Alea 1, 205–224 (2006)

Random Walk in Dynamic Markovian Random Environment

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Abstract. We consider a model, introduced by Boldrighini, Minlos and Pellegrinotti (1997, 2000), of random walks in *dynamical* random environments on the integer lattice $\mathbb{Z}^d$ with $d \geq 1$. In this model, the environment changes over time in a Markovian manner, independently across sites, while the walker uses the environment at its current location in order to make the next transition. In contrast with the cluster expansions approach of Boldrighini, Minlos and Pellegrinotti (2000), we follow a probabilistic argument based on regeneration times. We prove an annealed SLLN and invariance principle for any dimension, and provide a quenched invariance principle for dimension $d > 7$, providing for $d > 7$ an alternative to the analytical approach of Boldrighini, Minlos and Pellegrinotti (2000), with the added benefit that it is valid under weaker assumptions. The quenched results use, in addition to the regeneration times already mentioned, a technique introduced by Bolthausen and Sznitman (2002).

1. Introduction

In recent years there has been a great deal of study on *random walks in random environments* (RWRE) on $\mathbb{Z}^d$, $d \geq 1$, where first the *environment* is chosen at random and kept fixed throughout the time evolution, and then a walker moves randomly in such a way that given the environment, its position forms a time homogeneous Markov chain whose transition probabilities depend on the environment.

Received by the editors August 31, 2005, accepted March 27, 2006.

2000 *Mathematics Subject Classification.* 60J15,60F10,82C44,60J80.

Key words and phrases. Random walks. Random environment. Central Limit Theorem.

This work was begun while Antar Bandyopadhyay was a post-doctoral fellow at the IMA, University of Minnesota.

Ofer Zeitouni is partially supported by NSF grants DMS-0302230 and DMS-0503775.

Even though a lot is known about this model, there are many challenging problems left open, see Zeitouni (2004) and Sznitman (2004) for surveys.

In the current paper, we consider random walks in *dynamical* random environments, where along with the walker the environment changes over time. Such model was studied in an abstract setting by Kifer (1996). More relevant to us, the model we consider was first introduced by Boldrighini, Minlos and Pellegrinotti (1997), where they consider the case where the environment changes over time in an *i.i.d.* fashion. In Boldrighini, Minlos and Pellegrinotti (1997) and in Boldrighini, Minlos and Pellegrinotti (2004), they proved under certain assumptions (omitted here) that for almost every realization of the dynamic environment, the position of the random walk, properly centered and scaled, satisfies a *central limit theorem*, with covariance that does not depend on the particular realization of environment. Further, in Boldrighini, Minlos and Pellegrinotti (2000), they obtain similar results for $d \geq 3$ for certain environment evolving over time as a Markov chain, independently at each site. The proofs in these papers are based on *cluster expansion*, and involve a heavy analytic machinery. In the i.i.d. case, a somewhat simpler proof can be found in Stannat (2004).

Our goal in this paper is to describe a probabilistic treatment of this model, which is arguably simpler than those that have appeared in the literature. We recover most of the results of Boldrighini, Minlos and Pellegrinotti (2000), at least when $d > 7$, under weaker and more natural hypotheses. Our approach is based on the introduction of appropriate "regeneration times", borrowing this concept, if not the details of the construction, from the RWRE literature. Our approach to quenched results is based on a technique introduced in Bolthausen and Sznitman (2002).

The paper is divided as follows. The next section describes precisely the model and states our main results: a strong law of large numbers, an annealed CLT (in any dimension), and a quenched CLT (for $d > 7$). Section 3 constructs the regeneration times alluded to above, and derives their basic properties. Section 4 provides the proofs of our main results. Finally, we present in Section 5 several remarks and open problems.

2. Description of the model and main results

In what follows we will consider $\mathbb{Z}^d$, for a fixed $d \geq 1$ with nearest neighbor graph structure as our basic underlying graph. Let $N_{\mathbf{0}} := \{\pm \mathbf{e}_i\}_{i=1}^d \cup \{\mathbf{0}\}$ be the neighbors of the origin $\mathbf{0}$ (including itself), where $\mathbf{e}_i$ is the unit vector in the i^{th} coordinate direction. We use $N_{\mathbf{x}} := \mathbf{x} + N_{\mathbf{0}}$ to denote the collection of neighbors of an $\mathbf{x} \in \mathbb{Z}^d$.

Let $\mathcal{S} := \mathcal{S}_{\mathbf{0}}$ be a collection of probability measures on the $(2d+1)$ elements of $N_{\mathbf{0}}$. To simplify the presentation and avoid various measurability issues, we assume that $\mathcal{S}$ is a Polish space (including the possibilities that $\mathcal{S}$ is finite or countably infinite). For each $\mathbf{x} \in \mathbb{Z}^d$, $\mathcal{S}_{\mathbf{x}}$ denotes a copy of $\mathcal{S}$, with all elements of $\mathcal{S}$ shifted to have support on $N_{\mathbf{x}}$. Formally, an element $\omega(\mathbf{x}, \cdot)$ of $\mathcal{S}_{\mathbf{x}}$, is a probability measure satisfying

$$\omega(\mathbf{x}, \mathbf{y}) \geq 0 \;\; \forall \;\; \mathbf{x}, \mathbf{y} \in \mathbb{Z}^d \quad \text{and} \quad \sum_{\mathbf{y} \in N_{\mathbf{x}}} \omega(\mathbf{x}, \mathbf{y}) = 1$$

Let $\mathcal{B}_{\mathcal{S}}$ denote the Borel σ-field on $\mathcal{S}$. We let $K\colon \mathcal{S}\times\mathcal{B}_{\mathcal{S}} \to [0,1]$ denote a Markov transition probability on $\mathcal{S}$ with a unique stationary distribution π. Let P^π be the probability distribution on the standard product space $\mathcal{S}^{\mathbb{N}_0}$, where $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, giving the π-stationary Markov K-chain on $\mathcal{S}$. For each $\mathbf{x} \in \mathbb{Z}^d$, let $K_{\mathbf{x}}$, $\pi_{\mathbf{x}}$ and $P^{\pi_{\mathbf{x}}}$ be just the copies of respective quantities when we replace $\mathcal{S}$ by $\mathcal{S}_{\mathbf{x}}$. Let

$$\Omega := \prod_{\mathbf{x}\in\mathbb{Z}^d} \mathcal{S}_{\mathbf{x}}^{\mathbb{N}_0}, \quad \mathbf{P}^\pi := \bigotimes_{\mathbf{x}\in\mathbb{Z}^d} P^{\pi_{\mathbf{x}}}, \tag{2.1}$$

where the measure is defined on $\mathcal{F}$, the standard product σ-algebra on Ω. An element $\omega \in \Omega$ will be written as $\left\{ \left(\omega_t\left(\mathbf{x},\cdot\right)\right)_{t\geq 0} \,\middle|\, \mathbf{x} \in \mathbb{Z}^d \right\}$. We note that under $\mathbf{P}^\pi$, the canonical variable $\omega \in \Omega$ follows a distribution such that

- For each $\mathbf{x} \in \mathbb{Z}^d$, $\left(\omega_t\left(\mathbf{x},\cdot\right)\right)_{t\geq 0}$ is a stationary Markov chain with transition kernel K.[1]
- The chains $\left(\omega_t\left(\mathbf{x},\cdot\right)\right)_{t\geq 0}$ are i.i.d. as $\mathbf{x}$ varies over $\mathbb{Z}^d$.

We now turn to define a random walk $(X_t)_{t\geq 0}$. Given an environment $\omega \in \Omega$, $(X_t)_{t\geq 0}$ is a time inhomogeneous Markov chain taking values in $\mathbb{Z}^d$ with transition probabilities

$$\mathbf{P}_\omega \left(X_{t+1} = \mathbf{y} \,\middle|\, X_t = \mathbf{x} \right) = \omega_t\left(\mathbf{x},\mathbf{y}\right). \tag{2.2}$$

For each $\omega \in \Omega$, we denote by $\mathbf{P}_\omega^{\mathbf{x}}$ the law induced by $(X_t)_{t\geq 0}$ on $\left(\left(\mathbb{Z}^d\right)^{\mathbb{N}_0}, \mathcal{G}\right)$, where $\mathcal{G}$ is the σ-algebra generated by the cylinder sets, such that

$$\mathbf{P}_\omega^{\mathbf{x}}\left(X_0 = \mathbf{x}\right) = 1. \tag{2.3}$$

Naturally $\mathbf{P}_\omega^{\mathbf{x}}$ is called the *quenched law* of the random walk $\{X_t\}_{t\geq 0}$, starting at $\mathbf{x}$.

We note that for every $G \in \mathcal{G}$, the function

$$\omega \mapsto \mathbf{P}_\omega^{\mathbf{x}}\left(G\right)$$

is $\mathcal{F}$-measurable. Hence, we may define the measure $\mathbb{P}^{\mathbf{x}}$ on $\left(\Omega \times \left(\mathbb{Z}^d\right)^{\mathbb{N}_0}, \mathcal{F}\otimes\mathcal{G}\right)$ from the relation

$$\mathbb{P}^{\mathbf{x}}\left(F\times G\right) = \int_F \mathbf{P}_\omega^{\mathbf{x}}\left(G\right) \mathbf{P}^\pi\left(d\omega\right), \quad \forall\ F\in\mathcal{F},\, G\in\mathcal{G}.$$

With a slight abuse of notation, we also denote the marginal of $\mathbb{P}^{\mathbf{x}}$ on $\left(\mathbb{Z}^d\right)^{\mathbb{N}_0}$ by $\mathbb{P}^{\mathbf{x}}$, whenever no confusion occurs. This probability distribution is called the *annealed law* of the random walk $\{X_t\}_{t\geq 0}$, starting at $\mathbf{x}$. Note that under $\mathbb{P}^{\mathbf{x}}$, the random walk $\{X_t\}_{t\geq 0}$ is not, in general, a Markov chain. However, when $K(s,\cdot) = K(s_0,\cdot)$ for some $s_0 \in \mathcal{S}$, i.e. the environment is i.i.d. in time, then of course $\{X_t\}_{t\geq 0}$ is actually a random walk on $\mathbb{Z}^d$ under $\mathbb{P}^{\mathbf{x}}$, with deterministic increment distribution given by the mean of π.

Throughout this paper we will assume the followings hold.

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[1]By this we mean that $\left(\omega_t\left(\mathbf{x},\cdot\right)\right)_{t\geq 0}$ is a a stationary Markov chain on $\mathcal{S}_{\mathbf{x}}$ with transition probabilities $K_{\mathbf{x}}$ and stationary law $\pi_{\mathbf{x}}$. To alleviate notation, we identify in the sequel $\mathcal{S}_{\mathbf{x}}$ with $\mathcal{S}$, $\pi_{\mathbf{x}}$ with π, and $K_{\mathbf{x}}$ with K whenever no confusion occurs.

Assumption 2.1. (A1) *There exists* $0 < \kappa \leq 1$ *such that*

$$K\left(w, A\right) \geq \kappa\, \pi\left(A\right), \quad \forall \; w \in \mathcal{S}, A \in \mathcal{B}_{\mathcal{S}}. \tag{2.4}$$

(A2) *There exist* $0 < \varepsilon < 1$ *and a fixed translation invariant Markov kernel with only nearest neighbor transition* $q \colon \mathbb{Z}^d \times \mathbb{Z}^d \to [0,1]$ *with the property that* $q(\mathbf{x}, \mathbf{y}) = q(\mathbf{y} - \mathbf{x})$ *and*

$$\Bigg| \sum_{\mathbf{y} \in \mathbb{Z}^d : |y| = 1} q\left(0, \mathbf{y}\right) e^{\imath\, \ell \cdot \mathbf{y}} \Bigg| < 1, \quad \forall \; \ell \in \mathbb{Z}^d \setminus \{\mathbf{0}\}\,, \tag{2.5}$$

such that

$$\mathbf{P}^{\pi}\left(\omega_t\left(\mathbf{x}, \mathbf{y}\right) \geq \varepsilon\, q\left(\mathbf{x}, \mathbf{y}\right)\right) = 1, \quad \forall \; \mathbf{x}, \mathbf{y} \in \mathbb{Z}^d, \;\; t \geq 0. \tag{2.6}$$

(A3) $\kappa + \varepsilon^2 > 1$.

Remark 2.2. *Some comments regarding Assumption 2.1 are in order.*

(1) *Condition (A1) provides a uniform "fast mixing" rate for the environment chains. If, as in Boldrighini, Minlos and Pellegrinotti (2000),* $\mathcal{S}$ *is finite and* K *is irreducible and aperiodic, then while (A1) may fail, it does hold if* K *is replaced by* K^r *for some fixed* $r \geq 1$. *A slight modification of our arguments applies to that case, too.*

(2) *Condition (A2) is the same as that made in Boldrighini, Minlos and Pellegrinotti (1997, 2000) and later on in Stannat (2004). This condition essentially means that the random environment has a "deterministic" part* q, *which is non-degenerate. We thus refer to condition (A2) as an* ellipticity *condition.*

We note that the assumption that q *is translation invariant does not present a loss of generality. This is because under* $\mathbf{P}^{\pi}$ *the environment chains* $\left(\omega_t\left(\mathbf{x}, \cdot\right)\right)_{t \geq 0}$ *are assumed i.i.d. as* $\mathbf{x}$ *varies.*

Finally note that we could allow $\varepsilon = 1$ *in (2.6). In that case, the environment is given by the deterministic kernel* q. *This makes the walk a classical Markov chain which is well studied. This is the reason why we work with the restriction* $\varepsilon < 1$.

(3) *Condition (A3) is technical but absolutely crucial for our argument. It implies that there is a trade off between the environment Markov chain being fast mixing (*κ *close to* 1*) and the fluctuation in the environment being "small" (*ε *close to* 1*). The later is a condition assumed in Boldrighini, Minlos and Pellegrinotti (1997, 2000) also.*

We now formulate our main results as follows.

Theorem 2.3 (Annealed SLLN)**.** *Let Assumption 2.1 hold. Then there exists a deterministic* $\mathbf{v} \in \mathbb{R}^d$ *such that*

$$\frac{X_n}{n} \underset{n \to \infty}{\longrightarrow} \mathbf{v} \quad a.s. \quad \left[\mathbb{P}^{\mathbf{0}}\right] \tag{2.7}$$

Remark 2.4. *A (non-explicit) formula for the limit velocity* $\mathbf{v}$ *is given below, see (4.8). A consequence of that formula is that whenever the transition probabilities* $K(s, \cdot)$ *are invariant under lattice isometries (i.e.* $\int_{\mathcal{S}} K(s, ds') s'(e) = \int_{\mathcal{S}} K(s, ds')\, s'(Te)$ *for any lattice isometry* T *and any* $s \in \mathcal{S}$*), then* $\mathbf{v} = 0$.

We also prove

Theorem 2.5 (Annealed Invariance Principle)**.** *Let Assumption 2.1 hold. Then, there exists a* $(d \times d)$ *strictly positive definite matrix* Σ*, such that under* $\mathbb{P}^{\mathbf{0}}$*,*

$$\left(\frac{X_{\lfloor nt \rfloor} - nt\,\mathbf{v}}{\sqrt{n}}\right)_{t \geq 0} \xrightarrow[n\to\infty]{d} \mathrm{BM}_d\left(\Sigma\right)\,, \tag{2.8}$$

where $\mathrm{BM}_d\left(\Sigma\right)$ *denotes a* d*-dimensional Brownian motion with covariance matrix* Σ*,* $\xrightarrow{d}$ *denotes weak convergence of the laws, and* $\lfloor x \rfloor$ *denotes the integer part of* x*.*

Remark 2.6. *An implicit formula for the covariance matrix* Σ *is given in (4.16).*

In order to get a quenched invariance principle, we need some further restrictions. Set

$$\gamma = \log(1-\kappa)/\log\varepsilon\,. \tag{2.9}$$

Theorem 2.7 (Quenched Invariance Principle)**.** *Let Assumption 2.1 hold. Assume further that* $\gamma > 6$ *and also*

$$d > 1 + \frac{4 + \frac{2\gamma(d-1)}{\gamma(d-3)-2(d-1)} + \frac{8(d-1)}{\gamma(d-3)}}{1-6/\gamma} > 7\,. \tag{2.10}$$

Then, with the notation of Theorem 2.5, one has that (2.8) holds true under $\mathbf{P}^0_\omega$*, for* $\mathbf{P}^\pi$ *almost every* ω*.*

Remark 2.8. *Condition (2.10) is clearly not optimal, as the case* $\kappa = 1$ *demonstrates (recall that when* $\kappa = 1$*, the environment is i.i.d. in time, and the quenched CLT statement holds true in any dimension as soon as (A2) holds true, see Stannat (2004)). We do not know however whether the quenched CLT holds for Markov environments in low dimension.*

3. Construction of a "regeneration time"

Since for $\kappa = 1$ the annealed law on the random walk is Markovian with deterministic transition, we may and will in the sequel consider only the case $\kappa < 1$. Our approach is based on the construction of a sequence of a.s. finite (stopping) times $\{\tau_n\}_{n\geq 1}$ (on an enlarged probability space), such that, between two successions, say τ_n and τ_{n+1}, environment chains at each location visited by the walk go through a time of "regeneration", in the sense that they have started afresh from new states selected according to the stationary distribution π. This in turn provides a *renewal* structure for $\{\tau_n, X_{\tau_n}\}_{n\geq 1}$. We note that regeneration times have been extensively used in the RWRE context, see Sznitman (2004); Zeitouni (2004). What makes the situation considered here particularly simple is that the regeneration times constructed here are actually stopping times.

To introduce the regeneration times, we begin by constructing an extension of our probability space, which is similar to what is done in Zeitouni (2004).

Let $W = \{0,1\}$ and let $\mathcal{W}$ denote the σ-algebra on $W^{\mathbb{N}}$ generated by cylinder sets. For $\varepsilon > 0$ as in Condition (A2), let $\mathbf{Q}_\varepsilon$ be the product measure on $\left(W^{\mathbb{N}}, \mathcal{W}\right)$ such that the coordinate variables, say $(\epsilon_t)_{t\geq 1}$, are i.i.d. Bernoulli variables with $\mathbf{Q}_\varepsilon\left(\epsilon_1 = 1\right) = \varepsilon$.

For any $(\omega, \epsilon) \in \Omega \times W^{\mathbb{N}}$, we define a probability measure $\overline{\mathbb{P}}^{\mathbf{0}}_{\omega,\varepsilon}$ on $\left(\mathbb{Z}^d\right)^{\mathbb{N}_0}$ such that $\{X_t\}_{t\geq 0}$ is a Markov chain with state space $\mathbb{Z}^d$ and transition law

$$\overline{\mathbb{P}}^{\mathbf{0}}_{\omega,\varepsilon}\left(X_{t+1} = \mathbf{y} \,\middle|\, X_t = \mathbf{x}\right) = \mathbf{1}_{[\epsilon_{t+1}=1]}\, q\left(\mathbf{x},\mathbf{y}\right) + \frac{\mathbf{1}_{[\epsilon_{t+1}=0]}}{1-\varepsilon}\, \left[\omega_t\left(\mathbf{x},\mathbf{y}\right) - \varepsilon q\left(\mathbf{x},\mathbf{y}\right)\right], \tag{3.1}$$

where $\mathbf{y} \in \{\mathbf{x}\} \cup \{\mathbf{x} \pm \mathbf{e}_i\}_{i=1}^d$, and also

$$\overline{\mathbb{P}}^{\mathbf{0}}_{\omega,\varepsilon}\left(X_0 = \mathbf{0}\right) = 1. \tag{3.2}$$

Finally we define a measure $\overline{\mathbb{P}}^{\mathbf{0}}$ on $\left(\Omega \times W^{\mathbb{N}} \times \left(\mathbb{Z}^d\right)^{\mathbb{N}_0}, \mathcal{F} \otimes \mathcal{W} \otimes \mathcal{G}\right)$, such that the coordinate variables has the following distribution

$$(\omega, \epsilon) \sim \mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon \quad \text{and} \quad \{X_t\}_{t\geq 0} \sim \overline{\mathbb{P}}^{\mathbf{0}}_{\omega,\epsilon} \quad \text{given} \quad (\omega, \epsilon)\,.$$

It is immediate to check that under $\overline{\mathbb{P}}^{\mathbf{0}}$, the marginal distribution of the chain $\{X_t\}_{t\geq 0}$ is $\mathbb{P}^{\mathbf{0}}$ and also the conditional distribution of $\{X_t\}_{t\geq 0}$ given ω is $\mathbf{P}^{\mathbf{0}}_\omega$. We will call the ϵ-variables the ϵ-coin tosses. Heuristically, under $\overline{\mathbb{P}}^{\mathbf{0}}$, the evolution of the random walk at time t is done by first tossing the ε-coin ε_t, and depending on the outcome, taking a step either according to how the environment dictates it (if $\varepsilon_t = 0$) or taking a step according to the fixed transition kernel q (if $\varepsilon_t = 1$).

One more extension of the probability space is needed before we can define the "regeneration time" and it is for the Markov evolution of the environment at each sites. Notice under Condition (A1), a step in the K-chain on $\mathcal{S}$ can be taken as follows. First, toss a coin independently with probability κ of turning head, if it turns out head, then select a state according to the stationary distribution π independently; otherwise if the coin lands in tail, then take a step according to the Markov transition kernel

$$\tilde{K}\left(w, A\right) = \frac{K\left(w,A\right) - \kappa\pi\left(A\right)}{1-\kappa}\,, \quad w \in \mathcal{S}, A \in \mathcal{B}_{\mathcal{S}}.$$

A rigorous construction is as follows. Extend the measurable space $(\Omega, \mathcal{F})$ to accommodate i.i.d. Bernoulli$\,(\kappa)$ variables $\left\{\left(\alpha_t\left(\mathbf{x}\right)\right)_{t\geq 1} \,\middle|\, \mathbf{x} \in \mathbb{Z}^d\right\}$, such that under $\mathbf{P}^\pi$ the Markov evolution of the environment $\left(\omega_t\left(\mathbf{x},\cdot\right)\right)_{t\geq 0}$ at a site $\mathbf{x} \in \mathbb{Z}^d$ is obtained by the above description using the κ-coin tosses $\left(\alpha_t\left(\mathbf{x}\right)\right)_{t\geq 1}$. We note that in this construction

$$\left\{\left(\alpha_t\left(\mathbf{x}\right)\right)_{t\geq s+1} \,\middle|\, \mathbf{x} \in \mathbb{Z}^d\right\} \text{ is independent of } \left\{\left(\omega_t\left(\mathbf{x},\cdot\right)\right)_{0\leq t\leq s}; \left(X_t\right)_{0\leq t\leq s}\right\}, \tag{3.3}$$

for every $s \geq 0$.

For $t \geq 0$ and $\mathbf{x} \in \mathbb{Z}^d$, define

$$I_t\left(\mathbf{x}\right) := \sum_{s=0}^{t} \mathbf{1}\left(X_s = \mathbf{x}, \epsilon_s = 0\right), \tag{3.4}$$

which is the number of "proper" visits to the site $\mathbf{x}$ by the chain up to time t ("proper" visits are those visits in which the walker "learns" about the environment, that is those in which the next move depends on the random environment rather than on the auxiliary ε-coin). If $I_t(\mathbf{x}) > 0$ let

$$\gamma_t(\mathbf{x}) := \sup\left\{ s \le t \,\middle|\, X_s = \mathbf{x}, \epsilon_s = 0 \right\}, \tag{3.5}$$

be the time of last "proper" visit to $\mathbf{x}$ before time t, and

$$\eta_t(\mathbf{x}) := \inf\left\{ s \ge 0 \,\middle|\, \alpha_{\gamma_t(\mathbf{x})+s}(\mathbf{x}) = 1 \right\}, \tag{3.6}$$

be the first time after $\gamma_t(\mathbf{x})$ when the environment chain at site $\mathbf{x}$ takes a step according to the stationary measure π. For completeness we will take $\gamma_t(\mathbf{x}) = \eta_t(\mathbf{x}) = 0$ if $I_t(\mathbf{x}) = 0$. Finally we define

$$\tau_1 := \inf\left\{ t > 0 \,\middle|\, \gamma_t(\mathbf{x}) + \eta_t(\mathbf{x}) < t \ \ \forall\, \mathbf{x} \in \mathbb{Z}^d \right\}. \tag{3.7}$$

τ_1 will be called a "regeneration time", because from time τ_1 the environment chains at all sites look like they have started afresh from stationary distribution.

Proposition 3.1. *Let Assumption 2.1 hold. Then,* $\overline{\mathbb{E}}^{\mathbf{0}}\left(\tau_1^2\right) < \infty$.

Proof : Due to the strict inequality in Condition (A3), we can find $0 < \delta < 1$ such that $\varepsilon^2 > (1-\kappa)^{\delta}$. Define

$$L(t) = \lfloor -\frac{\delta \log t}{\log \varepsilon} \rfloor, \tag{3.8}$$

which is an increasing sequence of integers going to ∞ with $L(t) < t$ for large t.

For fixed $t \ge 1$, let β_t be the first time there is a run of length $L(t)$ of non-zero ϵ-coin tosses ending at it, that is

$$\beta_t := \inf\left\{ s \ge L(t) \,\middle|\, \epsilon_s = 1, \epsilon_{s-1} = 1, \cdots, \epsilon_{s-L(t)+1} = 1 \right\}. \tag{3.9}$$

From the definition of τ_1 we get

$$\begin{aligned} \overline{\mathbb{P}}^{\mathbf{0}}(\tau_1 > t) \ \le\ & \overline{\mathbb{P}}^{\mathbf{0}}(\beta_t > t) \\ & + \overline{\mathbb{P}}^{\mathbf{0}}\left(\beta_t \le t,\ \exists\, \mathbf{x} \in \mathbb{Z}^d \text{ s.t. } \eta_{\beta_t}(\mathbf{x}) \ge \beta_t - \gamma_{\beta_t}(\mathbf{x})\right) \end{aligned} \tag{3.10}$$

We will consider the first and second terms in the right hand side of (3.10) separately, as follows.

Consider first the second term, noting that for each $\mathbf{x} \in \mathbb{Z}^d$ and for any $t \ge 1$ the time $\eta_t(\mathbf{x})$ is nothing but a Geometric(κ) random variable. Thus,

$$\begin{aligned} & \overline{\mathbb{P}}^{\mathbf{0}}\left(\beta_t \le t,\ \exists\, \mathbf{x} \in \mathbb{Z}^d \text{ s.t. } \eta_{\beta_t}(\mathbf{x}) \ge \beta_t - \gamma_{\beta_t}(\mathbf{x})\right) \\ \le\ & C_0 \sum_{r=0}^{\infty} r^{d-1} \exp\left(-\lambda\, (L(t) \vee r)\right) \\ =\ & C_0 \left(\left(\sum_{r \le L(t)} r^{d-1} \right) \exp\left(-\lambda L(t)\right) + \sum_{r > L(t)} r^{d-1} \exp(-\lambda r) \right) \end{aligned} \tag{3.11}$$

where $C_0 = C_0(d) > 0$ is such that an L_1 ball in $\mathbb{Z}^d$ of radius r contains less than $C_0 r^{d-1}$ points, and $\lambda = -\log(1-\kappa)$. Indeed, the first inequality in (3.11) follows from the observation that

$$\beta_t - \gamma_{\beta_t}(\mathbf{x}) \ge L(t) \vee |\mathbf{x} - X_{\beta_t - L(t)}|,$$

with $|\cdot|$ denoting the L_1-norm on $\mathbb{Z}^d$, and then computing the probability by conditioning on $X_{\beta_t - L(t)}$.

Concerning the first term in (3.10), note that

$$\begin{aligned}
&\overline{\mathbb{P}}^{\mathbf{0}}\left(\beta_t > t\right) \\
\leq\ & \overline{\mathbb{P}}^{\mathbf{0}}\left(\bigcap_{j=0}^{\lfloor \frac{t-L(t)}{L(t)} \rfloor} \left[\epsilon_{jL(t)+1} = 1, \epsilon_{jL(t)+2} = 1, \cdots, \epsilon_{(j+1)L(t)} = 1\right]^c\right) \\
\leq\ & \left(1 - \varepsilon^{L(t)}\right)^{\frac{t}{L(t)} - 2}
\end{aligned} \tag{3.12}$$

Using the choice of $L(t)$ and equations (3.10), (3.11) and (3.12) one concludes that

$$\overline{\mathbb{P}}^{\mathbf{0}}\left(\tau_1 > t\right) \leq \frac{C_1}{t^{2+\zeta}}, \tag{3.13}$$

where $C_1, \zeta > 0$ are some constants (depending on δ, d). This completes the proof. ■

Remark 3.2. *An inspection of the proof reveals that in fact,*

$$\forall\ \gamma' < \frac{\log(1-\kappa)}{\log \varepsilon} =: \gamma\,, \quad \overline{\mathbb{E}}^{\mathbf{0}}\left[\tau_1^{\gamma'}\right] < \infty\,. \tag{3.14}$$

This will be useful when deriving the quenched invariance principle.

Define now a σ-algebra,

$$\mathcal{H}_1 := \sigma\left(\tau_1; \{X_t\}_{0 \leq t \leq \tau_1}; \left\{\left(\alpha_t\left(\cdot\right)\right)_{t \leq \tau_1}\right\}\right). \tag{3.15}$$

The following is the most crucial lemma.

Lemma 3.3. *For any measurable sets A, B, C we have,*

$$\begin{aligned}
&\overline{\mathbb{P}}^{\mathbf{0}}\left(\{X_{\tau_1+t} - X_{\tau_1}\}_{t\geq 0} \in A,\ \left\{\left(\omega_{\tau_1+t}\left(\cdot,\cdot\right)\right)_{t\geq 0}\right\} \in B,\ \left\{\left(\alpha_t\left(\cdot\right)\right)_{t\geq \tau_1+1}\right\} \in C \,\middle|\, \mathcal{H}_1\right) \\
=&\overline{\mathbb{P}}^{\mathbf{0}}\left(\{X_t\}_{t\geq 0} \in A,\ \left\{\left(\omega_t\left(\cdot,\cdot\right)\right)_{t\geq 0}\right\} \in B,\ \left\{\left(\alpha_t\left(\cdot\right)\right)_{t\geq 1}\right\} \in C\right).
\end{aligned} \tag{3.16}$$

Proof : Let h be a $\mathcal{H}_1$ measurable function. Write $\mathbf{1}_A := \mathbf{1}\left(\{X_t - X_0\}_{t\geq 0} \in A\right)$, $\mathbf{1}_B := \mathbf{1}\left(\left\{\left(\omega_t\left(\cdot,\cdot\right)\right)_{t\geq 0}\right\} \in B\right)$ and $\mathbf{1}_C := \mathbf{1}\left(\left\{\left(\alpha_t\left(\cdot\right)\right)_{t\geq 1}\right\} \in C\right)$. Note that for every $m \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{Z}^d$, since h is $\mathcal{H}_1$-measurable, there exists a random variable $h_{\mathbf{x},m}$ which is measurable with respect to $\sigma\left(\{X_t\}_{t\geq 0}; \left\{\left(\alpha_t\left(\cdot\right)\right)_{t\leq m}\right\}\right)$, such that $h = h_{\mathbf{x},m}$ on the event $[\tau_1 = m,\ X_{\tau_1} = \mathbf{x}]$. Writing θ for the time shift,

$$\begin{aligned}
&\overline{\mathbb{E}}^{\mathbf{0}}\left[\mathbf{1}_A \circ \theta^{\tau_1} \cdot \mathbf{1}_B \circ \theta^{\tau_1} \cdot \mathbf{1}_C \circ \theta^{\tau_1} \cdot h\right] \\
=&\sum_{m=1}^{\infty} \sum_{\mathbf{x}_m \in \mathbb{Z}^d} \overline{\mathbb{E}}^{\mathbf{0}}\left[\mathbf{1}_A \circ \theta^m \cdot \mathbf{1}_B \circ \theta^m \cdot \mathbf{1}_C \circ \theta^m \cdot \mathbf{1}\left(\tau_1 = m\right) \cdot \mathbf{1}\left(X_m = \mathbf{x}_m\right) \cdot h_{\mathbf{x}_m,m}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \sum_{\mathbf{x}_m \in \mathbb{Z}^d} \overline{\mathbb{E}}^{\mathbf{0}} \left[\mathbf{1}_B \circ \theta^m \cdot \mathbb{E}_\omega^{\mathbf{0}} \left[\mathbf{1}_A \circ \theta^m \cdot \mathbf{1}_C \circ \theta^m \cdot \mathbf{1}\left(\tau_1 = m\right) \cdot \mathbf{1}\left(X_m = \mathbf{x}_m\right) \cdot h_{\mathbf{x}_m, m} \right] \right] \\
&= \sum_{m=1}^{\infty} \sum_{\substack{\mathbf{x}_t \in \mathbb{Z}^d \\ 1 \le t \le m}} \sum_{\substack{e_t \in W \\ 1 \le t \le m}} \overline{\mathbb{E}}^{\mathbf{0}} \Big[\mathbf{1}_B \circ \theta^m \cdot \mathbb{E}_\omega^{\mathbf{0}} \Big[\mathbf{1}_A \circ \theta^m \cdot \mathbf{1}_C \circ \theta^m \cdot \mathbf{1}_{[\tau_1 = m]} \\
&\qquad\qquad \cdot \mathbf{1}_{[X_t = \mathbf{x}_t, 1 \le t \le m]} \cdot \mathbf{1}_{[\epsilon_t = e_t, 1 \le t \le m]} \cdot h_{\mathbf{x}_m, m} \Big] \Big] \\
&= \sum_{m=1}^{\infty} \sum_{\substack{\mathbf{x}_t \in \mathbb{Z}^d \\ 1 \le t \le m}} \sum_{\substack{e_t \in W \\ 1 \le t \le m}} \overline{\mathbb{E}}^{\mathbf{0}} \Big[\mathbf{1}_B \circ \theta^m \cdot \mathbb{E}_\omega^{\mathbf{0}} \Big[\mathbb{E}_\omega^{\mathbf{x}_m} \left[\mathbf{1}_A \cdot \mathbf{1}_C \right] \circ \theta^m \cdot \mathbf{1}_{[\tau_1 = m]} \\
&\qquad\qquad \cdot \mathbf{1}_{[X_t = \mathbf{x}_t, 1 \le t \le m]} \cdot \mathbf{1}_{[\epsilon_t = e_t, 1 \le t \le m]} \cdot h_{\mathbf{x}_m, m} \Big] \Big] \\
&= \sum_{m=1}^{\infty} \sum_{\substack{\mathbf{x}_t \in \mathbb{Z}^d \\ 1 \le t \le m}} \sum_{\substack{e_t \in W \\ 1 \le t \le m}} \overline{\mathbb{E}}^{\mathbf{0}} \Big[\mathbb{E}_\omega^{\mathbf{0}} \Big[\mathbb{E}_\omega^{\mathbf{x}_m} \left[\mathbf{1}_A \cdot \mathbf{1}_B \cdot \mathbf{1}_C \right] \circ \theta^m \cdot \mathbf{1}_{[\tau_1 = m]} \cdot \mathbf{1}_{[X_t = \mathbf{x}_t, 1 \le t \le m]} \\
&\qquad\qquad \cdot \mathbf{1}_{[\epsilon_t = e_t, 1 \le t \le m]} \cdot h_{\mathbf{x}_m, m} \Big] \Big] \\
&= \sum_{m=1}^{\infty} \sum_{\substack{\mathbf{x}_t \in \mathbb{Z}^d \\ 1 \le t \le m}} \sum_{\substack{e_t \in W \\ 1 \le t \le m}} \overline{\mathbb{E}}^{\mathbf{0}} \Big[\mathbb{E}_\omega^{\mathbf{x}_m} \left[\mathbf{1}_A \cdot \mathbf{1}_B \cdot \mathbf{1}_C \right] \circ \theta^m \cdot \mathbf{1}_{[\tau_1 = m]} \cdot \mathbf{1}_{[X_t = \mathbf{x}_t, 1 \le t \le m]} \\
&\qquad\qquad \cdot \mathbf{1}_{[\epsilon_t = e_t, 1 \le t \le m]} \cdot h_{\mathbf{x}_m, m} \Big] \\
&= \overline{\mathbb{E}}^{\mathbf{0}} \left[\mathbf{1}_A \cdot \mathbf{1}_B \cdot \mathbf{1}_C \right] \sum_{m=1}^{\infty} \sum_{\substack{\mathbf{x}_t \in \mathbb{Z}^d \\ 1 \le t \le m}} \sum_{\substack{e_t \in W \\ 1 \le t \le m}} \overline{\mathbb{E}}^{\mathbf{0}} \Big[\mathbf{1}_{[\tau_1 = m]} \cdot \mathbf{1}_{[X_t = \mathbf{x}_t, 1 \le t \le m]} \\
&\qquad\qquad \cdot \mathbf{1}_{[\epsilon_t = e_t, 1 \le t \le m]} \cdot h_{\mathbf{x}_m, m} \Big],
\end{aligned}
$$

where in the fourth equality we use the Markov property of the random walk given the environment and also the fact (3.3) with $s = m$; and the last equality uses the fact that on the event $[\tau_1 = m]$, the environment chains at every site have gone through "regeneration" before time m and after the last "proper" visit of the walk to that site, and hence under the law $\overline{\mathbb{P}}^{\mathbf{0}}$, at time m the environments at distinct sites are independent, π-distributed, and independent of the ϵ and α coins and the walk till time m. Substituting in the above the whole sample space in place of A, B and C, we conclude that

$$
\overline{\mathbb{E}}^{\mathbf{0}} \left[\mathbf{1}_A \circ \theta^{\tau_1} \cdot \mathbf{1}_B \circ \theta^{\tau_1} \cdot \mathbf{1}_C \circ \theta^{\tau_1} \cdot h \right] = \overline{\mathbb{E}}^{\mathbf{0}} \left[\mathbf{1}_A \cdot \mathbf{1}_B \cdot \mathbf{1}_C \right] \overline{\mathbb{E}}^{\mathbf{0}} \left[h \right], \tag{3.17}
$$

concluding the proof of the lemma. ∎

Consider now τ_1 as a function of $\left(\left(X_t\right)_{t \ge 0} ; \left\{ \left(\omega_t\left(\cdot,\cdot\right)\right)_{t \ge 0} \right\} ; \left\{ \left(\alpha_t\left(\cdot\right)\right)_{t \ge 1} \right\} \right)$, and set

$$
\tau_{n+1} := \tau_n + \tau_1 \left(\left(X_{\tau_n + t}\right)_{t \ge 0} ; \left\{ \left(\omega_{\tau_n + t}\left(\cdot,\cdot\right)\right)_{t \ge 0} \right\} ; \left\{ \left(\alpha_{\tau_n + t}\left(\cdot\right)\right)_{t \ge 1} \right\} \right), \tag{3.18}
$$

with $\tau_{n+1} = \infty$ on the event $[\tau_n = \infty]$. The following lemma gives the renewal sequence described earlier.

Lemma 3.4. $\overline{\mathbb{P}}^{\mathbf{0}} \left(\tau_n < \infty\right) = 1$, *for all* $n \ge 1$. *Moreover the sequence of random vectors* $\left\{ \left(\tau_{n+1} - \tau_n, X_{\tau_{n+1}} - X_{\tau_n}\right) \right\}_{n \ge 0}$, *where* $\tau_0 = 0$, *are i.i.d. under the law* $\overline{\mathbb{P}}^{\mathbf{0}}$.

Proof : Define

$$\mathcal{H}_n := \sigma\left(\tau_1, \tau_2, \ldots, \tau_n; (X_t)_{0\le t\le \tau_n}; \left\{(\alpha_t(\mathbf{x}))_{t\le\tau_n}\right\}\right), \tag{3.19}$$

then an obvious rerun of the proof of Lemma 3.4 yields that for measurable sets A, B and C,

$$\overline{\mathbb{P}}^{\mathbf{0}}\left(\{X_{\tau_n+t} - X_{\tau_n}\}_{t\ge 0} \in A,\ \left\{(\omega_{\tau_n+t}(\cdot,\cdot))_{t\ge 0}\right\} \in B,\ \left\{(\alpha_{\tau_n+t}(\cdot))_{t\ge 1}\right\} \in C \,\middle|\, \mathcal{H}_n\right)$$

$$= \overline{\mathbb{P}}^{\mathbf{0}}\left(\{X_t\}_{t\ge 0} \in A,\ \left\{(\omega_t(\cdot,\cdot))_{t\ge 0}\right\} \in B\ \left\{(\alpha_t(\cdot))_{t\ge 1}\right\} \in C\right). \tag{3.20}$$

So first of all we get from Proposition 3.1 that $\overline{\mathbb{P}}^{\mathbf{0}}(\tau_n < \infty) = 1$, for all $n \ge 1$, and also under $\overline{\mathbb{P}}^{\mathbf{0}}$

$$(\tau_1, X_{\tau_1}), (\tau_2 - \tau_1, X_{\tau_1} - X_{\tau_1}), \ldots, (\tau_{n+1} - \tau_n, X_{\tau_{n+1}} - X_{\tau_n}), \ldots$$

are i.i.d. of random vectors. ∎

4. Proofs of the main results

4.1. *Proof of Theorem 2.3.* Fix $\mathbf{a} \in \mathbb{R}^d$. From Proposition 3.1 we get that $\overline{\mathbb{E}}^{\mathbf{0}}[\tau_1] < \infty$. Since the random walk $\{X_t\}_{t\ge 0}$ has bounded increments, it follows that $\overline{\mathbb{E}}^{\mathbf{0}}[\mathbf{a}\cdot X_{\tau_1}] < \infty$ as well. Let $Y_n := \mathbf{a}\cdot\left(X_{\tau_n} - X_{\tau_{n-1}}\right)$, for $n \ge 1$, taking $\tau_0 = 0$. Using Lemma 3.4 we conclude

$$\frac{\tau_n}{n} \underset{n\to\infty}{\longrightarrow} \overline{\mathbb{E}}^{\mathbf{0}}[\tau_1] \quad \text{a.s.} \quad \left[\overline{\mathbb{P}}^{\mathbf{0}}\right], \tag{4.1}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} Y_k \underset{n\to\infty}{\longrightarrow} \overline{\mathbb{E}}^{\mathbf{0}}[\mathbf{a}\cdot X_{\tau_1}] \quad \text{a.s.} \quad \left[\overline{\mathbb{P}}^{\mathbf{0}}\right]. \tag{4.2}$$

We can find a (possibly random) sequence of numbers $\{k_n\}_{n\ge 1}$ increasing to ∞ such that for all $n \ge 1$ we have $\tau_{k_n} \le n < \tau_{k_n+1}$. Then from (4.1) we get

$$\frac{n}{k_n} \underset{n\to\infty}{\longrightarrow} \overline{\mathbb{E}}^{\mathbf{0}}[\tau_1] \quad \text{a.s.} \quad \left[\overline{\mathbb{P}}^{\mathbf{0}}\right]. \tag{4.3}$$

Since the increments of the random walk are bounded, we have for any $n \ge 1$ that

$$\begin{aligned} \left|\mathbf{a}\cdot\left(X_n - X_{\tau_{k_n}}\right)\right| &\le \|\mathbf{a}\|_2\,(n - \tau_{k_n}) \\ &\le \|\mathbf{a}\|_2\,(\tau_{k_n+1} - \tau_{k_n}). \end{aligned} \tag{4.4}$$

On the other hand, since by Lemma 3.4 the random variables $(\tau_n - \tau_{n-1})_{n\ge 1}$ are identically distributed and of finite mean, one gets for any $\delta > 0$ that

$$\sum_{n=0}^{\infty} \overline{\mathbb{P}}^{\mathbf{0}}(\tau_{n+1} - \tau_n > \delta n) = \sum_{n=0}^{\infty} \overline{\mathbb{P}}^{\mathbf{0}}(\tau_1 > \delta n) < \infty.$$

It follows from an application of the Borel-Cantelli Lemma that

$$\frac{\tau_{n+1} - \tau_n}{n} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \quad \left[\overline{\mathbb{P}}^{\mathbf{0}}\right]. \tag{4.5}$$

So using (4.4) and (4.3) we get

$$\frac{\left|\mathbf{a}\cdot\left(X_n - X_{\tau_{k_n}}\right)\right|}{n} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \quad \left[\overline{\mathbb{P}}^{\mathbf{0}}\right] \tag{4.6}$$

This together with (4.2) and (4.3) gives

$$\frac{\mathbf{a}\cdot X_n}{n} \underset{n\to\infty}{\longrightarrow} \frac{\overline{\mathbb{E}}^{\mathbf{0}}\left[\mathbf{a}\cdot X_{\tau_1}\right]}{\overline{\mathbb{E}}^{\mathbf{0}}\left[\tau_1\right]} \quad \text{a.s.} \quad \left[\overline{\mathbb{P}}^{\mathbf{0}}\right] \tag{4.7}$$

Finally taking $\mathbf{a} = \mathbf{e}_i$ for $i = 1, 2, \ldots, d$ we conclude

$$\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} \frac{\overline{\mathbb{E}}^{\mathbf{0}}\left[X_{\tau_1}\right]}{\overline{\mathbb{E}}^{\mathbf{0}}\left[\tau_1\right]} \quad \text{a.s.} \quad \left[\overline{\mathbb{P}}^{\mathbf{0}}\right] \tag{4.8}$$

which completes the proof. ∎

4.2. *Proof of Theorem 2.5.* Fix $\mathbf{a} \in \mathbb{R}^d$, let $(Y_n)_{n\geq 1}$ be as defined above. Put $\overline{Y}_n := Y_n - (\tau_n - \tau_{n-1})\,\mathbf{a}\cdot\mathbf{v}$, where $\mathbf{v} = \frac{\overline{\mathbb{E}}^{\mathbf{0}}[X_{\tau_1}]}{\overline{\mathbb{E}}^{\mathbf{0}}[\tau_1]}$. Let $S_n := \overline{Y}_1 + \overline{Y}_2 + \cdots + \overline{Y}_n$, for $n \geq 1$. By Proposition 3.1, τ_1 has finite second moment and, due to the boundedness of the increments of the random walk $(X_t)_{t\geq 0}$, so does X_{τ_1}. Further, by definition, $\overline{\mathbb{E}}^{\mathbf{0}}\left[S_n\right] = 0$. Thus by Lemma 3.4 and Donsker's invariance principle, see e.g. Billingsley (1999, Theorem 14.1), we have

$$\frac{S_{\lfloor nt\rfloor}}{\sigma_a\sqrt{n}} \overset{d}{\underset{n\to\infty}{\longrightarrow}} \mathrm{BM}\,(1)\,, \tag{4.9}$$

where $\mathrm{BM}(1) := \mathrm{BM}_1(1)$ and $\sigma_a^2 := \overline{\mathbb{E}}^{\mathbf{0}}\left[\overline{Y}_1^2\right] > 0$, where the last inequality is due to Condition (A2) and (2.5).

Put next $m_n := n/\overline{\mathbb{E}}^{\mathbf{0}}\left[\tau_1\right]$. Since $n \mapsto m_n$ is a deterministic scaling, it follows from (4.9) that

$$\frac{S_{\lfloor m_n t\rfloor}}{\sigma_a\sqrt{m_n}} \overset{d}{\underset{n\to\infty}{\longrightarrow}} \mathrm{BM}\,(1)\,. \tag{4.10}$$

Let $(k_n)_{n\geq 1}$ be as before. Our next step is to prove the analogue of (4.9) with $m_n t$ replaced by $k_{\lfloor nt\rfloor}$. A consequence of (4.3) is that for any fixed $T < \infty$,

$$\sup_{t\leq T}\left|\frac{k_{\lfloor nt\rfloor}}{n} - \frac{\lfloor m_n t\rfloor}{n}\right| \underset{n\to\infty}{\longrightarrow} 0\,, \quad \text{a.s.} \quad \left[\overline{\mathbb{P}}^{\mathbf{0}}\right] \tag{4.11}$$

Thus,

$$\frac{S_{k_{\lfloor nt\rfloor}}}{\sigma_a\sqrt{m_n}} \overset{d}{\longrightarrow} \mathrm{BM}\,(1)\,. \tag{4.12}$$

Note that for an appropriate $C = C(\mathbf{a}, d)$, using (4.4),

$$\sup_{t\leq T}\left|\frac{S_{k_{\lfloor nt\rfloor}} - \mathbf{a}\cdot X_{\lfloor nt\rfloor} - nt\,\mathbf{a}\cdot\mathbf{v}}{\sqrt{n}}\right| \leq C \max_{0\leq i\leq k_{\lfloor nT\rfloor}} \frac{\tau_{i+1}-\tau_i}{\sqrt{n}}\,. \tag{4.13}$$

Since $k_{\lfloor nT\rfloor} \leq \lfloor nT\rfloor$, we have that for any $\delta > 0$,

$$\overline{\mathbb{P}}^{\mathbf{0}}\left(\max_{0\leq i\leq k_{\lfloor nT\rfloor}} \frac{\tau_{i+1}-\tau_i}{\sqrt{n}} > \delta\right) \leq \sum_{i=1}^{\lfloor nT\rfloor} \overline{\mathbb{P}}^{\mathbf{0}}(\tau_1 > \delta\sqrt{n})\,. \tag{4.14}$$

Note that, since $\overline{\mathbb{E}}^{\mathbf{0}}\left[\tau_1^2\right] < \infty$, one has that

$$\sum_{i=1}^{\infty} \overline{\mathbb{P}}^{\mathbf{0}}(\tau_1 > \frac{\delta\sqrt{i}}{\sqrt{T}}) = \sum_{i=1}^{\infty} \overline{\mathbb{P}}^{\mathbf{0}}(\tau_1^2 > \frac{\delta^2 i}{T}) < \infty\,.$$

Hence, for each $\delta_1 > 0$ there is a deterministic constant I_{δ_1} depending on d, δ, T, δ_1 such that

$$\sum_{i=I_{\delta_1}}^{\infty} \overline{\mathbb{P}}^{\mathbf{0}}(\tau_1 > \frac{\delta\sqrt{i}}{\sqrt{T}}) < \delta_1 \,.$$

Therefore,

$$\begin{aligned} & \limsup_{n\to\infty} \sum_{i=1}^{\lfloor nT \rfloor} \overline{\mathbb{P}}^{\mathbf{0}}(\tau_1 > \delta\sqrt{n}) \\ \leq \; & \limsup_{n\to\infty} \left(\sum_{i=1}^{I_{\delta_1}} \overline{\mathbb{P}}^{\mathbf{0}}(\tau_1 > \delta\sqrt{n}) + \sum_{i=I_{\delta_1}+1}^{\infty} \overline{\mathbb{P}}^{\mathbf{0}}(\tau_1 > \frac{\delta\sqrt{i}}{\sqrt{T}}) \right) \leq \delta_1 \,. \end{aligned}$$

δ_1 being arbitrary, one concludes from the last limit and (4.14) that

$$\overline{\mathbb{P}}^{\mathbf{0}} \left(\max_{0 \leq i \leq k_{\lfloor nT \rfloor}} \frac{\tau_{i+1} - \tau_i}{\sqrt{n}} > \delta \right) \underset{n\to\infty}{\longrightarrow} 0 \,.$$

Together with (4.12) and (4.13), this implies that for every $\mathbf{a} \in \mathbb{R}^d \setminus 0$ we have

$$\frac{\mathbf{a} \cdot \left(X_{\lfloor nt \rfloor} - nt\,\mathbf{v} \right)}{\sigma_a \sqrt{m_n}} \xrightarrow{d} \mathrm{BM}\,(1) \,. \tag{4.15}$$

Since $\mathbf{a}$ is arbitrary, this completes the proof of the theorem, with

$$\Sigma := \frac{\mathrm{Var}_{\;\overline{\mathbb{P}}^{\mathbf{0}}} \left(X_{\tau_1} - \tau_1\,\mathbf{v} \right)}{\overline{\mathbb{E}}^{\mathbf{0}} \left[\tau_1 \right]} . \tag{4.16}$$

∎

4.3. *Proof of Theorem 2.7.* Our argument is based on the technique introduced in Bolthausen and Sznitman (2002), as developed in Bolthausen, Sznitman and Zeitouni (2003). Let $B^n_t = (X_{\lfloor nt \rfloor} - nt\mathbf{v})/\sqrt{n}$, and let $\mathcal{B}^n_t$ denote the polygonal interpolation of $(k/n) \to B^n_{k/n}$. Consider the space $\mathcal{C}_T := C([0,T], \mathbb{R}^d)$ of continuous $\mathbb{R}^d$-valued functions on $[0,T]$, endowed with the distance $d_T(u,u') = \sup_{t\leq T} |u(t) - u'(t)| \wedge 1$. By Bolthausen and Sznitman (2002, Lemma 4.1), Theorem 2.7 follows from Theorem 2.5 once we show that for all bounded Lipschitz function F on $\mathcal{C}_T$ and $b \in (1,2]$,

$$\sum_{m=1}^{\infty} \mathrm{Var}_{\;\mathbf{P}^{\pi}} \left(\mathbf{E}^0_\omega [F(\mathcal{B}^{\lfloor b^m \rfloor})] \right) < \infty \,. \tag{4.17}$$

In order to prove (4.17), we now follow the approach of Bolthausen, Sznitman and Zeitouni (2003). We construct the environment ω using the variables $\{\alpha_t(\mathbf{x}) : t \geq 1, \mathbf{x} \in \mathbb{Z}^d\}$ as described in Section 3. We next construct two independent sequences of i.i.d. Bernoulli(ε) random variables, that we denote by $(\varepsilon^{(1)}_t)_{t\geq 2}$ and $(\varepsilon^{(2)}_t)_{t\geq 1}$. Given these sequences and the environment ω, we construct two independent copies of the random walk, denoted $(X^{(1)}_t)_{t\geq 1}$ and $(X^{(2)}_t)_{t\geq 1}$, following the recipe of Section 3 (we of course use the sequence $\varepsilon^{(j)}$ to construct $X^{(j)}$, for $j = 1, 2$), and introduce the respective linear interpolations $(\mathcal{B}^{n,(j)}_t)_{t\geq 0}$. It is then clear that (4.17) is equivalent to

$$\sum_m \Big(\mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon \otimes \mathbf{Q}_\varepsilon \Big(\overline{\mathbb{P}}^{\mathbf{0}}_{\omega,\epsilon^{(1)}} \otimes \overline{\mathbb{P}}^{\mathbf{0}}_{\omega,\epsilon^{(2)}} \Big[F(\mathcal{B}^{\lfloor b^m \rfloor,(1)})F(\mathcal{B}^{\lfloor b^m \rfloor,(2)})\Big]\Big) \tag{4.18}$$
$$-\mathbf{P}^\pi \otimes \mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon \otimes \mathbf{Q}_\varepsilon \Big(\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\epsilon^{(1)}} \otimes \overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(2)},\epsilon^{(2)}} \Big[F(\mathcal{B}^{\lfloor b^m \rfloor,(1)})F(\mathcal{B}^{\lfloor b^m \rfloor,(2)})\Big]\Big)\Big)$$
$$< \infty\,,$$

where for any probability measure $\mathbf{P}$ and a measurable function f by $\mathbf{P}(f)$ we mean $\mathbf{E_P}[f]$. In the sequel, we write $\overline{\mathbb{P}}^{\mathbf{0}}_{\omega,\omega',\epsilon,\epsilon'}$ for $\overline{\mathbb{P}}^{\mathbf{0}}_{\omega,\epsilon} \otimes \overline{\mathbb{P}}^{\mathbf{0}}_{\omega',\epsilon'}$. Recalling the constant γ from (2.9), we next choose constants $\theta, \theta', \theta'', \mu, \alpha$ satisfying the following conditions:

$$\begin{aligned}
&0<\theta<1,\ 2/\gamma<\theta'<\theta/2,\ \theta>2(\theta'+1/(d-1)),\ \theta''<\theta',\ (\theta'-\theta'')\gamma>1,\\
&0<\mu<1/2,\ 1/2>\alpha>(1/\theta'+1)/\gamma,\\
&\theta''((d-1)-4-2\alpha(d-1))=\theta''(d-5-2\alpha d+2\alpha)>1\,.
\end{aligned} \tag{4.19}$$

It is not hard to verify that the assumptions of Theorem 2.7 imply that such constants can be found (indeed, verify, using that $d>7$ and $d-1>4/(1-6/\gamma)$ which implies that $\gamma>6(d-1)/(d-5)$, that taking $\theta'=(d-3)/2(d-1)$, $\alpha=1/\gamma+1/\gamma\theta'$ and $\theta''=\theta'(\gamma(d-3)-2(d-1))/\gamma(d-3)$ satisfies the constraints except for the last one with equality, and the last one with inequality due to (2.10), and argue by continuity). Fix then an integer m. We let $\bar\beta_m$ denote the first time there is a run of length $\lfloor m^\theta \rfloor$ of non-zero ϵ-coin tosses of both types ending at it, that is

$$\bar\beta_m = \inf\{s \geq \lfloor m^\theta \rfloor \,\big|\, \epsilon^{(1)}_s = \epsilon^{(2)}_s = \epsilon^{(1)}_{s-1} = \epsilon^{(2)}_{s-1} = \ldots = \epsilon^{(1)}_{s-\lfloor m^\theta \rfloor+1} = \epsilon^{(2)}_{s-\lfloor m^\theta \rfloor+1} = 1\}$$

For any $\mathbf{x} \in \mathbb{Z}^d$, set $D^{(j)}_m(\mathbf{x}) = |\mathbf{x} - X^{(j)}_{\bar\beta_m}|$. Setting $n = \lfloor b^m \rfloor$, define next the events

$$G_m = \{\bar\beta_m \leq n^\mu, X^{(1)}_{[\bar\beta_m,\infty)} \cap X^{(2)}_{[\bar\beta_m,\infty)} = \emptyset\}\,,$$

and for $j=1,2$,

$$R^{(j)}_m = \{\forall \mathbf{x} \in \mathbb{Z}^d, \exists t \in [\bar\beta_m - \lfloor m^\theta \rfloor, \bar\beta_m + D^{(j)}_m(\mathbf{x})] \,\text{such that}\; \alpha_t(\mathbf{x}) = 1\}\,.$$

Finally, set $\hat G_m = G_m \cap R^{(1)}_m \cap R^{(2)}_m$. The crucial element of the proof of Theorem 2.7 is contained in the following lemma, whose proof is postponed.

Lemma 4.1. *Under the assumptions of Theorem 2.7, the following estimates hold.*

$$\sum_{m=1}^\infty \mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon \otimes \mathbf{Q}_\varepsilon \left(\overline{\mathbb{P}}^{\mathbf{0}}_{\omega,\omega,\epsilon^{(1)},\epsilon^{(2)}}[(\hat G_m)^c]\right) < \infty\,. \tag{4.20}$$

$$\sum_{m=1}^\infty \mathbf{P}^\pi \otimes \mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon \otimes \mathbf{Q}_\varepsilon \left(\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(2)},\epsilon^{(1)},\epsilon^{(2)}}[(\hat G_m)^c]\right) < \infty\,. \tag{4.21}$$

Equipped with Lemma 4.1, let us complete the proof (4.18). Indeed, for any integer m,

$$\begin{aligned}\Delta_m :=& \\ &\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(1)},\epsilon^{(1)},\epsilon^{(2)}}\left[F(\mathcal{B}^{n,(1)})F(\mathcal{B}^{n,(2)})\right] - \overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(2)},\epsilon^{(1)}\epsilon^{(2)}}\left[F(\mathcal{B}^{n,(1)})F(\mathcal{B}^{n,(2)})\right] \\ &= \overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(1)},\epsilon^{(1)},\epsilon^{(2)}}\left[F(\mathcal{B}^{n,(1)})F(\mathcal{B}^{n,(2)});\hat{G}_m\right] \\ &\quad - \overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(2)},\epsilon^{(1)}\epsilon^{(2)}}\left[F(\mathcal{B}^{n,(1)})F(\mathcal{B}^{n,(2)});\hat{G}_m\right] + d_m \qquad (4.22)\end{aligned}$$

where

$$\sum_{m=1}^{\infty} \mathbf{P}^{\pi}\otimes\mathbf{P}^{\pi}\otimes\mathbf{Q}_{\varepsilon}\otimes\mathbf{Q}_{\varepsilon}(|d_m|) < \infty\,.$$

Let $\hat{\mathcal{B}}^{n,(j)}_t = \mathcal{B}^{n,(j)}_{t+\frac{\bar{\beta}_m}{n}} - \mathcal{B}^{n,(j)}_{\frac{\bar{\beta}_m}{n}}$ for $j=1,2$. Recall that on $\hat{G}_m$ one has $\bar{\beta}_m < n^{\mu}$ and hence, using the Lipschitz property of F, it holds a.s. with respect to $\overline{\mathbb{P}}^{\mathbf{0}}$ on $\hat{G}_m$ that

$$|F(\mathcal{B}^{n,(j)}) - F(\hat{\mathcal{B}}^{n,(j)})| \le 2n^{\mu-1/2}\,.$$

Substituting in (4.22), we get

$$\begin{aligned}\Delta_m =& \overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(1)},\epsilon^{(1)},\epsilon^{(2)}}\left[F(\hat{\mathcal{B}}^{n,(1)})F(\hat{\mathcal{B}}^{n,(2)});\hat{G}_m\right] \\ &- \overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(2)},\epsilon^{(1)}\epsilon^{(2)}}\left[F(\hat{\mathcal{B}}^{n,(1)})F(\hat{\mathcal{B}}^{n,(2)});\hat{G}_m\right] + e_m \qquad (4.23)\end{aligned}$$

where

$$\sum_{m=1}^{\infty} \mathbf{P}^{\pi}\otimes\mathbf{P}^{\pi}\otimes\mathbf{Q}_{\varepsilon}\otimes\mathbf{Q}_{\varepsilon}(|e_m|) < \infty\,. \qquad (4.24)$$

Conditioning on $X^{(j)}_{\bar{\beta}_m}$, $j=1,2$, one observes that

$$\begin{aligned}\mathbf{P}^{\pi}\otimes\mathbf{P}^{\pi}\otimes\mathbf{Q}_{\varepsilon}\otimes\mathbf{Q}_{\varepsilon}&\Big(\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(1)},\epsilon^{(1)},\epsilon^{(2)}}\left[F(\hat{\mathcal{B}}^{n,(1)})F(\hat{\mathcal{B}}^{n,(2)});\hat{G}_m\right] \\ &-\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(2)},\epsilon^{(1)}\epsilon^{(2)}}\left[F(\hat{\mathcal{B}}^{n,(1)})F(\hat{\mathcal{B}}^{n,(2)});\hat{G}_m\right]\Big) = 0\,. \qquad (4.25)\end{aligned}$$

Together with (4.23) and (4.24), one concludes that

$$\sum_{m=1}^{\infty} \mathbf{P}^{\pi}\otimes\mathbf{P}^{\pi}\otimes\mathbf{Q}_{\varepsilon}\otimes\mathbf{Q}_{\varepsilon}(\Delta_m) < \infty\,,$$

as claimed. ∎

Proof of lemma 4.1 We begin by considering the event $\{\bar{\beta}_m > n^{\mu}\}$. By the independence of the i.i.d. sequences $\epsilon^{(j)}$, we get the estimate

$$\begin{aligned}\mathbf{Q}_{\varepsilon}\otimes\mathbf{Q}_{\varepsilon}(\bar{\beta}_m > n^{\mu}) &\le (1-\varepsilon^{2\lfloor m^{\theta}\rfloor})^{n^{\mu}/\lfloor m^{\theta}\rfloor - 2} \\ &\le e^{-n^{\mu}\varepsilon^{2\lfloor m^{\theta}\rfloor}/\lfloor m^{\theta}\rfloor + 2\varepsilon^{2\lfloor m^{\theta}\rfloor}} \le c_1 e^{-e^{c_2 m}}\,, \qquad (4.26)\end{aligned}$$

for appropriate constants c_1, c_2, where the last estimate used that $\theta < 1$.

We next consider the event $(R^{(j)}_m)^c \cap \{\bar{\beta}_m \le n^{\mu}\}$. Decomposing according to the distance from $X^{(j)}_{\bar{\beta}_m}$, as in the proof of Proposition 3.1 (see (3.11)), one gets for some deterministic constants c_3, c_4 (that may depend on the choice of parameters)

$$\mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon((R_m^{(j)})^c \cap \{\bar\beta_m \le n^\mu\}) \le c_3 \sum_{r=1}^\infty r^{d-1}(1-\kappa)^{r+m^\theta} \le c_4(1-\kappa)^{m^\theta}. \quad (4.27)$$

We next turn to the crucial estimate of the probability of non-intersection after $\bar\beta_m$. For this we consider the cases $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$ separately.

First suppose $\mathbf{v} \neq \mathbf{0}$. We start by showing that at time $\bar\beta_m$, the two walkers are not likely to be too close in a $(d-1)$-dimensional sub-space. More precisely, let $V^\perp$ be the $(d-1)$ dimensional sub-space of $\mathbb{R}^d$ which is orthogonal to the vector $\mathbf{v}$. Let $\mathbb{L}^{d-1}$ be the $(d-1)$-dimensional lattice which is the projection of $\mathbb{Z}^d$ into $V^\perp$. Let $P_q^{\mathbf{x}}$ denote the law of a homogeneous Markov chain on $\mathbb{Z}^d$ with transition probabilities determined by q starting at $\mathbf{x} \in \mathbb{Z}^d$ and let $(M_t^{\tilde{\mathbf{x}}})_{t\ge 0}$ denote the projection of the associated walk into the lattice $\mathbb{L}^{d-1}$, where $\tilde{\mathbf{x}}$ is the projection of the point $\mathbf{x}$ into the lattice $\mathbb{L}^{d-1}$. In particular, $P_q^{\mathbf{x}}(M_0^{\tilde{\mathbf{x}}} = \tilde{\mathbf{x}}) = 1$. We use $M^{\tilde{\mathbf{x}}}, \bar M^{\tilde{\mathbf{y}}}$ to denote independent copies of such walks starting at $\mathbf{x}, \mathbf{y}$, respectively. For $j = 1, 2$, let $\widetilde X_{\bar\beta_m}^{(j)}$ denote the projection of the vector $X_{\bar\beta_m}^{(j)}$ into $V^\perp$. Then, with $A_m = \{|\widetilde X_{\bar\beta_m}^{(1)} - \widetilde X_{\bar\beta_m}^{(2)}| \ge \lfloor m^{\theta'} \rfloor\}$, we have,

$$\begin{aligned}
&\mathbf{P}^\pi \otimes \mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon \otimes \mathbf{Q}_\varepsilon \left(\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(1)},\epsilon^{(1)},\epsilon^{(2)}}(A_m^c) \right) \\
&\le \sup_{\mathbf{x},\mathbf{y}\in\mathbb{Z}^d} P_q^{\mathbf{x}} \otimes P_q^{\mathbf{y}} \left(|M^{\tilde{\mathbf{x}}}_{\lfloor m^\theta \rfloor} - M^{\tilde{\mathbf{y}}}_{\lfloor m^\theta \rfloor}| < m^{\theta'} \right) \\
&= \sup_{\mathbf{x},\mathbf{y}\in\mathbb{Z}^d} P_q^{\mathbf{y}} \left(P_q^{\mathbf{x}} \left(|M^{\tilde{\mathbf{x}}}_{\lfloor m^\theta \rfloor} - M^{\tilde{\mathbf{y}}}_{\lfloor m^\theta \rfloor}| < m^{\theta'} \,\middle|\, M^{\tilde{\mathbf{y}}}_{\lfloor m^\theta \rfloor} \right) \right) \\
&\le C m^{\theta'(d-1)} \sup_{\tilde{\mathbf{x}},\tilde{\mathbf{z}}\in\mathbb{L}^{d-1}} P_q^{\mathbf{x}} \left(M^{\tilde{\mathbf{x}}}_{\lfloor m^\theta \rfloor} = \tilde{\mathbf{z}} \right) \\
&\le C m^{\theta'(d-1)} \frac{1}{m^{\theta(d-1)/2}} = C m^{-\left(\frac{\theta}{2}-\theta'\right)(d-1)} \qquad (4.28)
\end{aligned}$$

where C is some deterministic constant and the last inequality is due to the local limit theorem for lattice distribution under $P_q^{\mathbf{x}}$, see Bhattacharya and Rao (1976, Theorem 22.1).

Letting

$$B_m = A_m \cap \{\bar\beta_m \le n^\mu\} \cap R_m^{(1)} \cap R_m^{(2)},$$

and repeating the estimate (4.28) without change for $\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(2)},\epsilon^{(1)},\epsilon^{(2)}}$ replacing $\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(1)},\epsilon^{(1)},\epsilon^{(2)}}$, and using that $(d-1)(\theta/2 - \theta') > 1$, we conclude that

$$\begin{aligned}
\max \Bigg(&\sum_{m=1}^\infty \mathbf{P}^\pi \otimes \mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon \otimes \mathbf{Q}_\varepsilon \left(\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(1)},\epsilon^{(1)},\epsilon^{(2)}}(B_m^c) \right), \qquad (4.29) \\
&\sum_{m=1}^\infty \mathbf{P}^\pi \otimes \mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon \otimes \mathbf{Q}_\varepsilon \left(\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(2)},\epsilon^{(1)},\epsilon^{(2)}}(B_m^c) \right) \Bigg) < \infty
\end{aligned}$$

Next, for $j = 1, 2$, we construct, using the recipe in Section 3, regeneration times $\tau_i^{(j)}$ corresponding to the walks $(X_t^{(j)})_{t\ge\bar\beta_m}$ and the chain on the environment. Note that under the measure $\mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon \otimes \mathbf{Q}_\varepsilon \otimes \overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(1)},\epsilon^{(1)},\epsilon^{(2)}}$, the sequences for $j = 1$ and $j = 2$ are not independent if the walks intersect. When no confusion occurs, we let P^τ denote the law of this sequence, noting that the sequence $(\tau_{i+1}^{(j)} - \tau_i^{(j)})_{i\geq 1}$ is i.i.d. under all measures involved in our construction, as well as, equation (3.14) holds. Fix an integer $k \geq 1$, for any fixed $2 \leq \gamma' < \gamma$ we then have

$$\begin{aligned}
& P^\tau\left(\left|\tau_k^{(j)} - kE^\tau[\tau_1^{(j)}]\right| > k/2\right) \\
= \; & P^\tau\left(\left|\tau_k^{(j)} - kE^\tau[\tau_1^{(j)}]\right|^{\gamma'} > (k/2)^{\gamma'}\right) \\
\leq \; & \frac{2^{\gamma'}\, E^\tau\left[\left|\tau_k^{(j)} - kE^\tau[\tau_1^{(j)}]\right|^{\gamma'}\right]}{k^{\gamma'}} \\
\leq \; & \frac{2^{\gamma'}\, B_{\gamma'}\, E^\tau\left[\left|\tau_1^{(j)} - E^\tau[\tau_1^{(j)}]\right|^{\gamma'}\right]}{k^{\gamma'/2}}\,,
\end{aligned} \tag{4.30}$$

where the last inequality follows from and application of Marcynkiewicz-Zygmund inequality Shiryayev (1984, Pg. 469) along with Hölder inequality, and $B_{\gamma'} > 0$ is an universal constant which depends on γ'. Recall, $\gamma\theta'/2 > 1$ and also $\gamma > 6$, thus we can choose $2 \leq \gamma' < \gamma$ such that $\gamma'\theta'/2 > 1$. Now choosing $k = c_5 m^{\theta'}$, for an appropriate c_5, one concludes the existence of a constant c_6 such that

$$\sum_{m=1}^{\infty} P^\tau\left(\tau^{(j)}_{\lfloor c_6 m^{\theta'}\rfloor} > m^{\theta'}/4\right) < \infty\,. \tag{4.31}$$

On the other hand,

$$\begin{aligned}
& P^\tau\left(\exists i > c_6 m^{\theta'} \,:\, \tau_{i+1}^{(j)} - \tau_i^{(j)} > i^\alpha\right) \\
\leq \; & \sum_{i=\lfloor c_6 m^{\theta'}\rfloor + 1}^{\infty} P^\tau\left(\tau_{i+1}^{(j)} - \tau_i^{(j)} > i^\alpha\right) \\
= \; & \sum_{i=\lfloor c_6 m^{\theta'}\rfloor + 1}^{\infty} P^\tau\left(\tau_1^{(j)} > i^\alpha\right) \\
\leq \; & \sum_{i=\lfloor c_6 m^{\theta'}\rfloor + 1}^{\infty} \frac{C}{i^{\alpha\gamma'}}\,,
\end{aligned}$$

for some deterministic constant C, and $\gamma' < \gamma$. Since our choice of constants implies that $\theta'(\alpha\gamma - 1) > 1$, we can choose $\gamma' < \gamma$ such that $\theta'(\alpha\gamma' - 1) > 1$, and so we conclude that

$$\sum_{m=1}^{\infty} P^\tau\left(\exists i > c_6 m^{\theta'} \,:\, \tau_{i+1}^{(j)} - \tau_i^{(j)} > i^\alpha\right) < \infty \tag{4.32}$$

Let $C_m^{(j)} = \{\tau^{(j)}_{\lfloor c_6 m^{\theta'}\rfloor} \leq m^{\theta'}/4\} \cap \{\forall i > c_6 m^{\theta'},\, \tau_{i+1}^{(j)} - \tau_i^{(j)} \leq i^\alpha\}$, and note that for all measures P^τ involved,

$$\sum_{m=1}^{\infty} P^\tau\left(\left(C_m^{(1)} \cap C_m^{(2)}\right)^c\right) < \infty\,. \tag{4.33}$$

Next, for $j=1,2$, define $S_t^{(j)} = \widetilde{X}^{(j)}_{\tau_t^{(j)}}$. Recall that $\widetilde{X}_s^{(j)}$ is the orthogonal projection of the vector $X_s^{(j)}$ into the space $\mathbb{L}^{d-1}$. Note that

$$\{X^{(1)}_{[\bar\beta_m,\infty)} \cap X^{(2)}_{[\bar\beta_m,\infty)} \neq \emptyset\} \cap A_m \cap C_m^{(1)} \cap C_m^{(2)} \subset \{|S_0^{(1)} - S_0^{(2)}| \geq m^{\theta'}, \exists \ell, k : |S_\ell^{(1)} - S_k^{(2)}| \leq \ell^\alpha + k^\alpha\} \tag{4.34}$$

Let Z_t denote a sum of (centered) t i.i.d. random vectors, each distributed according to $S_1^{(1)}$, thus taking values in $\mathbb{L}^{d-1}$. Let $(Z'_t)_{t\geq 1}$ denote an independent copy of $(Z_t)_{t\geq 1}$. Let $P_Z^{\mathbf{x},\mathbf{y}}$ denote the law of the sequences $((\mathbf{x}+Z_t),(\mathbf{y}+Z'_t))_{t\geq 1}$, where $\mathbf{x},\mathbf{y} \in \mathbb{L}^{d-1}$. Using that under all the measures involved in our computation, the paths $X^{(1)}$ and $X^{(2)}$ are independent until they intersect, one concludes that

$$\begin{aligned}
&\mathbf{P}^\pi \otimes \mathbf{P}^\pi \otimes \mathbf{Q}_\varepsilon \otimes \mathbf{Q}_\varepsilon \\
&\left(\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(1)},\epsilon^{(1)},\epsilon^{(2)}}\left(\{X^{(1)}_{[\bar\beta_m,\infty)} \cap X^{(2)}_{[\bar\beta_m,\infty)} \neq \emptyset\} \cap A_m \cap C_m^{(1)} \cap C_m^{(2)}\right)\right) \\
&\leq \max_{\substack{|\mathbf{x}|\geq m^{\theta'}\\ \mathbf{x}\in\mathbb{L}^{d-1}}} P_Z^{\mathbf{0},\mathbf{x}}\left(\exists \ell,k:\ |Z_\ell - Z'_k| \leq \ell^\alpha + k^\alpha\right),
\end{aligned} \tag{4.35}$$

with exactly the same estimate when $\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(2)},\epsilon^{(1)},\epsilon^{(2)}}$ replaces $\overline{\mathbb{P}}^{\mathbf{0}}_{\omega^{(1)},\omega^{(1)},\epsilon^{(1)},\epsilon^{(2)}}$. But, for some constant C_α whose value may change from line to line,

$$\begin{aligned}
&\max_{\substack{|\mathbf{x}|\geq m^{\theta'}\\ \mathbf{x}\in\mathbb{L}^{d-1}}} P_Z^{\mathbf{0},\mathbf{x}}\left(\exists \ell,k:\ |Z_\ell - Z'_k| \leq \ell^\alpha + k^\alpha\right) \\
&\leq \max_{\substack{|\mathbf{x}|\geq m^{\theta'}\\ \mathbf{x}\in\mathbb{L}^{d-1}}} P_Z^{\mathbf{0},\mathbf{x}}\left(\exists \ell,k:\ |Z_\ell - Z'_k| \leq C_\alpha(\ell+k)^\alpha\right) \\
&\leq \max_{\substack{|\mathbf{x}|\geq m^{\theta'}\\ \mathbf{x}\in\mathbb{L}^{d-1}}} P_Z^{\mathbf{0},\mathbf{x}}\left(\exists \ell,k:\ (\ell+k) < m^{2\theta''},\ |Z_\ell - Z'_k| \leq C_\alpha(\ell+k)^\alpha\right) \\
&\quad + \max_{\substack{|\mathbf{x}|\geq m^{\theta'}\\ \mathbf{x}\in\mathbb{L}^{d-1}}} P_Z^{\mathbf{0},\mathbf{x}}\left(\exists \ell,k:\ (\ell+k) \geq m^{2\theta''},\ |Z_\ell - Z'_k| \leq C_\alpha(\ell+k)^\alpha\right) \\
&=: I_1(m) + I_2(m).
\end{aligned} \tag{4.36}$$

For $I_1(m)$ in (4.36) we observe that because $\alpha < 1/2$, for any $\gamma' < \gamma$, we can find a constant $C_{\gamma'}$ such that for large m

$$\begin{aligned}
&\max_{\substack{|\mathbf{x}|\geq m^{\theta'}\\ \mathbf{x}\in\mathbb{L}^{d-1}}} P_Z^{\mathbf{0},\mathbf{x}}\left(\exists \ell,k:\ (\ell+k) < m^{2\theta''},\ |Z_\ell - Z'_k| \leq C_\alpha(\ell+k)^\alpha\right) \\
&\leq 2\, P_Z^{\mathbf{0},\mathbf{0}}\left(\max_{\ell \leq m^{2\theta''}} |Z_\ell| \geq \frac{m^{\theta'}}{3}\right) \\
&\leq \frac{C_{\gamma'}}{m^{(\theta'-\theta'')\gamma'}},
\end{aligned} \tag{4.37}$$

where the first inequality follows from the observation that

$$|Z_\ell - Z'_k| \geq |\mathbf{x}| - |Z_\ell| - |Z'_k - \mathbf{x}|,$$

and the last inequality is due again to the Marcynkiewicz-Zygmund inequality coupled with the Martingale maximal inequality of Burkholder and Gundy (see Shiryayev (1984, Pg. 469)). Therefore, since θ'' was chosen such that $(\theta'-\theta'')\gamma > 1$, it follows that

$$\sum_m I_1(m) < \infty\,. \tag{4.38}$$

Turning to the estimate of the term $I_2(m)$ in (4.36), we observe, with the value of the constants C_α, C'_α possibly changing from line to line, that

$$\begin{aligned}
&\sum_{\substack{\ell\geq k\\(\ell+k)\geq m^{2\theta''}}} \max_{\substack{|\mathbf{x}|\geq m^{\theta'}\\ \mathbf{x}\in\mathbb{L}^{d-1}}} P_Z^{\mathbf{0},\mathbf{x}}\left(|Z_\ell - Z'_k| \leq C_\alpha(\ell+k)^\alpha\right)\\
=& \sum_{\substack{\ell\geq k\\(\ell+k)\geq m^{2\theta''}}} \max_{\substack{|\mathbf{x}|\geq m^{\theta'}\\ \mathbf{x}\in\mathbb{L}^{d-1}}} E_Z^{\mathbf{0},\mathbf{x}}\left[P_Z^{\mathbf{0},\mathbf{x}}\left(|Z_\ell - Z'_k| \leq C_\alpha(\ell+k)^\alpha \,\middle|\, Z'_k\right)\right]\\
\leq& \sum_{\substack{\ell\geq k\\(\ell+k)\geq m^{2\theta''}}} \max_{\substack{|\mathbf{x}|\geq m^{\theta'}\\ \mathbf{x}\in\mathbb{L}^{d-1}}} C'_\alpha\,(\ell+k)^{\alpha(d-1)}\; E_Z^{\mathbf{0},\mathbf{x}}\left[\max_{|\mathbf{z}-Z'_k|\leq C_\alpha(\ell+k)^\alpha} P_Z^{\mathbf{0},\mathbf{x}}\left(Z_\ell=\mathbf{z}\right)\right]\\
\leq& \sum_{\substack{\ell\geq k\\(\ell+k)\geq m^{2\theta''}}} C'_\alpha\,(\ell+k)^{\alpha(d-1)}\;\frac{1}{\ell^{(d-1)/2}}\\
\leq& \sum_{\substack{\ell\geq k\\(\ell+k)\geq m^{2\theta''}}} C'_\alpha\,(\ell+k)^{\alpha(d-1)}\;\frac{1}{(\ell+k)^{(d-1)/2}}\,,
\end{aligned}$$

where the last but one inequality follows by applying the local limit estimate Bhattacharya and Rao (1976, Theorem 22.1). By symmetry, the same estimate holds with $k \geq \ell$ replacing the constraint $\ell \geq k$ in the summation. Thus we conclude that

$$\begin{aligned}
I_2\,(m) \;\leq&\; \sum_{\substack{\ell,k\\(\ell+k)\geq m^{2\theta''}}} \max_{\substack{|\mathbf{x}|\geq m^{\theta'}\\ \mathbf{x}\in\mathbb{L}^{d-1}}} P_Z^{\mathbf{0},\mathbf{x}}\left(|Z_\ell - Z'_k| \leq C_\alpha(\ell+k)^\alpha\right)\\
\leq&\; 2\sum_{\substack{\ell\geq k\\(\ell+k)\geq m^{2\theta''}}} C'_\alpha\,(\ell+k)^{\alpha(d-1)}\;\frac{1}{(\ell+k)^{(d-1)/2}}\\
\leq&\; \sum_{r=\lfloor m^{2\theta''}\rfloor+1}^{\infty} \frac{2C'_\alpha}{r^{\frac{d-1}{2}-\alpha(d-1)-1}} \leq \frac{2C'_\alpha}{m^{\theta''(d-2\alpha d+2\alpha-5)}}\,.
\end{aligned} \tag{4.39}$$

In particular, since $\theta''(d-2\alpha d+2\alpha-5) > 1$, one concludes that

$$\sum_m I_2(m) < \infty\,. \tag{4.40}$$

Combining (4.38) and (4.40), and substituting in (4.36) and then in (4.35), the conclusion of the lemma follows.

Finally for the case $\mathbf{v} = \mathbf{0}$ we can proceed in two ways. We could chose $V^\perp$ as an arbitrary subspace of co-dimension 1, and simply repeat the argument. Alternatively, we could proceed exactly the same manner except that one works directly with the walkers, rather than their projections. In this case we can replace $(d-1)$ by d in all the estimates above and the conclusions follows by observing that the inequalities in (4.19) holds when $(d-1)$ is replaced by d. The advantage of the second approach is then that when $\mathbf{v} = \mathbf{0}$, one may replace the constraint on d in (2.10) by the weaker constraint involving $d+1$.

This completes the proof of the lemma. ∎

5. Final Remarks and Open Problems

5.1. *The "regeneration" time.* We point out that our definition of the "regeneration" time τ_1 using the coin tosses $\left\{(\alpha_t(\cdot))_{t\geq 1}\right\}$, is quite arbitrary, and was tailored to Condition (A1). What we actually need for the argument is for every environment chain $(\omega_t(\cdot,\cdot))_{t\geq 0}$ a sequence of stopping times such that at these times, the chain starts from a stationary distribution and the times have "good" tail property. The assumption (A1) is used to ensure this is possible with our construction.

5.2. *The annealed CLT.* Our argument gives a trade-off between the strength of the random perturbation and the mixing rate of the environment. We suspect that such a trade-off is not needed, and that for $d \geq 3$, an annealed CLT holds true as soon as the environment is mixing enough in time. Our technique does not seem to resolve this question.

5.3. *The quenched CLT.* We have already emphasized that the condition (2.10) is not expected to be optimal. In particular, we suspect that if a critical dimension for the quenched CLT exists, it will be $d = 1$ (see also the comments at the end of Boldrighini, Minlos and Pellegrinotti (2000) hinting that such quenched CLT fails even for small perturbations of a fixed Markovian environment, and the numerical simulatons in Boldrighini et al. (2005) that are inconclusive).

Acknowledgment

We thank Amir Dembo for a remark that prompted us to modify our original construction of regeneration times. We are also grateful to the anonymous referee for a careful reading of the manuscript and for detecting a mistake in the original version.

References

R. N. Bhattacharya and R. Ranga Rao. *Normal approximation and asymptotic expansions*. John Wiley & Sons, New York-London-Sydney, 1976. Wiley Series in Probability and Mathematical Statistics.

P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

C. Boldrighini, G. Cosimi, S. Frigio, and A. Pellegrinotti. Computer simulations for some one-dimensional models of random walks in fluctuating random environment. *J. Stat. Phys*, 121, pages 361–372, 2005.

C. Boldrighini, R. A. Minlos, and A. Pellegrinotti. Almost-sure central limit theorem for a Markov model of random walk in dynamical random environment. *Probab. Theory Related Fields*, 109(2):245–273, 1997.

C. Boldrighini, R. A. Minlos, and A. Pellegrinotti. Random walk in a fluctuating random environment with Markov evolution. In *On Dobrushin's way. From probability theory to statistical physics*, volume 198 of *Amer. Math. Soc. Transl. Ser. 2*, pages 13–35. Amer. Math. Soc., Providence, RI, 2000.

C. Boldrighini, R. A. Minlos, and A. Pellegrinotti. Random walks in quenched i.i.d. space-time random environment are always a.s. diffusive. *Probab. Theory Related Fields*, 129(1):133–156, 2004.

E. Bolthausen and A.-S. Sznitman. On the static and dynamic points of view for certain random walks in random environment. *Methods Appl. Anal.*, 9(3):345–375, 2002.

E. Bolthausen, A.-S. Sznitman, and O. Zeitouni. Cut points and diffusive random walks in random environment. *Ann. Inst. H. Poincar. Probab. Statist.*, 39(3):527–555, 2003.

Y. Kifer. Perron-frobenius theorem, large deviations, and random perturbations in random environments. *Math. Z.*, 222(4):677–698, 1996.

A. N. Shiryayev. *Probability*, volume 95 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1984. Translated from the Russian by R. P. Boas.

W. Stannat. A remark on the CLT for a random walk in a random environment. *Probab. Theory Related Fields*, 130(3):377–387, 2004.

A.-S. Sznitman. Topics in random walk in random environment. In *Notes of course at School and Conference on probability*, ICTP Lecture Series, pages 203–266. Trieste, 2004.

O. Zeitouni. Random walks in random environment. In *Lectures on probability theory and statistics*, volume 1837 of *Lecture Notes in Math.*, pages 189–312. Springer, Berlin, 2004.