

# On the Cluster Size Distribution for Percolation on Some General Graphs

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## Abstract

We show that for any Cayley graph, the probability (at any  $p$ ) that the cluster of the origin has size  $n$  decays at a well-defined exponential rate (possibly 0). For general graphs, we relate this rate being positive in the supercritical regime with the amenability/nonamenability of the underlying graph.

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## 1 Introduction

Percolation is perhaps the most widely studied statistical physics model for modeling random media. In addition, it is a source of many challenging open problems and beautiful conjectures which are easy to state but often are very difficult to settle; see [13] for a survey and introduction. The classical literature concentrates on studying the model on Euclidean lattices  $\mathbb{Z}^d$ ,  $d \geq 2$  and on trees. However in recent years, there has been a great deal of interest in studying percolation on other infinite, locally finite, connected graphs; see [9, 7, 8, 15, 16, 14, 17, 26].

Our first theorem states that for any Cayley graph, the probability that the cluster of the origin has size  $n$  decays at a well-defined exponential rate. For  $\mathbb{Z}^d$ , this is Theorem 6.75 in [13].

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Throughout this paper,  $C$  will denote the connected component of a fixed vertex (the origin for Cayley graphs) for Bernoulli percolation.

**Theorem 1** *If  $G$  is a Cayley graph, then*

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_p(|C| = n)$$

*exists for every  $p \in (0, 1)$ .*

Our method for proving this result combines a randomized version of the usual method using subadditivity (as in for  $\mathbb{Z}^d$ ) together with a proof that any two finite subgraphs of  $G$  have disjoint translates that are at distance  $\leq \delta$  from each other where  $\delta$  is an appropriate function of the sizes of the subgraphs. One expects perhaps that one should be able to take  $\delta$  being a constant, depending only on the graph. See Question 3 for the statement of this problem.

**Remark:** Interesting, as we point out later, there is a concept of an ordered group and for such groups, the proof of Theorem 6.75 in [13] can be extended. However, for general groups, it seems that this proof cannot be applied.

It is of course of interest to know if the limit above is positive or 0. As will be pointed out later, it is positive below the critical value for all transitive graphs and so we restrict discussion to the supercritical regime. In this case, for  $\mathbb{Z}^d$ , the limit is 0 (see Theorem 8.61 in [13]) while for trees it is positive (although 0 at the critical value). Equation (10.12) in [13] has an explicit formula for these probabilities for the rooted infinite 3-ary tree.

One of the key issues studied in percolation is the difference in the behavior of percolation depending on whether the underlying graph is *amenable* or *nonamenable* [7, 8, 17, 26]. For example, for amenable transitive graphs, there is uniqueness of the infinite cluster for all values of  $p$  while for nonamenable transitive graphs, it is conjectured that there is nonuniqueness of the infinite cluster for some values of  $p$ . Here it is also worthwhile to point out that it is well known that properties of other probabilistic models associated with a graph differ depending on whether the graph is amenable or not. Perhaps the most classical of all is the relation with simple random walk on a graph, first studied by Kesten [22] where it was shown that there is a positive spectral gap in the transition operator if and only if the group is nonamenable. Similar relationships have been investigated with respect to other statistical physics models (see e.g. [19, 20, 17, 10]).

For the nonamenable case, we state the following question.

**Question 1** *Is it true that for a general transitive nonamenable graph  $G$  we have*

$$\mathbb{P}_p(|C| = n) \leq \exp(-\gamma(p)n) \quad \forall n \geq 1 \tag{1}$$

*for some  $\gamma(p) > 0$  whenever  $p \neq p_c(G)$  ?*

Consider a general weakly nonamenable graph  $G := (\mathcal{V}, \mathcal{E})$  (not necessarily transitive) with bounded degree. Using a not so difficult argument of counting *lattice animals*, one can prove that if  $v_0$  is a fixed vertex of  $G$  and  $C$  is the open connected component of  $v_0$ , then for sufficiently large  $p$  there is a function  $\gamma(p) > 0$ , such that

$$\mathbb{P}_p(|C| = n) \leq e^{-\gamma(p)n} \quad \forall n \geq 1. \quad (2)$$

In fact, in the appendix by Gábor Pete in [10] (see equation (A.3)), it is shown by a slightly more involved argument, that the exponential decay (2) holds whenever  $p > 1/(1 + \kappa')$  where  $\kappa' = \kappa'(G, v_0)$  is the anchored Cheeger constant. This is certainly in contrast to the  $\mathbb{Z}^d$  case and also, as we will see later in Section 4, to what happens for a large class of transitive amenable graphs.

Using classical branching process arguments, one can conclude that for any infinite regular tree (which are prototypes for transitive nonamenable graphs), we must have an exponential tail bound for the cluster size distribution, when  $p$  is not equal to the critical probability.

The assumption of transitivity is however needed for Question 1 to have a positive answer as the following example illustrates. The graph obtained by taking  $\mathbb{Z}^d$  and attaching a regular rooted tree with degree  $r + 1$  at each vertex with  $p_c(\mathbb{Z}^d) < \frac{1}{r}$  is a nontransitive, nonamenable graph which possesses an intermediate regime (above the critical value) of sub-exponential decay as next stated in detail.

**Theorem 2** *Consider the graph just described and suppose  $p_c(\mathbb{Z}^d) < \frac{1}{r}$ .*

- (a) *If  $p \in (0, p_c(\mathbb{Z}^d)) \cup (\frac{1}{r}, 1)$  then there are functions  $\phi_1(p) < \infty$  and  $\phi_2(p) > 0$ , such that for all  $n \geq 1$ ,*

$$\exp(-\phi_1(p)n) \leq \mathbb{P}_p(n \leq |C| < \infty) \leq \exp(-\phi_2(p)n). \quad (3)$$

- (b) *If  $p \in (p_c(\mathbb{Z}^d), \frac{1}{r})$  then there are functions  $\psi_1(p) < \infty$  and  $\psi_2(p) > 0$ , such that for all  $n \geq 1$ ,*

$$\exp(-\psi_1(p)n^{(d-1)/d}) \leq \mathbb{P}_p(n \leq |C| < \infty) \leq \exp(-\psi_2(p)n^{(d-1)/d}). \quad (4)$$

- (c) *For  $p = \frac{1}{r}$  we have constants  $c_1 > 0$  and  $c_2(\varepsilon) < \infty$  such that every  $\varepsilon > 0$  and for all  $n \geq 1$ ,*

$$\frac{c_1}{n^{1/2}} \leq \mathbb{P}_p(n \leq |C| < \infty) \leq \frac{c_2}{n^{1/2-\varepsilon}}. \quad (5)$$

As also explained in Section 5, if  $p_c(\mathbb{Z}^d) > \frac{1}{r}$ , this intermediate regime disappears.

An interesting class of graphs to investigate in regard to Question 1 are products of  $\mathbb{Z}^d$  with a homogeneous tree.

**Question 2** *Is there exponential decay in the supercritical regime for  $\mathbb{Z}^d \times T_r$  where  $T_r$  is the homogeneous  $r$ -ary tree?*

We now move to the amenable case.

**Conjecture 1** *Let  $G := (\mathcal{V}, \mathcal{E})$  be a transitive amenable graph. Then there is a sequence  $\alpha_n = o(n)$ , such that for  $p > p_c(G)$*

$$\mathbb{P}_p(n \leq |C| < \infty) \geq \exp(-\eta(p)\alpha_n) \quad \forall n \geq 1, \quad (6)$$

where  $\eta(p) < \infty$ .

It turns out that the argument of Aizenman, Delyon and Aouillard [2, 13] for proving this sub-exponential behavior for  $\mathbb{Z}^d$  can be successfully carried out for a large class of transitive amenable graphs. For  $\mathbb{Z}^d$ , the sequence  $\{\alpha_n\}$  can be taken to be  $\{n^{\frac{d-1}{d}}\}$ .

**Theorem 3** *If  $G := (\mathcal{V}, \mathcal{E})$  is a Cayley graph of a finitely presented amenable group with one end, then there is a sequence  $\alpha_n = o(n)$  such that for  $p > p_c(G)$ , there is  $\eta(p) < \infty$  such that*

$$\mathbb{P}_p(n \leq |C| < \infty) \geq \exp(-\eta(p)\alpha_n) \quad \forall n \geq 1. \quad (7)$$

We finally point out that transitivity is a necessary condition in Conjecture 1.

**Proposition 4** *There is an amenable nontransitive graph for which one has exponential decay of the cluster size distribution at all  $p \neq p_c$ .*

This paper concerns itself mostly with the supercritical case. It therefore seems appropriate to end this introduction with a few comments concerning the subcritical case. It was shown independently in [24] and [1] that for  $\mathbb{Z}^d$  in the subcritical regime, the size of the cluster of the origin has a finite expected value. While it seems that the argument in [24] does not work for all transitive graphs as it seems that it is needed that the balls in the graph grow slower than  $e^{n^\gamma}$  for some  $\gamma < 1$ , it is stated in [26] that the argument in [1] goes through for any transitive graph. Theorem 6.75 in [13] (due to [3]) states that for  $\mathbb{Z}^d$ , if the expected size of the cluster is finite, then exponential decay of the tail of the cluster size follows. As stated in [3], this result holds quite generally in transitive situations and so, in combination with the statement in [26] referred to above, for all transitive graphs, one has exponential decay of the cluster size in the subcritical regime.

We point out however, not surprisingly, that transitivity is again needed here. An example of a graph which does not have exponential decay in (a portion of) the subcritical regime is obtained by taking the positive integers, planting a binary tree of depth  $a_k$  (sufficiently large) at  $k$  for  $k \geq 1$  and also attaching to the origin a graph whose critical value is say  $3/4$ . This graph has  $p_c = 3/4$  but for some  $p < 3/4$ , exponential decay fails.

We mention that Questions 3 and 4 which appear later on and arise naturally in our study could also be of interest to people in geometric group theory.

The rest of the paper is organized as follows. In Section 2, we provide all the necessary definitions and notations. In Section 3, we prove Theorem 1. In Section 4, we prove Theorem 3 and Proposition 4. Finally, in Section 5, we prove Theorem 2 as well as study the variant of the example in Theorem 2 obtained by taking  $p_c(\mathbb{Z}^d) > \frac{1}{r}$  instead.

## 2 Definitions and notations

Let  $G = (\mathcal{V}, \mathcal{E})$  be an infinite, connected graph. We will say  $G$  is *locally finite* if every vertex has finite degree.

The i.i.d. Bernoulli bond percolation with probability  $p \in [0, 1]$  on  $G$  is a probability measure on  $\{0, 1\}^{\mathcal{E}}$ , such that the coordinate variables are i.i.d. with Bernoulli ( $p$ ) distribution. This measure will be denoted by  $\mathbb{P}_p$ . For a given configuration in  $\{0, 1\}^{\mathcal{E}}$ , it is customary to say that an edge  $e \in \mathcal{E}$  is *open* if it is in state 1, otherwise it is said to be *closed*. Given a configuration, write  $\mathcal{E} = \mathcal{E}_o \cup \mathcal{E}_c$ , where  $\mathcal{E}_o$  is the set of all open edges and  $\mathcal{E}_c$  is the set of all closed edges. The connected components of the subgraph  $(\mathcal{V}, \mathcal{E}_o)$  are called the *open connected components* or *clusters*.

One of the fundamental quantities in percolation theory is the critical probability  $p_c(G)$  defined by

$$p_c(G) := \inf \left\{ p \in [0, 1] \mid \mathbb{P}_p(\exists \text{ an infinite cluster}) = 1 \right\}. \quad (8)$$

The percolation model is said to be *subcritical*, *critical* or *supercritical* regime depending on whether  $p < p_c(G)$ ,  $p = p_c(G)$  or  $p > p_c(G)$  respectively.

For a fixed vertex  $v \in \mathcal{V}$ , let  $C(v)$  be the open connected component containing the vertex  $v$ . Let

$$\theta_G^v(p) := \mathbb{P}_p(C(v) \text{ is infinite}). \quad (9)$$

For a connected graph  $G$ , it is easy to show that irrespective of the choice of the vertex  $v$

$$p_c(G) = \inf \left\{ p \in [0, 1] \mid \theta_G^v(p) > 0 \right\}. \quad (10)$$

**Definition 1** *We will say a graph  $G = (\mathcal{V}, \mathcal{E})$  is transitive if for every pair of vertices  $u$  and  $v$  there is an automorphism of  $G$ , which sends  $u$  to  $v$ . In other words, a graph  $G$  is transitive if its automorphism group  $\text{Aut}(G)$  acts transitively on  $\mathcal{V}$ .*

Observe that if  $G$  is transitive then we can drop the dependency on the vertex  $v$  in (9), and then we can write  $\theta_G(p) = \mathbb{P}_p(C(v_0) \text{ is infinite})$  for a fixed vertex  $v_0$  of  $G$ .  $\theta_G(\cdot)$  is called the *percolation function* for a transitive graph  $G$ .

We now give definitions of some of the qualitative properties of a graph  $G$  which are important for our study.

**Definition 2** Let  $G := (\mathcal{V}, \mathcal{E})$  be an infinite, locally finite, connected graph. The Cheeger constant of  $G$ , denoted by  $\kappa(G)$ , is defined by

$$\kappa(G) := \inf \left\{ \frac{|\partial W|}{|W|} \mid \emptyset \neq W \subseteq \mathcal{V} \text{ and } |W| < \infty \right\} \quad (11)$$

where  $\partial W := \left\{ u \notin W \mid \exists v \in W, \text{ such that } \{u, v\} \in \mathcal{E} \right\}$  is the external vertex boundary. The graph  $G$  is said to be amenable if  $\kappa(G) = 0$ ; otherwise it is called nonamenable.

A variant and weaker property than the above is the following.

**Definition 3** Let  $G := (\mathcal{V}, \mathcal{E})$  be an infinite, locally finite, connected graph. We define the anchored Cheeger constant of  $G$  with respect to the vertex  $v_0$  by

$$\kappa'(G, v_0) := \inf \left\{ \frac{|\partial W|}{|W|} \mid v_0 \in W \subseteq \mathcal{V}, W \text{ connected and } |W| < \infty \right\} \quad (12)$$

where  $\partial W$  is defined as above. The graph  $G$  is said to be strongly amenable if  $\kappa'(G, v_0) = 0$ , otherwise it is called weakly nonamenable.

It is easily argued that for a connected graph  $G$ ,  $\kappa'(G, v_0) = 0$  implies that  $\kappa'(G, v) = 0$  for every vertex  $v$  and so the definition of strong amenability (or weak nonamenability) does not depend on the choice of the vertex  $v_0$ . Of course, the value of the constant  $\kappa'(G, v_0)$  may depend on the choice of  $v_0$  in the weakly nonamenable case. It follows by definition that  $\kappa(G) \leq \kappa'(G, v_0)$  for any  $v_0$  and so strong amenability implies amenability. On the other hand, it is easy to show that the two notions are not equivalent although if  $G$  is transitive then they are equivalent.

A special class of transitive graphs which are associated with finitely generated groups are the so-called *Cayley graphs*.

**Definition 4** Given a finitely generated group  $\bar{G}$  and a symmetric generating set  $\mathcal{S}$  (symmetric meaning that  $\mathcal{S} = \mathcal{S}^{-1}$ ), a graph  $G := (\mathcal{V}, \mathcal{E})$  is called the left-Cayley graph of  $\bar{G}$  obtained using  $\mathcal{S}$  if the vertex set of  $G$  is  $\bar{G}$  and the edge set is  $\left\{ \{u, v\} \mid v = su \text{ for some } s \in \mathcal{S} \right\}$ .

Similarly we can also define a *right-Cayley graph* of the group  $\bar{G}$  obtained using  $\mathcal{S}$ . Observe that the left- and right-Cayley graphs obtained using the same symmetric generating set are isomorphic, where an isomorphism is given by the group involution  $u \mapsto u^{-1}$ ,  $u \in \bar{G}$ . It is also easy to see that multiplication on the right by any element in  $\bar{G}$  is a graph automorphism of any left-Cayley graph.

If not explicitly mentioned otherwise, by a *Cayley graph* of a finitely generated group  $\bar{G}$ , we will always mean a left-Cayley graph with respect to some symmetric generating set.

**Definition 5** A group is finitely presented if it is described by a finite number of generators and relations.

**Definition 6** A graph is one-ended if when one removes any finite subset of the vertices, there remains only one infinite component. A group is one-ended if its Cayley graph is; it can be shown that this is then independent of the generators used to construct the Cayley graph.

### 3 Limit of the tail of the cluster size distribution for Cayley graphs

In this section, we prove Theorem 1. Throughout this section  $o$  will denote the identity element of our group.

A Cayley graph is said to have *polynomial growth* if the size of a ball is bounded by some polynomial (in its radius). Given a finitely generated group, its Cayley graph having polynomial growth does not depend on the choice of the finite symmetric generating set. It is well known (see [11]) that the growth of a Cayley graph of polynomial growth is always between  $\alpha r^k$  and  $\frac{1}{\alpha} r^k$ , for some  $k \in \mathbb{N}$  and  $\alpha \in (0, 1)$  and that if a Cayley graph is not of polynomial growth, then for any polynomial  $p(n)$ , the ball of radius  $n$  around  $o$  is larger than  $p(n)$  for all but at most finitely many  $n$ .

Let  $G$  be a Cayley graph with degree  $d$ . Denote by  $C_x$  the open component of vertex  $x$ ;  $C$  will stand for the open component of  $o$ . As usual, for a (not necessarily induced) subgraph  $H$  of  $G$ ,  $E(H)$  is the edge set and  $V(H)$  is the vertex set of  $H$ . Given some  $p \in [0, 1]$ , let  $\pi_n := \mathbb{P}_p(|C| = n)$ .

**Lemma 5** If  $G$  is a Cayley graph of linear or of quadratic growth, then

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_p(|C| = n)$$

exists for every  $p \in (0, 1)$ .

*Proof:* If  $G$  has quadratic growth then the vertices of  $G$  can be partitioned into finite classes, so-called *blocks of imprimitivity*, in such a way that the group of automorphisms restricted to the classes is  $\mathbb{Z}^2$ , see [27]. Now we can mimic the proof of the claim for  $\mathbb{Z}^2$ , see [13]: use the subadditive theorem and the fact that for any two connected finite subgraphs of  $G$ , one of them has a translate that is disjoint from the other, but at bounded distance from it. For Cayley graphs of linear growth, one can proceed along the same arguments, since a partition into blocks of imprimitivity, as above, exists (see [11]). ■

Before starting the proof of Theorem 1, we first prove the following lemma which gives an important estimate for Cayley graphs with at least cubic growth. Using the simple structure of Cayley graphs of linear or quadratic growth, Lemma 6 is true for every Cayley graph. (In the latter two cases,  $(|A| + |B|)^{3/4}$  can be replaced by 1.)

**Lemma 6** *Let  $G$  be a Cayley graph of at least cubic growth and  $A, B \subset V(G)$ . Then there is a  $\gamma \in \text{Aut}(G)$  such that the translate  $\gamma A$  is disjoint from  $B$  and  $\text{dist}(\gamma A, B) \leq (|A| + |B|)^{3/4}$ .*

*Proof:* Let  $\mathcal{A}_n$  be the set of all connected subgraphs of size  $n$  in  $G$  that contain the  $o$ . Fix some group  $\Gamma$  of automorphisms of  $G$  such that  $\Gamma$  is vertex-transitive on  $G$  and only the identity of  $\Gamma$  has a fixed point. (If  $G$  is a left Cayley graph then choosing  $\Gamma$  to be the group itself acting with right multiplication works.) For a vertex  $x$  of  $G$ , let  $\gamma_x \in \Gamma$  be the (unique) element of  $\Gamma$  that takes  $o$  to  $x$ . Finally, for a subgraph  $H$  of  $G$  denote by  $H'$  the 1-neighborhood of  $H$  (that is, the set of vertices at distance  $\leq 1$  from  $H$ ). Note that if  $H$  is connected and  $|V(H)| > 1$ , then  $|V(H')| \leq d|V(H)|$ , because every point of  $H$  has at most  $d - 1$  neighbors outside of  $H$ .

Let  $n, m > 1$  and  $A \in \mathcal{A}_n, B \in \mathcal{A}_m$ . Suppose that for some  $\gamma \neq \gamma' \in \Gamma$  there is a point  $x$  in  $A'$  such that  $\gamma B'$  and  $\gamma' B'$  both contain  $x$ . Then, by the choice of  $\Gamma$ ,  $\gamma^{-1}x \neq \gamma'^{-1}x$ . Since  $\gamma^{-1}x, \gamma'^{-1}x \in B'$ , we conclude that every  $x \in A'$  is contained in at most  $|V(B')|$  translates of  $B'$  by  $\Gamma$ . Hence there are at most  $|V(A')||V(B')|$  translates of  $B'$  that intersect  $A'$ . Since  $G$  has at least cubic growth, so there is a constant  $\alpha > 0$  such that, the ball of radius  $(n + m)^{3/4}$  around  $o$  contains at least  $\alpha(n + m)^{9/4}$  points, which is greater than  $|V(A')||V(B')| \leq d^2nm$  for  $m, n$  sufficiently large. Therefore there exists a vertex  $x_{A,B}$  in this ball of radius  $(n + m)^{3/4}$  such that  $\gamma_{x_{A,B}} B'$  does not intersect  $A'$ . Fix such an  $x_{A,B}$ . Fix some path  $P(A, B)$  of minimal length between  $A$  and  $x_{A,B}$ , denote its length by  $|P(A, B)|$ . By the choice of  $x_{A,B}$  we have  $|P(A, B)| \leq (n + m)^{3/4}$ . Taking  $\gamma := \gamma_{x_{A,B}}^{-1}$  completes the proof. ■

*Proof of Theorem 1:* For graphs of linear or quadratic growth, the theorem follows from Lemma 5.

Assume now that our group has at least cubic growth and so the ball of radius  $r$  has volume  $\geq \alpha r^3$  with some  $\alpha > 0$  by the facts about Cayley graphs that we mentioned earlier. Fix  $\Gamma$  as in the proof of the previous lemma.

The generalized subadditive limit theorem (see Theorem II.6 in the Appendix of [13]) gives the result if we can show that

$$\pi_{m+n} \geq \pi_m \pi_n c^{(m+n)^{3/4} \log_2(m+n)} \quad (13)$$

whenever  $m$  and  $n$  are sufficiently large, where  $0 < c = c(d, p) < 1$  is some constant depending only on  $d$  and  $p$ .

We will first show that

$$\begin{aligned} & 2^{(m+n)^{3/4}((1+d) \log_2(m+n) + c_1(d))} \pi_{m+n} \\ & \geq \sum_{A \in \mathcal{A}_n} \sum_{B \in \mathcal{A}_m} \mathbb{P}_p(C = A) \mathbb{P}_p(C_{x_{A,B}} = \gamma_{x_{A,B}}(B)) p^{(n+m)^{3/4}} (1-p)^{2d(n+m)^{3/4}} \end{aligned}$$

where  $c_1(d)$  is a constant depending only on  $d$ . We will then show that the theorem will follow easily from here.

To prove the above inequality let  $A \in \mathcal{A}_n$ ,  $B \in \mathcal{A}_m$ . Define  $x_{A,B}$  and  $\gamma_{x_{A,B}}$  as in the proof of Lemma 6. Let  $U(A, B)$  be defined as the union of three graphs:  $U(A, B) := A \cup \gamma_{x_{A,B}} B \cup P(A, B)$ . Fix some arbitrary  $\tilde{X}(A, B) \subset U(A, B)$  set of vertices not containing  $o$  such that the subgraph  $K(A, B) := U(A, B) \setminus \tilde{X}(A, B)$  is connected and  $|V(K(A, B))| = n + m$ . Then let  $X(A, B)$  be the subgraph of  $U(A, B)$  consisting of the edges incident to some element of  $\tilde{X}(A, B)$ .

For fixed  $A \in \mathcal{A}_n$  and  $B \in \mathcal{A}_m$  we obtain

$$\begin{aligned} & \mathbb{P}_p(C = K(A, B)) \\ & \geq \mathbb{P}_p(C = A) \mathbb{P}_p(C_{x_{A,B}} = \gamma_{x_{A,B}}(B) \mid C = A) p^{|P(A,B)|} (1-p)^{2d|P(A,B)|} \end{aligned}$$

by first opening the edges of  $P(A, B)$ , closing the other edges incident to the inner vertices of  $P(A, B)$  but not in  $A \cup \gamma_{x_{A,B}}(B)$ , and finally closing every edge incident to some element of  $X(A, B)$ , whenever it is necessary. The events  $\{C = A\}$  and  $\{C_{x_{A,B}} = \gamma_{x_{A,B}}(B)\}$  are independent because they are determined by disjoint sets of edges, since  $x_{A,B}$  was chosen such that  $A'$  and  $\gamma_{x_{A,B}}(B')$  are disjoint. Hence the previous inequality can be rewritten as

$$\begin{aligned} & \mathbb{P}_p(C = K(A, B)) \\ & \geq \mathbb{P}_p(C = A) \mathbb{P}_p(C_{x_{A,B}} = \gamma_{x_{A,B}}(B)) p^{(m+n)^{3/4}} (1-p)^{2d(m+n)^{3/4}} \quad (14) \end{aligned}$$

also using  $|P(A, B)| \leq (n+m)^{3/4}$ .

Now we will show that a given  $K \in \mathcal{A}_{m+n}$  can be equal to  $K(A, B)$  for at most  $2^{(m+n)^{3/4}((1+d)\log_2(m+n)+c_1(d))}$  pairs  $(A, B)$ , where  $c_1(d)$  is a constant depending only on  $d$ . First, given  $m$  and  $n$ ,  $U(A, B)$  determines  $(A, B)$  up to a factor

$$2^{(\log_2(m+n)+1+\log_2 d)(m+n)^{3/4}} m.$$

This is because of the following reason. An upper bound for the number of choices for the edges of  $P(A, B) \setminus (A \cup \gamma_{x_{A,B}} B)$  from  $U(A, B)$  is

$$|E(U(A, B))|^{(m+n)^{3/4}} \leq (2d(m+n))^{(m+n)^{3/4}} = 2^{(\log_2(m+n)+1+\log_2 d)(m+n)^{3/4}},$$

using  $|P(A, B)| \leq (n+m)^{3/4}$  and  $|E(U(A, B))| \leq (n+m+(n+m)^{3/4})d$ . If we delete the edges of  $P(A, B) \setminus (A \cup \gamma_{x_{A,B}} B)$  from  $U(A, B)$ , we get back  $A \cup \gamma_{x_{A,B}} B$ . This has two components, so one of them is  $A$  and the other one is  $\gamma_{x_{A,B}} B$ . The set  $\gamma_{x_{A,B}} B$  may coincide for at most  $|V(B)| = m$  many different  $B$ 's (all being  $\Gamma$ -translates of  $\gamma_{x_{A,B}} B$  to  $o$ , using again the choice of  $\Gamma$ ). We conclude that the number of  $(A, B)$  pairs that give the same  $U(A, B)$  is at most  $2^{(m+n)^{3/4}(\log_2(m+n)+1+\log_2 d)} m$ . Now,  $X(A, B)$  is  $U(A, B) \setminus E(K(A, B))$  without its isolated points (points of degree 0) and so for a given  $K \in \mathcal{A}_{m+n}$ ,

$$\begin{aligned} & |\{(A, B) : K(A, B) = K\}| \\ & \leq 2^{(m+n)^{3/4}(\log_2(m+n)+1+\log_2 d)} m |\{U(A, B) : K(A, B) = K\}| \\ & = 2^{(m+n)^{3/4}(\log_2(m+n)+1+\log_2 d)} m |\{X(A, B) : K(A, B) = K\}|. \end{aligned}$$

We will bound the cardinality of the set on the right side, with this fixed  $K$ . Given  $A$  and  $B$ ,  $X(A, B)$  is such a graph that the union  $K(A, B) \cup X(A, B)$  is connected, and  $|V(X(A, B))| \leq d(n+m)^{3/4}$  (since  $X$  is contained in the 1-neighborhood of  $\tilde{X}$ , and  $|\tilde{X}| \leq (n+m)^{3/4}$ ). To find an upper bound for the number of possible  $X(A, B)$ 's with these two properties (and hence where possibly  $K(A, B) = K$ ), we first specify the vertices of  $K(A, B)$  that are also in  $X(A, B)$  (at most  $\binom{n+m}{d(n+m)^{3/4}}$  possibilities). If  $X(A, B)$  has  $k$  components, with some arbitrary fixed ordering of the vertices of  $G$ , let  $x_i$  be the first element of  $K(A, B) \cap X(A, B)$  in the  $i$ 'th component. Then for each  $x_i$  choose the size of the component of  $X(A, B)$  that contains it. There are at most  $2^{d(n+m)^{3/4}+1}$  total ways to do this because the number of ways to express an integer  $k$  as the ordered sum of positive integers (which would be representing the sizes of the different components) is at most  $2^k$  and then we can sum this up from 1 to  $d(n+m)^{3/4}$  corresponding to the different possible sizes for the vertex size of  $X(A, B)$ . Next, we finally choose the components themselves. It is known that the number of lattice animals on  $\ell$  vertices is at most  $7^{2d\ell}$  (see (4.24) in [13]) which gives us a total bound of  $7^{2d^2(n+m)^{3/4}}$  for the number of ways to choose all the components. Note that we did not have to choose  $x_i$ , since  $x_i$  is determined by  $X(A, B) \cap K(A, B)$  as soon as we know the components of the  $x_j$  for all  $j < i$ . Calculations similar to the above can be found in [28]. We obtain an upper bound of

$$2^{(m+n)^{3/4}(\log_2(m+n)+1+\log_2 d)} m \binom{n+m}{d(n+m)^{3/4}} 2^{d(n+m)^{3/4}+1} 7^{2d^2(n+m)^{3/4}}$$

for the number of all possible pairs  $(A, B)$  that define the same  $K = K(A, B)$  for some connected subgraph  $K$  with  $n+m$  vertices, whenever  $m$  and  $n$  are not too small. Bounding the binomial coefficient by  $(n+m)^{d(n+m)^{3/4}}$ , it is easy to see that this is at most  $2^{(m+n)^{3/4}((1+d)\log_2(m+n)+c_1(d))}$  for some constant  $c_1(d)$ .

Since every  $K(A, B)$  is in  $\mathcal{A}_{m+n}$ , the first inequality below follows from this last estimation on the overcount. The second one is a consequence of (14)

$$\begin{aligned} & 2^{(m+n)^{3/4}((1+d)\log_2(m+n)+c_1(d))} \pi_{m+n} \\ \geq & \sum_{A \in \mathcal{A}_n} \sum_{B \in \mathcal{A}_m} \mathbb{P}_p(C = K(A, B)) \\ \geq & \sum_{A \in \mathcal{A}_n} \sum_{B \in \mathcal{A}_m} \mathbb{P}_p(C = A) \mathbb{P}_p(C_{x_{A,B}} = \gamma_{x_{A,B}}(B)) p^{(n+m)^{3/4}} (1-p)^{2d(n+m)^{3/4}} \\ \geq & \pi_n \pi_m \beta^{(n+m)^{3/4}} \\ \geq & \pi_n \pi_m \beta^{(n+m)^{3/4} \log_2(m+n)}, \end{aligned}$$

where  $\beta := p(1-p)^{2d} \in (0, 1)$ , whenever  $m$  and  $n$  are large enough. This yields Equation (13) with an appropriate choice of  $c(d, p)$  as desired and proves the theorem.  $\blacksquare$

**Remarks:** The following claim seems intuitively clear, but ‘‘continuity’’ argu-

ments that work for  $\mathbb{Z}^d$  (or more generally, for so-called ordered groups) fail for arbitrary groups. If it were true, then the proof of Theorem 1 would become significantly simpler: the subadditive theorem could be applied almost right away.

**Question 3** *Let  $G$  be a transitive graph. Is there a constant  $c$  depending on  $G$  such that for any finite subgraphs  $A$  and  $B$  there is an automorphism  $\gamma$  such that  $\gamma A$  and  $B$  are disjoint and at distance  $c$  from each other?*

Our Lemma 6 only shows that there exists a  $\gamma$  such that  $A$  and  $\gamma B$  are at distance  $\leq (|A| + |B|)^{3/4}$ . As observed by Iva Kozáková (personal communication), one cannot have a positive answer to Question 3 with  $c = 1$  for all groups. An example showing this is the free product of a cycle of length 3 and a cycle of length 4.

It is worth noting that for a Cayley graph of a so-called *ordered group*, the proof of Theorem 1 is rather straightforward. This is primarily because of the remarks made above. In this case the proof is really a generalization of the proof for  $\mathbb{Z}^d$ . Interesting enough one can also show that on the infinite regular tree with degree 3 (with is not a Cayley graph of an ordered group) such an argument does not work. Still, Theorem 1 holds of course for it and there is in fact an affirmative answer to Question 3 in this case.

## 4 Sub-exponential decay for certain transitive amenable graphs in the supercritical regime

While Question 1 and Conjecture 1 propose a characterization of amenability via cluster size decay in the supercritical regime (assuming, for completeness, the widely believed conjecture [9], that  $p_c < 1$  whenever  $G$  grows faster than linear), a conjecture of Pete suggests that this sharp contrast vanishes from a slightly different point of view. Instead of the size of the cluster, consider the size of its boundary. It is known from Kesten and Zhang [21] that when  $G = \mathbb{Z}^d$ , for all  $p > p_c$ , there exists a  $k$  such that the probability that the exterior boundary of the  $k$ -closure (see Definition 7) of a finite supercritical cluster has size  $\geq n$  decays exponentially in  $n$ . (This is not true without taking the closure, as also shown in [21] for  $p \in (p_c, 1 - p_c)$ .) This led Pete to conjecture that for any transitive graph and supercritical  $p$ , there exists a constant  $k = k(p)$  such that  $\mathbb{P}(n < |\partial_k^+ C(o)| < \infty) \leq \exp(-cn)$ , where  $\partial_k^+ C(o)$  denotes the exterior boundary of the  $k$ -closure of the cluster of  $o$ . See [25] for applications.

Before starting on the proof of Theorem 3, we prove the following (technical) lemma which will be needed in the proof.

**Lemma 7** *Let  $G$  be an amenable Cayley graph. Then there is a sequence  $\{W_n\}_{n \geq 1}$  of subsets of  $\mathcal{V}$  such that for every  $n \geq 1$  the induced graph on  $W_n$  is connected and*

$$\lim_{n \rightarrow \infty} \frac{|\partial W_n|}{|W_n|} = 0. \quad (15)$$

Moreover,  $\sup_n \frac{|W_{n+1}|}{|W_n|} < \infty$ .

*Proof:* For groups of linear or quadratic growth, define  $W_n$  to be the ball of radius  $n$  and it is immediate. (In fact, for all groups of polynomial growth, the (nontrivial) facts we mentioned earlier concerning them implies that we can take  $W_n$  to be the ball of radius  $n$  in these cases as well.)

We now assume that the group has at least cubic growth rate. Since  $G$  is amenable, there exists a sequence  $\{W_n\}_{n \geq 1}$  of nonempty finite subsets of  $\mathcal{V}$  such that for every  $n \geq 1$  the induced subgraph on  $W_n$  is connected and satisfies equation (15). (In the definition of amenability, the  $W_n$ 's are not necessarily connected, but it is easy to check that they may be taken to be.) Without loss of generality, we can also assume  $|W_n| \leq |W_{n+1}|$ .

Now, whenever  $|W_{n+1}|/|W_n| > 3$  we will add a new set  $E$  in the Følner sequence, after  $W_n$ , with the property that  $E$  is connected, that  $|E|/|W_n| \leq 3$ , and such that  $|\partial E|/|E| \leq |\partial W_n|/|W_n| + 2d/|W_n|^{1/4}$ . The lemma then can be proved by repeating this procedure as long as there are two consecutive sets in the sequence whose sizes have ratio greater than 3.

So all what is left, is to show the existence of such an  $E$ . Now, apply Lemma 6 with  $A$  and  $B$  both chosen to be  $W_n$ . Take the union of  $A$ ,  $\gamma B$ , and the path of length  $\leq (|A| + |B|)^{3/4}$  between  $A$  and  $\gamma(B)$ . Let the resulting graph be  $E$ . Clearly  $E$  satisfies the condition about its size. It also satisfies the isoperimetric requirement, because  $|\partial E| \leq 2|\partial W_n| + (2d|W_n|)^{3/4}$  and  $|E| \geq 2|W_n|$ , where  $d$  is the degree of a vertex in  $G$ . This completes the proof. ■

*Proof of Theorem 3:* Let  $\{W_n\}_{n \geq 1}$  be a sequence of subsets of  $\mathcal{V}$  satisfying the conditions of Lemma 7.

For a finite set  $W \subseteq V(G)$ , let  $\partial_{\text{Ext}} W$  be the set of  $v \in \partial W$  for which there exists a path from  $v$  to  $\infty$  which lies (other than  $v$ ) in  $V(G) \setminus (W \cup \partial W)$ . It is easy to see that if the induced graph on  $W$  is connected, then for any vertex  $w \in W$  the set  $\partial_{\text{Ext}} W$  is a *minimal cutset* between  $w$  and  $\infty$ . From [6, 28] we know that, since we are assuming the graph  $G$  is a Cayley graph of a finitely presented group with one end, there exists a positive integer  $t_0$ , such that any minimal cutset  $\Pi$  between any vertex  $v$  and  $\infty$  must satisfy

$$\forall A, B \text{ with } \Pi = A \cup B, \text{dist}_G(A, B) \leq t_0. \quad (16)$$

Letting  $U^t := \left\{ v \in V(G) \mid \text{dist}_G(v, U) \leq t \right\}$  for any  $U \subseteq V(G)$ , and  $t \in \mathbb{N}$ , it is not hard to deduce from the above that for any connected finite subset of vertices  $W$ , we have that the induced subgraph on  $(\partial_{\text{Ext}} W)^{t_0}$  is connected. In particular, it follows that for each  $n \geq 1$  the induced graph on  $(\partial_{\text{Ext}} W_n)^{t_0}$  is connected, and further by using (15) we get

$$\lim_{n \rightarrow \infty} \frac{|(\partial_{\text{Ext}} W_n)^{t_0}|}{|W_n|} = 0. \quad (17)$$

Now the proof by Aizenman, Delyon and Souillard [2] as given in [13] (see page 218), essentially goes through when we replace a “ $n$ -ball” of  $\mathbb{Z}^d$  by  $W_n$ ,

and the “boundary of a  $n$ -ball” by  $(\partial_{\text{Ext}} W_n)^{t_0}$ , leading to the sub-exponential bound (7). The point of Lemma 7 is that we need to obtain the claim in the theorem for all  $n$ ; without Lemma 7, we could only make the conclusion for a sequence of  $n$  going to  $\infty$ . ■

**Remarks:** Note that to carry out the above proof, we do not need that (16) holds for all minimal cutsets but only for some fixed Følner sequence, i.e. for a sequence of connected  $W_n$ 's satisfying (15). Thus a positive answer to the following question would imply Theorem 3 for an arbitrary amenable group.

**Definition 7** *The  $k$ -closure of a graph  $G$  is defined to be the graph on the vertex set of  $G$  with an edge between two vertices if and only if their distance in  $G$  is at most  $k$ .*

Hence (16) is equivalent to saying that any minimal cutset  $\Pi$  of  $G$  is connected in the  $t_0$ -closure of  $G$ .

**Question 4** *Does every amenable graph have a Følner sequence  $\{W_n\}$  (that is, a sequence satisfying Equation (15)), such that for some  $k$  the  $k$ -closure of  $\partial W_n$  is connected for every  $n$ ?*

In [28], an example of a Cayley graph (coming from the so-called the lamplighter group) with one end, whose “usual” Følner sequence does not satisfy the above property for its minimal cutsets, is given. However, the lamplighter group is not a counterexample to Question 4 as shown by the following construction:

**Example 1** Recall that, informally, the lamplighter group  $G$  is defined as follows. An element of  $G$  is a labeling of  $\mathbb{Z}$  with labels “on” or “off”, with only finitely many on, together with one specified element of  $\mathbb{Z}$ , the position of the lamplighter. Take the element when we move the lamplighter one step to the right (corresponding to multiplication from the right by the element with all the lamps off and the lamplighter at 1), and the element when we switch the lamp where the lamplighter is (corresponding to the element when the lamplighter is in 0 and the lamp there is the only one on), as a set of generators for the right-Cayley graph that we consider now. This way we defined multiplication for any two elements. See e.g. [28] for a more formal definition.

Given  $x \in G$ , let  $\pi(x)$  be the position of the lamplighter.

To construct the desired Følner sequence  $W_n$ , let  $B_n$  be the set of elements  $x$  with  $\pi(x) \in [1, n]$  and all the lamps outside  $[1, n]$  are off. We shall add paths to  $B_n$  to get  $W_n$ , in the following way. For each element  $x$  of the inner boundary of  $B_n$  we will define a path  $P_x$ . Note that since  $x$  is on the boundary of  $B_n$ ,  $\pi(x)$  is either 1 or  $n$ . If  $\pi(x) = n$ , and if the rightmost lamp that is on is at place  $n - k$ ,  $P_x$  will be the following. Start from  $x$ , then the lamplighter moves to the  $n + k + 1$ 'th place, switches the lamp on there, then moves back to place  $n - k$ , switch the lamp, then move to  $n + k + 1$  again and switch, and then move back to  $n$ . The endpoints of  $P_x$  are in the boundary of  $B_n$  and the interior of  $P_x$  is disjoint from  $B_n$ . For those  $x$ , where  $\pi(x) = 1$ , use the above definition

but “reflected”. Finally, define a path from the point where all lamps are off and the lamplighter is in  $n$  to the one where all lamps are off and he is in 1, by sending the lamplighter to  $2n$ , switch, go to  $-n$ , switch, back to  $2n$ , switch, back to  $-n$ , switch, and then go to 1. Define  $W_n$  as the union of  $B_n$  and all the  $P_x$ , where  $x$  is some boundary point of  $B_n$ .

We only sketch the proof of that  $W_n$  is a Følner sequence with boundaries that are connected in the 2-closure. We leave it for the interested reader to fill out the details.

Look at the 2-closure  $G_2$  of  $G$ . Define a graph on the connected components of the inner boundary  $\partial B_n$  in this thickening: put an edge between two if some points of the two are connected by a path  $P_x$  defined above. One can show that the paths were defined so that this graph is connected. The boundary of a path is clearly connected in the 2-closure, and (one can show that) these path-boundaries are (basically) contained in  $\partial W_n$ . One concludes that  $W_n$  has a connected boundary in  $G_2$ .  $W_n$  is Følner, because the paths added (and hence their boundaries, and the boundary of  $W_n$ ) were constructed so as to have total length constant times  $2^n$ , while  $W_n$  has size of order  $n2^n$ .

We complete this section with the

*Proof of Proposition 4:*

Let  $\mathbb{T}_2^\rho$  be the infinite rooted binary tree, with root  $\rho$ . Consider the graph  $G$  obtained by attaching an infinite *ray* (a copy of  $\mathbb{Z}_+$ ) at the vertex  $\rho$  of the graph  $\mathbb{T}_2^\rho$ . It is easy to see that  $G$  is amenable even though  $\mathbb{T}_2^\rho$  is not, but of course is not transitive.

Observe that for the i.i.d. bond percolation on  $G$ , the critical probability  $p_c(G) = p_c(\mathbb{T}_2) = \frac{1}{2}$ . Let  $C$  be the open connected component containing the vertex  $\rho$ . Fix  $0 < p < 1$ , it is immediate that under the measure  $\mathbb{P}_p$  we have

$$|C| \stackrel{d}{=} X + Y,$$

where  $X \sim \text{Geometric}(p)$  and  $Y$  has a distribution same as the total size of a Galton-Watson branching process with progeny distribution  $\text{Binomial}(2, p)$ , and  $X$  and  $Y$  are independent.

Now for every  $p \in (0, 1)$  we know that  $X$  has an exponential tail. Moreover from classical branching process theory we know that when  $p \neq \frac{1}{2}$  we must also have an exponential tail for  $Y$  on the event  $[Y < \infty]$ . This is because when  $p < \frac{1}{2}$  the process is subcritical and we can use Lemma 8(a) (given in Section 5); and when  $p > \frac{1}{2}$ , the process is supercritical, but on the event  $[Y < \infty]$  it is distributed according to a subcritical Galton-Watson process (see Theorem I.D.3 on page 52 of [5]). These facts together prove that for every  $p \neq \frac{1}{2}$  there is a constant  $\lambda(p) > 0$ , which may depend on  $p$ , such that

$$\mathbb{P}_p(|C| = n) \leq \exp(-\lambda(p)n) \quad \forall n \geq 1. \tag{18}$$

■

## 5 A special non-transitive, nonamenable graph

In this section we will study a particular non-transitive graph, which is also nonamenable; namely, the  $d$ -dimensional integer lattice  $\mathbb{Z}^d$  with rooted regular trees planted at each vertex of it. More precisely, for each  $\mathbf{x} \in \mathbb{Z}^d$ , let  $\mathbb{T}_r^{\mathbf{x}}$  be an infinite rooted regular tree with degree  $(r + 1)$  which is rooted at  $\mathbf{x}$ . Thus each vertex of  $\mathbb{T}_r^{\mathbf{x}}$  has degree  $(r + 1)$  except for the root  $\mathbf{x}$ , which has degree  $r$ . We consider the graph

$$G := \mathbb{Z}^d \cup \left( \bigcup_{\mathbf{x} \in \mathbb{Z}^d} \mathbb{T}_r^{\mathbf{x}} \right). \quad (19)$$

Observe that  $p_c(G) = \min \{p_c(\mathbb{Z}^d), p_c(\mathbb{T}_r^{\mathbf{0}})\}$  and recall  $p_c(\mathbb{T}_r^{\mathbf{0}}) = \frac{1}{r}$ .

Before we prove Theorem 2, we note that for this particular graph  $G$  the two critical points  $p_c(\mathbb{Z}^d)$  and  $\frac{1}{r}$  play two different roles.  $p_c(\mathbb{Z}^d)$  is the critical point for the i.i.d. Bernoulli bond percolation on  $G$ , but when  $p$  is between  $p_c(\mathbb{Z}^d)$  and  $\frac{1}{r}$  the cluster size of the origin behaves like the cluster size of the origin for supercritical bond percolation on  $\mathbb{Z}^d$ . This is of course intuitively clear, because in this region the planted trees are all subcritical. On the other hand when  $p > \frac{1}{r}$  then the tree components take over and we have the exponential decay of the cluster size of the origin, conditioned to be finite (see Lemma 8 below).

The following lemma will be needed to prove Theorem 2. This result is classical in the branching process literature, for a proof see [23, 4].

**Definition 8** *we say that a nonnegative random variable has an exponential tail if there exists  $c$  so that  $P(X \geq t) \leq e^{-ct}$  for  $t \geq 1$ .*

**Lemma 8** *Consider a subcritical or critical branching process with progeny distribution  $N$ . Let  $S$  be the total size of the population starting with one individual.*

- (a) *If  $N$  has an exponential tail and the process is subcritical, then  $S$  has an exponential tail.*
- (b) *If the process is critical and  $N$  has a finite variance, then there is a constant  $c$  (depending on the distribution of  $N$ ), such that*

$$\mathbf{P}(S = n) \leq \frac{c}{n^{3/2}} \quad (20)$$

*and if the offspring distribution is non-lattice then*

$$\mathbf{P}(S = n) \sim \frac{c}{n^{3/2}}. \quad (21)$$

*Proof of Theorem 2:* (a) Let  $C_{\mathbb{Z}^d} := C \cap \mathbb{Z}^d$  and  $C_{\mathbb{T}_r^{\mathbf{x}}} := C \cap \mathbb{T}_r^{\mathbf{x}}$  for  $\mathbf{x} \in \mathbb{Z}^d$ . By definition

$$C = \bigcup_{\mathbf{x} \in C_{\mathbb{Z}^d}} C_{\mathbb{T}_r^{\mathbf{x}}}, \quad (22)$$

where the union is a disjoint union. Thus

$$|C| = \sum_{\mathbf{x} \in C_{\mathbb{Z}^d}} |C_{\mathbb{T}_{\mathbf{x}}}|. \quad (23)$$

First take  $p < p_c(\mathbb{Z}^d)$  in which case  $\mathbb{P}_p(|C| < \infty) = 1$ . Using the special structure of this particular graph  $G$ , we observe that when conditioned on the random cluster  $C_{\mathbb{Z}^d}$ , the *tree-components*,  $\{C_{\mathbb{T}_{\mathbf{x}}}\}_{\mathbf{x} \in C_{\mathbb{Z}^d}}$ , are independent and identically distributed, each being a family tree of a subcritical Galton-Watson branching process with progeny distribution Binomial( $r, p$ ). So using the representation (23) we conclude that

$$|C| \stackrel{d}{=} \sum_{j=1}^N S_j, \quad (24)$$

where  $N \stackrel{d}{=} |C_{\mathbb{Z}^d}|$  and  $(S_1, S_2, \dots)$  are i.i.d. random variables each distributed according to the total size of a subcritical Galton-Watson branching process with progeny distribution Binomial( $r, p$ ) and are independent of  $N$ . Now using Lemma 8 we obtain that each  $S_j$  has an exponential tail, and moreover from Theorem 6.75 of Grimmett [13] we know that the random variable  $N$  has an exponential tail. Then using the following (easy) lemma, whose proof is given later, we conclude that the size of the cluster  $C$  also has an exponential tail, proving the required upper bound.

**Lemma 9** *Let  $(S_j)_{j \geq 1}$  be i.i.d. non-negative random variables, which are independent of  $N$ , which is a positive integer valued random variable. Let*

$$Z := \sum_{j=1}^N S_j. \quad (25)$$

*Then if  $S_j$ 's and  $N$  have exponential tails, so does  $Z$ .*

The exponential lower bound holds trivially by observing that  $\mathbb{P}_p(n \leq |C| < \infty) \geq \mathbb{P}_p(|C_{\mathbb{Z}^d}| \geq n)$ .

Next we take  $p > \frac{1}{r}$ , in which case  $\mathbb{P}_p(|C| < \infty) < 1$ . Observe that

$$[|C| < \infty] = [|C_{\mathbb{Z}^d}| < \infty, \text{ and } |C_{\mathbb{T}_{\mathbf{x}}}| < \infty \ \forall \ \mathbf{x} \in C_{\mathbb{Z}^d}]. \quad (26)$$

Once again using the special structure of this particular graph  $G$ , we observe that when conditioned on the random cluster  $C_{\mathbb{Z}^d}$  and the event  $[|C| < \infty]$ , the *tree-components*  $\{C_{\mathbb{T}_{\mathbf{x}}}\}_{\mathbf{x} \in C_{\mathbb{Z}^d}}$  are independent and identically distributed with distribution being the same as a supercritical Galton-Watson tree with progeny distribution Binomial( $r, p$ ) conditioned to be finite. As discussed earlier in the proof of Proposition 4, it is known that any supercritical Galton-Watson branching process, conditioned to be finite, has the same distribution as that of a subcritical Galton-Watson branching process (see Theorem I.D.3 of [5]). So

again by Lemma 8 the cluster sizes  $\{|C_{\mathbb{T}_r^{\mathbf{x}}}| \}_{\mathbf{x} \in C_{\mathbb{Z}^d}}$ , conditioned on  $C_{\mathbb{Z}^d}$  and on  $|C_{\mathbb{Z}^d}| < \infty$ , have an exponential tail.

Now we will show that  $|C_{\mathbb{Z}^d}|$  also has an exponential tail under the conditional measure  $\mathbb{P}_p(\cdot | |C| < \infty)$ . For that we observe

$$\begin{aligned} \mathbb{P}_p(|C_{\mathbb{Z}^d}| \geq n | |C| < \infty) &= \frac{\mathbb{P}_p(\infty > |C_{\mathbb{Z}^d}| \geq n \text{ and } |C_{\mathbb{T}_r^{\mathbf{x}}}| < \infty \forall \mathbf{x} \in C_{\mathbb{Z}^d})}{\mathbb{P}_p(|C| < \infty)} \\ &\leq \frac{(\mathbb{P}_p(|C_{\mathbb{T}_r^{\mathbf{0}}}| < \infty))^n}{\mathbb{P}_p(|C| < \infty)}. \end{aligned}$$

Since  $\mathbb{P}_p(|C_{\mathbb{T}_r^{\mathbf{0}}}| < \infty) < 1$ ,  $|C_{\mathbb{Z}^d}|$  also has exponential tail under the conditional measure  $\mathbb{P}_p(\cdot | |C| < \infty)$ .

Thus again from the representation (23) we conclude that under the conditional measure  $\mathbb{P}_p(\cdot | |C| < \infty)$ , we have

$$|C| \stackrel{d}{=} \sum_{i=1}^{\bar{N}} \bar{S}_i, \quad (27)$$

where  $(\bar{S}_1, \bar{S}_2, \dots)$  are i.i.d. random variables with exponential tails and are independent of  $\bar{N}$  which also has an exponential tail. So finally using Lemma 9 again, we get the required upper bound.

Once again the exponential lower bound can be obtained trivially by observing

$$\begin{aligned} \mathbb{P}_p(n \leq |C| < \infty) &\geq \mathbb{P}_p(n \leq |C_{\mathbb{T}_r^{\mathbf{0}}}| < \infty, \text{ and } C_{\mathbb{Z}^d} = \{\mathbf{0}\}) \\ &= (1-p)^{2d} \mathbb{P}_p(n \leq |C_{\mathbb{T}_r^{\mathbf{0}}}| < \infty). \end{aligned}$$

(b) First we obtain the lower bound for  $p \in (p_c(\mathbb{Z}^d), \frac{1}{r})$ . We observe that  $\mathbb{P}_p(|C_{\mathbb{T}_r^{\mathbf{x}}}| < \infty) = 1$  for all  $\mathbf{x} \in \mathbb{Z}^d$ . Thus from (26) we conclude that under  $\mathbb{P}_p$  the events  $[|C| < \infty]$  and  $[|C_{\mathbb{Z}^d}| < \infty]$  are a.s. equal. So

$$\begin{aligned} \mathbb{P}_p(n \leq |C| < \infty) &= \mathbb{P}_p(|C| \geq n, |C_{\mathbb{Z}^d}| < \infty) \\ &\geq \mathbb{P}_p(n \leq |C_{\mathbb{Z}^d}| < \infty) \\ &\geq \exp(-\psi_1(p) n^{(d-1)/d}), \end{aligned}$$

where  $\psi_1(p) < \infty$  is a constant. The last inequality follows from Theorem 8.61 of Grimmett [13], but as explained therein it is much easier to derive (see page 218 of [13]).

Since the two events  $[|C| < \infty]$  and  $[|C_{\mathbb{Z}^d}| < \infty]$  are a.s. equal, we have that for every fixed  $L > 0$  such that  $n/L \in \mathbb{N}$ ,

$$\mathbb{P}_p(n \leq |C| < \infty) \leq \mathbb{P}_p\left(\frac{n}{L} \leq |C_{\mathbb{Z}^d}| < \infty\right) + \mathbf{P}\left(\sum_{j=1}^{n/L} S_j \geq n\right) \quad (28)$$

where  $(S_j)_{j \geq 1}$  are i.i.d. random variables distributed as the total size of a subcritical Galton-Watson branching process with Binomial  $(r, p)$  progeny distribution.

Now from Lemma 8 we get that  $\mu := \mathbf{E}[S_1] < \infty$  and moreover the moment generating function  $\mathfrak{M}_{S_1}(s) := \mathbf{E}[\exp(sS_1)] < \infty$  for some  $s > 0$ . Thus using the large deviation estimate Lemma 9.4 of [12] we will get an exponential upper bound for the second summand on the right hand side of (28), by choosing  $L > \mu$ .

Moreover it follows from Theorem 8.65 of Grimmett [13] that the first summand on the right hand side of (28) must satisfy an upper bound of the form

$$\mathbb{P}_p \left( \frac{n}{L} \leq |C_{\mathbb{Z}^d}| < \infty \right) \leq \exp \left( -\frac{\eta(p)}{L^{(d-1)/d}} n^{(d-1)/d} \right), \quad (29)$$

where  $\eta(p) > 0$ .

This proves the required upper bound.

(c) Finally we will prove the polynomial bounds when  $p = \frac{1}{r}$ . First to get the upper bound, we observe that for any  $0 < \beta < 1$  we have

$$\mathbb{P}_p (n \leq |C| < \infty) \leq \mathbb{P}_p (\lfloor n^\beta \rfloor \leq |C_{\mathbb{Z}^d}| < \infty) + \mathbf{P} \left( \sum_{j=1}^{\lfloor n^\beta \rfloor} \bar{S}_j \geq n \right) \quad (30)$$

where  $(\bar{S}_j)_{j \geq 1}$  are i.i.d. random variables distributed as the size of a critical Galton-Watson branching process with Binomial  $(r, \frac{1}{r})$  progeny distribution.

Now consider the second summand on the right hand side of (30),

$$\begin{aligned} \mathbf{P} \left( \sum_{j=1}^{\lfloor n^\beta \rfloor} \bar{S}_j \geq n \right) &\leq \mathbf{P} \left( \bar{S}_j \geq \frac{n}{\lfloor n^\beta \rfloor} \text{ for some } 1 \leq j \leq \lfloor n^\beta \rfloor \right) \\ &\leq n^\beta \mathbf{P} (S_1 \geq n^{1-\beta}) \\ &\leq n^\beta \frac{c'}{(n^{1-\beta})^{1/2}} = \frac{c'}{n^{\frac{1}{2} - \frac{3}{2}\beta}}, \end{aligned} \quad (31)$$

where the last inequality follows from Lemma 8(b) where  $c' \equiv c'(\beta) > 0$  is a constant.

Now once again from Theorem 8.65 of Grimmett [13] we get that the first summand on the right hand side of (30) satisfies an upper bound of the form

$$\mathbb{P}_p (\lfloor n^\beta \rfloor \leq |C_{\mathbb{Z}^d}| < \infty) \leq \exp \left( -\frac{1}{2} \eta(p) n^{\beta(d-1)/d} \right), \quad (32)$$

where  $\eta(p) > 0$ . Now for fixed  $\varepsilon > 0$  we take  $\beta = \frac{2}{3}\varepsilon > 0$ , then the required upper bound follows using (30), (31) and (32).

Finally to get the lower bound, we note that as in case (b), we also have  $|C_{\mathbb{T}_x^*}| < \infty$  a.s. with respect to  $\mathbb{P}_p$ , for all  $\mathbf{x} \in \mathbb{Z}^d$  and so  $[|C| < \infty]$  and

$[|C_{\mathbb{Z}^d}| < \infty]$  are a.s. equal. Thus

$$\begin{aligned} \mathbb{P}_p(n \leq |C| < \infty) &= \mathbb{P}_p(|C| \geq n, |C_{\mathbb{Z}^d}| < \infty) \\ &\geq \mathbb{P}_p(|C_{\mathbb{Z}^d}| < \infty \text{ and } |C_{\mathbb{T}_r^{\mathbf{o}}}| \geq n) \\ &= (1 - \theta_{\mathbb{Z}^d}(p)) \mathbb{P}_p(|C_{\mathbb{T}_r^{\mathbf{o}}}| \geq n) \\ &= \frac{c''}{n^{1/2}} \end{aligned}$$

where  $c'' > 0$  is a constant. The last equality follows from Lemma 8(b).  $\blacksquare$

**Remark:** The above theorem does not cover the case  $p = p_c(\mathbb{Z}^d)$  and for that we would need exact tail behavior of the cluster-size distribution for critical i.i.d. bond percolation on  $\mathbb{Z}^d$ . Unfortunately, except for  $d = 2$  (see Theorem 11.89 of [13]) and for large  $d$  (see [18]), such results are largely unknown.

We now provide a proof of Lemma 9 which is presumably well known.

*Proof of Lemma 9:* By assumption, there exists  $\gamma > 1$  such that the generating function  $\phi_N(s) := \mathbf{E}[s^N] < \infty$  for all  $s < \gamma$ . Similarly, there exists  $c > 0$ , such that the moment generating function  $\mathfrak{M}_{S_1}(\lambda) := \mathbf{E}[\exp(\lambda S_1)] < \infty$  for all  $\lambda < c$ . By the Lebesgue dominated convergence theorem,  $\mathfrak{M}_{S_1}(\lambda) \rightarrow 1$  as  $\lambda \downarrow 0$  and so we can find  $\lambda_0 > 0$  such that  $1 \leq \mathfrak{M}_{S_1}(\lambda_0) < \gamma$ .

Now by definition (25), the moment generating function of  $Z$  is given by

$$\mathfrak{M}_Z(s) = \phi(\mathfrak{M}_{S_1}(s)) .$$

So in particular  $\mathfrak{M}_Z(\lambda_0) < \infty$ . Then by Markov inequality we get

$$\mathbf{P}(Z > z) \leq \mathfrak{M}_Z(\lambda_0) \exp(-\lambda_0 z) ,$$

which completes the proof.  $\blacksquare$

The following theorem covers the case when  $p_c(\mathbb{Z}^d) > \frac{1}{r}$ , in which case  $p_c(G) = \frac{1}{r}$ . It is not surprising that the intermediate regime of sub-exponential decay does not appear in this case.

**Theorem 10** *Suppose  $p_c(\mathbb{Z}^d) > \frac{1}{r}$  and let  $C$  be the open connected component of the origin  $\mathbf{o}$  of  $G$ .*

(a) *For  $p \neq \frac{1}{r}$  we have*

$$\exp(-\nu_1(p)n) \leq \mathbb{P}_p(n \leq |C| < \infty) \leq \exp(-\nu_2(p)n) \quad \forall n \geq 1, \quad (33)$$

*where  $\nu_1(p) < \infty$  and  $\nu_2(p) > 0$ .*

(b) *For  $p = \frac{1}{r}$  the lower bound in equation (5) holds with the same constant  $c_1 > 0$ , and the upper bound holds for every  $\varepsilon > 0$  but with possibly a different constant  $c'_2 \equiv c'_2(\varepsilon) < \infty$ .*

*Proof:* (a) First observe that when  $p < \frac{1}{r} = p_c(G)$  then the same argument of the first part of the proof of Theorem 2(a) applies to get the upper and lower bounds. Moreover when  $p \geq p_c(\mathbb{Z}^d) > \frac{1}{r}$ , the tree components are in supercritical regime, the second part of the proof of Theorem 2(a) applies for both the upper and lower bounds.

So all remains is to prove the exponential bounds in the intermediate case when  $\frac{1}{r} < p < p_c(\mathbb{Z}^d)$ . The argument of the second part of the proof of Theorem 2(a) again goes through here, although in this case the only real difference is  $\mathbb{P}_p(|C_{\mathbb{Z}^d}| < \infty) = 1$ , which does not affect the proof.

(b) For this we follow exactly the same steps of the proof of part (c) of Theorem 2 for both the upper and lower bounds. The only difference here is in equation (32), which in this case should be

$$\mathbb{P}_p(\lfloor n^\beta \rfloor \leq |C_{\mathbb{Z}^d}| < \infty) \leq \exp(-\frac{1}{2}\eta(p)n^\beta), \quad (34)$$

where  $\eta(p) > 0$ . So we may need a different constant than  $c_2$  in the upper bound in equation (5). ■

**Remark:** Once again, the case  $p = p_c(\mathbb{Z}^d) = \frac{1}{r}$  is left open, because of similar reason as mentioned in the remark after the proof of Theorem 2. (Of course, one would be surprised if there were any  $d$  and  $r$  (other than  $d = r = 2$ ) where the above held.)

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