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Random walks in I.I.D. random environment on Cayley trees

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ABSTRACT

We consider the random walk in an *i.i.d.* random environment on the infinite d-regular tree for $d \geq 3$. We consider the tree as a Cayley graph of the free product of finitely many copies of $\mathbb{Z}$ and $\mathbb{Z}_2$ and define the i.i.d. environment as invariant under the action of this group. Under a mild non-degeneracy assumption we show that the walk is always transient.

1. Introduction

In this short note we consider a random walk in random environment (RWRE) model on a regular tree with degree $d \geq 3$, where the environment at the vertices is *independent* and is also "*identically distributed*" (i.i.d.). We make this notion of *i.i.d. environment* rigorous by first defining a translation invariant model on a group G which is a free product of finitely many groups, $G_1, G_2, \ldots, G_k$ and $H_1, H_2, \ldots, H_r$, where each $G_i \cong \mathbb{Z}$ and each $H_j \cong \mathbb{Z}_2$ with $d = 2k + r$. Observing the fact that the Cayley graph of G is a regular tree with degree d, we transfer back the model on the d-regular tree we started with. We prove that under a mild non-degeneracy assumption such a walk is always transient.

1.1. Basic setup

Cayley graph: Let G be a group defined above, that is, G is a free product of $k + r \geq 2$ groups, namely $G_1, G_2, \ldots, G_k$ with $k \geq 0$ and $H_1, H_2, \ldots, H_r$ with $r \geq 0$, where each $G_i \cong \mathbb{Z}$ and each $H_j \cong \mathbb{Z}_2$ and $d = 2k + r \geq 3$. Suppose $G_i = \langle a_i \rangle$ for $1 \leq i \leq k$ and $H_j = \langle b_j \rangle$ where $b_j^2 = e$ for $1 \leq j \leq r$. Here by $\langle a \rangle$ we mean the group generated by a single element a. Let $S := \{a_1, a_2, \ldots, a_k\} \cup \left\{a_1^{-1}, a_2^{-1}, \ldots, a_k^{-1}\right\} \cup \{b_1, b_2, \ldots, b_r\}$ be a generating set for G. We note that S is a symmetric set, that is, $s \in S \iff s^{-1} \in S$.

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We now define a graph $\bar{G}$ with vertex set G and edge set $E := \left\{\{x, y\} \,\middle|\, yx^{-1} \in S\right\}$. Such a graph $\bar{G}$ is called a *(left) Cayley Graph* of G with respect to the generating set S. Since G is a free product of groups which are isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2$, it is easy to see that $\bar{G}$ is a graph with no cycles and is regular with degree d, thus it is isomorphic to the d-regular infinite tree which we will denote by $\mathbb{T}_d$. We will abuse the terminology a bit and will write $\mathbb{T}_d$ for the Cayley graph of G. We will consider the identity element e of G as the root of $\mathbb{T}_d$. We will write $N(x)$ for the set of all neighbors of a vertex $x \in \mathbb{T}_d$. Notationally, $N(x) = \left\{y \in G \,\middle|\, yx^{-1} \in S\right\}$. Observe that from definition $N(e) = S$. For $x \in G$, define the mapping $\theta_x : G \to G$ by $\theta_x(y) = yx$, then θ_x is an automorphism of $\mathbb{T}_d$. We will call θ_x the *translation by* x. For a vertex $x \in \mathbb{T}_d$ and $x \neq e$, we denote by $|x|$, the length of the unique path from the root e to x and $|e| = 0$. Further, if $x \in \mathbb{T}_d$ and $x \neq e$, then we define $\overleftarrow{x}$ as the *parent* of x, that is, the penultimate vertex on the unique path from e to x.

Random Environment: Let $\mathcal{S} := \mathcal{S}_e$ be a collection of probability measures on the d elements of $N(e) = S$. To simplify the presentation and avoid various measurability issues, we assume that $\mathcal{S}$ is a Polish space (including the possibilities that $\mathcal{S}$ is finite or countably infinite). For each $x \in \mathbb{T}_d$, $\mathcal{S}_x$ is the push-forward of the space $\mathcal{S}$ under the translation θ_x, that is, $\mathcal{S}_x := \mathcal{S} \circ \theta_x^{-1}$. Note that the probabilities on $\mathcal{S}_x$ have support on $N(x)$. That is to say, an element $\omega(x, \cdot)$ of $\mathcal{S}_x$, is a probability measure satisfying

$$\omega(x, y) \geq 0 \quad \forall y \in \mathbb{T}_d \quad \text{and} \quad \sum_{y \in N(x)} \omega(x, y) = 1.$$

Let $\mathcal{B}_{\mathcal{S}_x}$ denote the Borel σ-algebra on $\mathcal{S}_x$. The *environment space* is defined as the measurable space $(\Omega, \mathcal{F})$ where

$$\Omega := \prod_{x \in \mathbb{T}_d} \mathcal{S}_x, \qquad \mathcal{F} := \bigotimes_{x \in \mathbb{T}_d} \mathcal{B}_{\mathcal{S}_x}. \tag{1}$$

An element $\omega \in \Omega$ will be written as $\left\{\omega(x, \cdot) \,\middle|\, x \in \mathbb{T}^d\right\}$. An environment distribution is a probability P on $(\Omega, \mathcal{F})$. We will denote by E the expectation taken with respect to the probability measure P.

Random Walk: Given an environment $\omega \in \Omega$, a random walk $(X_n)_{n \geq 0}$ is a time homogeneous Markov chain taking values in $\mathbb{T}_d$ with transition probabilities

$$\mathbf{P}_\omega\left(X_{n+1} = y \,\middle|\, X_n = x\right) = \omega(x, y).$$

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each $\omega \in \Omega$, we denote by $\mathbf{P}_\omega^x$ the law induced by $(X_n)_{n \geq 0}$ on $\left((\mathbb{T}_d)^{\mathbb{N}_0}, \mathcal{G}\right)$, where $\mathcal{G}$ is the σ-algebra generated by the cylinder sets, such that

$$\mathbf{P}_\omega^x(X_0 = x) = 1. \tag{2}$$

The probability measure $\mathbf{P}_\omega^x$ is called the *quenched law* of the random walk $(X_n)_{n \geq 0}$, starting at x. We will use the notation $\mathbf{E}_\omega^x$ for the expectation under the quenched measure $\mathbf{P}_\omega^x$.

Following Zeitouni (2004), we note that for every $B \in \mathcal{G}$, the function

$$\omega \mapsto \mathbf{P}_\omega^x(B)$$

is $\mathcal{F}$-measurable. Hence, we may define the measure $\mathbb{P}^x$ on $\left(\Omega \times (\mathbb{T}_d)^{\mathbb{N}_0}, \mathcal{F} \otimes \mathcal{G}\right)$ by the relation

$$\mathbb{P}^x(A \times B) = \int_A \mathbf{P}_\omega^x(B)\, P(d\omega), \quad \forall A \in \mathcal{F}, B \in \mathcal{G}.$$

With a slight abuse of notation, we also denote the marginal of $\mathbb{P}^x$ on $(\mathbb{T}_d)^{\mathbb{N}_0}$ by $\mathbb{P}^x$, whenever no confusion occurs. This probability distribution is called the *annealed law* of the random walk $(X_n)_{n \geq 0}$, starting at x. We will use the notation $\mathbb{E}^x$ for the expectation under the annealed measure $\mathbb{P}^x$.

1.2. Main results

Throughout this paper we will assume that the following holds:

(A1) P is a product measure on $(\Omega, \mathcal{F})$ with "*identical*" marginals, that is, under P the random probability laws $\left\{\omega(x, \cdot) \,\middle|\, x \in \mathbb{T}^d\right\}$ are independent and "identically" distributed in the sense that

$$P \circ \theta_x^{-1} = P, \tag{3}$$

for all $x \in G$.

(A2) For all $1 \leq i \leq d$,

$$E\left[|\log \omega(e, s_i)|\right] < \infty. \tag{4}$$

It is worth noting that under this assumption $\omega(x, y) > 0$ almost surely (a.s.) with respect to the measure P for all $x \in \mathbb{T}_d$ and $y \in N(x)$.

The following is our main result.

Theorem 1. *Under assumptions* (A1) *and* (A2) *the random walk* $(X_n)_{n\geq 0}$ *is transient* $\mathbb{P}^e$*-a.s., that is,* $\mathbb{P}^e\left(\lim_{n\to\infty}|X_n| = \infty\right) = 1$.

An immediate question that arises is whether the above walk has a speed which may be zero. The following result provides a partial answer to this question with (A2) replaced by the usual *uniform ellipticity* condition.

(A3) There exists $\epsilon > 0$ such that

$$P\left(\omega\left(e, s_i\right) > \epsilon \ \forall 1 \leq i \leq d\right) = 1. \tag{5}$$

Theorem 2. *Under assumptions* (A1) *and* (A3) *with* $\epsilon > \frac{1}{2(d-1)}$ *we have* $\mathbb{P}^e$*-a.s.*

$$\liminf_{n\to\infty} \frac{|X_n|}{n} > 0. \tag{6}$$

Note that the condition $\epsilon > \frac{1}{2(d-1)}$ is compatible with the ellipticity condition (5) as $d \geq 3$.

1.3. Remarks

Random walk in Random Environment (RWRE) model on the one dimensional integer lattice $\mathbb{Z}$ was first introduced by Solomon (1975) where he gave explicit criteria for the recurrence and transience of the walk for i.i.d. environment. Since then a large variety of results have been discovered for RWRE in $\mathbb{Z}^d$, yet there are many challenging problems which are still left open (see Bogachev et al. (2006) for a non-technical survey and Zeitouni (2004), Sznitman (2004) for more technical details).

Perhaps the earliest known results for RWRE on trees are by Lyons and Pemantle (1992). In that paper they consider a model on rooted trees known as *random conductance model*. In that model, the random conductances along each path from vertices to the root are assumed to be independent and identically distributed. The random walk is then shown to be recurrent or transient depending on how large the value of the average conductance is.

In our setup, the assumption (A1) essentially says that the random transition laws $\left\{\omega\left(x,\cdot\right)\middle| x \in \mathbb{T}^d\right\}$ are *independent and identically distributed (i.i.d.)*. On $\mathbb{T}_d$ we introduced the group structure to define *identically distributed* and we made the probability law P invariant under *translations* by the group elements. Hence the RWRE model in this article is different from the *random conductance model* discussed above. It is interesting to note that the only example where the two models agree is the deterministic environment of the *simple symmetric walk* on $\mathbb{T}_d$.

Perhaps the model closest to ours was introduced by Rozikov (2001) where the author considered the case with $k = 0$ and $r = d \geq 3$, that is, the group G is a free product for d copies of $\mathbb{Z}_2$. Our model is slightly more general from this perspective, but more importantly in Rozikov (2001) to prove transience, it was assumed that

$$E\left[\left|\log\frac{\omega\left(x, sx\right)}{\omega\left(x, s'x\right)}\right|\right] < \infty \tag{7}$$

and

$$E\left[\log\frac{\omega\left(x, sx\right)}{\omega\left(x, s'x\right)}\right] \neq 0, \tag{8}$$

for every $x \in \mathbb{T}_d$ and for two different elements $s, s' \in S$ (see the assumption made in the beginning of Section 7 and Theorem 5 of Rozikov (2001)). The first assumption (7) made in Rozikov (2001) is more general than our assumption (A2). However, the second assumption made in Rozikov (2001), namely Eq. (8), may not be satisfied by certain environments (be it random or non-random) which are covered by our formulation, for example, the condition (8) is not satisfied by the *simple symmetric random walk* on $\mathbb{T}_d$. So neither our model is a subclass of the models studied by Rozikov (2001), nor our model covers all cases discussed in there. So we believe our work is an important addition to the earlier work of Rozikov (2001) and none makes the other redundant. We would also like to point out that the techniques used in our work are entirely different from that of Rozikov (2001).

There have also been several other contributions on random trees, particularly on random walk on Galton–Watson trees (Lyons, 1990; Lyons et al., 1995, 1996; Dembo et al., 2002; Peres and Zeitouni, 2008). It is worth pointing out here that a random walk on a Galton–Watson tree (Lyons, 1990) satisfies the assumption (A1) and so does a random walk on a multi-type Galton–Watson tree (Dembo and Sun, 2012).

Our last result (Theorem 2) is certainly far from satisfactory. We strongly believe that under the assumption (A1) and (A3) the sequence of random variables $\left(\frac{|X_n|}{n}\right)_{n\geq 0}$ has a $\mathbb{P}^e$-almost sure limit which is non-random and strictly positive. A similar conclusion has been derived for the special case of random walk on Galton–Watson trees (Lyons et al., 1995). This and the central limit theorem for such walks will be studied in future work.

2. Proofs of the main results

2.1. Proof of Theorem 1

Given an environment $\omega \in \Omega$, and a vertex $\sigma \in \mathbb{T}_d$ which is not the root, we define the conductance of the edge $\left\{\sigma, \overleftarrow{\sigma}\right\}$ as

$$C\left(\sigma, \overleftarrow{\sigma}\right) = \omega(e, x_1) \left(\prod_{k=1}^{|\sigma|-1} \frac{\omega(x_k, x_{k+1})}{\omega(x_k, x_{k-1})} \right), \tag{9}$$

where $e = x_0, x_1, x_2, \ldots, x_{|\sigma|-1} = \overleftarrow{\sigma}, x_{|\sigma|} = \sigma$ is the unique path from the root e to the vertex σ. Further we define $\Phi\left(\sigma\right) := C\left(\sigma, \overleftarrow{\sigma}\right)^{-1}$. Suppose $\sigma = \alpha_n\alpha_{n-1}\cdots\alpha_1$ where $\alpha_i \in S$ and $\alpha_{i+1} \neq \alpha_i^{-1}$ and $n = |\sigma|$. More generally $x_k = \alpha_k\alpha_{k-1}\cdots\alpha_1$ for $1 \le k \le n$. Note that $x_k = \alpha_k x_{k-1}$ for all $1 \le k \le n$. We can now rewrite $\Phi\left(\sigma\right)$ as

$$\Phi\left(\sigma\right) = \frac{1}{\omega(e, x_1)} \prod_{k=1}^{n-1} \frac{\omega(x_k, x_{k-1})}{\omega(x_k, x_{k+1})} = \left(\prod_{k=1}^{n-1} \frac{\omega_k\left(\alpha_k^{-1}\right)}{\omega_{k-1}\left(\alpha_k\right)} \right) \frac{1}{\omega_{n-1}\left(\alpha_n\right)}, \tag{10}$$

where we write $\omega_k\left(s\right) := \omega\left(x_k, sx_k\right)$ for any $s \in S$.

We will now show that there is a (non-random) sequence of positive real numbers $(\beta_n)_{n\ge1}$ such that $\sum_{n=1}^{\infty} \beta^n < \infty$ and P-a.s.

$$\lim_{n\to\infty} \sum_{\sigma_n \in \mathbb{T}_d^n} \beta_n \left(\Phi\left(\sigma_n\right)\right)^{-1} = \infty, \tag{11}$$

where $\mathbb{T}_d^n := \left\{ x \in \mathbb{T}_d \,\middle|\, |x| = n \right\}$. Then by Corollary 4.2 in Lyons (1990), the random walk has to be transient. For this we will show that P-a.s., there is a subset of vertices of $\mathbb{T}_d^n$ with size $O\left((d-1)^{n-1}\right)$ such that the Φ-value of these vertices is strictly smaller than $(d-1)^{\frac{n}{2}}$.

Let $\mathcal{B}_{\mathbb{N}_0}$ denote the product σ-algebra on $S^{\mathbb{N}_0}$, and μ be a probability measure on $\left(S^{\mathbb{N}_0}, \mathcal{B}_{\mathbb{N}_0}\right)$ such that $(Y_n)_{n\ge0} \in S^{\mathbb{N}_0}$ forms a Markov chain on S with

$$\mu\left(Y_n = s \,\middle|\, Y_{n-1} = t\right) = \frac{1}{d-1}, \quad s, t \in S \text{ with } s \neq t^{-1}. \tag{12}$$

It is easy to see that the chain $(Y_n)_{n\ge0}$ is an aperiodic, irreducible and finite state Markov chain and its stationary distribution is the uniform distribution on S. We shall assume that Y_0 is uniformly distributed on S. Thus each Y_n is also uniform on S.

Let $\eta_n = Y_nY_{n-1}\cdots Y_1$. From Eq. (12) it follows that η_n is uniformly distributed on the set of vertices $\mathbb{T}_d^n$. Now

$$\begin{aligned} \frac{1}{n}\log\Phi\left(\eta_n\right) &= o\left(1\right) + \frac{1}{n}\sum_{k=1}^{n-1}\left(\log\omega_k\left(Y_k^{-1}\right) - \log\omega_{k-1}\left(Y_k\right)\right) \\ &= o\left(1\right) + \frac{1}{n}\sum_{s\in S}\sum_{j=1}^{N_{n-1}\left(s^{-1}\right)}\left(\log\omega_{k_j\left(s^{-1}\right)}\left(s^{-1}\right) - \log\omega_{k_j\left(s^{-1}\right)-1}\left(s\right)\right), \end{aligned} \tag{13}$$

where for each $s \in S$, $k_1\left(s^{-1}\right), k_2\left(s^{-1}\right)\ldots, k_{N_{n-1}\left(s^{-1}\right)}\left(s^{-1}\right)$ are the time points k when $Y_k = s^{-1}$ and

$$N_n\left(s\right) = \sum_{k=1}^{n} \mathbf{1}\left(Y_k = s\right). \tag{14}$$

Now consider the product space $\left(\Omega \times S^{\mathbb{N}_0}, \mathcal{B}_\Omega \otimes \mathcal{B}_{\mathbb{N}_0}, P \otimes \mu\right)$. By Theorems 6.5.5 and 6.6.1 of Durrett (2010) we have $P \otimes \mu$-a.s. for all $s \in S$,

$$\lim_{n\to\infty} \frac{N_n\left(s\right)}{n} = \frac{1}{d}. \tag{15}$$

Further under assumption (A2) and using the Strong Law of Large Numbers for i.i.d. random variables we have P-a.s., for every fixed $s \in S$,

$$\lim_{n\to\infty} \frac{1}{N_{n-1}\left(s^{-1}\right)} \sum_{j=1}^{N_{n-1}\left(s^{-1}\right)} \log\omega_{k_j\left(s^{-1}\right)}\left(s^{-1}\right) = E\left[\log\omega_1\left(s^{-1}\right)\right],$$

and also

$$\lim_{n\to\infty} \frac{1}{N_{n-1}\left(s^{-1}\right)} \sum_{j=1}^{N_{n-1}\left(s^{-1}\right)} \log\omega_{k_j\left(s^{-1}\right)-1}\left(s\right) = E\left[\log\omega_1\left(s\right)\right].$$

As S is a symmetric set of generators for G, therefore $P \otimes \mu$-a.s.,

$$\lim_{n\to\infty} \frac{1}{n} \log \Phi(\eta_n) = \frac{1}{d} \sum_{s\in S} E\left[\log \omega_1\left(s^{-1}\right) - \log \omega_1(s)\right] = 0. \tag{16}$$

So by Fubini's theorem, it follows that Eq. (16) holds μ-a.s., for every $\omega \in \Omega$ a.s. with respect to P. Fix such an $\omega \in \Omega$. As $d \geq 3$, find $\frac{1}{d-1} < \Delta < 1$. Since almost sure convergence implies convergence in probability, so $\exists M_\mu^\omega \in \mathbb{N}$ such that for all $n \geq M_\mu^\omega$,

$$\mu\left(\Phi(\eta_n) < \left(\frac{1}{\sqrt{\Delta}}\right)^n\right) > \frac{1}{2}. \tag{17}$$

But recall that under μ, the distribution of η_n is uniform on the vertices of $\mathbb{T}_d^n$, so

$$\frac{\#\left\{\sigma_n \in \mathbb{T}_d \middle| \Phi(\sigma_n) < \left(\frac{1}{\sqrt{\Delta}}\right)^n\right\}}{d(d-1)^{n-1}} > \frac{1}{2} \tag{18}$$

for all $n \geq M_\mu^\omega$. Let $\beta_n = \Delta^{\frac{n}{2}}$. Observe that $\sum_{n=1}^\infty \beta^n < \infty$. Now for $n \geq M_\mu^\omega$,

$$\sum_{\sigma_n \in \mathbb{T}_d^n} \beta_n \left(\Phi(\sigma_n)\right)^{-1} \geq \sum_{\substack{\sigma_n \in \mathbb{T}_d^n \\ \Phi(\sigma_n) < \left(\frac{1}{\sqrt{\Delta}}\right)^n}} \beta_n \left(\Phi(\sigma_n)\right)^{-1} \geq \frac{1}{2} d(d-1)^{n-1} \Delta^n. \tag{19}$$

By the choice of Δ it follows that P-a.s. Eq. (11) holds, which completes the proof. □

2.2. Proof of Theorem 2

Let $D_n := |X_n|$, then

$$\begin{aligned} D_n &= \sum_{i=1}^n (D_i - D_{i-1}) \\ &= \sum_{i=1}^n \left(D_i - D_{i-1} - \mathbb{E}_\omega^e\left[D_i - D_{i-1} \middle| X_0, \ldots, X_{i-1}\right]\right) + \sum_{i=1}^n \mathbb{E}_\omega^e\left[D_i - D_{i-1} \middle| X_0, \ldots, X_{i-1}\right]. \end{aligned} \tag{20}$$

But then $M_n := \sum_{i=1}^n \left(D_i - D_{i-1} - \mathbb{E}_\omega^e\left[D_i - D_{i-1} \middle| X_0, \ldots, X_{i-1}\right]\right)$ is a martingale with zero mean and bounded increments, so by Theorem 3 of Azuma (1967)

$$\frac{M_n}{n} \to 0 \quad \mathbb{P}^e\text{-a.s.} \tag{21}$$

Further it is easy to see that

$$D_i - D_{i-1} = \begin{cases} +1 & \text{if } X_{i-1} = e \\ +1 & \text{if } X_{i-1} \not\in \left\{e, \overleftarrow{X}_{i-1}\right\} \\ -1 & \text{if } X_{i-1} = \overleftarrow{X}_{i-1}. \end{cases}$$

Thus,

$$\mathbb{E}_\omega^e\left[D_i - D_{i-1} \middle| X_0, \ldots, X_{i-1}\right] = 1 - 2 \times \mathbf{1}(X_{i-1} \neq e)\, \omega\left(X_{i-1}, \overleftarrow{X}_{i-1}\right).$$

Now under our assumption (A3) with $\epsilon > \frac{1}{2(d-1)}$ there exists $\delta > 0$ such that P-a.s.

$$\omega\left(x, \overleftarrow{x}\right) < \frac{1}{2} - \delta(d-1) \quad \forall x \in \mathbb{T}_d.$$

This is because $\omega\left(x, \overleftarrow{x}\right) = 1 - \sum_{x\sim y, y\neq \overleftarrow{x}} \omega(x, y)$. Thus $\mathbb{P}^e$-a.s.

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \left(1 - 2 \times \mathbf{1}(X_{i-1} \neq e)\, \omega\left(X_{i-1}, \overleftarrow{X}_{i-1}\right)\right) > 2\delta(d-1) > 0. \tag{22}$$

Finally, by (20) $D_n = M_n + \sum_{i=1}^n \left(1 - 2 \times \mathbf{1}(X_{i-1} \neq e)\, \omega(X_{i-1}, \overleftarrow{X}_{i-1})\right)$, so using Eqs. (21) and (22) we conclude that (6) holds. □

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