Optimal Sequential Assignment with Capacity Constrained Verification

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Abstract

A principal has k identical objects to assign to n sequentially arriving and impatient agents. Each agent desires to be allocated the object, and the principal's value from allocating the object to an agent is the agent's private information. The principal can perfectly verify the value of up to m agents where $m < k$. The paper is concerned with the optimal policy that maximizes the principal's total expected value. We show that the optimal policy consists of deterministic mechanisms for each possible contingency. The deterministic mechanism is of the following kind: each arriving agent is verified and allocated the object if his type is above a cut-off. The optimal policy is characterized, and a recursive equation is presented to determine the cut-offs of this optimal policy.

[1]

1 Introduction

We consider an allocation problem of assigning identical objects to a group of impatient agents who arrive in a sequential order. Each agent desires to be allocated the object, and the principal's

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value from allocating the object to an agent is the agent's private information. The principal cannot use monetary transfers to elicit the agent's private information, though this information is based on verifiable evidence. The principal has a verification technology that can perfectly check this evidence, but there is a capacity constraint, and it can check fewer than the number of objects to allocate. The optimal policy that maximizes the principal's total expected value must consider the "frictions" resulting from sequentiality and the capacity constraint on the verification technology. Allocating an object today by screening the agent using the verification technology means that the object cannot be allocated in the future to an agent who values it more, and also, other objects cannot be allocated to agents with a higher value because of the verification check used.

To consider an example of such a setting, suppose a firm is building a team of identical positions. Offers from applicants arrive in a sequential order for these positions. The value to the firm from assigning a position to the applicant is private information but is verifiable (eg, education, work experience, skills). The firm has limited resources to do the verification checks, fewer than the number of positions available. The firm aims to design an optimal policy to allocate the positions.

We assume that agents' values are independently distributed. This helps in classifying the past decisions in terms of states, where each state represents the set of agents yet to arrive, the number of objects yet to be allocated, and the remaining number of verification checks. The optimal policy of the principal specifies a mechanism for each possible state that may arise. The mechanism design question is: given past decisions, if any, how to optimally check the arriving agent based on her report and use the result (if verified) along with her report to allocate an object, taking into account its impact on optimally allocating to the agents yet to arrive.

We show that the optimal policy consists of deterministic mechanisms for each possible state. Given a state, the arriving agent is asked to report her type. If the reported type is above a cut-off value, then the agent is verified and is allocated the position if she reports the truth. Otherwise, the agent is not verified and not allocated the position. In a deterministic mechanism, incentive compatibility forces an agent type to be verified if she is allocated the object, as the lowest type will never be allocated the object. Also, as there are limited verification checks, it is not optimal for the principal to verify and not allocate the object to an agent type. Hence, in the optimal policy, deterministic mechanisms have a cut-off structure. For the optimal cut-off type, the principal is indifferent between verifying and allocating the object to the cut-off type and not verifying and not allocating to that type.

The optimal cut-off at each state is strictly greater than the optimal cut-off if the principal had complete information of the type of agents when they arrive. Due to limited verification checks, the principal has to balance the trade-off between not allocating the object to relatively higher types and increasing the probability of verifying the future arriving agents. Interestingly, the principal cannot gain by randomly verifying certain types of agent and randomly allocating the object to certain types. Random verification may help the principal in increasing the probability of verifying the agents yet to arrive. The principal may also gain by randomly allocating the object to agent types between the cut-offs of the first-best policy and the optimal deterministic policy. However, incentive compatibility then forces the principal to allocate all agent types, who are not verified, with the same probability, which is strictly positive. This results in the optimal deterministic policy being the optimal policy.

2 Literature Review

The literature on sequential assignment began with the paradigmatic analysis by Derman et al. (1972). They consider a model of complete information, where several heterogeneous objects are allocated over time to sequentially arriving agents. Our model differs from theirs as we assume that the arriving agents' types are private information. Also, we restrict attention to the sequential assignment of identical objects. Gershkov and Moldovanu (2010) also study the sequential assignment problem where arriving agents' types are private information, but they allow for monetary transfers. They show that the dynamic efficient policy can be implemented by a dynamic analogue of the VCG (Vickrey-Clarke-Groves) mechanism. Our model does not allow for monetary transfers but assumes that private information is based on verifiable evidence.

Our paper relates to the recent work on allocation problems where private information is based on verifiable evidence. Ben-Porath et al. (2014) study an optimal allocation problem where a principal desires to allocate a single indivisible good among a group of agents where agent types can be learnt at a cost. Subsequent work has studied costly verification without transfers in collective decisions (Erlanson and Kleiner (2020)), delegation (Halac and Yared (2020)), or general mechanism design problems (Ben-Porath et al. (2019)). Erlanson and Kleiner (2024) study a static allocation problem with capacity-constrained verification. They consider a setting where the principal has to allocate a set of identical objects among a group of agents who do not arrive sequentially.

In dynamic contexts, Popov (2016) studies a model of repeated risk sharing with costly verification. Li and Libgober (2023) study a dynamic principal-agent problem with costly verification. Epitropou and Vohra (2019) is the closest work to ours. They consider an allocation problem where a single indivisible good has to be allocated among a group of sequentially arriving agents where agent types can be learnt at a cost. In our model, several identical objects are to be allocated, and there is a capacity constraint on the verification technology rather than the cost of verification.

The rest of the paper is organized as follows: we present the model in section 3. In section 4, we determine the optimal deterministic policy. We study the properties of the optimal deterministic policy in Section 6. Finally, in section 7, we study whether the principal can gain by using stochastic mechanisms.

3 Model

Suppose there are $k < n$ identical positions or indivisible goods to be allocated. The offers from n agents for these positions arrive in a sequential order: first offer from agent 1 appears, then offer from agent 2, $\dots$, and then offer from agent n. Agent $j's$ type is a random draw T_j from a CDF F on $\mathcal{R}_+$ whose density f is strictly positive on a compact interval $[0, \bar{t}]$ and 0 outside it. Let $E(F)$ be denoted by μ. All agent types are independent. All agents like the position and are expected utility maximizers. The principal's payoff from allocating the position to agent j is t_j, and the payoff from not assigning the good is 0. Each agent knows their type, but the principal does not. As agents are impatient, the principal has to decide whether to accept or reject the offer for a position when it arrives. She cannot wait for all the offers to arrive before making a decision. However, the principal can perfectly verify the type of m number of agents where $m < k$. The principal's problem is to determine the optimal policy that maximizes the expected payoff from allocating k positions to n agents who arrive sequentially with a verification technology that can learn the types of m agents perfectly. Given the sequential arrival of agents with private information, the optimal policy must specify a mechanism for every possible history of past decisions. For the optimal policy, without loss of generality, we may restrict attention to direct mechanisms [2] of the following kind: each agent on her arrival is asked to report her type and truth-telling is incentive compatible for each type. Based on the reports, it is perhaps randomly determined whether an agent is verified. Finally,

[2] This can be shown using arguments similar in Ben-Porath et al. (2014).

the good is allocated also perhaps randomly based on the reports and if there was one., the verification check and its outcome. Observe that as agents' types are independently distributed, all possible histories of past decisions can be classified in terms of different states, where each state is represented by the ordered set of agents yet to arrive, the number of positions yet to be allocated, and the number of verification checks remaining. For example, the initial history can be represented by the state (n, k, m). Let $V(n', k', m')$ be the expected value to the principal at state (n', k', m') when the principal follows the optimal policy. The Principal's objective is to maximize $V(n, k, m)$.

4 Optimal policy with deterministic mechanisms

In this section, we determine the optimal policy where we restrict attention to direct deterministic mechanisms, ie, where the agent based on his report is either verified or not verified and is allocated or not allocated the position based on the reports, and if there was one, the verification check and its outcome.

Before proving the theorem, we will make an observation related to the optimal policy.

Observation 1 *At state $(n' = 1, k', m')$ where $k' \geq 1$ and $m' \geq 0$, the optimal policy for the principal is to allocate the agent with the position. Otherwise, the principal will lose because of vacant positions. Hence, the optimal value for the principal at the state $(n' = 1, k', m')$ is μ. At state $(n' = k', k', m')$, the optimal policy for the principal is to allocate each agent when she arrives with the position. The optimal value for the principal at the state (k', k', m') is $k'\mu$. Also, at state $(n', k', m' = 0)$, as there are no verification checks available to screen the arriving agents, the optimal policy for the principal is to allocate k' positions to the first k arriving agents.*

Theorem 1 *For the optimal policy, for each state (n', k', m'), where $2 \leq n' \leq n$, $1 \leq k' \leq k$ and $1 \leq m' \leq m$ there exists a cut-off $c(n', k', m') \in [\underline{t}, \overline{t}]$. If the arriving agent reports a type strictly greater than $c(n', k', m')$, then the agent is verified and is allocated the position if she reports the truth. Otherwise, the agent is not verified and not allocated the position. The cut-off $c(n', k', m')$ is given by $V(n' - 1, k', m') - V(n' - 1, k' - 1, m' - 1)$ where $V(n' - 1, k' - 1, m' - 1)$ and $V(n' - 1, k' - 1, m' - 1)$ are the continuation value from the optimal policy at state $(n' - 1, k' - 1, m' - 1)$ and $(n' - 1, k', m')$ respectively.*

Proof :

We first claim that in the optimal policy, we must have

$$V(n', k', m') > V(n', k', m' - 1) > V(n', k' - 1, m' - 1) \tag{1}$$

where $2 \leq n' \leq n$, $1 \leq k' \leq k$ and $1 \leq m' \leq m$.

First note that $V(n', k', 1) > V(n', k', 0)$ where $n' \leq n$ and $k' \leq k$. The optimal policy at state $V(n', k', 0)$ is to allocate the position randomly to any of the k' agents. Hence, $V(n', k', 0) = k'\mu$. Consider the following policy (that may not be optimal) at state $(n', k', 1)$. The arriving agent is asked to report her type; if the reported type is above μ, then the agent is verified and allocated the object. Otherwise, the agent is not verified and not allocated the object. Based on the realization of the arrival type, either state $(n' - 1, k', 1)$ or state $(n' - 1, k' - 1, 0)$ is reached. Irrespective of the state reached, the policy randomly allocates the remaining objects without verifying even if a verification check is available. The expected value if the principal follows this policy:

$$F(\mu)k\mu + (1 - F(\mu))((k - 1)\mu + E(X|X > \mu))$$

which is strictly greater than $k\mu$. Hence, we have $V(n', k', 1) > V(n', k', 0)$.

Now let ϕ be the optimal policy for the principal at state $(n', k', m'-1)$, such that $\phi(n'', k'', m'')$ denotes the mechanism at state (n'', k'', m'') where (n'', k'', m'') is the state that can be reached when the principal employs the optimal policy at state $(n', k', m' - 1)$. Now, consider a policy ψ at state (n', k', m') (that may not be optimal) such that $\psi(n'', k'', m'') = \phi(n'', k'', m'' - 1)$ for all states (n'', k'', m'') where $m'' \geq 2$. As, show above, $V(n'', k'', 1) > V(n'', k'', 0)$ for all $n'' \leq n'$ and $k'' \leq k'$. This implies that the expected value at state (n', k', m') from the optimal policy is strictly greater than $V(n', k', m' - 1)$. Using analogous arguments, we can show that $V(n', k', m'-1) > V(n', k'-1, m'-1)$, i.e. the optimal value for the principal is strictly greater when there exists an additional position to allocate.

In the optimal policy, for any state (n', k', m'), by equation 1, the principal does not verify and allocate the position to the lowest type, i.e. type 0. Observe that for the mechanism to be incentive-compatible, it must be the case that an agent who reports a type, t, is allocated the position only if she is verified and reports truthfully. Otherwise, a type that is not verified and not allocated the position has an incentive to misreport her type as t.

Consider that in the optimal policy, at the state (n', k', m'), the arriving agent with type $t' \in [0, \bar{t}]$ is verified and allocated the position. This implies,

$$t' + V(n' - 1, k' - 1, m' - 1) > V(n' - 1, k', m') \tag{2}$$

where $V(n'-1,k'-1,m'-1)$ and $V(n'-1,k'-1,m'-1)$ are the continuation value from the optimal policy at state $(n'-1,k'-1,m'-1)$ and $(n'-1,k',m')$ respectively. The left-hand side of inequality 2 is the payoff to the principal from verifying and allocating the object to type t', whereas the right-hand side represents the payoff from not allocating and not verifying. Now, consider any type $t'' > t'$. By 2, we have

$$t'' + V(n'-1,k'-1,m'-1) > V(n'-1,k',m')$$

In the optimal policy at state (n',k',m'), any type greater than t' must be verified and allocated the position. Now, suppose the agent with type t' is not verified and hence not allocated the position. This implies;

$$t' + V(n'-1,k'-1,m'-1) < V(n'-1,k',m') \tag{3}$$

where $V(n'-1,k'-1,m'-1)$ and $V(n'-1,k'-1,m'-1)$ are the continuation value from the optimal policy at state $(n'-1,k'-1,m'-1)$ and $(n'-1,k',m')$ respectively. Now, consider any type $t'' < t'$. By 3, we have;

$$t'' + V(n'-1,k'-1,m'-1) < V(n'-1,k',m')$$

In the optimal policy at state (n',k',m'), any type strictly lower than t' is not verified and not allocated the position. Hence, to determine the optimal policy, for each state, it is without loss of generality to restrict attention to cut-off-based mechanisms of the form above, where the optimal cut-off is given by

$$c(n',k',m') = V(n'-1,k',m') - V(n'-1,k'-1,m'-1) \tag{4}$$

For the optimal cut-off type, the principal is indifferent between verifying and allocating the object to the cut-off type and not verifying and not allocating to that type. Using the optimal cut-off, the principal's optimal value at the state (n',k',m'):

$$V(n',k',m') = \int_{c(n',k',m')}^{\bar{t}} [x + V(n'-1,k'-1,m'-1)]f(x)dx \quad + F(c(n',k',m'))V(n'-1,k',m') \tag{5}$$

Using observation 1 and equation 5, we can recursively determine the cut-offs for each possible state and, hence, the optimal deterministic policy. ■

A recursive equation to determine the optimal cut-offs is derived in section 6, where we study the properties of the optimal deterministic policy.

Note, by observation 1 and Theorem 1, the optimal cutoff for states where $n' = 1$ and where $n' = k'$ is 0.

5 Example

Consider an environment where the principal has 3 identical objects. These objects are to be allocated among a group of 4 impatient agents who arrive in a sequential order. The principal can perfectly verify the value of up to 2 agents. All the arriving agents' value is uniformly distributed over the interval $[0, 1]$. We aim to determine the optimal deterministic policy for this environment that maximizes the total expected value. Note that the initial state for the problem is $(4, 3, 2)$. By Theorem 1, we can restrict attention to cut-off mechanisms. Based on the realized type of first agent, two possible states may be reached. If the realized type is greater than the cut-off $c(4, 3, 2)$, then state (3,2,1) is reached as the agent is verified and allocated the object. Otherwise, state $(3, 3, 2)$ is reached. At state $(3, 3, 2)$, the number of agents yet to arrive is equal to the number of objects remaining. Hence, the optimal policy for the principal is to allocate each agent with the position. We have $V(3, 3, 2) = 3\mu = 1.5$. Now, for state $(3, 2, 1)$, based on the realization of the value of the second agent, two possible states may be reached. If the realized type is greater the the cut-off $c(3, 2, 1)$, then state $(2, 1, 0)$ is reached as the second arriving agent is verified and allocated the object. Otherwise, state $(2, 2, 1)$ is reached. At state $(2, 1, 0)$, no verification checks are left. Hence, the principal cannot screen the remaining agents. The optimal policy for the principal at this state is to allocate the remaining object to the 3rd arriving agent. We have, $V(2, 1, 0) = 0.5$. At state $(2, 2, 1)$, the optimal policy for the principal is to allocate the 2 objects to the two remaining arriving agents. Hence, we have $v(2, 2, 1) = 1$. By equation 4, we know that $c(3, 2, 1) = V(2, 2, 1) - V(2, 1, 0)$. Hence, $c(3, 2, 1) = 0.5$. Using equation 5, we have $V(3, 2, 1) = 1.125$. This determines the cut-off for the initial state, $c(4, 3, 2) = V(3, 3, 2) - V(3, 2, 1) = 0.375$.

6 Properties of the Optimal Deterministic Policy

We now seek to understand how the optimal policy changes as we perturb the parameters of our model. In particular, for any given initial state, we will show the directional change in the optimal cut-off for three cases: (i) the number of individuals yet to arrive increases, (ii). The number of positions to allocate increases, (iii) The number of available verification checks increases.

Theorem 2 *For state (n', k', m') where $n' > k' > m'$, suppose $c(n', k', m')$ is the cut-off using the optimal policy. Then,*

1. $c(n'+1, k', m') > c(n', k', m')$
2. $c(n', k'+1, m') < c(n', k', m')$
3. $c(n', k', m'+1) < c(n', k', m')$

where $c(n'+1, k', m'), c(n', k'+1, m')$ and $c(n', k', m'+1)$ are the optimal cut-offs respectively for states $(n'+1, k', m'), (n', k'+1, m')$ and $(n', k', m'+1)$

Proof : We will prove this using induction simultaneously on n', k', and m'. We will first prove the base case for 1. Note that the valid state for this case is $(n', k', m') = (3, 2, 1)$. Using observation 1 and equation 4, we have $c(3, 2, 1) = V(2, 2, 1) - V(2, 1, 0) = \mu$. Using equation 5, we have $V(3, 2, 1) = 2\mu F(\mu) + \int_{\mu}^{\bar{t}} x f(x)dx + (1 - F(\mu))\mu$. Using equation 4 and 1, we have $c(4, 2, 1) = V(3, 2, 1) - V(3, 1, 0) = \mu F(\mu) + \int_{\mu}^{\bar{t}} x f(x)dx > \mu$. Hence, we have $c(4, 2, 1) > c(3, 2, 1)$. We will now prove the base case for 2. Note that we cannot use the state $(3, 2, 1)$ for the base case because the state (3,3,1) can never be the initial state. Hence, the valid state for this case is $(n', k', m') = (4, 2, 1)$. As shown above, we have $c(4, 2, 1) = \mu F(\mu) + \int_{\mu}^{\bar{t}} x f(x)dx > \mu$. By equation 4, we have $c(4, 3, 1) = V(3, 3, 1) - V(3, 2, 0) = \mu$. Hence, we have $c(4, 2, 1) > c(4, 3, 1)$. Lastly, we will prove the base case for 3. Note that due to the reasons as above, the valid state to consider for the base case is $(n', k', m') = (4, 3, 1)$. Using 4 and 1, we have $c(4, 3, 1) = V(3, 3, 1) - V(3, 2, 0) = \mu$. Also, $c(4, 3, 2) = V(3, 3, 2) - V(3, 2, 1)$. As shown above, $V(3, 2, 1) > 2\mu$. Hence, $c(4, 3, 2) = 3\mu - V(3, 2, 1) < \mu = c(4, 3, 1)$. Thereby, proving the base case for 1, 2 and 3. Now, suppose the hypothesis holds for all possible (n, k, m) where $n < n'$, for all possible (n, k, m) where $k < k'$, and for all possible (n, k, m), where $m < m'$. We will now prove it for the state (n', k', m'). We can write the optimal cut-off at the state (n', k', m') as:

$$\begin{aligned}
& c(n',k',m') \\
= \;& V(n'-1,k',m') - V(n'-1,k'-1,m'-1) && \textit{using equation 4} \\
= \;& F(c(n'-1,k',m'))V(n'-2,k',m') + \int_{c(n'-1,k',m')}^{\bar{t}} xf(x)dx \\
& +(1-F((c(n'-1,k',m'))V(n'-2,k'-1,m'-1) - V(n'-1,k'-1,m'-1) && \textit{using equation 5} \\
= \;& \int_{c(n'-1,k',m')}^{\bar{t}} xf(x)dx + F(c(n'-1,k',m'))(c(n'-1,k',m') \\
& -(V(n'-1,k'-1,m'-1) - V(n'-2,k'-1,m'-1)) && \textit{using equation 4}
\end{aligned}$$

Now, again using equation 4 and equation 5, we can write,

$$\begin{aligned}
& V(n'-1,k'-1,m'-1) - V(n'-2,k'-1,m'-1) \\
= \;& \int_{c(n'-1,k'-1,m'-1)}^{\bar{t}} xf(x)dx - (1-F(c(n'-1,k',m'))c(n'-1,k'-1,m'-1)
\end{aligned} \tag{6}$$

By the induction hypothesis, we have $c(n'-1,k'-1,m'-1) > c(n'-1,k',m')$. Using equation 6 in the expression for $c(n',k',m')$, we have

$$\begin{aligned}
& c(n',k',m') \\
= \;& F(c(n'-1,k',m'))c(n'-1,k',m') + \int_{c(n'-1,k',m')}^{c(n'-1,k'-1,m'-1)} xf(x)dx \\
& +(1-F(c(n'-1,k'-1,m'-1))c(n'-1,k'-1,m'-1)
\end{aligned} \tag{7}$$

We will first prove that $c(n'+1,k',m') > c(n',k',m')$, where using equation 7, we can write

$$\begin{aligned}
& c(n'+1,k',m') \\
= \;& F(c(n',k',m'))c(n',k',m') + \int_{c(n',k',m')}^{c(n',k'-1,m'-1)} xf(x)dx \\
& +(1-F(c(n',k'-1,m'-1))c(n',k'-1,m'-1)
\end{aligned}$$

Note that by the induction hypothesis, we have $c(n'-1,k',m') < c(n'-1,k'-1,m'-1)$ and $c(n'-1,k',m') < c(n',k',m') < c(n',k'-1,m'-1)$. Using the expression for $c(n'+1,k',m')$ and equation 7, if $c(n'-1,k'-1,m'-1) < c(n',k',m')$, then we have $c(n'+1,k',m') > c(n',k',m')$.

Now, consider the case, where $c(n'-1,k'-1,m'-1) \geq c(n',k',m')$. Then,

$$
\begin{aligned}
& c(n'+1,k',m') - c(n',k',m') \\
= \; & F(c(n'-1,k',m'))(c(n',k',m') - c(n'-1,k',m')) + \int_{c(n'-1,k',m')}^{c(n',k',m')} (c(n',k',m') - x) f(x) dx \\
& + \int_{c(n'-1,k'-1,m'-1)}^{c(n',k'-1,m'-1)} (x - c(n',k',m')) f(x) dx \\
& +(1 - F(c(n',k'-1,m'-1))(c(n',k'-1,m'-1) - c(n'-1,k'-1,m'-1)) \\
> \; & 0
\end{aligned}
$$

Now, we will prove that $c(n',k',m') > c(n',k'+1,m')$. Note that by the induction hypothesis, we have $c(n'-1,k'+1,m') < c(n'-1,k',m') < c(n'-1,k',m'-1) < c(n'-1,k'-1,m'-1)$. Then,

$$
\begin{aligned}
& c(n',k',m') - c(n',k'+1,m') \\
= \; & F(c(n'-1,k'+1,m'))(c(n'-1,k',m') - c(n'-1,k+1,m')) \\
& + \int_{c(n'-1,k'+1,m')}^{c(n'-1,k',m')} (c(n'-1,k',m') - x) f(x) dx \\
& + \int_{c(n'-1,k',m'-1)}^{c(n',k'-1,m'-1)} (x - c(n'-1,k',m'-1)) f(x) dx \\
& +(1 - F(c(n'-1,k'-1,m'-1))(c(n'-1,k'-1,m'-1) - c(n'-1,k',m'-1)) \\
> \; & 0
\end{aligned}
$$

Finally, we will prove that $c(n',k',m') > c(n',k',m'+1)$. Note that by the induction hypothesis, we have $c(n'-1,k',m'+1) < c(n'-1,k',m') < c(n'-1,k'-1,m') < c(n'-1,k'-1,m'-1)$. Then,

$$
\begin{aligned}
& c(n',k',m') - c(n',k',m'+1) \\
= \; & F(c(n'-1,k',m'+1))(c(n'-1,k',m') - c(n'-1,k,m'+1)) \\
& + \int_{c(n'-1,k',m'+1)}^{c(n'-1,k',m')} (c(n'-1,k',m') - x) f(x) dx \\
& + \int_{c(n'-1,k'-1,m')}^{c(n',k'-1,m'-1)} (x - c(n'-1,k'-1,m')) f(x) dx \\
& +(1 - F(c(n'-1,k'-1,m'-1))(c(n'-1,k'-1,m'-1) - c(n'-1,k'-1,m')) \\
> \; & 0
\end{aligned}
$$

Hence, proved. ■

Consider the first best environment, i.e. where the principal is informed of the agent's type on her arrival. The optimal policy for this environment again consists of cut-off mechanisms, as shown in Derman et al. (1972). As there is complete information regarding the agent's type on their arrival, all possible contingencies can be classified in terms of states (n, k), where n is the number of agents yet to arrive and k is the number of positions available. Equivalently, the states can be represented as $(n, k, m = n)$, where the number of verification checks available, m, is exactly equal to the number of individuals yet to arrive, n. Using Theorem 2, we have the optimal cut-off at each state when the principal does not know the agent's type is strictly greater than the optimal cut-off in the first best environment.

7 Can the Principal Gain by using Stochastic Mechanisms?

In the optimal deterministic policy, for any given state, all types of the arriving agents above the cut-off are verified with probability 1, and all types below the cut-off are allocated the position with probability 0. Given the capacity constraint on verification checks and as the optimal cutoff for deterministic policy is strictly greater than the first best environment, we consider whether the principal can gain by using stochastic mechanisms. We restrict attention to direct mechanisms of the following kind. For each state, each agent on her arrival is asked to report her type and truth-telling is incentive compatible. An agent who reports type x is verified with probability $p(x)$ and is allocated the position only if he reports truthfully. If the agent with type x is not verified (with probability$(1 - p(x))$, then she is allocated the position with probability $q(x)$. We aim to determine the optimal policy when the principal can use the larger class of mechanisms.

Theorem 3 *The optimal deterministic policy is the optimal policy.*

Proof : Fix state (n, k, m). We will first show that in the optimal mechanism, $p(0) = 0$, the agent with the lowest type should be verified with probability 0. Suppose that in the optimal mechanism $p(0) > 0$. Principal's payoff if agent with type 0 realizes:

$$p(0)V(n-1,k-1,m-1)+(1-p(0))q(0)V(n-1,k-1,m)+(1-p(0))(1-q(0))V(n-1,k,m) \quad (8)$$

We first claim that in the optimal policy, we must have

$$V(n-1,k,m) > V(n-1,k,m-1) > V(n-1,k-1,m-1) \quad (9)$$

First note that $V(n', k', 1) > V(n', k', 0)$ where $n' \leq n$ and $k' \leq k$. The optimal policy at state $V(n', k', 0)$ is to allocate the position to the first k' agents. Using the optimal deterministic policy at state $(n', k', 1)$, we have the payoff from this policy strictly greater than $V(n', k', 0)$. Thus we have $V(n', k', 1) > V(n', k', 0)$. Now let ϕ be the optimal policy for the principal at state $(n, k, m-1)$, such that $\phi(n', k', m')$ denotes the mechanism at state (n', k', m') where (n', k', m') is the state that can be reached when the principal employs the optimal policy at state $(n, k, m-1)$. Now, consider a policy ψ at state (n, k, m) (that may not be optimal) such that $\psi(n', k', m') = \phi(n', k', m'-1)$ for all states $(n', k', m)'$ where $m \geq 2$ and let the $\psi(n', k', 1)$ be the mechanism from the optimal deterministic policy. As, $V(n', k', 1) > V(n', k', 0)$ for all $n' \leq n$ and $k' \leq k$, the value at state (n, k, m) from policy ψ is strictly greater than $V(n-1, k, m-1)$. Hence, we have $V(n, k, m) > V(n, k, m-1)$. Using analogous arguments, we can show that $V(n, k-1, m) > V(n-1, k-1, m-1)$, i.e., the optimal value for the principal is strictly greater when there exists an additional position to allocate.

This implies that equation 8 is strictly decreasing in $p(0)$. Hence, $p(0) = 0$.

We will now show that if in the optimal mechanism, $p(c) = 0$, then for all $c' < c$, $p(c') = 0$. Note that, $p(c) = 0$ implies that:

$$\begin{aligned} & q(c)(c + V(n-1, k-1, m) + (1-q(c))V(n-1, k, m) \\ > \; & p(c + V(n-1, k-1, m-1)) + (1-p)q(c + V(n-1, k-1, m)) \\ + \; & (1-p)(1-q)V(n-1, k, m) \end{aligned} \tag{10}$$

for all $p \in (0, 1]$ and for all $q \in [0, 1]$ where p represents the probability of verifying and q represents the probability that the object is allocated if type c is not verified. For any p, if $q < q(c)$, then by inequality 9, equation 10 holds for all $c \in [0, \bar{t}]$. If $q \geq q(c)$, rerwriting inequality 10,

$$\begin{aligned} & (q(c) - (1-p)q)V(n-1, k-1, m) + ((1 - q(c) - (1-p)(1-q))V(n-1, k, m) \\ + \; & pV(n-1, k-1, m-1) \\ > \; & c(p + (1-p)q - q(c)) \end{aligned}$$

for all $p \in (0, 1]$ and for all $q \in [0, 1]$. This implies that for every $c' < c$,

$$\begin{aligned} & (q(c) - (1-p)q)V(n-1, k-1, m) + ((1 - q(c)) - (1-p)(1-q))V(n-1, k, m) \\ + \; & pV(n-1, k-1, m-1) \\ > \; & c'(p + (1-p)q - q(c)) \end{aligned}$$

for all $p \in (0, 1]$ and for all $q \in [0, 1]$. Hence, $p(c') = 0$ for all $c' < c$.

We will now show that if in the optimal mechanism $p(c) > 0$ for type c, then for all $c' > c$, $p(c') > 0$. In the optimal policy $p(c) > 0$ implies that;

$$\begin{aligned} & p(c)(c + V(n-1, k-1, m-1)) + (1 - p(c))q(c + V(n-1, k-1, m)) \\ + & (1 - p(c))(1 - q)V(n-1, k, m) \\ > & q(c + V(n-1, k-1, m)) + (1 - q)V(n-1, k, m) \end{aligned}$$

for all $q \in [0, 1]$ where q is the probability of allocating the object when the agent is not verified. Rewriting the above inequality as,

$$\begin{aligned} & (q - (1 - p(c))q)V(n-1, k-1, m) + ((1 - q) - (1 - p(c))(1 - q))V(n-1, k, m) \\ + & p(c)V(n-1, k-1, m-1) \\ < & c(p(c) + (1 - p(c))q - q) \end{aligned}$$

for all $q \in [0, 1]$. This implies that for every $c' > c$;

$$\begin{aligned} & (q - (1 - p(c))q)V(n-1, k-1, m) + ((1 - q) - (1 - p(c))(1 - q))V(n-1, k, m) \\ + & p(c)V(n-1, k-1, m-1) \\ < & c'(p(c) + (1 - p(c))q - q) \end{aligned}$$

for all $q \in [0, 1]$. Hence, in the optimal policy for all $c' > c$, $p(c') > 0$.

Without loss, we can restrict attention to mechanisms where there exists a cut-off $c \geq 0$ such that $p(c') = 0$ for all $c' \leq c$ and $p(c') > 0$ for all $c' > c$.

To determine the optimal policy, the principal has to choose the cut-off c, $p(x)$ for all $x > c$ and $q(x)$ for all $x \in [0, \bar{t}]$ to maximize

$$\begin{aligned} & \int_0^c q(x)(x + V(n-1, k-1, m))f(x)dx \\ + & \int_0^c (1 - q(x))V(n-1, k, m)f(x)dx \\ + & \int_c^{\bar{t}} p(x)(x + V(n-1, k-1, m-1))f(x)dx \\ + & \int_c^{\bar{t}} (1 - p(x))q(x)(x + V(n-1, k-1, m))f(x)dx \\ + & \int_c^{\bar{t}} (1 - p(x))(1 - q(x))V(n-1, k, m)f(x)dx \end{aligned} \tag{11}$$

subject to incentive compatibility conditions.

We will now characterize incentive-compatible mechanisms within this class of mechanisms. Suppose in the optimal mechanism, we have $p(c') = p(c'') = 0$ where c' *and* $c'' < c$. Then, we must have $q(c') = q(c'')$ by incentive compatibility. If $q(c') > q(c'')$, then c'' has an incentive to misreport to c'. Hence, in the optimal mechanism with cut-off c, we have $q(c') = q(c)$ for all $c' < c$. Let us denote $q(c) = q^*$. Hence, we can rewrite the maximization problem as:

$$\begin{aligned}
& F(c)(q^*(E(x|x \leq c) + V(n-1, k-1, m)) + (1-q^*)V(n-1, k, m)) \\
+ & \int_c^{\bar{t}} p(x)(x + V(n-1, k-1, m-1))f(x)dx \\
+ & \int_c^{\bar{t}} (1-p(x))q(x)(x + V(n-1, k-1, m))f(x)dx \\
+ & \int_c^{\bar{t}} (1-p(x))(1-q(x))V(n-1, k, m)f(x)dx
\end{aligned} \tag{12}$$

To ensure that a type below c does not misreport to a type strictly greater than c, in the optimal mechanism, we must have

$$q^* \geq (1-p(x))q(x) \tag{13}$$

for all $x > c$.

We first claim that, equation 13 must be binding for every $x > c$. Suppose not. Consider any type $x > c$ such that in the optimal mechanism, $q^* > (1-p(x))q(x)$. For this to hold, it must be that

$$E(x|x \leq c) \geq V(n-1, k, m) - V(n-1, k-1, m) \tag{14}$$

otherwise, the principal can reduce q^* and improve its payoff. Also, we must have:

$$x \leq V(n-1, k, m) - V(n-1, k-1, m) \quad for\ all\ x > c \tag{15}$$

otherwise, the principal can increase $q(x)$ and improve its payoff. However, condition 14 and 15 cannot simultaneously hold as $x > c$.

Thus, in the optimal mechanism, we have

$$q^* = (1-p(x))q(x) \tag{16}$$

for all $x > c$.

For any type $x > c$, incentive compatibility requires that $p(x) + (1 - p(x))q(x) \geq q^*$ and $p(x) + (1 - p(x))q(x) \geq (1 - p(y))q(y)$ for all $y > c$. Equation 16 and $p(x) > 0$ ensure that the incentive constraints for types strictly greater than c are satisfied.

Using equation 16, we can rewrite the maximization problem as where the principal has to choose the cut-off c, q^* and $p(x)$ for all $x \in [c, \bar{t}]$ to maximize

$$\begin{aligned} & F(c)(q^*(E(x|x \leq c) + V(n-1, k-1, m)) + (1 - q^*)V(n-1, k, m))) \\ + & \int_c^{\bar{t}} p(x)(x + V(n-1, k-1, m-1))f(x)dx \\ + & \int_c^{\bar{t}} q^*(x + V(n-1, k-1, m))f(x)dx \\ + & \int_c^{\bar{t}} (1 - p(x))V(n-1, k, m)f(x)dx \\ - & (1 - F(c))q^*V(n-1, k, m) \end{aligned} \tag{17}$$

As argued above at the cut-off c, $p(c) = 0$. For the principal not to have an incentive to increase $p(c)$ for type c, we must have

$$c + V(n-1, k-1, m-1) \leq V(n-1, k, m) \tag{18}$$

As c is the optimal cutoff, inequality 18 must be binding. Hence, we have

$$c = V(n-1, k, m) - V(n-1, k-1, m-1) \tag{19}$$

Now, consider type $x > c$. Differentiating expression 17 with respect to $p(x)$, we have using equation 19

$$x + V(n-1, k-1, m-1) - V(n-1, k, m) > 0$$

Thus, for all $x > c$, we must have $p(x) = 1$. This also implies from equation 16 that $q^* = 0$. Note that the optimal cut-off c is the same as in the optimal deterministic policy. Hence, proved. ■

References

Ben-Porath, E., Dekel, E., and Lipman, B. L. (2014). Optimal allocation with costly verification. *American Economic Review*, 104(12):3779–3813.

Ben-Porath, E., Dekel, E., and Lipman, B. L. (2019). Mechanisms with evidence: Commitment and robustness. *Econometrica*, 87(2):529–566.

Derman, C., Lieberman, G. J., and Ross, S. M. (1972). A sequential stochastic assignment problem. *Management Science*, 18(7):349–355.

Epitropou, M. and Vohra, R. (2019). Optimal on-line allocation rules with verification. In *Algorithmic Game Theory: 12th International Symposium, SAGT 2019, Athens, Greece, September 30–October 3, 2019, Proceedings 12*, pages 3–17. Springer.

Erlanson, A. and Kleiner, A. (2020). Costly verification in collective decisions. *Theoretical Economics*, 15(3):923–954.

Erlanson, A. and Kleiner, A. (2024). Optimal allocations with capacity constrained verification. *arXiv preprint arXiv:2409.02031*.

Gershkov, A. and Moldovanu, B. (2010). Efficient sequential assignment with incomplete information. *Games and Economic Behavior*, 68(1):144–154.

Halac, M. and Yared, P. (2020). Commitment versus flexibility with costly verification. *Journal of Political Economy*, 128(12):4523–4573.

Li, Z. and Libgober, J. (2023). The dynamics of verification when searching for quality. *Available at SSRN 4430184*.

Popov, L. (2016). Stochastic costly state verification and dynamic contracts. *Journal of Economic Dynamics and Control*, 64:1–22.