Endogeny for the
Logistic Recursive Distributional Equation

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Abstract. In this article we prove the endogeny and bivariate uniqueness property for a particular "max-type" recursive distributional equation (RDE). The RDE we consider is the so called logistic RDE, which appears in the proof of the $\zeta(2)$-limit of the random assignment problem using the local weak convergence method proved by D. Aldous [Probab. Theory Related Fields 93 (1992)(4), 507 – 534]. This article provides a non-trivial application of the general theory developed by D. Aldous and A. Bandyopadhyay [Ann. Appl. Probab. 15 (2005)(2), 1047 – 1110]. The proofs involves analytic arguments, which illustrate the need to develop more analytic tools for studying such max-type RDEs.

1. Introduction and the main result

Fixed-point equations have found many applications in various fields of mathematics. In probability theory they are usually referred as distributional identities. A recent article by Aldous and Bandyopadhyay [5] provides a general framework to study certain type of distributional identities which arise in a variety of settings.

Given a measurable space $(S, \mathcal{S})$ write $\mathcal{P}(S)$ for the set of all probabilities on S. According to [5], a *recursive distributional equation* (RDE) is a fixed-point equation on $\mathcal{P}(S)$ defined as

$$X \stackrel{d}{=} g\left(\xi; (X_j : 1 \leq j \leq^* N)\right) \quad \text{on } S, \tag{1}$$

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where it is assumed that $(X_j)_{j\geq 1}$ are S-valued random variables with identical distribution and are (stochastically) independent, and their common distribution is same as that of the random variable X, and are independent of the pair (ξ, N). Here ξ is a measurable function taking value on some measurable space, say $(\Lambda, \mathcal{Z})$, N is a non-negative integer valued random variable, which may take the value ∞ and g is a given S-valued function. (In the above equation by "$\leq^* N$" we mean the left hand side is "$\leq N$" if $N < \infty$, and "$< N$" otherwise). In (1) the distribution of X is unknown, while the distribution of the pair (ξ, N) and the function g are assumed to be known quantities. Perhaps a more conventional (analytic) way of writing the equation (1) would be

$$\mu = T\left(\mu\right), \tag{2}$$

where $T : \mathcal{P}\left(S\right) \to \mathcal{P}\left(S\right)$ is a function defined as

$$T\left(\mu\right) := \mathrm{dist}\big(g\left(\xi; \left(X_j : 1 \leq j \leq^* N\right)\right)\big)$$

where (ξ, N) is as above and $(X_j)_{j\geq 1}$ are independent and identically distributed (i.i.d.) with common distribution $\mu \in \mathcal{P}\left(S\right)$ and these sets of random variables are stochastically independent.

As outlined in [5] in many applications RDEs play a very crucial role. Examples include study of Galton-Watson branching processes and related random trees, probabilistic analysis of algorithms with suitable recursive structure [11, 15, 16], statistical physics models on trees [2, 3, 6, 12], and statistical physics and algorithmic questions in the mean-field model of distance [1, 2, 4, 7, 8, 19]. In many of these applications, particularly in the last two types mentioned above, often one needs to construct a particular tree indexed stationary process related to a given RDE, which is called a *recursive tree process* (RTP) [5]. More precisely, suppose the RDE (1) has a solution, say μ. Then as shown in [5], using the consistency theorem of Kolmogorov [9], one can construct a process, say $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$, indexed by $\mathcal{V} := \left(\cup_{d\geq 1}\mathbb{N}^d\right) \cup \{\emptyset\}$, such that

$$
\begin{array}{ll}
\text{(i)} & X_{\mathbf{i}} \sim \mu \quad \forall\, \mathbf{i} \in \mathcal{V} \\
\text{(ii)} & \text{for each } d \geq 0,\ (X_{\mathbf{i}})_{|\mathbf{i}|=d} \text{ are independent} \\
\text{(iii)} & X_{\mathbf{i}} = g\left(\xi_{\mathbf{i}}; \left(X_{\mathbf{i}j} : 1 \leq j \leq^* N_{\mathbf{i}}\right)\right) \quad \forall\, \mathbf{i} \in \mathcal{V} \\
\text{(iv)} & X_{\mathbf{i}} \text{ is independent of } \left\{ \left(\xi_{\mathbf{i}'}, N_{\mathbf{i}'}\right) \,\big|\, |\mathbf{i}'| < |\mathbf{i}| \right\} \quad \forall\, \mathbf{i} \in \mathcal{V},
\end{array}
\tag{3}
$$

where $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ are taken to be i.i.d. copies of the pair (ξ, N), and by $|\cdot|$ we mean the length of a finite word. The process $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ is called an invariant *recursive tree process* (RTP) with marginal μ. The i.i.d. random variables $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ are called the *innovation process*. In some sense, an invariant RTP with marginal μ is an almost sure representation of a solution μ of the RDE (1). Here we note that there is a natural tree structure on $\mathcal{V}$. Taking $\mathcal{V}$ as the vertex set, we join two words $\mathbf{i}, \mathbf{i}' \in \mathcal{V}$ by an edge, if and only if, $\mathbf{i}' = \mathbf{i}j$ or $\mathbf{i} = \mathbf{i}'j$, for

some $j \in \mathbb{N}$. We will denote this tree by $\mathbb{T}_\infty$. The empty-word $\emptyset$ will be taken as the root of the tree $\mathbb{T}_\infty$. For simplicity we will write $\emptyset j = j$ for $j \in \mathbb{N}$.

In the applications mentioned above the variables $(X_\mathbf{i})_{\mathbf{i} \in \mathcal{V}}$ of a RTP are often used as auxiliary variables to define or to construct some useful random structures. In those cases typically the innovation process defines the "internal" variables while the RTP is constructed "externally" using the consistency theorem. It is then natural to ask whether the RTP is measurable only with respect to the i.i.d. innovation process $(\xi_\mathbf{i}, N_\mathbf{i})$.

Definition 1. An invariant RTP with marginal μ is called *endogenous*, if the root variable $X_\emptyset$ is almost surely measurable with respect to the completion of the σ-algebra

$$\mathcal{G} := \sigma \left(\left\{ (\xi_\mathbf{i}, N_\mathbf{i}) \,\middle|\, \mathbf{i} \in \mathcal{V} \right\} \right).$$

This notion of endogeny has been the main topic of discussion in [5]. The authors provide a necessary and sufficient condition for endogeny in the general setup [5, Theorem 11]. Some other concepts similar to endogeny can be found in [6].

In this article we provide a non-trivial application of the theory developed in [5]. The example we consider here arise from the study of the asymptotic limit of random assignment problem using local-weak convergence method [4]. A detailed background of this example is given in Section 2.

1.1. Main result. The following RDE plays the central role in deriving the asymptotic limit of the random assignment problem [4],

$$X \stackrel{d}{=} \min_{j \geq 1} \left(\xi_j - X_j \right) \quad \text{on } \mathbb{R}, \tag{4}$$

where $(X_j)_{j \geq 1}$ are i.i.d. with same law as X and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson point process of rate 1 on $(0, \infty)$, that is, $\xi_1 \leq \xi_2 \leq \cdots \leq \xi_k \leq \cdots$ a.s. and the collection $(\xi_i - \xi_{i-1})_{i \geq 1}$ are i.i.d. random variables with Exponential (1) distribution. Here we write $\xi_0 \equiv 0$. It is known [4] that the RDE (4) has a unique solution as the logistic distribution, given by

$$\mathbf{P} \left(X \leq x \right) = \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R}. \tag{5}$$

For this reason we will call RDE (4) the *logistic RDE*. The following is our main result.

Theorem 1. *The invariant recursive tree process with logistic marginals associated with the RDE* (4) *is endogenous.*

This result though looks technical, but provides a concrete example falling under the general theory developed in [5]. The proof of Theorem 1 involves analytic techniques, thus this work also demonstrate the need of developing analytic tools for studying max-type RDEs in general.

1.2. Outline of rest of the paper. The next section provides the background and motivation for deriving our main result. In Sections 3 we prove the bivariate uniqueness property of the logistic RDE (4) and finally Section 4 gives the proof the main result.

2. Background and motivation for logistic RDE

For a given $n \times n$ matrix of costs (C_{ij}), consider the problem of assigning n jobs to n machines in the most "cost effective" way. Thus the task is to find a permutation π of $\{1, 2, \ldots, n\}$, which solves the following minimization problem

$$A_n := \min_{\pi} \sum_{i=1}^{n} C_{i,\pi(i)}. \tag{6}$$

This problem has been extensively studied in literature for a fixed cost matrix, and there are various algorithms to find the optimal permutation π. A probabilistic model for the assignment problem can be obtained by assuming that the costs are independent random variables each with Uniform$[0, 1]$ distribution. Although this model appears to be quite simple, careful investigations of it in the last few decades have shown that it has enormous richness in its structure. See [2, 18] for survey and other related works.

Our interest in this problem is from another perspective. In 2001, Aldous [4] showed

$$\lim_{n \to \infty} \mathbf{E}[A_n] = \zeta(2) = \frac{\pi^2}{6}, \tag{7}$$

confirming the earlier work of Mézard and Parisi [13], where they computed the same limit using non-rigorous arguments based on the replica method [14]. It is also known that the limit exists and it is universal for any i.i.d. cost distribution with density f such that $f(0) = 1$ (see [1]). So for calculation of the limiting constant one can assume that C_{ij}'s are i.i.d. with Exponential distribution with mean n and then write the objective function A_n in the normalized form,

$$A_n \stackrel{d}{=} \min_{\pi} \frac{1}{n} \sum_{i=1}^{n} C_{i,\pi(i)}. \tag{8}$$

In [4], the limit $\zeta(2)$ was identified in terms of an optimal matching problem on an infinite tree with random edge weights, described as follows:

> Let $\mathbb{T}_\infty := (\mathcal{V}, \mathcal{E})$ be the canonical infinite rooted labeled tree, as before, where $\emptyset$ is the root. For every vertex $\mathbf{i} \in \mathcal{V}$, let $(\xi_{\mathbf{i}j})_{j \geq 1}$ be points of a Poisson point process of rate 1 on $(0, \infty)$, and they are independent as $\mathbf{i}$ varies. Define the weight of the edge $e = (\mathbf{i}, \mathbf{i}j) \in \mathcal{E}$ as $\xi_{\mathbf{i}j}$.

This structure is called *Poisson weighted infinite tree* and henceforth abbreviated as PWIT.

Let $K_{n,n}^r$ be the complete bipartite graph on $2n$ vertices with a root selected uniformly at random. Suppose we also equip it with i.i.d. Exponential edge weights with mean n. Then one can show [2,4] that in the sense of Aldous-Steel local weak convergence $K_{n,n}^r$ converges to the PWIT. Moreover heuristically the random assignment problem on $K_{n,n}^r$ has a "natural" analog to the limit structure, which is to consider the "optimal" (in sense of minimizing the "total cost") matching problem on PWIT. Naturally PWIT being an infinite graph with edge weights each having mean at least 1, the "total cost" of any matching is infinite a.s., and hence minimization "total cost" is not quite meaningful. However, Aldous [4] showed that it is possible to make a sensible definition of "optimal matching" on PWIT which minimizes the "average edge weight". This construction is quite hard, and we refer the readers to [2, 4] for the technical details. Here we only provide the basic essentials to understand the motivation for our work.

Consider the heuristic description of the "optimal" matching problem on PWIT and suppose we define variables $X_{\mathbf{i}}$ for each vertex $\mathbf{i}$ as follows

$$
\begin{aligned}
X_{\mathbf{i}} = {} & \text{Total cost of a maximal matching on the subtree } \mathbb{T}_\infty^{\mathbf{i}} \\
& - \text{Total cost of a maximal matching on the forest } \mathbb{T}_\infty^{\mathbf{i}} \setminus \{\mathbf{i}\},
\end{aligned}
\tag{9}
$$

where $\mathbb{T}_\infty^{\mathbf{i}}$ is the subtree rooted at the vertex $\mathbf{i}$. Here by "total cost" we mean the sum total of all the edge weights in the matching. As noted above, both the "total costs" appearing in (9) are infinity almost surely. Thus rigorously speaking $X_{\mathbf{i}}$ is not well defined. But at the heuristic level if we ignore this, then simple manipulation yields that they must satisfy the following recurrence relation (see [4, Section 4.2])

$$
X_{\mathbf{i}} = \min_{j \geq 1} \left(\xi_{\mathbf{i}j} - X_{\mathbf{i}j} \right).
\tag{10}
$$

This is of course the recurrence relation for a RTP associated with the logistic RDE (4). Thus one can now construct the $X_{\mathbf{i}}$-variables externally as the RTP associated with the logistic RDE and then use them to redefine the optimal matching on PWIT. This is the construction given in [4] which also provides a characterization of the optimal matching on the PWIT. Finally one can then derive the $\zeta(2)$-limit for the random assignment problem.

Once again a natural question would be to figure out whether the random variables $X_{\mathbf{i}}$'s are truly external or not, in other words to see whether the RTP is endogenous or not (see [4, Remarks (4.2.d) and (4.2.e)]). This is our main motivation for this work. Theorem 1 proves that the $X_{\mathbf{i}}$-variables can be defined using only the edge-weights. In other words they have no "external randomness" in them. Some other significance of this result has also been pointed out in [5, Section 7.5].

3. Bivariate uniqueness for the logistic RDE

In this section we prove the bivariate uniqueness property for the logistic RDE (4). ▮

Theorem 2. *Consider the following bivariate RDE*

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \min_{j \geq 1} (\xi_j - X_j) \\ \min_{j \geq 1} (\xi_j - Y_j) \end{pmatrix}, \tag{11}$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d. pairs with same joint distribution as (X, Y) and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson process of rate 1 on $(0, \infty)$. Then the unique solution of this RDE is given by a "diagonal measure", namely, $\mu^\nearrow := \text{dist}(X, X)$ where X has the logistic distribution.

3.1. Proof of Theorem 2. First observe that if the equation (11) has a solution then, the marginal distributions of X and Y solve the logistic RDE (4), and hence they are both logistic. Further by inspection $\mu^\nearrow$ is a solution of (11). So it is enough to prove that $\mu^\nearrow$ is the only solution of (11).

Let $\mu^{(2)}$ be a solution of (11). Now consider the points $\{(\xi_j; (X_j, Y_j)) \mid j \geq 1\}$ as a point process, say $\mathcal{P}$ on $(0, \infty) \times \mathbb{R}^2$. Since $(\xi_j)_{j \geq 1}$ is a Poisson point process on $(0, \infty)$ of rate 1 and $(X_j, Y_j)_{j \geq 1}$ are i.i.d. random vectors on $\mathbb{R}^2$ with distribution $\mu^{(2)}$ which are independent of the Poisson process $(\xi_j)_{j \geq 1}$, thus $\mathcal{P}$ is a Compound Poisson process on $(0, \infty) \times \mathbb{R}^2$ with mean intensity $\rho(t; (x, y)) \, dt \, d(x, y) := dt \, \mu^{(2)}(d(x, y))$ (see [10, Lemma 6.4.VI]). Thus if $G(x, y) := \mathbf{P}(X > x, Y > y)$, for $x, y \in \mathbb{R}$, then

$$\begin{aligned} G(x, y) &= \mathbf{P}\left(\min_{j \geq 1} (\xi_j - X_j) > x, \text{ and, } \min_{j \geq 1} (\xi_j - Y_j) > y \right) \\ &= \mathbf{P}\left(\begin{array}{c} \text{No points of } \mathcal{P} \text{ are in} \\ \{(t; (u, v)) \mid t - u \leq x, \text{ or, } t - v \leq y\} \end{array} \right) \\ &= \exp\left(- \iiint_{\{(t;(u,v)) \mid t-u \leq x, \text{ or, } t-v \leq y\}} \rho(t; (u, v)) \, dt \, d(u, v) \right) \\ &= \exp\left(- \int_0^\infty \left[\bar{H}(t - x) + \bar{H}(t - y) - G(t - x, t - y) \right] dt \right) \\ &= \bar{H}(x) \, \bar{H}(y) \exp\left(\int_0^\infty G(t - x, t - y) \, dt \right), \end{aligned} \tag{12}$$

where $\bar{H}$ is the right tail of logistic distribution, defined as

$$\bar{H}(x) = \frac{e^{-x}}{(1 + e^{-x})}$$

for $x \in \mathbb{R}$. The last equality follows from properties of the logistic distribution (see Proposition 3(c)). For notational convenience in this paper we will write $\bar{F}(\cdot) := 1 - F(\cdot)$ for any distribution function F.

The following simple lemma reduces the bivariate problem to a univariate problem.

Lemma 1. *For any two random variables U and V, $U = V$ a.s. if and only if* $U \overset{d}{=} V \overset{d}{=} U \wedge V$.

Proof. First of all, if $U = V$ a.s. then $U \wedge V = U$ a.s.

Conversely suppose that $U \overset{d}{=} V \overset{d}{=} U \wedge V$. Fix a rational q, then under our assumption,

$$\mathbf{P}(U \leq q < V) = \mathbf{P}(V > q) - \mathbf{P}(U > q, V > q) = \mathbf{P}(V > q) - \mathbf{P}(U \wedge V > q) = 0$$

A similar calculation will show that $\mathbf{P}(V \leq q < U) = 0$. These are true for any rational q, thus $\mathbf{P}(U \neq V) = 0$. $\square$

Thus if we can show that $X \wedge Y$ also has logistic distribution, then from the lemma above we will be able to conclude that $X = Y$ a.s., and hence the proof will be complete. Put $g(\cdot) := \mathbf{P}(X \wedge Y > \cdot)$, we will show $g = \bar{H}$. Now, for every fixed $x \in \mathbb{R}$, by definition $g(x) = G(x, x)$. So using (12) we get

$$g(x) = \bar{H}^2(x) \exp\left(\int_{-x}^{\infty} g(s)\, ds\right), \quad x \in \mathbb{R}. \tag{13}$$

Notice that from (26) (see Proposition 3(c)) $g = \bar{H}$ is a solution of this nonlinear integral equation (13), which corresponds to the solution $\mu^{(2)} = \mu^{\nearrow}$ of the original equation (11). To complete the proof of Theorem 2 we need to show that this is the only solution. For that we will prove that the operator associated with (13) (defined on an appropriate space) is monotone and has unique fixed-point as $\bar{H}$. The techniques we will use here are similar to Eulerian recursion [17], and are heavily based on analytic arguments.

Let $\mathfrak{F}$ be the set of all functions $f : \mathbb{R} \to [0, 1]$ such that

- $\bar{H}^2(x) \leq f(x) \leq \bar{H}(x)$ for all $x \in \mathbb{R}$,
- f is continuous and non-increasing.

Observe that by definition $\bar{H} \in \mathfrak{F}$. Further from (13) it follows that $g(x) \geq \bar{H}^2(x)$, as well as, $g(x) = \mathbf{P}(X \wedge Y > x) \leq \mathbf{P}(X > x) = \bar{H}(x)$ for all $x \in \mathbb{R}$. Note also that g being the tail of the random variable $X \wedge Y$, is continuous (because both X and Y are continuous random variables) and non-increasing. So it is appropriate to search for solutions of (13) in $\mathfrak{F}$.

Let $T : \mathfrak{F} \to \mathfrak{F}$ be defined as

$$T(f)(x) := \bar{H}^2(x) \exp\left(\int_{-x}^{\infty} f(s)\, ds\right), \quad x \in \mathbb{R}. \tag{14}$$

Note that this operator T is not same as the general operator defined in Section 1, henceforth by T we will mean the specific operator defined above. Proposition 1 of Section 3.2 shows that T does indeed map $\mathfrak{F}$ into itself. Observe that the equation (13) is nothing but the fixed-point equation associated with the operator T, that is,

$$g = T(g) \quad \text{on } \mathfrak{F}. \tag{15}$$

We here note that using (26) (see Proposition 3(c)) T can also be written as

$$T(f)(x) := \bar{H}(x) \, \exp\left(-\int_{-x}^{\infty} \left(\bar{H}(s) - f(s)\right) ds\right), \quad x \in \mathbb{R}, \tag{16}$$

which will be used in the subsequent discussion.

Define a partial order $\preccurlyeq$ on $\mathfrak{F}$ as, $f_1 \preccurlyeq f_2$ in $\mathfrak{F}$ if $f_1(x) \leq f_2(x)$ for all $x \in \mathbb{R}$, then the following result holds.

Lemma 2. *T is a monotone operator on the partially ordered set $(\mathfrak{F}, \preccurlyeq)$.*

Proof. Let $f_1 \preccurlyeq f_2$ be two elements of $\mathfrak{F}$, so from definition $f_1(x) \leq f_2(x)$, for all $x \in \mathbb{R}$. Hence

$$\int_{-x}^{\infty} f_1(s) \, ds \leq \int_{-x}^{\infty} f_2(s) \, ds \quad \forall \, x \in \mathbb{R}$$
$$\Rightarrow \quad T(f_1)(x) \leq T(f_2)(x) \qquad \forall \, x \in \mathbb{R}$$
$$\Rightarrow \quad T(f_1) \preccurlyeq T(f_2). \qquad \qquad \square$$

Put $f_0 = \bar{H}^2$, and for $n \in \mathbb{N}$, define $f_n \in \mathfrak{F}$ recursively as, $f_n = T(f_{n-1})$. Now from Lemma 2 we get that if g is a fixed-point of T in $\mathfrak{F}$ then,

$$f_n \preccurlyeq g \quad \forall \, n \geq 0. \tag{17}$$

If we can show $f_n \to \bar{H}$ point wise, then using (17) we will get $\bar{H} \preccurlyeq g$, so from definition of $\mathfrak{F}$ it will follow that $g = \bar{H}$, and our proof will be complete. For that, the following lemma gives an explicit recursion for the functions $\{f_n\}_{n \geq 0}$.

Lemma 3. *Let $\beta_0(s) = 1 - s$, $0 \leq s \leq 1$. Define recursively*

$$\beta_n(s) := \int_s^1 \frac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)}\right) dw, \quad 0 < s \leq 1. \tag{18}$$

Then for $n \geq 1$,

$$f_n(x) = \bar{H}(x) \, \exp\left(-\beta_{n-1}(\bar{H}(x))\right), \quad x \in \mathbb{R}. \tag{19}$$

Proof. We will prove this by induction on n. Fix $x \in \mathbb{R}$, for $n = 1$ we get

$$f_1(x) = T(f_0)(x)$$
$$= \bar{H}(x) \exp\left(-\int_{-x}^{\infty} \left(\bar{H}(s) - \bar{H}^2(s)\right) ds\right) \qquad \text{[using (16)]}$$
$$= \bar{H}(x) \exp\left(-\int_{-x}^{\infty} \bar{H}(s) H(s) \, ds\right)$$
$$= \bar{H}(x) \exp\left(-\int_{-x}^{\infty} H'(s) \, ds\right) \qquad \text{[using Proposition 3(a)]}$$
$$= \bar{H}(x) \exp\left(-H(x)\right)$$
$$= \bar{H}(x) \exp\left(-\beta_0(\bar{H}(x))\right)$$

Now, assume that the assertion of the Lemma is true for $n \in \{1, 2, \ldots, k\}$, for some $k \geq 1$, then from definition we have

$$f_{k+1}(x) = T(f_k)(x)$$
$$= \bar{H}(x) \exp\left(-\int_{-x}^{\infty} \left(\bar{H}(s) - f_k(s)\right) ds\right) \qquad \text{[using (16)]}$$
$$= \bar{H}(x) \exp\left(-\int_{-x}^{\infty} \bar{H}(s) \left(1 - e^{-\beta_{k-1}(\bar{H}(s))}\right) ds\right) \qquad (20)$$
$$= \bar{H}(x) \exp\left(-\int_{\bar{H}(x)}^{1} \frac{1}{w} \left(1 - e^{-\beta_{k-1}(1-w)}\right) dw\right)$$

The last equality follows by substituting $w = H(s)$ and thus from parts (a) and (b) of Proposition 3 we get that $\frac{dw}{w} = \bar{H}(s) \, ds$ and $H(-x) = \bar{H}(x)$. Finally by definition of β_n's and using (20) we get $f_{k+1} = T(f_k)$. $\square$

To complete the proof it is now enough to show that $\beta_n \to 0$ point wise, which will imply by Lemma 3 that $f_n \to \bar{H}$ point wise, as $n \to \infty$. Using Proposition 2 (see Section 3.2) we get the following characterization of the point wise limit of these β_n's.

Lemma 4. *There exists a function $L : [0, 1] \to [0, 1]$ with $L(1) = 0$, such that*

$$L(s) = \int_{s}^{1} \frac{1}{w} \left(1 - e^{-L(1-w)}\right) dw \quad \forall \; s \in [0, 1), \qquad (21)$$

and $L(s) = \lim_{n \to \infty} \beta_n(s)$ for all $0 \leq s \leq 1$.

Proof. From the Proposition 2 we know that for any $s \in [0, 1]$ the sequence $\{\beta_n(s)\}$ is decreasing, and hence there exists a function $L : [0, 1] \to [0, 1]$ such

that $L(s) = \lim_{n \to \infty} \beta_n(s)$. Now observe that $\beta_n(1 - w) \leq \beta_0(1 - w) = w$ for all $0 \leq w \leq 1$, and hence

$$0 \leq \frac{1}{w}\left(1 - e^{-\beta_n(1-w)}\right) \leq \frac{\beta_n(1 - w)}{w} \leq 1 \quad \forall\, 0 \leq w \leq 1.$$

Thus by taking limit as $n \to \infty$ in (18) and using the dominated convergence theorem along with part (a) of Proposition 2 we get that

$$L(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-L(1-w)}\right) dw \quad \forall\, 0 \leq s < 1. \qquad \square$$

The above lemma basically translates the non-linear integral equation (13) to the non-linear integral equation (21), where the solution $g = \bar{H}$ of (13) is given by the solution $L \equiv 0$ of (21). So at first sight this may not lead us to the conclusion. But fortunately, something nice happens for equation (21), and we have the following result which is enough to complete the proof of Theorem 2.

Lemma 5. *If $L : [0, 1] \to [0, 1]$ is a function which satisfies the non-linear integral equation* (21), *namely,*

$$L(s) = \int_s^1 \frac{1}{w}\left(1 - e^{-L(1-w)}\right) dw \quad \forall\, 0 \leq s < 1,$$

and if $L(1) = 0$, then $L \equiv 0$.

Proof. First note that $L \equiv 0$ is a solution. Now let L be any solution of (21), then L is infinitely differentiable on the open interval $(0, 1)$, by repetitive application of Fundamental Theorem of Calculus.

Consider $\eta(w) := (1 - w)e^{L(1-w)} + we^{-L(w)} - 1$, $w \in [0, 1]$. Observe, that $\eta(0) = \eta(1) = 0$ as $L(1) = 0$. Now, from (21) we get that

$$L'(w) = -\frac{1}{w}\left(1 - e^{-L(1-w)}\right), \quad w \in (0, 1).$$

Thus differentiating the function η we get

$$\eta'(w) = e^{-L(w)}\left[2 - \left(e^{L(1-w)} + e^{-L(1-w)}\right)\right] \leq 0 \quad \forall\, w \in (0, 1). \qquad (22)$$

So the function η is decreasing in $(0, 1)$ and is continuous in $[0, 1]$ with boundary values as 0, hence $\eta \equiv 0$. Thus we must have $\eta' \equiv 0$, so from equation (22) we get that $e^{L(s)} + e^{-L(s)} = 2$ for all $s \in (0, 1)$. This implies $L \equiv 0$ on $[0, 1]$. $\qquad \square$

3.2. Some technical details. This section provides some of the technical results which were needed in the previous section.

Proposition 1. *The operator T maps $\mathfrak{F}$ into $\mathfrak{F}$.*

Proof. First note that if $f \in \mathfrak{F}$, then by definition $T(f)(x) \geq \bar{H}^2(x)$ for all $x \in \mathbb{R}$. Next by definition of $\mathfrak{F}$ we get that $f \in \mathfrak{F} \Rightarrow f \preccurlyeq \bar{H}$, thus

$$\int_{-x}^{\infty} f(s)\, ds \leq \int_{-x}^{\infty} \bar{H}(s)\, ds \quad \forall\, x \in \mathbb{R}$$

$$\Rightarrow \quad T(f)(x) \leq \bar{H}^2(x) \exp\left(\int_{-x}^{\infty} \bar{H}(s)\, ds\right) = \bar{H}(x) \quad \forall\, x \in \mathbb{R}.$$

The last equality follows from (26) (see Proposition 3(c)). So,

$$\bar{H}^2(x) \leq T(f)(x) \leq \bar{H}(x) \quad \forall\, x \in \mathbb{R}. \tag{23}$$

Now we need to show that for any $f \in \mathfrak{F}$ we must have $T(f)$ continuous and non-increasing. From the definition $T(f)$ is continuous (in fact, infinitely differentiable). Moreover if $x \leq y$ be two real numbers, then

$$\int_{-x}^{\infty} \left(\bar{H}(s) - f(s)\right) ds \leq \int_{-y}^{\infty} \left(\bar{H}(s) - f(s)\right) ds,$$

because $f \preccurlyeq \bar{H}$. Also $\bar{H}(x) \geq \bar{H}(y)$, thus using (16) we get

$$T(f)(x) \geq T(f)(y) \tag{24}$$

So using (23) and (24) we conclude that $T(f) \in \mathfrak{F}$ if $f \in \mathfrak{F}$. $\qquad\square$

Proposition 2. *The following are true for the sequence of functions $\{\beta_n\}_{n \geq 0}$ defined in (18).*

(a) *For every fixed $s \in (0, 1]$, the sequence $\{\beta_n(s)\}$ is decreasing.*

(b) *For every $n \geq 1$, $\lim_{s \to 0+} \beta_n(s)$ exists, and is given by*

$$\int_0^1 \frac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)}\right) dw,$$

we will write this as $\beta_n(0)$.

(c) *The sequence of numbers $\{\beta_n(0)\}$ is also decreasing.*

Proof. (a) Notice that $\beta_0(s) = 1 - s$ for $s \in [0, 1]$, thus

$$\beta_1(s) = \int_s^1 \frac{1 - e^{-w}}{w}\, dw < 1 - s = \beta_0(s) \quad \forall\, s \in (0, 1].$$

Now assume that for some $n \geq 1$ we have $\beta_n(s) \leq \beta_{n-1}(s) \leq \cdots \leq \beta_0(s)$ for all $s \in (0,1]$, if we show that $\beta_{n+1}(s) \leq \beta_n(s)$ for all $s \in (0,1]$ then by induction the proof will be complete. For that, fix $s \in (0,1]$ then

$$\beta_{n+1}(s) = \int_s^1 \frac{1}{w} \left(1 - e^{-\beta_n(1-w)}\right) dw \leq \int_s^1 \frac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)}\right) dw = \beta_n(s).$$

This proves the part (a).

(b, c) First note that by trivial induction $\beta_n(s) \geq 0$ for every $s \in (0,1]$, $n \geq 0$. Thus from definition for every $n \geq 0$, the limit $\lim_{s\to 0+} \beta_n(s)$ exists in $[0,\infty]$ and is given by $\int_0^1 \frac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)}\right) dw$. Now using (a) above we conclude

$$\beta_{n+1}(0) = \lim_{s\to 0+} \beta_{n+1}(s) \leq \lim_{s\to 0+} \beta_n(s) = \beta_n(0), \tag{25}$$

for every $n \geq 0$. Since $\beta_0(0) = 1$, so we get $\beta_n(0) < \infty$ for all $n \geq 0$, and the sequence is decreasing. This proves parts (b) and (c). $\square$

Finally, the following proposition lists some basic facts about the logistic distribution function.

Proposition 3. *Suppose X is a random variable with logistic distribution and $H(x) := \mathbf{P}(X \leq x)$ for $x \in \mathbb{R}$ be the distribution function of X. Then*

(a) *H is infinitely differentiable and $H'(\cdot) = H(\cdot)\bar{H}(\cdot)$, where $\bar{H}(\cdot) = 1 - H(\cdot)$.*
(b) *H is symmetric around 0, that is, $H(-x) = \bar{H}(x)$ for all $x \in \mathbb{R}$.*
(c) *$\bar{H}$ is the unique solution of the non-linear integral equation*

$$\bar{H}(x) = \exp\left(-\int_{-x}^{\infty} \bar{H}(s)\, ds\right) \quad \forall\, x \in \mathbb{R}. \tag{26}$$

Proof. (a) and (b) trivially follows from the definition of H. For (c) notice that the equation (26) is nothing but the Logistic RDE, this is because

$$\mathbf{P}\left(\min_{j\geq 1} (\xi_j - X_j) > x\right) = \exp\left(-\int_{-x}^{\infty} \bar{H}(s)\, ds\right) \quad \forall\, x \in \mathbb{R}$$

where $(X_j)_{j\geq 1}$ are i.i.d. with distribution function H and are independent of $(\xi_j)_{j\geq 1}$, which are points of a Poisson point process of rate 1 on $(0,\infty)$. Thus from the fact that $\bar{H}$ is the unique solution of logistic RDE (see [4, Lemma 5]) we conclude that $\bar{H}$ is unique solution of equation (26). $\square$

4. Proof of Theorem 1

We note that by Theorem 2 the logistic RDE (4) has bivariate uniqueness property. Thus by [5, Theorem 11 (b)] to prove endogeny for the logistic RDE (4) all remains is to check the following technical continuity assumption.

Proposition 4. *Let $\mathfrak{S}$ be the set of all probabilities on $\mathbb{R}^2$ and let $\Gamma : \mathfrak{S} \to \mathfrak{S}$ be the operator associated with the RDE (11), that is,*

$$\Gamma\left(\mu^{(2)}\right) \overset{d}{=} \left(\begin{array}{c} \min_{j \geq 1}\left(\xi_j - X_j\right) \\ \min_{j \geq 1}\left(\xi_j - Y_j\right) \end{array} \right), \tag{27}$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d. with joint law $\mu^{(2)} \in \mathfrak{S}$ and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson point process of rate 1 on $(0, \infty)$. Then Γ is continuous with respect to the weak convergence topology when restricted to the subspace $\mathfrak{S}^*$ defined as

$$\mathfrak{S}^* := \left\{ \mu^{(2)} \ \middle| \ both\ the\ marginals\ of\ \mu^{(2)}\ are\ logistic\ distribution \right\}. \tag{28}$$

Before we prove this proposition, it is worth mentioning that the operator Γ is not continuous with respect to the weak convergence topology on the whole space $\mathfrak{S}$. In fact, it is not difficult to see that the operator Γ is every where discontinuous on $\mathfrak{S}$. But fortunately for applying [5, Theorem 11 (b)] we only need the continuity of Γ when restricted to the subspace $\mathfrak{S}^*$.

Proof of Proposition 4. Let $\{\mu_n^{(2)}\}_{n=1}^{\infty} \subseteq \mathfrak{S}^{\star}$ be such that $\mu_n^{(2)} \overset{d}{\longrightarrow} \mu^{(2)}$ where $\mu^{(2)} \in \mathfrak{S}^{\star}$. We will show that $\Gamma(\mu_n^{(2)}) \overset{d}{\longrightarrow} \Gamma(\mu^{(2)})$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space such that there exists $\{(X_n, Y_n)\}_{n=1}^{\infty}$ and (X, Y) random vectors taking values in $\mathbb{R}^2$, with $(X_n, Y_n) \sim \mu_n^{(2)}$, $n \geq 1$, and $(X, Y) \sim \mu^{(2)}$. Notice that by definition $X_n \overset{d}{=} Y_n \overset{d}{=} X \overset{d}{=} Y$, and each has logistic distribution.

Fix $x, y \in \mathbb{R}$, let $G_n(x, y) := \Gamma(\mu_n^{(2)})\left((x, \infty) \times (y, \infty)\right)$, then by exactly the same argument as in the derivation of (12) we get

$$\begin{aligned}
G_n(x, y) &= \bar{H}(x)\bar{H}(y) \exp\left(-\int_0^{\infty} G_n\left(t - x, t - y\right) dt \right) \\
&= \bar{H}(x)\bar{H}(y) \exp\left(-\int_0^{\infty} \mathbf{P}\left(X_n > t - x, Y_n > t - y\right) dt \right) \\
&= \bar{H}(x)\bar{H}(y) \exp\left(-\int_0^{\infty} \mathbf{P}\left((X_n + x) \wedge (Y_n + y) > t\right) dt \right) \\
&= \bar{H}(x)\bar{H}(y) \exp\left(-\mathbf{E}\left[(X_n + x)^+ \wedge (Y_n + y)^+\right] \right),
\end{aligned}$$

and a similar calculation will also give that

$$\begin{aligned}
G(x, y) &:= \Gamma(\mu^{(2)})\left((x, \infty) \times (y, \infty)\right) \\
&= \bar{H}(x)\bar{H}(y) \exp\left(-\mathbf{E}\left[(X + x)^+ \wedge (Y + y)^+\right] \right).
\end{aligned}$$

Now to complete the proof all we need is to show

$$\mathbf{E}\left[(X_n + x)^+ \wedge (Y_n + y)^+\right] \longrightarrow \mathbf{E}\left[(X + x)^+ \wedge (Y + y)^+\right].$$

Since we assumed that $(X_n, Y_n) \xrightarrow{d} (X, Y)$ it follows

$$(X_n + x)^+ \wedge (Y_n + y)^+ \xrightarrow{d} (X + x)^+ \wedge (Y + y)^+ \quad \forall\ x, y \in \mathbb{R}.$$

Fix $x, y \in \mathbb{R}$, define $Z_n^{x,y} := (X_n + x)^+ \wedge (Y_n + y)^+$, and $Z^{x,y} := (X + x)^+ \wedge (Y + y)^+$. Observe that

$$0 \le Z_n^{x,y} \le (X_n + x)^+ \le |X_n + x| \quad \forall\ n \ge 1.$$

But, $|X_n + x| \overset{d}{=} |X + x|$ for all $n \ge 1$. So clearly $\{Z_n^{x,y}\}_{n=1}^{\infty}$ is uniformly integrable. Hence we conclude (using the theorem of Billingsley [9, Theorem 25.12]) that $\mathbf{E}\left[Z_n^{x,y}\right] \longrightarrow \mathbf{E}\left[Z^{x,y}\right]$. This completes the proof. $\qquad\square$

References

[1] Aldous, D. Asymptotics in the random assignment problem. *Probab. Theory Related Fields* 93 (1992)(4), 507 – 534.

[2] Aldous, D. and Steele, J. M., The objective method: probabilistic combinatorial optimization and local weak convergence. In: *Probability on Discrete Structures* (ed.: H. Kesten). Encyclopaedia Math. Sci. 110. Berlin: Springer 2004, pp. 1 – 72.

[3] Aldous, D. J., The percolation process on a tree where infinite clusters are frozen. *Math. Proc. Cambridge Philos. Soc.* 128 (2000)(3), 465 – 477.

[4] Aldous, D. J., The $\zeta(2)$ Limit in the Random Assignment Problem. *Random Structures Algorithms* 18 (2001)(4), 381 – 418.

[5] Aldous, D. J. and Bandyopadhyay, A., A survey of max-type recursive distributional equations. *Ann. Appl. Probab.* 15 (2005)(2), 1047 – 1110.

[6] Bandyopadhyay, A., A necessary and sufficient condition for the tail-triviality of a recursive tree process. *Sankhyā* 68 (2006), 1 – 23.

[7] Bandyopadhyay, A. and Gamarnik, D., Counting without sampling. New algorithms for enumeration problems using statistical physics. In: *Proc. Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms*. Philadelphia (PA): SIAM 2006, pp. 890 – 899.

[8] Bandyopadhyay, A. and Gamarnik, D., Counting without sampling: asymptotics of the log-partition function for certain statistical physics models. *Random Structures Algorithms* 33 (2008)(4), 452 – 479.

[9] Billingsley, P., *Probability and Measure.* Third ed. New York: Wiley 1995.

[10] Daley, D. J. and Vere-Jones, D., *An Introduction to the Theory of Point Processes. Vol. I.* Elementary theory and methods. Sec. ed. Probab. Appl. (N. Y.). New York: Springer-Verlag 2003.

[11] Fill, J. A. and Janson, S., A characterization of the set of fixed points of the Quicksort transformation. *Electron. Comm. Probab.* 5 (2000), 77 – 84 (electronic).

[12] Gamarnik, D., Nowicki, T. and Swirszcz, G., Maximum weight independent sets and matchings in sparse random graphs. Exact results using the local weak convergence method. *Random Structures Algorithms* 28(2006), 76 – 106.

[13] Mézard, M. and Parisi, G., On the solution of the random link matching problem. *J. Physique* 48 (1987), 1451 – 1459.

[14] Mézard, M., Parisi, G. and Virasoro, M. A., *Spin Glass Theory and Beyond.* Singapore: World Scientific 1987.

[15] Rösler, U. and Rüschendorf, L., The contraction method for recursive algorithms. *Algorithmica* 29 (2001)(1-2), 3 – 33.

[16] Rösler, U., A fixed point theorem for distributions. *Stochastic Process. Appl.* 42 (1992)(2), 195 – 214.

[17] Simmons, G. F., *Differential Equations with Applications and Historical Notes.* New York: McGraw-Hill 1972.

[18] Steele, J. M., *Probability Theory and Combinatorial Optimization.* Philadelphia (PA): SIAM 1997.

[19] Wästlund, J., The mean field traveling salesman and related problems. *Acta Math.* 204 (2010), 91 – 150.