

Bivariate Uniqueness and Endogeny for the Logistic Recursive Distributional Equation

Antar Bandyopadhyay

Department of Mathematics
Chalmers University of Technology
SE - 412 96, Göteborg
SWEDEN
E-Mail : antar@math.chalmers.se

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Abstract

In this article we prove the *bivariate uniqueness* property for a particular “max-type” *recursive distributional equation* (RDE). Using the general theory developed in [5] we then show that the corresponding *recursive tree process* (RTP) has no *external* randomness, more precisely, the RTP is *endogenous*. The RDE we consider is so called the *Logistic RDE*, which appears in the proof of the $\zeta(2)$ -limit of the random assignment problem [4] using the local weak convergence method. Thus this work provides a non-trivial application of the general theory developed in [5].

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1 Introduction and the Main Result

Fixed-point equations or distributional identities have appeared in the probability literature for quite a long time in a variety of settings. The recent survey of Aldous and Bandyopadhyay [5] provides a general framework to study certain type of distributional equations.

Given a space S write $\mathcal{P}(S)$ for the set of all probabilities on S . A *recursive distributional equation* (RDE) [5] is a fixed-point equation on $\mathcal{P}(S)$ defined as

$$X \stackrel{d}{=} g(\xi; (X_j : 1 \leq j \leq^* N)) \text{ on } S, \quad (1)$$

where it is assumed that $(X_j)_{j \geq 1}$ are i.i.d. S -valued random variables with same distribution as X , and are independent of the pair (ξ, N) . Here N is a non-negative integer valued random variable, which may take the value ∞ , and g is a given S -valued function. (In the above equation by " $\leq^* N$ " we mean the left hand side is " $\leq N$ " if $N < \infty$, and " $< N$ " otherwise). In (1) the distribution of X is *unknown*, while the distribution of the pair (ξ, N) and the function g are the *known* quantities. Perhaps a more conventional (analytic) way of writing the equation (1) would be

$$\mu = T(\mu), \quad (2)$$

where $T : \mathcal{P} \rightarrow \mathcal{P}(S)$ is a function defined on $\mathcal{P} \subseteq \mathcal{P}(S)$ such that $T(\mu)$ is the distribution of the right-hand side of the equation (1), when $(X_j)_{j \geq 1}$ are i.i.d. $\mu \in \mathcal{P}$.

As outlined in [5] in many applications RDEs play a very crucial role. Examples include study of Galton-Watson branching processes and related random trees, probabilistic analysis of algorithms with suitable recursive structure [18, 10, 17], statistical physics models on trees [3, 2, 11, 6, 7, 8], and statistical physics and algorithmic questions in the mean-field model of distance [1, 4, 2]. In many of these applications, particularly in the last two types mentioned above, often one needs to construct a particular tree indexed *stationary* process related to a given RDE, which is called a *recursive tree process* (RTP) [5]. More precisely, suppose the RDE (1) has a solution, say μ . Then as shown in [5], using the consistency theorem of Kolmogorov [9], one can construct a process, say $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$, indexed by $\mathcal{V} := \left(\bigcup_{d \geq 1} \mathbb{N}^d \right) \cup \{\emptyset\}$, such that

- (i) $X_{\mathbf{i}} \sim \mu \quad \forall \mathbf{i} \in \mathcal{V}$,
- (ii) For each $d \geq 0$, $(X_{\mathbf{i}})_{|\mathbf{i}|=d}$ are independent,
- (iii) $X_{\mathbf{i}} = g(\xi_{\mathbf{i}}; (X_{\mathbf{i}j} : 1 \leq j \leq^* N_{\mathbf{i}})) \quad \forall \mathbf{i} \in \mathcal{V}$,
- (iv) $X_{\mathbf{i}}$ is independent of $\left\{ (\xi_{\mathbf{i}'}, N_{\mathbf{i}'}) \mid |\mathbf{i}'| < |\mathbf{i}| \right\} \quad \forall \mathbf{i} \in \mathcal{V}$,

where $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ are taken to be i.i.d. copies of the pair (ξ, N) , and by $|\cdot|$ we mean the length of a finite word. The process $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ is called an invariant *recursive tree process* (RTP) with marginal μ . The i.i.d. random variables $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ are called the *innovation process*. In some sense an invariant RTP with marginal μ , is an almost sure representation of a solution μ of the RDE

(1). Here we note that there is a natural tree structure on \mathcal{V} . Taking \mathcal{V} as the vertex set, we join two words $\mathbf{i}, \mathbf{i}' \in \mathcal{V}$ by an edge, if and only if, $\mathbf{i}' = \mathbf{i}j$ or $\mathbf{i} = \mathbf{i}'j$, for some $j \in \mathbb{N}$. We will denote this tree by \mathbb{T}_∞ . The empty-word \emptyset will be taken as the root of the tree \mathbb{T}_∞ , and we will write $\emptyset j = j$ for $j \in \mathbb{N}$.

In the applications mentioned above the variables $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ of a RTP are often used as auxiliary variables to define or to construct some useful random structures. In those cases typically the innovation process defines the “internal” variables while the RTP is constructed “externally” using the consistency theorem. It is then natural to ask whether the RTP is measurable only with respect to the i.i.d. innovation process $(\xi_{\mathbf{i}}, N_{\mathbf{i}})$.

Definition 1 *An invariant RTP with marginal μ is called endogenous, if the root variable X_\emptyset is almost surely measurable with respect to the σ -algebra*

$$\mathcal{G} := \sigma \left(\left\{ (\xi_{\mathbf{i}}, N_{\mathbf{i}}) \mid \mathbf{i} \in \mathcal{V} \right\} \right).$$

This notion of endogeny has been the main topic of discussion in [5]. The authors provide a necessary and sufficient condition for endogeny in the general setup [5, Theorem 11]. Some other concepts similar to endogeny can be found in [7].

In this article we provide a non-trivial application of the theory developed in [5]. The example we consider here arise from the study of the asymptotic limit of *random assignment problem* using local-weak convergence method [4]. A detailed background of this example is given in Section 2.

1.1 Main Result

The following RDE plays the central role in deriving the asymptotic limit of the random assignment problem [4],

$$X \stackrel{d}{=} \min_{j \geq 1} (\xi_j - X_j) \quad \text{on } \mathbb{R}, \quad (4)$$

where $(X_j)_{j \geq 1}$ are i.i.d with same law as X and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson point process of rate 1 on $(0, \infty)$. It is known [4] that the RDE (4) has a unique solution as the *Logistic* distribution, given by

$$\mathbf{P}(X \leq x) = \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R}. \quad (5)$$

For this reason we will call RDE (4) the *Logistic RDE*. The following is our main result.

Theorem 1 *The invariant recursive tree process with Logistic marginals associated with the RDE (4) is endogenous.*

This result though looks technical but, provides a concrete example falling under the general theory developed in [5]. The proof of Theorem 1 involves analytic techniques, thus this work also demonstrate the need of developing analytic tools for studying *max-type* RDEs in general.

1.2 Outline of Rest of the Paper

The next section provides the background and motivation for deriving our main result. In Section 3 we review some of the concepts from [5] and state a version of Theorem 11 of [5], which we will need to prove our main result. In Sections 4 and 5 we prove the main result. Finally Section 6 provides some further discussion. Some known facts about Logistic distribution which are needed for the proofs are given in the appendix.

2 Background and Motivation for Logistic RDE

For a given $n \times n$ matrix of costs (C_{ij}) , consider the problem of assigning n jobs to n machines in the most “cost effective” way. Thus the task is to find a permutation π of $\{1, 2, \dots, n\}$, which solves the following minimization problem

$$A_n := \min_{\pi} \sum_{i=1}^n C_{i,\pi(i)}. \quad (6)$$

This problem has been extensively studied in literature for a fixed cost matrix, and there are various algorithms to find the optimal permutation π . A probabilistic model for the assignment problem can be obtained by assuming that the costs are independent random variables each with Uniform $[0, 1]$ distribution. Although this model appears to be quite simple, careful investigations of it in the last few decades have shown that it has enormous richness in its structure. See [20, 2] for survey and other related works.

Our interest in this problem is from another perspective. In 2001 Aldous [4] showed

$$\lim_{n \rightarrow \infty} \mathbf{E}[A_n] = \zeta(2) = \frac{\pi^2}{6}, \quad (7)$$

confirming the earlier work of Mézard and Parisi [13], where they computed the same limit using some non-rigorous arguments based on the *replica method* [14]. In an earlier work Aldous [1] showed that the limit of $\mathbf{E}[A_n]$ as $n \rightarrow \infty$ exists for any i.i.d. cost distribution. He also proved that the final limit does not depend on the specifics of the cost distribution, except only on the value of the density at 0, provided it exists and is strictly positive. So for calculation of the limiting constant one can assume that C_{ij} 's are i.i.d. with Exponential distribution with mean n . Then we can redefine the objective function A_n in the normalized form,

$$A_n := \min_{\pi} \frac{1}{n} \sum_{i=1}^n C_{i,\pi(i)}. \quad (8)$$

From historical perspective it is worth mentioning that in 1998 Parisi [16] conjectured that in this case the following exact formula holds

$$\mathbf{E}[A_n] = 1 + \frac{1}{4} + \dots + \frac{1}{n^2}, \quad \forall n \geq 1.$$

Recently two separate groups Linusson and Wästlund [12] and C. Nair, B. Prabhakar and M. Sharma [15] have independently proved this conjecture using combinatorial techniques. Thus also proving the limit. However Aldous [4] used local-weak convergence techniques to identify the limit constant $\zeta(2)$ in terms of an optimal matching problem on an infinite tree with random edge weights, described as follows

Let $\mathbb{T}_\infty := (\mathcal{V}, \mathcal{E})$ be the canonical infinite rooted labeled tree, as before, where \emptyset is the root. For every vertex $\mathbf{i} \in \mathcal{V}$, let $(\xi_{ij})_{j \geq 1}$ be points of a Poisson point process of rate 1 on $(0, \infty)$, and they are independent as \mathbf{i} varies. Define the weight of the edge $e = (\mathbf{i}, \mathbf{i}j) \in \mathcal{E}$ as ξ_{ij} .

This structure is called *Poisson weighted infinite tree* and henceforth abbreviated as PWIT.

Let $K_{n,n}^r$ be the complete graph on n vertices with a root selected uniformly at random. Suppose we also equip it with i.i.d. Exponential edge weights with mean n . Then one can show [4, 2] that in the sense of Aldous-Steel local weak convergence $K_{n,n}^r$ converges to the PWIT. Moreover heuristically the random assignment problem on $K_{n,n}^r$ has a “natural” analog to the limit structure, which is to consider the “optimal” (in sense of minimizing the “total cost”) matching problem on PWIT. Naturally PWIT being an infinite graph with edge weights each having mean at least 1, the “total cost” of any matching is infinite a.s., and hence minimizing “total cost” is not quite meaningful. However Aldous [4] showed that it is possible to make a sensible definition of “optimal matching” on PWIT which is invariant with respect to the automorphism of the tree \mathbb{T}_∞ , and minimizes the “average edge weight”. This construction is quite hard, and we refer the readers to [4, 2] for the technical details. Here we only provide the basic essentials to understand the motivation for our work.

Consider the heuristic description of the “optimal” matching problem on PWIT and suppose we define variables $X_{\mathbf{i}}$ for each vertex \mathbf{i} as follows

$$\begin{aligned} X_{\mathbf{i}} &= \text{Total cost of a maximal matching on the subtree } \mathbb{T}_\infty^{\mathbf{i}} \\ &\quad - \text{Total cost of a maximal matching on the forest } \mathbb{T}_\infty^{\mathbf{i}} \setminus \{\mathbf{i}\}, \end{aligned} \quad (9)$$

where $\mathbb{T}_\infty^{\mathbf{i}}$ is the subtree rooted at the vertex \mathbf{i} . Here by “total cost” we mean the sum total of all the edge weights in the matching. As noted above, both the “total costs” appearing in (9) are infinity almost surely. Thus rigorously speaking $X_{\mathbf{i}}$ is not well defined. But at the heuristic level if we forget this important issue, and work with these $X_{\mathbf{i}}$ -variables as if they are well defined, then simple manipulation yields that they must satisfy the following recurrence relation (see Section 4.2 of [4])

$$X_{\mathbf{i}} = \min_{j \geq 1} (\xi_{ij} - X_{\mathbf{i}j}). \quad (10)$$

This is of course the recurrence relation for a RTP associated with the Logistic RDE (4). Having observe that one can now construct the $X_{\mathbf{i}}$ -variables *externally*

as the RTP associated with the Logistic RDE, and use them to redefine the optimal matching on PWIT. This is precisely what Aldous did in [4], and later on referred as *540-degree argument* by Aldous and Bandyopadhyay in [5]. This construction also provides a characterization of the optimal matching on the PWIT. Finally one can then derive the $\zeta(2)$ -limit for the random assignment problem.

Once again a natural question would be to figure out whether the random variables $X_{\mathbf{i}}$'s are truly external or not, in other words to see whether the RTP is endogenous or not (see remarks (4.2.d) and (4.2.e) in [4]). This is our main motivation for this work. Theorem 1 proves that the $X_{\mathbf{i}}$ -variables can be defined using only the edge-weights and hence they have no external randomness in them.

Other significance of this result has been pointed out in Section 7.5 of [5]. We would like to note that the endogeneity of the Logistic RTP helps to define approximately feasible solution for the finite n -matching problem by using the optimal solution of the matching problem on PWIT. Thus with the help of endogeneity one can write a possibly simpler proof of Aldous' original argument for the $\zeta(2)$ -limit of the random assignment problem. But such derivation for this particular problem is not quite illuminating, and hence we do not pursue in that direction. As indicated in Section 7.5 of [5] in general endogeneity is an essential ingredient to make rigorous argument for the *cavity method*, and this work is only to illustrate one such non-trivial proof of endogeneity.

3 Review of Bivariate Uniqueness and Endogeneity

In this section we review some of the concepts from [5] which will be needed to prove our main result, Theorem 1.

In the general setting of equation (1) the question of endogeneity is quite abstract. Aldous and Bandyopadhyay in [5] introduces a concept called *bivariate uniqueness* for an invariant RTP, and showed under certain conditions that is equivalent to endogeneity. In the general setting bivariate uniqueness is defined as follows.

Consider a general RDE given by (1) and let $T: \mathcal{P} \rightarrow \mathcal{P}(S)$ be the induced operator. We will consider a bivariate version of it. Write $\mathcal{P}^{(2)}$ for the space of probability measures on $S^2 = S \times S$, with marginals in \mathcal{P} . We can now define a map $T^{(2)}: \mathcal{P}^{(2)} \rightarrow \mathcal{P}(S^2)$ as follows

Definition 2 For a probability $\mu^{(2)} \in \mathcal{P}^{(2)}$, $T^{(2)}(\mu^{(2)})$ is the joint distribution of

$$\left(\begin{array}{l} g\left(\xi, X_j^{(1)}, 1 \leq j \leq^* N\right) \\ g\left(\xi, X_j^{(2)}, 1 \leq j \leq^* N\right) \end{array} \right)$$

where we assume

1. $(X_j^{(1)}, X_j^{(2)})_{j \geq 1}$ are independent with joint distribution $\mu^{(2)}$ on S^2 ;

2. the family of random variables $(X_j^{(1)}, X_j^{(2)})_{j \geq 1}$ are independent of the innovation pair (ξ, N) .

We note that we use the *same realization* of the pair (ξ, N) in both components. Immediately from the definition we have

- (a) If μ is a solution of the RDE then the associated *diagonal measure* μ^\nearrow is a fixed-point for the operator $T^{(2)}$, where

$$\mu^\nearrow := \text{dist}(X, X), \quad (11)$$

where $X \sim \mu$.

- (b) If $\mu^{(2)}$ is a fixed-point of the operator $T^{(2)}$ then each marginal is a solution of the original RDE.

So if μ is a solution of the RDE (1) then μ^\nearrow is a fixed point of $T^{(2)}$ and there may or may not be other fixed points of $T^{(2)}$ with marginals μ .

Definition 3 *An invariant RTP with marginal μ has the bivariate uniqueness property if μ^\nearrow is the unique fixed point of the operator $T^{(2)}$ with marginals μ .*

Sometimes with slight abuse of terminology we will say that a solution of the RDE (1) has bivariate uniqueness property, or even the RDE has bivariate uniqueness property, if it has unique solution, meaning that the invariant RTP associated with the solution has the bivariate uniqueness property. Similar abuse will be done for the term endogeny also.

Theorem 11 of [5] shows that under appropriate assumptions the two concepts, namely bivariate uniqueness and endogeny are equivalent. Rather than stating this general equivalence theorem, we here only state the part we will need to prove endogeny for the Logistic RDE.

Theorem 2 (Theorem 11(b) of [5]) *Let S be a Polish space. Consider an invariant RTP with marginal distribution μ . Suppose the bivariate uniqueness property holds. If also $T^{(2)}$ is continuous with respect to the weak convergence on the set of bivariate distributions with marginals μ , then the endogenous property holds.*

Thus to prove endogenous property for the Logistic RDE (4) we will show that the bivariate uniqueness property holds and also establish the technical condition of Theorem 2, these are done in the following two sections.

4 Bivariate Uniqueness for the Logistic RDE

In this section we prove the bivariate uniqueness property for the Logistic RDE (4).

Theorem 3 Consider the following bivariate RDE

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \min_{j \geq 1} (\xi_j - X_j) \\ \min_{j \geq 1} (\xi_j - Y_j) \end{pmatrix}, \quad (12)$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d. pairs with same joint distribution as (X, Y) and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson process of rate 1 on $(0, \infty)$. Then the unique solution of this RDE is given by the diagonal measure μ^\nearrow where μ is the Logistic distribution.

4.1 Proof of Theorem 3

First observe that if the equation (12) has a solution then, the marginal distributions of X and Y solve the Logistic RDE (4), and hence they are both Logistic. Further by inspection μ^\nearrow is a solution of (12). So it is enough to prove that μ^\nearrow is the only solution of (12).

Let $\mu^{(2)}$ be a solution of (12). Notice that the points $\{(\xi_j; (X_j, Y_j)) \mid j \geq 1\}$ form a Poisson point process, say \mathcal{P} , on $(0, \infty) \times \mathbb{R}^2$, with mean intensity $\rho(t; (x, y)) dt d(x, y) := dt \mu^{(2)}(d(x, y))$. Thus if $G(x, y) := \mathbf{P}(X > x, Y > y)$, for $x, y \in \mathbb{R}$, then

$$\begin{aligned} G(x, y) &= \mathbf{P}\left(\min_{j \geq 1} (\xi_j - X_j) > x, \text{ and, } \min_{j \geq 1} (\xi_j - Y_j) > y\right) \\ &= \mathbf{P}\left(\text{No points of } \mathcal{P} \text{ are in } \left\{(t; (u, v)) \mid t - u \leq x, \text{ or, } t - v \leq y\right\}\right) \\ &= \exp\left(-\int \int \int_{t-u \leq x, \text{ or, } t-v \leq y} \rho(t; (u, v)) dt d(u, v)\right) \\ &= \exp\left(-\int_0^\infty [\overline{H}(t-x) + \overline{H}(t-y) - G(t-x, t-y)] dt\right) \\ &= \overline{H}(x) \overline{H}(y) \exp\left(\int_0^\infty G(t-x, t-y) dt\right), \end{aligned} \quad (13)$$

where \overline{H} is the right tail of Logistic distribution, defined as $\overline{H}(x) = e^{-x} / (1 + e^{-x})$ for $x \in \mathbb{R}$. The last equality follows from properties of the Logistic distribution (see Fact 3 of appendix). For notational convenience in this paper we will write $\overline{F}(\cdot) := 1 - F(\cdot)$, for any distribution function F .

The following simple lemma reduces the bivariate problem to a univariate problem.

Lemma 4 For any two random variables U and V , $U = V$ a.s. if and only if $U \stackrel{d}{=} V \stackrel{d}{=} U \wedge V$.

Proof : First of all if $U = V$ a.s. then $U \wedge V = U$ a.s.

Conversely suppose that $U \stackrel{d}{=} V \stackrel{d}{=} U \wedge V$. Fix a rational q , then under our assumption,

$$\begin{aligned} \mathbf{P}(U \leq q < V) &= \mathbf{P}(V > q) - \mathbf{P}(U > q, V > q) \\ &= \mathbf{P}(V > q) - \mathbf{P}(U \wedge V > q) \\ &= 0 \end{aligned}$$

A similar calculation will show that $\mathbf{P}(V \leq q < U) = 0$. These are true for any rational q , thus $\mathbf{P}(U \neq V) = 0$. \blacksquare

Thus if we can show that $X \wedge Y$ also has Logistic distribution, then from the lemma above we will be able to conclude that $X = Y$ a.s., and hence the proof will be complete. Put $g(\cdot) := \mathbf{P}(X \wedge Y > \cdot)$, we will show $g = \overline{H}$. Now, for every fixed $x \in \mathbb{R}$, by definition $g(x) = G(x, x)$. So using (13) we get

$$g(x) = \overline{H}^2(x) \exp\left(\int_{-x}^{\infty} g(s) ds\right), \quad x \in \mathbb{R}. \quad (14)$$

Notice that from (A1) (see Fact 3 of appendix) $g = \overline{H}$ is a solution of this non-linear integral equation (14), which corresponds to the solution $\mu^{(2)} = \mu^{\nearrow}$ of the original equation (12). To complete the proof of Theorem 3 we need to show that this is the only solution. For that we will prove that the operator associated with (14) (defined on an appropriate space) is monotone and has unique fixed-point as \overline{H} . The techniques we will use here are similar to Eulerian recursion [19], and are heavily based on analytic arguments.

Let \mathfrak{F} be the set of all functions $f : \mathbb{R} \rightarrow [0, 1]$ such that

- $\overline{H}^2(x) \leq f(x) \leq \overline{H}(x)$, $\forall x \in \mathbb{R}$,
- f is continuous and non-increasing.

Observe that by definition $\overline{H} \in \mathfrak{F}$. Further from (14) it follows that $g(x) \geq \overline{H}^2(x)$, as well as, $g(x) = \mathbf{P}(X \wedge Y > x) \leq \mathbf{P}(X > x) = \overline{H}(x)$, $\forall x \in \mathbb{R}$. Note also that g being the tail of the random variable $X \wedge Y$, is continuous (because both X and Y are continuous random variables) and non-increasing. So it is appropriate to search for solutions of (14) in \mathfrak{F} .

Let $T : \mathfrak{F} \rightarrow \mathfrak{F}$ be defined as

$$T(f)(x) := \overline{H}^2(x) \exp\left(\int_{-x}^{\infty} f(s) ds\right), \quad x \in \mathbb{R}. \quad (15)$$

Note that this operator T is not same as the general operator defined in Section 1, henceforth by T we will mean the specific operator defined above. Proposition 9 of Section 4.2 shows that T does indeed map \mathfrak{F} into itself. Observe that the equation (14) is nothing but the fixed-point equation associated with the operator T , that is,

$$g = T(g) \text{ on } \mathfrak{F}. \quad (16)$$

We here note that using (A1) (see Fact 3 of appendix) T can also be written as

$$T(f)(x) := \overline{H}(x) \exp\left(-\int_{-x}^{\infty} (\overline{H}(s) - f(s)) ds\right), \quad x \in \mathbb{R}, \quad (17)$$

which will be used in the subsequent discussion.

Define a partial order \preceq on \mathfrak{F} as, $f_1 \preceq f_2$ in \mathfrak{F} if $f_1(x) \leq f_2(x)$, $\forall x \in \mathbb{R}$, then the following result holds.

Lemma 5 T is a monotone operator on the partially ordered set (\mathfrak{F}, \preceq) .

Proof: Let $f_1 \preceq f_2$ be two elements of \mathfrak{F} , so from definition $f_1(x) \leq f_2(x)$, $\forall x \in \mathbb{R}$. Hence

$$\begin{aligned} \int_{-x}^{\infty} f_1(s) ds &\leq \int_{-x}^{\infty} f_2(s) ds, \quad \forall x \in \mathbb{R} \\ \Rightarrow \overline{T}(f_1)(x) &\leq \overline{T}(f_2)(x), \quad \forall x \in \mathbb{R} \\ \Rightarrow T(f_1) &\preceq T(f_2). \end{aligned}$$

Put $f_0 = \overline{H}^2$, and for $n \in \mathbb{N}$, define $f_n \in \mathfrak{F}$ recursively as, $f_n = T(f_{n-1})$. Now from Lemma 5 we get that if g is a fixed-point of T in \mathfrak{F} then, ■

$$f_n \preceq g, \quad \forall n \geq 0. \quad (18)$$

If we can show $f_n \rightarrow \overline{H}$ pointwise, then using (18) we will get $\overline{H} \preceq g$, so from definition of \mathfrak{F} it will follow that $g = \overline{H}$, and our proof will be complete. For that, the following lemma gives an explicit recursion for the functions $\{f_n\}_{n \geq 0}$.

Lemma 6 Let $\beta_0(s) = 1 - s$, $0 \leq s \leq 1$. Define recursively

$$\beta_n(s) := \int_s^1 \frac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)}\right) dw, \quad 0 < s \leq 1. \quad (19)$$

Then for $n \geq 1$,

$$f_n(x) = \overline{H}(x) \exp(-\beta_{n-1}(\overline{H}(x))), \quad x \in \mathbb{R}. \quad (20)$$

Proof: We will prove this by induction on n . Fix $x \in \mathbb{R}$, for $n = 1$ we get

$$\begin{aligned} f_1(x) &= T(f_0)(x) \\ &= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} (\overline{H}(s) - \overline{H}^2(s)) ds\right) \quad [\text{using (17)}] \\ &= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} \overline{H}(s) (1 - \overline{H}(s)) ds\right) \\ &= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} \overline{H}(s) H(s) ds\right) \\ &= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} H'(s) ds\right) \quad [\text{using Fact 1 of appendix}] \\ &= \overline{H}(x) \exp(-H(x)) \\ &= \overline{H}(x) \exp(-\beta_0(\overline{H}(x))) \end{aligned}$$

Now, assume that the assertion of the Lemma is true for $n \in \{1, 2, \dots, k\}$, for some $k \geq 1$, then from definition we have

$$\begin{aligned}
f_{k+1}(x) &= T(f_k)(x) \\
&= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} (\overline{H}(s) - f_k(s)) ds\right) \quad [\text{using (17)}] \\
&= \overline{H}(x) \exp\left(-\int_{-x}^{\infty} \overline{H}(s) \left(1 - e^{-\beta_{k-1}(\overline{H}(s))}\right) ds\right) \\
&= \overline{H}(x) \exp\left(-\int_{\overline{H}(x)}^1 \frac{1}{w} \left(1 - e^{-\beta_{k-1}(1-w)}\right) dw\right) \quad (21)
\end{aligned}$$

The last equality follows by substituting $w = H(s)$ and thus from Fact 1 and Fact 2 of the appendix we get that $\frac{dw}{w} = \overline{H}(s) ds$ and $H(-x) = \overline{H}(x)$. Finally by definition of β_n 's and using (21) we get $f_{k+1} = T(f_k)$. ■

To complete the proof it is now enough to show that $\beta_n \rightarrow 0$ pointwise, which will imply by Lemma 6 that $f_n \rightarrow \overline{H}$ pointwise, as $n \rightarrow \infty$. Using Proposition 10 (see Section 4.2) we get the following characterization of the pointwise limit of these β_n 's.

Lemma 7 *There exists a function $L : [0, 1] \rightarrow [0, 1]$ with $L(1) = 0$, such that*

$$L(s) = \int_s^1 \frac{1}{w} \left(1 - e^{-L(1-w)}\right) dw, \quad \forall s \in [0, 1], \quad (22)$$

and $L(s) = \lim_{n \rightarrow \infty} \beta_n(s)$, $\forall 0 \leq s \leq 1$.

Proof : From the Proposition 10 we know that for any $s \in [0, 1]$ the sequence $\{\beta_n(s)\}$ is decreasing, and hence \exists a function $L : [0, 1] \rightarrow [0, 1]$ such that $L(s) = \lim_{n \rightarrow \infty} \beta_n(s)$. Now observe that $\beta_n(1-w) \leq \beta_0(1-w) = w$, $\forall 0 \leq w \leq 1$, and hence

$$0 \leq \frac{1}{w} \left(1 - e^{-\beta_n(1-w)}\right) \leq \frac{\beta_n(1-w)}{w} \leq 1, \quad \forall 0 \leq w \leq 1.$$

Thus by taking limit as $n \rightarrow \infty$ in (19) and using the *dominated convergence theorem* along with part (a) of Proposition 10 we get that

$$L(s) = \int_s^1 \frac{1}{w} \left(1 - e^{-L(1-w)}\right) dw, \quad \forall 0 \leq s < 1. \quad \blacksquare$$

The above lemma basically translates the non-linear integral equation (14) to the non-linear integral equation (22), where the solution $g = \overline{H}$ of (14) is given by the solution $L \equiv 0$ of (22). So at first sight this may not lead us to the conclusion. But fortunately, something nice happens for equation (22), and we have the following result which is enough to complete the proof of Theorem 3.

Lemma 8 *If $L : [0, 1] \rightarrow [0, 1]$ is a function which satisfies the non-linear integral equation (22), namely,*

$$L(s) = \int_s^1 \frac{1}{w} \left(1 - e^{-L(1-w)}\right) dw, \quad \forall 0 \leq s < 1,$$

and if $L(1) = 0$, then $L \equiv 0$.

Proof : First note that $L \equiv 0$ is a solution. Now let L be any solution of (22), then L is infinitely differentiable on the open interval $(0, 1)$, by repetitive application of Fundamental Theorem of Calculus.

Consider,

$$\eta(w) := (1-w)e^{L(1-w)} + we^{-L(w)} - 1, \quad w \in [0, 1]. \quad (23)$$

Observe that $\eta(0) = \eta(1) = 0$ as $L(1) = 0$. Now, from (22) we get that

$$L'(w) = -\frac{1}{w} \left(1 - e^{-L(1-w)}\right), \quad w \in (0, 1). \quad (24)$$

Thus differentiating the function η we get

$$\eta'(w) = e^{-L(w)} \left[2 - \left(e^{L(1-w)} + e^{-L(1-w)}\right)\right] \leq 0, \quad \forall w \in (0, 1). \quad (25)$$

So the function η is decreasing in $(0, 1)$ and is continuous in $[0, 1]$ with boundary values as 0, hence $\eta \equiv 0$. Thus we must have $\eta' \equiv 0$, so from equation (25) we get that

$$e^{L(s)} + e^{-L(s)} = 2 \quad \text{for all } s \in (0, 1).$$

This implies $L \equiv 0$ on $[0, 1]$. ■

4.2 Some Technical Details

This section provides some of the technical results which were needed in the previous section.

Proposition 9 *The operator T maps \mathfrak{F} into \mathfrak{F} .*

Proof : First note that if $f \in \mathfrak{F}$, then by definition $T(f)(x) \geq \overline{H}^2(x)$, $\forall x \in \mathbb{R}$. Next by definition of \mathfrak{F} we get that $f \in \mathfrak{F} \Rightarrow f \preceq \overline{H}$, thus

$$\begin{aligned} \int_{-x}^{\infty} f(s) ds &\leq \int_{-x}^{\infty} \overline{H}(s) ds, \quad \forall x \in \mathbb{R} \\ \Rightarrow T(f)(x) &\leq \overline{H}^2(x) \exp\left(\int_{-x}^{\infty} \overline{H}(s) ds\right) = \overline{H}(x), \quad \forall x \in \mathbb{R} \end{aligned}$$

The last equality follows from (A1) (see Fact 3 of appendix). So,

$$\overline{H}^2(x) \leq T(f)(x) \leq \overline{H}(x), \quad \forall x \in \mathbb{R}. \quad (26)$$

Now we need to show that for any $f \in \mathfrak{F}$ we must have $T(f)$ continuous and non-increasing. From the definition $T(f)$ is continuous (in fact, infinitely differentiable). Moreover if $x \leq y$ be two real numbers, then

$$\int_{-x}^{\infty} (\overline{H}(s) - f(s)) ds \leq \int_{-y}^{\infty} (\overline{H}(s) - f(s)) ds,$$

because $f \preceq \overline{H}$. Also $\overline{H}(x) \geq \overline{H}(y)$, thus using (17) we get

$$T(f)(x) \geq T(f)(y) \tag{27}$$

So using (26) and (27) we conclude that $T(f) \in \mathfrak{F}$ if $f \in \mathfrak{F}$. ■

Proposition 10 *The following are true for the sequence of functions $\{\beta_n\}_{n \geq 0}$ defined in (19).*

- (a) *For every fixed $s \in (0, 1]$, the sequence $\{\beta_n(s)\}$ is decreasing.*
- (b) *For every $n \geq 1$, $\lim_{s \rightarrow 0^+} \beta_n(s)$ exists, and is given by*

$$\int_0^1 \frac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)}\right) dw,$$

we will write this as $\beta_n(0)$.

- (c) *The sequence of numbers $\{\beta_n(0)\}$ is also decreasing.*

Proof : (a) Notice that $\beta_0(s) = 1 - s$ for $s \in [0, 1]$, thus

$$\beta_1(s) = \int_s^1 \frac{1 - e^{-w}}{w} dw < 1 - s = \beta_0(s), \quad \forall s \in (0, 1].$$

Now assume that for some $n \geq 1$ we have $\beta_n(s) \leq \beta_{n-1}(s) \leq \dots \leq \beta_0(s)$, $\forall s \in (0, 1]$, if we show that $\beta_{n+1}(s) \leq \beta_n(s)$, $\forall s \in (0, 1]$ then by induction the proof will be complete. For that, fix $s \in (0, 1]$ then

$$\begin{aligned} \beta_{n+1}(s) &= \int_s^1 \frac{1}{w} \left(1 - e^{-\beta_n(1-w)}\right) dw \\ &\leq \int_s^1 \frac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)}\right) dw \\ &= \beta_n(s). \end{aligned}$$

This proves the part (a).

(b, c) First note that by trivial induction $\beta_n(s) \geq 0$ for every $s \in (0, 1]$, $n \geq 0$. Thus from definition for every $n \geq 0$, the limit $\lim_{s \rightarrow 0^+} \beta_n(s)$ exists in $[0, \infty]$ and is given by

$$\int_0^1 \frac{1}{w} \left(1 - e^{-\beta_{n-1}(1-w)}\right) dw.$$

Now using (a) above we conclude

$$\beta_{n+1}(0) = \lim_{s \rightarrow 0^+} \beta_{n+1}(s) \leq \lim_{s \rightarrow 0^+} \beta_n(s) = \beta_n(0), \quad (28)$$

for every $n \geq 0$. Since $\beta_0(0) = 1$, so we get $\beta_n(0) < \infty$ for all $n \geq 0$, and the sequence is decreasing. Proving parts (b) and (c). ■

5 Proof of Theorem 1

Once again we will use the general Theorem 11(b) of [5], stated here as Theorem 2. We note that by Theorem 3 the Logistic RDE (4) has bivariate uniqueness property and hence all remains is to check the technical continuity condition.

Proposition 11 *Let \mathfrak{S} be the set of all probabilities on \mathbb{R}^2 and let $\Gamma : \mathfrak{S} \rightarrow \mathfrak{S}$ be the operator associated with the RDE (12), that is,*

$$\Gamma(\mu^{(2)}) \stackrel{d}{=} \left(\begin{array}{c} \min_{j \geq 1} (\xi_j - X_j) \\ \min_{j \geq 1} (\xi_j - Y_j) \end{array} \right), \quad (29)$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d with joint law $\mu^{(2)} \in \mathfrak{S}$ and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson point process of rate 1 on $(0, \infty)$. Then Γ is continuous with respect to the weak convergence topology when restricted to the subspace \mathfrak{S}^* defined as

$$\mathfrak{S}^* := \left\{ \mu^{(2)} \mid \text{both the marginals of } \mu^{(2)} \text{ are Logistic distribution} \right\}. \quad (30)$$

Before we prove this proposition, it is worth mentioning that the operator Γ is not continuous with respect to the weak convergence topology on the whole space \mathfrak{S} . In fact, as it turns out it is every where discontinuous on \mathfrak{S} (see Section 6). But fortunately for applying Theorem 2 we only need the continuity of Γ when restricted to the subspace \mathfrak{S}^* .

Proof of Proposition 11 : Let $\left\{ \mu_n^{(2)} \right\}_{n=1}^{\infty} \subseteq \mathfrak{S}^*$ and suppose that $\mu_n^{(2)} \xrightarrow{d} \mu^{(2)} \in \mathfrak{S}^*$. We will show that $\Gamma(\mu_n^{(2)}) \xrightarrow{d} \Gamma(\mu^{(2)})$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space such that, $\exists \{(X_n, Y_n)\}_{n=1}^{\infty}$ and (X, Y) random vectors taking values in \mathbb{R}^2 , with $(X_n, Y_n) \sim \mu_n^{(2)}$, $n \geq 1$, and $(X, Y) \sim \mu^{(2)}$. Notice that by definition $X_n \stackrel{d}{=} Y_n \stackrel{d}{=} X \stackrel{d}{=} Y$, and each has Logistic distribution.

Fix $x, y \in \mathbb{R}$, then using similar calculations as in (13) we get

$$\begin{aligned} G_n(x, y) &:= \Gamma(\mu_n^{(2)})((x, \infty) \times (y, \infty)) \\ &= \overline{H}(x) \overline{H}(y) \exp \left(- \int_0^{\infty} \mathbf{P}(X_n > t - x, Y_n > t - y) dt \right) \\ &= \overline{H}(x) \overline{H}(y) \exp \left(- \int_0^{\infty} \mathbf{P}((X_n + x) \wedge (Y_n + y) > t) dt \right) \\ &= \overline{H}(x) \overline{H}(y) \exp(-\mathbf{E}[(X_n + x)^+ \wedge (Y_n + y)^+]), \end{aligned} \quad (31)$$

and a similar calculation will also give that

$$\begin{aligned} G(x, y) &:= \Gamma(\mu^{(2)})((x, \infty) \times (y, \infty)) \\ &= \overline{H}(x)\overline{H}(y) \exp(-\mathbf{E}[(X+x)^+ \wedge (Y+y)^+]). \end{aligned} \quad (32)$$

Now to complete the proof all we need is to show

$$\mathbf{E}[(X_n+x)^+ \wedge (Y_n+y)^+] \longrightarrow \mathbf{E}[(X+x)^+ \wedge (Y+y)^+].$$

Since we assumed that $(X_n, Y_n) \xrightarrow{d} (X, Y)$ thus

$$(X_n+x)^+ \wedge (Y_n+y)^+ \xrightarrow{d} (X+x)^+ \wedge (Y+y)^+, \quad \forall x, y \in \mathbb{R}. \quad (33)$$

Fix $x, y \in \mathbb{R}$, define $Z_n^{x,y} := (X_n+x)^+ \wedge (Y_n+y)^+$, and $Z^{x,y} := (X+x)^+ \wedge (Y+y)^+$. Observe that

$$0 \leq Z_n^{x,y} \leq (X_n+x)^+ \leq |X_n+x|, \quad \forall n \geq 1. \quad (34)$$

But, $|X_n+x| \xrightarrow{d} |X+x|$, $\forall n \geq 1$. So clearly $\{Z_n^{x,y}\}_{n=1}^\infty$ is uniformly integrable. Hence we conclude (using Theorem 25.12 of Billingsley [9]) that

$$\mathbf{E}[Z_n^{x,y}] \longrightarrow \mathbf{E}[Z^{x,y}].$$

This completes the proof. ■

6 Final Remarks

(a) Intuitively, a natural approach to show that the fixed-point equation $\Gamma(\mu^{(2)}) = \mu^{(2)}$ on \mathfrak{S} has unique solution, would be to specify a metric ρ on \mathfrak{S} such that the operator Γ becomes a contraction with respect to it. Unfortunately, this approach seems rather hard or may even be impossible. Perhaps the reason being the Logistic RDE (4) itself does not have a contractive property, in fact, it does not have a full domain of attraction (see [5]). However its exact domain of attraction is not yet known (see open problem 62 of [5]). On the other hand from the proof of Theorem 3 it is clear that equation (14) has the whole of \mathfrak{F} within its domain of attraction. So it is possible to have a suitable metric of contraction for T but, we have been unable to find it.

(b) Although at first glance it seems that the operator T as defined in (15) is just an analytic tool to solve the equation (14) but, it has a nice interpretation through Logistic RDE (4). Suppose \mathfrak{A} is the operator associated with Logistic RDE, that is,

$$\mathfrak{A}(\mu) \stackrel{d}{=} \min_{j \geq 1} (\xi_j - X_j), \quad (35)$$

where $(\xi_j)_{j \geq 1}$ are points of a Poisson point process of mean intensity 1 on $(0, \infty)$, and are independent of $(X_j)_{j \geq 1}$, which are i.i.d with distribution μ on \mathbb{R} . It is easy to check that the domain of definition of \mathfrak{A} is the space

$$\mathcal{A} := \left\{ F \mid F \text{ is a distribution function on } \mathbb{R} \text{ and } \int_0^\infty \overline{F}(s) ds < \infty \right\}. \quad (36)$$

Note that the condition $\int_0^\infty \overline{F}(s) ds < \infty$ means $\mathbf{E}_F[X^+] < \infty$. Now it is easy to see that \mathfrak{F} can be embedded into \mathcal{A} and definition of T can be naturally extended on whole of \mathcal{A} . In that case the following identity holds

$$\frac{\overline{T(\mu)}(\cdot)}{\overline{H}(\cdot)} \times \frac{\overline{\mathfrak{A}(\mu)}(\cdot)}{\overline{H}(\cdot)} = 1, \quad \forall \mu \in \mathcal{A}. \quad (37)$$

This at least explains the monotonicity of T through anti-monotonicity property of the Logistic operator \mathfrak{A} (easy to check).

(c) It is interesting to note that the operator \mathfrak{A} is every where discontinuous with respect to the weak convergence topology on \mathcal{A} . This is because, given any distribution $F_0 \in \mathcal{A}$, we can construct a sequence of distributions $\{F_n\}_{n \geq 1} \subseteq \mathcal{A}$ converging in distribution to F_0 , such that

$$\mathbf{E}_{F_n} \left[(x + X_n)^+ \right] \rightarrow \infty, \quad \text{for all } x \in \mathbb{R}.$$

Note $F_0 \in \mathcal{A} \Rightarrow \mathbf{E}_{F_0} \left[(x + X_0)^+ \right] < \infty$, for all $x \in \mathbb{R}$. On the other hand we know that for any distribution function F ,

$$\mathfrak{A}(F)(x) = 1 - \exp \left(-\mathbf{E}_F \left[(x + X)^+ \right] \right), \quad \text{for all } x \in \mathbb{R}$$

(see the proof of Fact 3 in the appendix). Thus for every $x \in \mathbb{R}$,

$$\mathfrak{A}(F_n)(x) \not\rightarrow \mathfrak{A}(F_0)(x).$$

So \mathfrak{A} is discontinuous at F_0 for every $F_0 \in \mathcal{A}$ with respect to the weak convergence topology. This also indicates that the same phenomenon is true for the bivariate operator Γ .

Appendix

Here we provide some known facts about the Logistic distribution which are used in the Sections 4 and 5. First recall that we say a real valued random variable X has Logistic distribution if its distribution function is given by (5), namely,

$$H(x) = \mathbf{P}(X \leq x) = \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R}.$$

The following facts hold for the function H .

Fact 1 H is infinitely differentiable, and $H'(\cdot) = H(\cdot)\overline{H}(\cdot)$, where $\overline{H}(\cdot) = 1 - H(\cdot)$.

Proof : From the definition it follows that H is infinitely differentiable on \mathbb{R} . Further,

$$\begin{aligned} H'(x) &= \frac{1}{1+e^{-x}} \times \frac{e^{-x}}{1+e^{-x}} \\ &= H(x)\overline{H}(x) \quad \forall x \in \mathbb{R} \end{aligned}$$

■

Fact 2 H is symmetric around 0, that is, $H(-x) = \overline{H}(x) \quad \forall x \in \mathbb{R}$.

Proof : From the definition we get that for any $x \in \mathbb{R}$,

$$H(-x) = \frac{1}{1+e^x} = \frac{e^{-x}}{1+e^{-x}} = \overline{H}(x).$$

■

Fact 3 \overline{H} is the unique solution of the non-linear integral equation

$$\overline{H}(x) = \exp\left(-\int_{-x}^{\infty} \overline{H}(s) ds\right), \quad \forall x \in \mathbb{R}. \quad (\text{A1})$$

Proof : Notice that the equation (A1) is nothing but Logistic RDE, this is because

$$\mathbf{P}\left(\min_{j \geq 1} (\xi_j - X_j) > x\right) = \exp\left(-\int_{-x}^{\infty} \overline{H}(s) ds\right), \quad \forall x \in \mathbb{R}$$

where $(X_j)_{j \geq 1}$ are i.i.d. with distribution function H and are independent of $(\xi_j)_{j \geq 1}$, which are points of a Poisson point process of rate 1 on $(0, \infty)$. Thus from the fact that \overline{H} is the unique solution of Logistic RDE (Lemma 5 of [4]) we conclude that \overline{H} is unique solution of equation (A1). ■

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