMS-Feasibility

Although both size monotonicity and MMS-feasibility impose structural constraints on valuations, they are fundamentally incomparable. We illustrate this by providing explicit examples.

Size monotonicity does not imply MMS-feasibility.

Let $E = \{a, b, c, d\}$ and consider a valuation function v that is $\langle \ell, \ell + 1 \rangle$-size monotonic for $\ell \in \{1, 2\}$. Suppose:

$$v(\{a, b\}) > v(\{c, d\}) > v(\{b, c\}) > v(\{a, d\}).$$

This function satisfies the size monotonicity property but violates MMS-feasibility. Specifically, comparing the partitions $\{\{a, b\}, \{c, d\}\}$ and $\{\{b, c\}, \{a, d\}\}$, we observe:

$$\max\{v(\{b, c\}), v(\{a, d\})\} < \min\{v(\{a, b\}), v(\{c, d\})\},$$

contradicting Definition 3.1.

MMS-feasibility does not imply size monotonicity.

Let v be an *additive* valuation function, i.e., $v(S) = \sum_{s \in S} v(\{s\})$ for all $S \subseteq E$. Such functions are always MMS-feasible (Akrami et al., 2023). Suppose:

$$v(\{a\}) > v(\{b, c, d\}).$$

Then, for all subsets $S \subseteq \{b, c, d\}$, we have $v(S) \leq v(\{b, c, d\}) < v(\{a\})$, while for any S with $a \in S$, $v(S) \geq v(\{a\})$. Thus:

- $v(\{a\}) > v(\{b, c\})$, violating $\langle 1, 2 \rangle$-size monotonicity.
- $v(\{a, b\}) > v(\{b, c, d\})$, violating $\langle 2, 3 \rangle$-size monotonicity.

Hence, additive (and MMS-feasible) valuations need not be size monotonic.

In conclusion, the structural assumption of size monotonicity yields existence results for EFX allocations that are independent of those obtained under MMS-feasibility. In particular, Theorem 3.3 demonstrates that EFX allocations can be guaranteed under conditions that do not rely on MMS-feasibility, thereby highlighting a broader landscape of sufficient conditions for achieving fairness in the allocation of indivisible goods.

A. Appendix

A.1 Proof of Theorem 3.2

Proof. In view of Observation 2.2, it suffices to prove Theorem 3.2 under the assumption that the valuation profile v_N is strict. We provide a unified argument that applies to all parts of the theorem.

Let $\hat{v}_N$ be a strict valuation profile satisfying the assumptions of the theorem. Define $r = m \bmod n$ and $\ell = \left\lfloor \frac{m}{n} \right\rfloor$. By the assumptions of the theorem, r agent(s) have $\langle k, k+1 \rangle$-size monotonic valuation functions and $\langle j, j+1 \rangle$-set monotonic valuation functions at $\hat{v}_N$, where $k, j \in \{\ell - 1, \ell\}$, corresponding to the respective parts of the theorem. Without loss of generality, let $\{n-r+1, \ldots, n\}$ be the set of these r agents. Also, by the assumptions of the theorem, $n-1-r$ agents have $\langle \ell-1, \ell \rangle$-size monotonic valuation functions and the remaining agent has $\langle \ell-1, \ell \rangle$-set monotonic valuation function or has a valuation function without any restriction, corresponding to the respective parts of the theorem. Without loss of generality, assume that Agent 1 has $\langle \ell-1, \ell \rangle$-set monotonic valuation function or has a valuation function without any restriction, corresponding to the respective parts of the theorem.

Let $\mathcal{X}_0 = \emptyset$, and for each $i \geq 1$, define $\mathcal{X}_i = \bigcup_{j=1}^{i} X_j$. We construct an allocation $X_N = (X_1, \ldots, X_n)$ as follows:

(i) For each $1 \leq i \leq n-r$, let

$$X_i = \arg \max_{\substack{S \subseteq E \setminus \mathcal{X}_{i-1} \\ |S| = \ell}} \hat{v}_i(S),$$

where the argmax is well-defined (i.e., unique) due to the assumption that $\hat{v}_N$ is strict.[4]

(ii) For each $n-r < i \leq n$, choose $X_i \subseteq E \setminus \mathcal{X}_{i-1}$ such that $|X_i| = \ell + 1$.

By construction, we have $X_i \cap X_j = \emptyset$ for all $i, j \in N$ with $i \neq j$, and

$$\sum_{i=1}^{n} |X_i| = (n-r)\ell + r(\ell+1) = n\ell + r = m,$$

[4]That is, each agent i has a strict preference ordering over bundles of size ℓ.

which confirms that X_N constitutes a valid allocation of all m goods. We now prove that the allocation X_N is EFX at the valuation profile $\hat{v}_N$.

Fix an arbitrary agent $i \in N$, and let $j \in N \setminus \{i\}$ be any other agent. Consider an arbitrary good $x_j \in X_j$. We claim that

$$\hat{v}_i(X_i) \geq \hat{v}_i(X_j \setminus \{x_j\}).$$

Suppose $i = 1$. By the construction of the allocation X_N, we have $|X_1| = \ell$ and $|X_j| \in \{\ell, \ell+1\}$. First, consider the case where $|X_j| = \ell + 1$.[5] By definition of X_1, we have

$$\hat{v}_1(X_1) \geq \hat{v}_1(S) \quad \text{for all } S \subseteq E \text{ with } |S| = \ell.$$ [6]

Since $|X_j \setminus \{x_j\}| = \ell$, it follows that

$$\hat{v}_1(X_1) \geq \hat{v}_1(X_j \setminus \{x_j\}).$$

Now suppose $|X_j| = \ell$.[7] Since $\hat{v}_1$ is strict and X_1 is selected to maximize $\hat{v}_1$ over all subsets of size ℓ, we have

$$\hat{v}_1(X_1) > \hat{v}_1(X_j).$$

Furthermore, by the $\langle \ell - 1, \ell \rangle$-set monotonicity of $\hat{v}_1$, it holds that

$$\hat{v}_1(X_j) > \hat{v}_1(X_j \setminus \{x_j\}).$$

Combining the two inequalities yields

$$\hat{v}_1(X_1) > \hat{v}_1(X_j \setminus \{x_j\}),$$

as desired.

Suppose $i \in \{2, \ldots, n - r\}$.[8] By the construction of the allocation X_N, we have $|X_i| = \ell$ and $|X_j| \in \{\ell, \ell + 1\}$. First, consider the case where $j \in \{1, \ldots, n - r\}$. Then, by the construction of the allocation X_N, $|X_j| = \ell$. Since $\hat{v}_i$ is $\langle \ell - 1, \ell \rangle$-size monotonic, it follows that $\hat{v}_i(X_i) > \hat{v}_i(X_j \setminus \{x_j\})$.

[5] Note that for part (iii) of the theorem, by construction of X_N, we have $|X_i| = \ell + 1$ for all $i \in N \setminus \{1\}$.

[6] That is, $X_1 = \arg \max_{S \subseteq E : |S| = \ell} \hat{v}_1(S)$, so $\hat{v}_1(X_1) \geq \hat{v}_1(S)$ for all $S \subseteq E$ with $|S| = \ell$.

[7] Note that part (iii) of the theorem is excluded in this case.

[8] Note that this set is empty for part (iii) of the theorem because $r = n - 1$.

Now, suppose $j \in N \setminus \{1, \ldots, n-r\}$. Then, by the construction of the allocation X_N, $|X_j| = \ell + 1$, and hence $|X_j \setminus \{x_j\}| = \ell$. Since $1 \leq i \leq n-r$ and $j > n-r$, which in particular means that $i < j$, we have that both X_i and $X_j \setminus \{x_j\}$ are subsets of $E \setminus \mathcal{X}_{i-1}$, where $\mathcal{X}_{i-1}$ is as defined in the construction of X_N. Since $X_i = \operatorname{argmax}_{S \subseteq E \setminus \mathcal{X}_{i-1} : |S| = \ell} \hat{v}_i(S)$ and $|X_i| = |X_j \setminus \{x_j\}| = \ell$, it follows that $\hat{v}_i(X_i) > \hat{v}_i(X_j \setminus \{x_j\})$.

Finally, suppose $i \in N \setminus \{1, \ldots, n-r\}$. Then, by definition, $|X_i| = \ell + 1$. Moreover, by the construction of X_N, we have $|X_k| \in \{\ell, \ell+1\}$ for all $k \in N$.

For part (i) of the theorem, if $r = 0$, then there is nothing to prove. If $r = 1$, then $i = n$ and $j \in \{1, \ldots, n-1\}$, implying $|X_j| = \ell$. Since $\hat{v}_i$ is a $\langle \ell - 1, \ell \rangle$-size monotonic and $\langle \ell, \ell+1 \rangle$-set monotonic valuation function, it follows that

$$\hat{v}_i(X_i) > \hat{v}_i(X_j \setminus \{x_j\}).$$

For part (ii) and (iii) of the theorem, note that $|X_j| \in \{\ell, \ell+1\}$, and hence $|X_j \setminus \{x_j\}| \in \{\ell - 1, \ell\}$. Since $\hat{v}_i$ is a $\langle \ell, \ell+1 \rangle$-size monotonic and $\langle \ell - 1, \ell \rangle$-set monotonic valuation function, we again conclude that

$$\hat{v}_i(X_i) > \hat{v}_i(X_j \setminus \{x_j\}).$$

This completes the proof. ∎

A.2 Proof of Theorem 3.3

Proof. As explained in the proof of Theorem 3.2, by Observation 2.2, it suffices to establish the existence of an EFX allocation for each *strict* valuation profile v_N that satisfies the assumptions of Theorem 3.3. As before, we present a unified argument that proves both parts of the theorem.

Let $\hat{v}_N$ be a strict valuation profile that satisfies the assumptions of the theorem. Define $r = (m-1) \bmod (n-1)$ and $\ell = \left\lfloor \frac{m-1}{n-1} \right\rfloor$. By the theorem's assumption, there exist $n-2$ agents whose valuation functions are $\langle 1, \ell \rangle$-size monotonic at $\hat{v}_N$. Without loss of generality, let $M = \{3, \ldots, n\}$ denote the set of these $n-2$ agents. In addition, r agents out of M have either $\langle \ell, \ell+1 \rangle$-set monotonic or $\langle \ell, \ell+1 \rangle$-size monotonic valuation functions, depending on the part of the theorem under consideration. Without loss of generality, let $\{n-r+1, \ldots, n\}$ be the set of these r agents. Finally, the remaining two agents (Agent 1 and Agent 2) have $\langle 1, \ell \rangle$-set monotonic valuation functions.

Let $W^1 = \operatorname{argmax}_{S \subseteq E : |S|=1} \hat{v}_1(S)$. Define $Y^1 = \operatorname{argmax}_{S \subseteq E \setminus W^1 : |S|=\ell} \hat{v}_2(S)$. Let $\mathcal{X}_0^1 = \emptyset$ and, for each $i \geq 1$, define $\mathcal{X}_i^1 = \bigcup_{j=1}^{i} X_j^1$. We now define the allocation X_N^1 as follows:

(i) $X_1^1 = W^1$,

(ii) $X_2^1 = Y^1$,

(iii) For each $3 \leq i \leq n - r$, let

$$X_i^1 = \operatorname{argmax}_{S \subseteq E \setminus \mathcal{X}_{i-1}^1 : |S|=\ell} \hat{v}_i(S),$$

(iv) For each $n - r < i \leq n$, choose $X_i^1 \subseteq E \setminus \mathcal{X}_{i-1}^1$ such that $|X_i^1| = \ell + 1$.

By construction, $X_i^1 \cap X_j^1 = \emptyset$ for all $i, j \in N$, and

$$\sum_{i=1}^{n} |X_i^1| = |X_1^1| + \sum_{i=2}^{n} |X_i^1| = 1 + (n-1)\ell + r = 1 + (m-1) = m,$$

which verifies that X_N^1 constitutes a valid allocation. Define

$$\max(X_N^1) = \max\{|X_i^1| : i \in N\}.$$

Observe that for any $i \in N \setminus \{1\}$, we have $\max(X_N^1) \in \{|X_i^1|, |X_i^1| + 1\}$. Furthermore, note that $|E| > \max(X_N^1)$ since $n \geq 2$.

Now consider the subset of agents $\{2, \ldots, n\}$, which has cardinality $n - 1$. Among these, agent 2 has $\langle 1, \ell \rangle$-set monotonic valuation function, whereas the remaining $n - 2$ agents have valuation functions according to different parts of the theorem under consideration. Therefore, the conditions of Theorem 3.2 are satisfied when the set of agents is restricted to $\{2, \ldots, n\}$ and the set of goods is restricted to $\cup_{i \in \{2,\ldots,n\}} X_i^1$.

Therefore, by the proof of Theorem 3.2, it follows that $X_i^1 \; E_{\hat{v}_i} \; X_j^1$ for every $i, j \in \{2, \ldots, n\}$. Moreover, for every $i \in \{2, \ldots, n\}$, we also have $X_i^1 \; E_{\hat{v}_i} \; X_1^1$ since $|X_1^1| = 1$.[9]

Hence, either the allocation X_N^1 is EFX at $\hat{v}_N$, or it fails to satisfy $X_1^1 \; E_{\hat{v}_1} \; X_i^1$ for some $i \in \{2, \ldots, n\}$. It is worth noting that if $\ell = 1$ or $\ell = 2$ and $r = 0$, then X_N^1 must be EFX at $\hat{v}_N$ because

[9] Since we are studying division of *goods*, we assume that the empty set is the least preferred bundle for any valuation function.

$X_1^1 \ E_{\hat{v}_1} \ X_i^1$ holds for all $i \in \{2, \ldots, n\}$; indeed, $|X_i^1| \leq 2$ for all $i \in \{2, \ldots, n\}$ and, by definition, X_1^1 is the most preferred singleton according to $\hat{v}_1$, thereby completing the proof in this case.

We thus restrict our attention to the cases where either $\ell = 2$ and $r > 0$, or $\ell > 2$ with $r \geq 0$. In both cases, we observe that $\max(X_N^1) \geq 3$, and in particular, $|X_1^1| < |X_i^1|$ for every $i \in \{2, \ldots, n\}$. If the allocation X_N^1 is EFX at $\hat{v}_N$, then the proof is complete. We now consider the remaining case. Suppose that there exists $q \in \{2, \ldots, n\}$ such that $X_1^1 \ E_{\hat{v}_1} \ X_q^1$. This implies that there exists a good $x_q \in X_q^1$ such that

$$\hat{v}_1(X_q^1 \setminus \{x_q\}) > \hat{v}_1(X_1^1).$$

Thus, there exists a subset $Z \subseteq E \setminus X_1^1$ with $|Z| = \max(X_N^1) - 1$ such that $\hat{v}_1(Z) > \hat{v}_1(X_1^1)$. Such a set Z always exists, since $\max(X_N^1)$ is either $|X_q^1|$ or $|X_q^1| + 1$. If $|X_q^1| = \max(X_N^1)$, then we may take $Z = X_q^1 \setminus \{x_q\}$. Otherwise, if $|X_q^1| + 1 = \max(X_N^1)$, then it must be the case that $|X_q^1| = \ell$. Since $\hat{v}_1$ is $\langle 1, \ell \rangle$-set monotonic, it must be the case that

$$\hat{v}_1(X_q^1) > \hat{v}_1(X_q^1 \setminus \{x_q\}) > \hat{v}_1(X_1^1).$$

Therefore, we may take $Z = X_q^1$.

Moreover, observe that $E \neq Z \cup X_1^1$, because

$$|Z \cup X_1^1| = (\max(X_N^1) - 1) + 1 = \max(X_N^1) < |E|,$$

since $|E| > \max(X_N^1)$. Therefore, $E \setminus (Z \cup X_1^1) \neq \emptyset$.

Define the set

$$G^1 = \left\{ S \subseteq E : |S| = 2 \text{ and } \exists T \subseteq E \setminus S \text{ with } |T| = \max(X_N^1) - 1 \text{ such that } \hat{v}_1(T) > \hat{v}_1(X_1^1) \right\}.$$

Note that $G^1 \neq \emptyset$, since for any $g \in E \setminus (Z \cup X_1^1)$, we have $X_1^1 \cup \{g\} \in G^1$, as $|X_1^1 \cup \{g\}| = 2$ and $Z \subseteq E \setminus (X_1^1 \cup \{g\})$ with $|Z| = \max(X_N^1) - 1$ and $\hat{v}_1(Z) > \hat{v}_1(X_1^1)$. Define

$$W^2 = \arg\max_{S \in G^1} \hat{v}_1(S).$$

Next, define

$$H^1 = \left\{ S \subseteq E \setminus W^2 : |S| = \max(X_N^1) - 1 \text{ and } \hat{v}_1(S) > \hat{v}_1(X_1^1) \right\}.$$

Since $W^2 \in G^1$ by definition, it follows that $H^1 \neq \emptyset$. Define

$$Y^2 = \arg\max_{S \in H^1} \hat{v}_1(S).$$

Let $\mathcal{X}_0^2 = \emptyset$ and define $\mathcal{X}_i^2 = \bigcup_{j=1}^{i} X_j^2$ for $i \geq 1$. We now define the allocation X_N^2 as follows:

(i) $X_2^2 = \arg\max_{S \in \{W^2, Y^2\}} \hat{v}_2(S)$,

(ii) $X_1^2 = \{W^2, Y^2\} \setminus X_2^2$,

(iii) For $3 \leq i \leq n - r + 1$[10], define

$$X_i^2 = \arg\max_{\substack{S \subseteq E \setminus \mathcal{X}_{i-1}^2 \\ |S| = \max(X_N^1) - 1}} \hat{v}_i(S),$$

(iv) For $n - r + 1 < i \leq n$, let $X_i^2 \subseteq E \setminus \mathcal{X}_{i-1}^2$ with $|X_i^2| = \max(X_N^1)$.

By construction, we have $X_i^2 \cap X_j^2 = \emptyset$ for every $i \neq j \in N$, and

$$\sum_{i=1}^{n} |X_i^2| = m,$$

since $|X_1^2| = |X_1^1| + 1$, $|X_{n-r+1}^2| = |X_{n-r+1}^1| - 1$, and all other bundle sizes are unchanged from X_N^1. Therefore, X_N^2 is a valid allocation.

Now, consider the subset of agents $\{3, \ldots, n\}$, of size $n - 2$. The valuation functions of these agents satisfy the conditions of Theorem 3.2 when the set of agents is restricted to $\{3, \ldots, n\}$ and the set of goods is restricted to $\cup_{i \in \{3,\ldots,n\}} X_i^2$. By the proof of Theorem 3.2, it follows that

$$X_i^2 \, E_{\hat{v}_i} \, X_j^2 \quad \text{for all } i, j \in \{3, \ldots, n\}.$$

Furthermore, for each $i \in \{3, \ldots, n\}$ and $j \in \{1, 2\}$, we also have $X_i^2 \, E_{\hat{v}_i} \, X_j^2$, because of the assumption on $\hat{v}_i$ and the fact that $|X_j^2| \leq |X_i^2|$. Therefore,

$$X_i^2 \, E_{\hat{v}_i} \, X_j^2 \quad \text{for all } i \in \{3, \ldots, n\} \text{ and } j \in N.$$

[10] With the interpretation that for $r = 0$, $n + 1 \equiv 2$.

Additionally, by construction, $X_2^2 \, E_{\hat{v}_2} \, X_1^2$. We now distinguish two cases.

CASE A.

Suppose $X_2^2 = W^2$. Then, by the definition of Y^2, we have

$$X_1^2 \, E_{\hat{v}_1} \, X_i^2 \quad \text{for all } i \in \{2, \ldots, n\}.$$

Thus, either: - X_N^2 is EFX at $\hat{v}_N$, and the proof is complete; or - $X_2^2 \, E_{\hat{v}_2} \, X_i^2$ fails for some $i \in \{3, \ldots, n\}$.

Assume the latter. Then, define:

$$\bar{\mathcal{X}}_0^2 := \emptyset, \qquad \bar{\mathcal{X}}_i^2 := \bigcup_{j=1}^{i} \bar{X}_j^2 \quad \text{for } i \geq 1.$$

Construct a new allocation $\bar{X}_N^2$ as follows:

1. $\bar{X}_1^2 := W^2$,
2. $\bar{X}_2^2 := \arg \max_{\substack{S \subseteq E \setminus W^2 \\ |S| = \max(X_N^1) - 1}} \hat{v}_2(S)$,
3. For $3 \leq i \leq n - r + 1$[11], set

$$\bar{X}_i^2 := \arg \max_{\substack{S \subseteq E \setminus \bar{\mathcal{X}}_{i-1}^2 \\ |S| = \max(X_N^1) - 1}} \hat{v}_i(S),$$

4. For $n - r + 1 < i \leq n$, choose any

$$\bar{X}_i^2 \subseteq E \setminus \bar{\mathcal{X}}_{i-1}^2 \quad \text{such that } |\bar{X}_i^2| = \max(X_N^1).$$

This construction ensures that $\bar{X}_i^2 \cap \bar{X}_j^2 = \emptyset$ for $i \neq j$ and that

$$\sum_{i=1}^{n} |\bar{X}_i^2| = \sum_{i=1}^{n} |X_i^2| = m,$$

[11] For $r = 0$, interpret $n + 1 \equiv 2$.

so $\bar{X}^2_N$ is indeed a valid allocation.

Now consider the set $\{2, \ldots, n\}$. Agent 2 has $\langle 1, \ell \rangle$-set monotonic valuation function, whereas the remaining $n-2$ agents have valuation functions according to different parts of the theorem under consideration. Therefore, the conditions of Theorem 3.2 are satisfied when the set of agents is restricted to $\{2, \ldots, n\}$ and the set of goods is restricted to $\cup_{i \in \{2,\ldots,n\}} \bar{X}^2_i$. It follows that:

$$\bar{X}^2_i \, E_{\hat{v}_i} \, \bar{X}^2_j \quad \text{for all } i, j \in \{2, \ldots, n\}.$$

We now argue that $\bar{X}^2_2 \, E_{\hat{v}_2} \, \bar{X}^2_1$ also holds. Since we assumed $X^2_2 \, E_{\hat{v}_2} \, X^2_i$ fails for some $i \in \{3, \ldots, n\}$, there exists a set $U \subseteq E \setminus W^2$ such that $|U| = \max(X^1_N) - 1$ and $\hat{v}_2(U) > \hat{v}_2(W^2)$. Hence, by construction of $\bar{X}^2_2$ as the set maximizing $\hat{v}_2$ over all such subsets, we obtain:

$$\bar{X}^2_2 \, E_{\hat{v}_2} \, \bar{X}^2_1.$$

Finally, for each $i \in \{3, \ldots, n\}$, note that $|\bar{X}^2_1| = 2 \leq \max(X^1_N) - 1$, and $\hat{v}_i$ is $\langle 1, \ell \rangle$-size monotonic. Therefore, we also have:

$$\bar{X}^2_i \, E_{\hat{v}_i} \, \bar{X}^2_1.$$

Thus, in the allocation $\bar{X}^2_N$, we have:

$$\bar{X}^2_i \, E_{\hat{v}_i} \, \bar{X}^2_j \quad \text{for all } i \in \{2, \ldots, n\}, \ j \in N.$$

This completes the analysis of Case A.

CASE B.

Suppose $X^2_1 = W^2$. Consider the allocation $\bar{X}^2_N$ as defined in Case A.

Thus, in both Case A and Case B, we have constructed an allocation $\bar{X}^2_N$ such that:

(i) $\bar{X}^2_1 = W^2$,

(ii) $|\bar{X}^2_i| \in \{\max(X^1_N) - 1, \max(X^1_N)\}$ for all $i \in \{2, \ldots, n\}$, and

(iii) $\bar{X}^2_i \, E_{\hat{v}_i} \, \bar{X}^2_j$ for all $i \in \{2, \ldots, n\}$ and all $j \in \{1, \ldots, n\}$.

Therefore, either $\bar{X}^2_N$ is EFX at $\hat{v}_N$, in which case the proof is complete, or $\bar{X}^2_1 \, E_{\hat{v}_1} \, \bar{X}^2_i$ fails for some $i \in \{2, \ldots, n\}$. Assume the latter.

Since $|X_1^1| < |X_i^1|$ for all $i \in \{2, \ldots, n\}$, it follows that $|\bar{X}_1^2| = |X_1^1| + 1 \leq \max(X_N^1) - 1$. If $\bar{X}_1^2 \, E_{\hat{v}_1} \, \bar{X}_i^2$ fails for some i, then $|\bar{X}_1^2| < \max(X_N^1) - 1$. To see this, suppose instead that $|\bar{X}_1^2| = \max(X_N^1) - 1 = 2$, so $\max(X_N^1) = 3$. If $\bar{X}_1^2 \, E_{\hat{v}_1} \, \bar{X}_j^2$ fails for some $j \in \{2, \ldots, n\}$, then there exists $x_j \in \bar{X}_j^2$ such that

$$\hat{v}_1(\bar{X}_j^2 \setminus \{x_j\}) > \hat{v}_1(\bar{X}_1^2).$$

If $|\bar{X}_j^2| = 3$, then $|\bar{X}_j^2 \setminus \{x_j\}| = 2$ and if $|\bar{X}_j^2| = 2$, then the fact that $\hat{v}_1$ is $\langle 1, \ell \rangle$-set monotonic implies that $\hat{v}_1(\bar{X}_j^2) > \hat{v}_1(\bar{X}_j^2 \setminus \{x_j\}) > \hat{v}_1(\bar{X}_1^2)$. Thus, it follows that there exists $Z \subseteq E \setminus W^2$ with $|Z| = 2$ such that $\hat{v}_1(Z) > \hat{v}_1(W^2)$, contradicting the definition of W^2 as the maximizing set in G^1. Therefore, $|\bar{X}_1^2| < \max(X_N^1) - 1$.

We now slightly abuse notation and write $X_N^2 := \bar{X}_N^2$. Then, by a similar argument, there exists a set $Z \subseteq E \setminus X_1^2$ with $|Z| = \max(X_N^2) - 1$ and $v_1(Z) > v_1(X_1^2)$. Define new sets W^3 and Y^3 analogously with $|W^3| = 3$. Write $m - 2 = (n-1)\ell + r$ for some $\ell \in \mathbb{N}$ and $0 \leq r < n - 1$.[12]

By repeating the same logic (as in Cases A and B), either an EFX allocation exists at $\hat{v}_N$, or we obtain a new allocation X_N^3 such that:

(i) $X_1^3 = W^3$,

(ii) $|X_i^3| \in \{\max(X_N^2) - 1, \max(X_N^2)\}$ for all $i \in \{2, \ldots, n\}$,

(iii) $X_i^3 \, E_{\hat{v}_i} \, X_j^3$ for all $i \in \{2, \ldots, n\}$ and all $j \in \{1, \ldots, n\}$.

Again, either X_N^3 is EFX at $\hat{v}_N$, or $X_1^3 \, E_{\hat{v}_1} \, X_i^3$ fails for some i. In the latter case, using the same reasoning, we conclude that $|X_1^3| < \max(X_N^2) - 1$.

We continue this process. Since E and N are finite, this results in the fact that either an EFX allocation exists at $\hat{v}_N$, or for some $c \in \mathbb{N}$, we obtain sets W^c, Y^c and an allocation X_N^c such that:

(i) $c \leq |X_i^c| \leq c + 1$ for each $i \in N$,

(ii) $X_1^c = W^c$,

(iii) $|X_i^c| \in \{\max(X_N^{c-1}) - 1, \max(X_N^{c-1})\}$ for all $i \in \{2, \ldots, n\}$,

(iv) $X_i^c \, E_{\hat{v}_i} \, X_j^c$ for all $i \in \{2, \ldots, n\}$ and $j \in N$.

We claim that X_N^c is EFX at $\hat{v}_N$.

Let us first summarize the key inductive observations:

[12] Here ℓ and r may differ from those in earlier steps.

- $|X_1^1| < |X_1^2| < \cdots < |X_1^c|$, with $|X_1^j| = j$ for each $j \in \{1, \ldots, c\}$,
- $\hat{v}_1(X_1^1) < \hat{v}_1(X_1^2) < \cdots < \hat{v}_1(X_1^c)$,
- $\max(X_N^c) \leq \max(X_N^{c-1}) \leq \cdots \leq \max(X_N^1)$,
- If $\max(X_N^{j+1}) < \max(X_N^j)$, then $\max(X_N^{j+1}) + 1 = \max(X_N^j)$.

Now we prove that $X_1^c \, E_{\hat{v}_1} \, X_j^c$ holds for all $j \in \{2, \ldots, n\}$. Suppose not. Then for some j, there exists $x_j \in X_j^c$ such that:

$$\hat{v}_1(X_j^c \setminus \{x_j\}) > \hat{v}_1(X_1^c).$$

Then, there exists $Z \subseteq E \setminus W^c$ with $|Z| = c$ and

$$\hat{v}_1(Z) > \hat{v}_1(W^c) > \hat{v}_1(W^{c-1}),$$

contradicting the definition of W^c, since $Z \in G^{c-1}$ and $W^c = \arg\max_{S \in G^{c-1}} \hat{v}_1(S)$, where:

$$G^{c-1} = \left\{ S \subseteq E \;:\; |S| = c \text{ and } \exists T \subseteq E \setminus S \text{ with } |T| = \max(X_N^{c-1}) - 1 \text{ and } \hat{v}_1(T) > \hat{v}_1(X_1^{c-1}) \right\}.$$

Such a set Z exists: if $|X_j^c| = c$, then $Z = X_j^c$; if $|X_j^c| = c + 1$, then $Z = X_j^c \setminus \{x_j\}$.

Therefore, $X_1^c \, E_{\hat{v}_1} \, X_j^c$ holds for all $j \in \{2, \ldots, n\}$. Hence, X_N^c is EFX at $\hat{v}_N$.

This concludes the proof of the theorem. ■

References

[1] Hannaneh Akrami, Noga Alon, Bhaskar Ray Chaudhury, Jugal Garg, Kurt Mehlhorn, and Ruta Mehta. Efx: a simpler approach and an (almost) optimal guarantee via rainbow cycle number. In *Proceedings of the 24th ACM Conference on Economics and Computation*, pages 61–61, 2023.

[2] Georgios Amanatidis, Georgios Birmpas, Aris Filos-Ratsikas, Alexandros Hollender, and Alexandros A Voudouris. Maximum nash welfare and other stories about efx. *Theoretical Computer Science*, 863:69–85, 2021.

[3] Ben Berger, Avi Cohen, Michal Feldman, and Amos Fiat. Almost full efx exists for four agents. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 36, pages 4826–4833, 2022.

[4] Ioannis Caragiannis, David Kurokawa, Hervé Moulin, Ariel D Procaccia, Nisarg Shah, and Junxing Wang. The unreasonable fairness of maximum nash welfare. *ACM Transactions on Economics and Computation (TEAC)*, 7(3):1–32, 2019.

[5] Bhaskar Ray Chaudhury, Telikepalli Kavitha, Kurt Mehlhorn, and Alkmini Sgouritsa. A little charity guarantees almost envy-freeness. *SIAM Journal on Computing*, 50(4):1336–1358, 2021.

[6] Pratik Ghosal, Vishwa Prakash HV, Prajakta Nimbhorkar, and Nithin Varma. (almost full) efx for three (and more) types of agents. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 39, pages 13889–13896, 2025.

[7] Vishwa Prakash HV, Pratik Ghosal, Prajakta Nimbhorkar, and Nithin Varma. Efx exists for three types of agents. *arXiv preprint arXiv:2410.13580*, 2024.

[8] Ryoga Mahara. Existence of efx for two additive valuations. *Discrete Applied Mathematics*, 340:115–122, 2023.

[9] Ryoga Mahara. Extension of additive valuations to general valuations on the existence of efx. *Mathematics of operations research*, 49(2):1263–1277, 2024.

[10] Benjamin Plaut and Tim Roughgarden. Almost envy-freeness with general valuations. *SIAM Journal on Discrete Mathematics*, 34(2):1039–1068, 2020.