Negatively Reinforced Balanced Urn Models

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Notations and Few Basic Definitions

- $\mathbb{R}^+ = \{x \in \mathbb{R} | x \geq 0\}$
- $\Delta_k = \{x \in \mathbb{R}^K : \sum_{i=1}^K x_i = 1\}$
- $\mathbf{u}_i$ is the row vector of length K, with i^{th} element equal to 1 and rest all equal to 0.
- $I \equiv I_K$ is the $K \times K$-identity matrix.
- $J \equiv J_K := \mathbf{1}^T \mathbf{1}$ is the $K \times K$ matrix with all entries equal to 1.
- $\| \cdot \|_p$ denotes the $\mathcal{L}^p$ norm on $\mathbb{R}^d$ for $p \geq 1$ that is for $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$ $\|x\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}$. $\| \cdot \|$ without any subscript will denote the $\mathcal{L}^2$ norm.
- A function $f : M \to \mathbb{R}^d$ is called *Lipschitz* if for every $x, y \in M \subseteq \mathbb{R}^d$

  $$\|f(x) - f(y)\| \leq L\|x - y\|$$

  for some $0 < L < \infty$. For a Lipschitz function f, its Lipschitz constant M is defined as

  $$M := \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{\|x - y\|} \tag{0.0.1}$$

  and such functions will be referred as $Lip(M)$. The function f is called a *contraction* if $M < 1$.
- For a complex number λ, $\Re(\lambda)$ denotes the real part of λ.
- A matrix A is called *normal* if

  $$AA^T = A^T A.$$

- The exponential of a matrix A is defined as

$$e^A := \sum_{l=0}^{\infty} \frac{A^l}{l!}.$$

- For $x > 0$ and a matrix A,

$$x^A := \exp\left((\log x)A\right).$$

- $S = \{1, \cdots, K\}$ denotes the set of K colors.

- *Urn configuration* at time n is denoted by $U_n = \left(U_{n,1}, \cdots, U_{n,K}\right), \forall n \geq 0$.

- Y_n denotes the vector of *color proportions* at time n, defined as

$$Y_n := \frac{U_n}{\sum_{j=1}^{K} U_{n,j}}, \forall n \geq 0.$$

- Z_n denotes the random color selected at time $n+1$, $\forall n \geq 0$.

- For every $n \geq 0$, χ_{n+1} is the random vector of length K, with j^{th} entry equal to 1 if color j is selected at time $n+1$, and 0 otherwise, that is

$$\chi_{n+1} = \left(\mathbf{1}_{\{Z_n=j\}}\right)_{j\in S}.$$

- $N_n = (N_{n,1}, \cdots, N_{n,K})$, is the vector of color counts, that is for every $j \in S$

$$N_{n,j} := \sum_{m=1}^{n} \mathbf{1}_{\{Z_m=j\}}.$$

Chapter 1

Introduction

Various kinds of *random reinforcement models* have been of much interest in recent years [36, 58, 49, 10, 40, 59, 66, 24, 31, 35, 33, 55, 30, 28]. *Urn schemes*, which were first studied by Pólya [67], are perhaps the simplest reinforcement models, which have many applications and generalisations [67, 42, 41, 7, 9, 65, 44, 48, 49, 10, 40, 24, 25, 35, 35, 33, 57, 30, 55, 28, 18, 16, 17, 63]. In general, reinforcement models typically adhere to the structure of "*rich get richer*", which can also be termed as *positive reinforcement*. However, there have been some studies on *negative reinforcements* models in the context of percolation, such as the *forest fire*-type models from the point-of-view of *self-destruction* [73, 77, 74, 75, 76, 37, 38, 26, 27, 68, 32, 2, 1] and *frozen percolation*-type models from the point-of-view of *stagnation* [4, 11, 79, 78, 80]. For urn schemes, a type of "negative reinforcement" have been studied when balls can be thrown away from the urn, as well as, added [39, 81, 51, 52, 33, 63]. In such models, it is usually assumed that the model is *tenable*, that is, regardless of the stochastic path taken by the process, it is never required to remove a ball of a colour not currently present in the urn. Perhaps the most famous of such scheme is the *Ehrenfest urn* [39, 63], which models the diffusion of an ideal gas between two chambers of a box. There are some models without tenability, such as the *OK Corral Model* [81, 51, 52] or *Simple Harmonic Urn* in two colours [33]. Typically these are used for modelling *destructive competition*.

In recent days, there has been some work on negative reinforcements, random graphs [70, 71, 15] from a different point-of-view, where attachment probabilities of a new vertex are decreasing functions of the degree of the existing vertices. Such models have also been referred as "*de-preferential attachment*" [15] as opposed to usual *"preferential"* attachment models [19, 3]. In particular, they consider two types of random graphs namely linear de-preferential and inverse de-preferential, such that, a new vertex joins to an existing vertex with probability proportional to a linear or an inverse non-increasing weight function. They have obtained the degree distribution of such graphs and showed that the degree of a fixed vertex grows at an order of $\log n$ for the linear de-preferential random graph and $\sqrt{\log n}$ for the inverse de-preferential random graph. Motivated by this later set of works, in this thesis, we investigate some models of *negatively reinforced urn schemes*, where the selection probabilities are proportional to a decreasing function of the proportion of colours.

Negatively reinforced urn schemes are also natural models for modelling problems with resource constrains. In particular, multi-server queuing systems with capacity constrains [61, 62]. For such cases, it is desirable that at the steady state limit, all agents are having equal loads. This gives us another motivation to study negatively reinforced urn models in order to study the *load balancing problems* in resource constraint systems. In an urn model with finitely many colours if we consider urn configuration or the colour counts as a vector of load among the K different resources, then the negative reinforcement scheme with a non-increasing weight function and a fixed replacement matrix can be considered as a possible solution to the load balancing problem with finitely many resources. The objective in such problems is to attain balance between the resources in the long run. In this thesis, we show that with a negatively reinforced urn scheme such a limit is obtained under fairly general conditions on the replacement mechanism.

1.1 Model Description

In this thesis, we will only consider *balanced* urn schemes with K-colours, which we will index by the set $S := \{1, 2, \ldots, K\}$. More precisely, if $R := ((R_{i,j}))_{1 \leq i,j \leq K}$ denotes the *replacement matrix*, that is, $R_{i,j} \geq 0$ is the *number of balls of colour j to be placed in the the urn when the colour of the selected ball is i*, then for a balanced urn, all row sums of R are constant. In this case, dividing all entries by the common row total, we may assume R is a *stochastic matrix*. For simplicity, will also assume that the starting configuration $U_0 := (U_{0,j})_{1\leq j\leq K}$ is a probability distribution on the set of colours S. As we will see from the proofs of our main results, this apparent loss of generality can easily be removed.

Denote by $U_n := (U_{n,j})_{1\leq j\leq K} \in [0,\infty)^K$ the random configuration of the urn at time n. Also let $\mathcal{F}_n := \sigma(U_0, U_1, \cdots, U_n)$ be the natural filtration. We define a random variable Z_n by

$$\mathbb{P}\left(Z_n = j \,\middle|\, \mathcal{F}_n\right) \propto w\left(\frac{U_{n,j}}{\sum_{i=1}^{K} U_{n,i}}\right), \ 1 \leq j \leq K. \tag{1.1.1}$$

where $w : [0,1] \to \mathbb{R}^+$ is a *non-increasing function.* Some examples of non-increasing weight functions are:

1. Linear: $w(x) = \alpha - \beta x, \quad for \quad \alpha > 0, \beta > 0,$ (1.1.2)

2. Inverse: $w(x) = \dfrac{1}{\alpha + \beta x}, \quad for \ \alpha > 0, \beta > 0,$ (1.1.3)

3. Exponential: $w(x) = \exp(\alpha - \beta x), \quad for \quad \alpha \geq 0, \beta > 0.$ (1.1.4)

Note that, if Z_n represents the randomly chosen colour at the $(n+1)$-th draw, then starting with U_0, we can define $(U_n)_{n\geq 0}$ recursively as follows

$$U_{n+1} = U_n + R(Z_n, \cdot), \tag{1.1.5}$$

where $R(Z_n, \cdot)$ is the Z_n-th row of the replacement matrix R. Taking

$$\chi_{n+1} := (\mathbf{1}(Z_n = j))_{1\leq j\leq K}, \tag{1.1.6}$$

the equation (1.1.5) can be written in the vector notation as

$$U_{n+1} = U_n + \chi_{n+1} R. \tag{1.1.7}$$

We note here that in this thesis, all vectors will always be written as row vectors unless mentioned otherwise. We call the process $(U_n)_{n\geq 0}$, a *negatively reinforced urn scheme* with initial configuration U_0 and replacement matrix R. In this thesis, we will discuss asymptotic properties of the following two types of statistics.

1.1.1 Random Configuration of the Urn

As defined earlier $U_n := (U_{n,j})_{1\leq j\leq K} \in [0,\infty)^K$ denotes the random configuration of the urn at time n. Let

$$Y_n \coloneqq \frac{U_n}{\sum_{i=1}^K U_{n,i}} \tag{1.1.8}$$

to be the *vector of colour proportions* at time n. Note that, Y_n is a probability mass function since, and for all $n \geq 0$,

$$\sum_{i=1}^{K} U_{n,i} = n+1. \tag{1.1.9}$$

This holds because we assume that R is a stochastic matrix and U_0 is a probability vector. Thus the *proportion* of the colours at time n, is given by

$$Y_n \coloneqq \frac{U_n}{n+1}. \tag{1.1.10}$$

Further,

$$\mathbb{P}\left(Z_n = j \,\middle|\, \mathcal{F}_n\right) = \frac{w(Y_{n,j})}{S_w(Y_n)} \quad \text{for } 1 \leq j \leq K. \tag{1.1.11}$$

We define

$$\mathbf{w}(Y_n) \coloneqq (w(Y_{n,1}), \cdots, w(Y_{n,K})) \tag{1.1.12}$$

and

$$S_w(Y_n) \coloneqq \sum_{j=1}^{K} w(Y_{n,j}). \tag{1.1.13}$$

Thus, $\dfrac{\mathbf{w}(Y_n)}{S_w(Y_n)}$ is the conditional distribution of the $(n+1)$-th selected colour, namely Z_n, given $\mathcal{F}_n = \sigma\left(U_0, U_1, \cdots, U_n\right)$

1.1.2 Colour Count Statistics

Let $N_n := (N_{n,1}, \ldots, N_{n,K})$ be the vector of length K, whose j-th element is the number of times colour j was selected in the first n trials, that is

$$N_{n,j} = \sum_{m=0}^{n-1} \mathbf{1}\left(Z_m = j\right), \;\; 1 \leq j \leq K. \tag{1.1.14}$$

Note that,

$$\sum_{j=1}^{K} N_{n,j} = n, \;\; \forall\, n \geq 1 \tag{1.1.15}$$

and therefore we define the proportion of colour counts as

$$\tilde{Y}_n := \frac{N_n}{n}. \tag{1.1.16}$$

Observe that by equation (1.1.7)

$$U_{n+1} = U_0 + N_{n+1}R. \tag{1.1.17}$$

1.2 A Motivating Example: Negatively Reinforced Urn Model with Two Colours

In this section, we consider an urn model with only two colours, that is when $K = 2$. This is our starting example. Recall that,

$$P(Z_n = j|\mathcal{F}_n) \propto w(Y_{n,j}), \quad \text{for } j \; \in \; \{1, 2\}.$$

where Z_n is the colour selected at time $(n + 1)$.

1.2.1 Linear Weight Function

We first consider the linear weight function, $w(x) = \theta - x$, for $\theta \geq 1$. In this case,

$$P(Z_n = j|\mathcal{F}_n) = \frac{\theta - Y_{n,j}}{2\theta - 1}, \quad \text{for } j \in \{1, 2\}. \tag{1.2.1}$$

This model will be studied extensively in Chapter 2. For two colours the following theorem is a consequence of Theorem 2.3.1 and Theorem 2.3.2.

Theorem 1.2.1. *Consider a linear but negatively reinforced urn model $(U_n)_{n\geq 0}$ with weight function $w(x) = \theta - x$, for $\theta \geq 1$, such that $\frac{1}{2\theta-1}(\theta J - I)R$ is irreducible, then*

$$\left(\frac{U_{n,1}}{n+1}, \frac{U_{n,2}}{n+1}\right) \to (\mu, 1-\mu) \quad a.s. \tag{1.2.2}$$

where $(\mu, 1-\mu)$ is the unique stationary distribution of the stochastic matrix $\frac{1}{2\theta-1}(\theta J - I)R$, and there exists a 2×2 variance-co-variance matrix Σ, such that,

$$\frac{U_n - n\mu}{\sigma_n} \implies N_2(0, \Sigma), \tag{1.2.3}$$

where

$$\sigma_n = \begin{cases} \sqrt{n \log n} & \text{if the eigenvalues of } R \text{ are } 1 \text{ and } \lambda = \frac{1-2\theta}{2} \text{ with } \theta \in \left[1, \frac{3}{2}\right]; \\ \\ \sqrt{n} & \text{if the eigenvalues of } R \text{ are } 1 \text{ and } \lambda > \frac{1-2\theta}{2} \text{ with } \theta \in \left[1, \frac{3}{2}\right]; \text{or } \theta > \frac{3}{2}. \end{cases} \tag{1.2.4}$$

Remark 1.2.1. For $K = 2$, the matrix $\frac{1}{2\theta-1}(\theta J - I)R$ is reducible only when $\theta = 1$ and $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This case is discussed in Section 1.3.

1.2.2 General Weight Function

We begin by observing that for $K = 2$ a linear negatively reinforced model with $w(x) = \theta - x$ and the negatively reinforced model with an inverse weight function $\tilde{w}(x) = \frac{1}{\theta - 1 + x}$, will generate the same urn processes, provided the starting configurations are same. This is because

$$\begin{aligned} \frac{\tilde{w}(y)}{S_{\tilde{w}}(\mathbf{y})} &= \frac{\frac{1}{\theta-1+y}}{\frac{1}{\theta-1+y} + \frac{1}{\theta-1+(1-y)}} \\ &= \frac{\theta - y}{2\theta - 1} \\ &= \frac{w(y)}{S_w(y)} \quad \forall\, y \in [0,1]. \end{aligned} \tag{1.2.5}$$

Therefore, from equation (1.1.11) the two urn processes are same in distribution, provided their initial configurations are same. Such similarities are rare to be observed in the urn models with more than two colours. In fact, as we will see in the sequel, the negatively reinforced model with an inverse weight function is technically hard for $K = 3$ or more. In fact, in Chapter 4, we will only provide a conjecture on the asymptotics of the urn configuration.

In the next theorem, we obtain a necessary and sufficient condition under which a large class of negatively reinforced urn processes with a general weight function are same as the the linear negatively reinforced urn process for $K = 2$. In particular, as shown above the inverse weight function satisfies the required condition.

Theorem 1.2.2. *Let $K = 2$ and suppose $(U_n)_{n\geq 0}$ is a linear negative reinforcement process with weight function $w(y) = \theta - y$, for $\theta \geq 1$, and $\left(\tilde{U}_n\right)_{n\geq 0}$ is an urn process with weight function $\tilde{w}$. Then*

$$\left(\tilde{U}_n\right)_{n\geq 0} \stackrel{d}{=} (U_n)_{n\geq 0} \tag{1.2.6}$$

if and only if, $\tilde{U}_0 = U_0$ and the weight function $\tilde{w}$ is of the form:

$$\tilde{w}(y) = \frac{\xi(y)}{\theta - (1-y)}, \tag{1.2.7}$$

where $\xi : [0,1] \to \mathbb{R}^+$ is a symmetric function around $\frac{1}{2}$, that is, $\xi(y) = \xi(1-y), \ \forall\, y \in [0,1]$.

Proof: From equation (1.1.11), the two urn processes are same in distribution, if and only if, for every $y \in [0,1]$

$$\frac{\tilde{w}(y)}{\tilde{w}(y) + \tilde{w}(1-y)} = \frac{\theta - y}{2\theta - 1}$$

$\iff$

$$(\theta - (1-y))\tilde{w}(y) = (\theta - y)\tilde{w}(1-y)$$

Now let, $\xi(y) = \tilde{w}(y)(\theta - (1-y))$, to get that

$$\xi(y) = \xi(1-y). \tag{1.2.8}$$

□

1.3 Negative and Positive Reinforcements

In this section, we will observe that even with a non-increasing weight function the urn model can behave like a classical urn model, where typically the idea is "rich get richer", leading to what one may call positive reinforcement. If the replacement scheme is not in the favour of negative reinforcement, then a negative reinforcement may become a positive reinforcement, essentially because two negatives makes a positive. The following example with $K = 2$ colours, illustrates this phenomenon.

Example 1.3.1. Consider a two colour linear negatively reinforced urn process $(U_n)_{n\geq 0}$ with $\theta = 1$ and a Freedman's replacement scheme that is

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.3.1}$$

then we have

$$\mathbf{E}\left[U_{n+1}|\mathcal{F}_n\right] = U_n + \frac{U_n}{n+1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.3.2}$$

that is, the resulting urn scheme is of the Pólya urn type, and therefore the proportion of a colour converges almost surely to a random variable which follows a Beta distribution, with parameters given by the initial composition.

It is worth to note here that the above example along with the Theorem 1.2.1, covers all cases for two colours.

One sufficient condition to ensure the negative reinforcement with a non-increasing weight function is when the replacement matrix is doubly stochastic (that is its column sums are also equal to 1) and all the diagonal elements are more than $\frac{1}{2}$. In this thesis, we will study doubly stochastic cases extensively.

1.4 Applications of Negative Reinforcement

In this section, we briefly describe two possible applications of the negatively reinforced urn model.

1.4.1 Power of Two Choices

Suppose that n balls are placed into n bins sequentially, and each ball is placed in a bin chosen uniformly at random. Then the maximum load in any bin is approximately $\frac{\log n}{\log\log n}$ [43]. If instead, each ball is placed in the least loaded of $d \geq 2$ bins chosen independently and uniformly at random, then the maximum load is of order $\frac{\log\log n}{\log d}$ [8]. The important implication of this result is that having only two choices yields a large reduction in the maximum load. This phenomenon is often referred as two-choice paradigm.

Now, if we consider bins as different colours, then such a model with finitely many bins can be studied for an urn model as well. In Chapter 5, we study a softer version of this model by making weighted reinforcement, which in spirit is similar to the above mentioned two choice model. However, we still observe similar phenomenon in the weighted reinforcement version. This observation, in particular, gives a possible direction for the further applications of the negatively reinforced urn models.

1.4.2 Extension of OK Corral to Border Aggregation Model

The OK Corral process as introduced in [81], is a $\mathbb{Z}^2$ valued process $(X_n, Y_n)_{n\geq 0}$, with transition probabilities

$$P\left((X_{n+1}, Y_{n+1}) = (X_n - 1, Y_n) | (X_n, Y_n)\right) = \frac{Y_n}{X_n + Y_n}$$
$$P\left((X_{n+1}, Y_{n+1}) = (X_n, Y_n - 1) | (X_n, Y_n)\right) = \frac{X_n}{X_n + Y_n}.$$

The process starts with $X_0 = Y_0 = N$ and stops when X_n or Y_n is zero. Let

$$\tau := \inf\{n : X_n = 0 \, or \, Y_n = 0\} \quad \text{and} \quad S = X_\tau + Y_\tau.$$

Kingman and Volkov [52] showed that the OK Corral model is associated with Friedman's urn, and obtained asymptotic results including exact expression of the probability of the process ending at S.

Similar to this, we believe that the negatively reinforced model with inverse weight function for $K \geq 3$, can be used to study an extension of OK Corral model to border aggregation model [72]. However, we have only been able to give analytical proofs for negatively reinforced urn models with inverse weight function for 2 colours, simulation results presented in Chapter 4 suggest that the similar results would hold for the urn models with 3 or more colours.

1.5 Outline and Brief Sketch of the Results

A chapter wise outline of this thesis is given as follows:

1.5.1 Urn Models with Negative but Linear Reinforcement

In Chapter 2, which is based on our work in [14], we consider the *linear non-increasing weight function* as mentioned in example (see equation (1.1.2)). In case of linear weight function, we observe that the asymptotic properties of $(U_n)_{n\geq 0}$ depend on whether a new replacement matrix $\hat{R} := RA$ is irreducible or not (where A is the matrix of linear transformation). In the case when $\hat{R}$ is irreducible, we establish a coupling between the classical urn model and the linear negatively reinforced model, and obtain the almost sure convergence and the central limit theorems for the random urn configuration vector U_n and colour count statistics N_n. We also give a necessary and sufficient condition for matrix $\hat{R}$ to be irreducible. For the case when $\hat{R}$ is reducible, we mainly use martingale arguments to get almost sure convergence and the rate of convergence for a given balanced replacement matrix R.

1.5.2 Negatively Reinforced Urn Model with Lipschitz Weight Function

In Chapter 3, which is based on our work in [13], we consider general weight functions. For such models it is not possible to give a coupling with the classical urn model and also it is difficult to find a martingale for a general non-linear weight function, even for a simplest replacement rule say $R = I$. For such models, we use the method of *stochastic approximation* (*SA*) to obtain the almost sure convergence and the central limit theorem type results. The idea behind using stochastic approximation methods, is to approximate the recurrence process with the solution of an ordinary differential equation, under certain assumptions. Using SA method, we obtain almost sure convergence results and different scaling limits in the case when the replacement matrix is doubly stochastic for a general weight functions, under certain assumptions.

As an example in this chapter we show that the linear weight function case can also be analysed using stochastic approximation, but using SA method we only get asymptotics mostly for doubly stochastic replacement matrices, whereas in Chapter 2 using coupling argument and martingale technique we could obtain the asymptotics for a general replacement matrix.

1.5.3 Inverse Weight Function

In this chapter, we consider the inverse weight function for the negative reinforcement. The stochastic approximation techniques used for a general weight function in Chapter 3, do not hold for the inverse weight function, as it does not satisfy the required criterion of being Lipschitz. As we saw in the previous section, inverse weight function is equivalent to a linear weight function in case of two colours, therefore we expect the similar results to hold for $K \geq 3$ as well. In this chapter, we present simulation results which strongly suggest the convergence of urn configuration to uniform vector and asymptotic normality holds for $R = I$. Based on the simulation results, we conjecture that the almost sure convergence and central limit theorem should also hold for inverse weight function.

1.5.4 Choice of Two in Weighted Negative Reinforcement

In Chapter 5, which is based on our work in [12], we introduce a new urn model namely choice of two model. This can be considered as an easy implementation of a negative reinforcement rule in order to achieve a balanced system. In this model, at every time $n \geq 1$, two or more *colours* are chosen uniformly at random without replacement from the set of colours, and then reinforcement is done according to a non-increasing weight function w. The two fold interest in this models is to understand:

(a) the asymptotic properties of this urn model,

(b) optimal number of choices.

We obtain the limiting configuration of this urn model as a uniform vector. Further we compare the model of two choices with other models with either no choice or more than two choices, and show that two choices are enough for significant improvement in the efficiency of the model. In literature such models are also referred as the *power of two choices* algorithms which are well studied for an infinite system of balls and bins as mentioned in the survey article on *the power of two choices* [64].

1.5.5 Review of Stochastic Approximation

In Appendix A, we present the results obtained in the stochastic approximation theory, which we have used in Chapter 3 and Chapter 5. Here we mainly give results obtained in [20, 23, 82].

Chapter 2

Urn Models with Negative but Linear Reinforcement[1]

2.1 Model

The main focus in this chapter is on the linear type negatively reinforced urn models for $K \geq 2$ colors. In this chapter, we derive the almost sure convergence and central limit theorems for the urn configuration for a general class of replacement matrices. Recall that $(U_n)_{n\geq 0}$ denotes the urn configuration at time n, where $U_n = (U_{n,1}, \cdots, U_{n,K})$ and U_0 is the vector of initial configuration. Also as mentioned in Chapter 1, Z_n denotes the randomly chosen color at the $(n+1)$-th draw. Recall that the dynamics of $(U_n)_{n\geq 0}$ is given by the recursion (1.1.7), and for the linear negatively reinforced urn model, the distribution of Z_n is defined as

$$\mathbb{P}\left(Z_n = j \,\middle|\, \mathcal{F}_n\right) \propto w\left(\frac{U_{n,j}}{n+1}\right), \;\; 1 \leq j \leq K. \tag{2.1.1}$$

where we take

$$w(x) = \alpha - \beta x, \tag{2.1.2}$$

for $\alpha > 0$ and $\beta > 0$ which are considered as the parameters of the model. Note that, for every $j \in \{1, 2, \cdots, K\}$

$$\mathbb{P}\left(Z_n = j \,\middle|\, \mathcal{F}_n\right) = \frac{\alpha}{K\alpha - \beta} - \frac{\beta}{K\alpha - \beta}\frac{U_{n,j}}{n+1} \tag{2.1.3}$$

[1]This chapter is based on the paper entitled "*Linear de-preferential urn models* " accepted for publication in *Advances in Applied Probability 50.4 (December 2018)* [14].

which can also be written as:

$$\mathbb{P}\left(Z_n = j \,\middle|\, \mathcal{F}_n\right) = \frac{\theta}{K\theta - 1} - \frac{1}{K\theta - 1}\frac{U_{n,j}}{n+1}, \ \ 1 \leq j \leq K. \tag{2.1.4}$$

for $\theta = \dfrac{\alpha}{\beta} \geq 1$ and the respective weight function is given by

$$w_\theta(x) = \theta - x. \tag{2.1.5}$$

Thus, the weight function given in equation (2.1.2) with two parameters α, β, is equivalent to the weight function in equation (2.1.5), for $\theta = \dfrac{\alpha}{\beta}$. Thus, in this chapter we will only consider the weight function w_θ.

Note that, the conditional distribution of the $(n+1)$-th selected color, namely Z_n, given $\mathcal{F}_n$, is given by $\dfrac{U_n A}{n+1}$, where A is a $K \times K$ matrix given by

$$A = \frac{\theta}{K\theta - 1} J - \frac{1}{K\theta - 1} I. \tag{2.1.6}$$

We begin by observing the following fact which follows from equations (1.1.7), (2.1.4) and (2.1.6)

$$\mathbf{E}\left[U_{n+1} \,\middle|\, \mathcal{F}_n\right] = U_n + \mathbf{E}\left[\chi_{n+1} \,\middle|\, \mathcal{F}_n\right] R = U_n + \frac{U_n}{n+1} AR. \tag{2.1.7}$$

Thus,

$$\mathbf{E}\left[U_{n+1}A \,\middle|\, \mathcal{F}_n\right] = U_n A + \frac{U_n}{n+1} ARA. \tag{2.1.8}$$

Let $\hat{U}_n := U_n A, n \geq 0$, then

$$\hat{U}_{n+1} = \hat{U}_n + \chi_{n+1} RA, \tag{2.1.9}$$

and

$$\mathbf{E}\left[\hat{U}_{n+1} \,\middle|\, \mathcal{F}_n\right] = \hat{U}_n + \frac{\hat{U}_n}{n+1} RA.$$

Therefore $\left(\hat{U}_n\right)_{n\geq 0}$ is a classical urn scheme (uniform selection of balls, as in the classical Pólya type urns), with replacement matrix RA. The construction $\left(\hat{U}_n\right)_{n\geq 0}$ is essentially a coupling of a negative but linearly reinforced urn $(U_n)_{n\geq 0}$ with replacement matrix R, to a classical urn $(\hat{U}_n)_{n\geq 0}$ with replacement matrix $\hat{R}$. Note that, we get a one to one correspondence, as A is always invertible. We define the new $K \times K$ stochastic matrix as:

$$\hat{R} := RA, \tag{2.1.10}$$

where A is as defined in (2.1.6). As we state in the sequel, the asymptotic properties of the urn configuration $(U_n)_{n\geq 0}$ and the color count statistics $(N_n)_{n\geq 0}$ depends on whether the stochastic matrix $\hat{R}$, is *irreducible* or *reducible*. We first state a necessary and sufficient condition for that.

2.2 A Necessary and Sufficient Condition for $\hat{R}$ to be Irreducible

Definition 2.2.1. A directed graph $\mathcal{G} = (\mathcal{V}, \vec{\mathcal{E}})$ is called the graph associated with a $K \times K$ stochastic $R = ((R_{i,j}))_{1\leq i,j\leq K}$, if

$$\mathcal{V} = \{1, 2, \ldots, K\} \text{ and } \vec{\mathcal{E}} = \{(\overrightarrow{i,j}) | R_{i,j} > 0; i, j \in \mathcal{V}\}.$$

Definition 2.2.2. A stochastic matrix R is called a *star*, if there exists a $j \in \{1, 2, \cdots, K\}$, such that,

$$R_{i,j} = 1 \quad \text{for all } i \neq j,$$

and in that case, we say j is the central vertex.

By definition, for the graph associated with a star replacement matrix, there is a central vertex such that each vertex other than the central vertex has only one outgoing edge and that is towards the central vertex. We note that in the definition of a star we allow the central vertex to have a self loop. Graph associated with a star matrix with 5 vertices is given below.

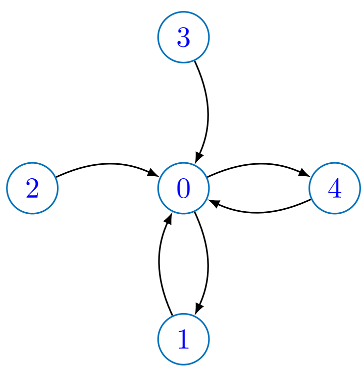


Figure 2.1: Graph of a star matrix, with 0 as the central vertex.

As we will see in the sequel, the asymptotic properties will depend on the irreducibility of the (new) stochastic matrix $\hat{R}$, as defined in (2.1.10). Following Theorem provides a necessary and sufficient condition for $\hat{R}$ to be irreducible.

Theorem 2.2.1. *Let R be a $K \times K$ stochastic matrix with $K \geq 2$, then $\hat{R}$ is irreducible, if and only if either $\theta > 1$ or $\theta = 1$ but R is not a star.*

Proof: Suppose G and $\hat{G}$ are the directed graphs associated with the matrices R and $\hat{R}$ respectively, as defined earlier. Observe that, $\hat{R}$ is the product of two stochastic matrices, R and A. The underlying Markov chain of $\hat{R}$ can be seen as a two step Markov chain where the first step is taken according to R and the second step is taken according to A. Recall from equation (2.1.10) that

$$\hat{R} = \frac{1}{K\theta - 1} (\theta J - R) . \tag{2.2.1}$$

Now, to show that the Markov chain associated with $\hat{R}$ is irreducible, it is enough to show that there exist a directed path between any two fixed vertices say u and v, in $\hat{G}$.

Clearly for $\theta > 1$, $\hat{R}_{uv} > 0$ for all u, v, and thus $\hat{R}$ is irreducible. Therefore, we only have to verify irreducibility for $\theta = 1$ case. From equation (2.2.1) we get

$$\hat{R}_{uv} = \frac{1 - R_{u,v}}{K - 1} \quad \forall u, v\{1, 2, \cdots, K\}. \tag{2.2.2}$$

To complete the proof, we will show that there is a path from any two fied vertices u and v of length at most 2. We consider the following two cases:

Case 1 $R_{u,v} < 1$**:** In this case, from equation (2.2.2) we get, $\hat{R}_{uv} > 0$. Therefore (u, v) is an edge in $\hat{G}$ and trivially there is a path of length 1 from u to v in $\hat{G}$.

Case 2 $R_{u,v} = 1$**:** In this case, u has no R-neighbor other than v, that is (u, v) is the only incoming edge to v in G and from equation (2.2.2), we have

$$\hat{R}_{uv} = 0.$$

Note that, for $\theta = 1$ and $K = 2$, $\hat{R}$ is reducible only when R is the Friedman urn scheme, which is a star with two vertices. Thus in the rest of the proof we take $K > 2$, and show that $\hat{R}^2_{uv} > 0$, that is there is a path of length 2.

Now, if R is not a star then there must exists a vertex l such that it leads to a vertex other than the central vertex, say m that is $R_{l,m} > 0$ $(m \neq v)$.

Now, according to $\hat{R}$ chain, there is a positive probability of going from u to l in one step (first take a R-step from u to v which happens with probability 1 is this case, as $R_{u,v} = 1$, and then take a A-step to l with probability $1/(K-1)$) and a positive probability of going from l to v in one step (first take a R-step from l to m with probability $R_{l,m}$, and then take a A-step to v with probability $1/(K-1)$). Therefore, there is path of length two in $\hat{G}$ from u to v and thus the chain is irreducible.

□

2.3 Asymptotics of the Random Configuration of the Urn

2.3.1 $\hat{R}$ is Irreducible.

Theorem 2.3.1. *Let $\hat{R}$ be irreducible. Then, for every starting configuration U_0,*

$$\frac{U_{n,j}}{n+1} \longrightarrow \mu_j, \quad a.s. \quad \forall\, 1 \leq j \leq K, \tag{2.3.1}$$

where μ is the unique solution of the following matrix equation

$$(\theta \mathbf{1} - \mu)\, R = (K\theta - 1)\, \mu. \tag{2.3.2}$$

Remark 2.3.1. Notice that, $\nu = \mu A$ is the unique solution of the matrix equation $\nu \hat{R} = \nu$. Also note that from equation (2.3.2) we get, $\mu = \nu R$.

Remark 2.3.2. Since $\frac{U_{n,j}}{n+1}$ is a bounded random variable, we get

$$\frac{\mathbf{E}\left[U_{n,j}\right]}{n+1} \longrightarrow \mu_j, \quad a.s., \quad \forall\, 1 \leq j \leq K, \tag{2.3.3}$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_K)$, is given in equation (2.3.2).

Remark 2.3.3. It is worth to note here that, the stochastic matrices R and $\hat{R}$ both have uniform distribution as their unique stationary distribution, if and only if, R is doubly stochastic, that is when $\mathbf{1}R = \mathbf{1}$.

Proof of Theorem 2.3.1 : Recall that, $\hat{U}_n = U_n A$ is the configuration of a classical urn model with replacement matrix $\hat{R}$. Since by our assumption, $\hat{R}$ is irreducible therefore by Theorem 2.2. of [10], the limit of $\frac{1}{n+1}\hat{U}_n$ is the normalized left eigenvector of $\hat{R}$ associated with the maximal eigenvalue 1. That is

$$\frac{\hat{U}_n}{n+1} \longrightarrow \nu, \; a.s.$$

where ν satisfies

$$\nu \hat{R} = \nu.$$

Since $U_n = \hat{U}_n A^{-1}$, we have

$$\frac{U_n}{n+1} \longrightarrow \mu, \quad a.s.,$$

where $\mu = \nu A^{-1}$, and it satisfies the following matrix equation:

$$(\theta \mathbf{1} - \mu) R = (K\theta - 1) \mu.$$

This completes the proof. □

Theorem 2.3.2. *Suppose $\hat{R}$ is irreducible then there exists a $K \times K$ variance-co-variance matrix Σ, such that,*

$$\frac{U_n - n\mu}{\sigma_n} \implies N_K(0, \Sigma), \tag{2.3.4}$$

where for $K \geq 3$,

$$\sigma_n = \begin{cases} \sqrt{n \log n} & \text{if } K = 3, \theta = 1 \text{ and one of the eigenvalue of } R \text{ is } -1, \\ \\ \sqrt{n} & \text{otherwise.} \end{cases} \tag{2.3.5}$$

and for $K = 2$

$$\sigma_n = \begin{cases} \sqrt{n \log n} & \text{if the eigenvalues of } R \text{ are } 1 \text{ and } \lambda = \frac{1-2\theta}{2} \text{ with } \theta \in \left[1, \frac{3}{2}\right]; \\ \\ \sqrt{n} & \text{if the eigenvalues of } R \text{ are } 1 \text{ and } \lambda > \frac{1-2\theta}{2} \text{ with } \theta \in \left[1, \frac{3}{2}\right]; \text{or } \theta > \frac{3}{2}. \end{cases} \tag{2.3.6}$$

Remark 2.3.4. Note that Σ is a *positive semi-definite* matrix because of (1.1.9).

Proof of Theorem 2.3.2: Let $1, \lambda_1, \ldots \lambda_s$ be the distinct eigenvalues of R, such that, $1 \geq \Re(\lambda_1) \geq \cdots \geq \Re(\lambda_s) \geq -1$, where $\Re(\lambda)$ denotes the real part of the eigenvalue λ. Recall from equation (2.1.10) that $\hat{R} = \dfrac{1}{K\theta - 1} (\theta J - R)$. So the eigenvalues of $\hat{R}$ are $1, b\lambda_1, \cdots, b\lambda_s$, where $b = \dfrac{-1}{K\theta - 1}$. Let $\tau = \max\{0, b\Re(\lambda_s)\}$. Since $\hat{U}_n = U_n A$, is a classical urn scheme with replacement matrix $\hat{R}$, using Theorem 3.2 of [10], if

$$b\Re(\lambda_s) \leq \frac{1}{2} \tag{2.3.7}$$

then there exists a variance-co-variance matrix $\hat{\Sigma}$, such that

$$\frac{\hat{U}_n - n\nu}{\sigma_n} \implies N\left(0, \hat{\Sigma}\right)$$

where

$$\sigma_n = \begin{cases} \sqrt{n \log n} & \text{if } b\Re(\lambda_s) = \frac{1}{2}, \\ \\ \sqrt{n} & \text{if } b\Re(\lambda_s) < \frac{1}{2}. \end{cases} \tag{2.3.8}$$

Since $b \neq 0$,

$$b\Re(\lambda_s) \leq \frac{1}{2} \iff \Re(\lambda_s) \geq \frac{-1}{2}(K\theta - 1). \tag{2.3.9}$$

Now since $\theta \geq 1$ and $\Re(\lambda_s) \geq -1$ the above equation (2.3.9) holds whenever $K \geq 3$. Further, for $K \geq 3$, equality in (2.3.9) holds if and only if, $\theta = 1$, and $K = 3$. Moreover, for $K = 2$, the condition is equivalent to $\Re(\lambda_s) \geq \dfrac{1-2\theta}{2}$. Thus, σ_n is given in (2.3.5) and (2.3.6) Therefore,

$$\frac{U_n - n\mu}{\sigma_n} \implies N(0, \Sigma)$$

where $\Sigma = A^T \Sigma' A$. □

Remark 2.3.5. In the above theorem for 2 colours, we give central limit theorem only for case when $\theta > \frac{3}{2}$ and when $\theta \in \left[1, \frac{3}{2}\right]$ and $\lambda \geq \frac{1-2\theta}{2}$ The case when $\theta \in \left[1, \frac{3}{2}\right]$ and $\lambda < \frac{1-2\theta}{2}$ is covered in Chapter 3, but only for a doubly stochastic relacement matrix R.

2.3.2 $\hat{R}$ is Reducible.

By Theorem 2.2.1, we know that $\hat{R}$ can be reducible, if and only if, R is star and $\theta = 1$. Suppose R is a star with $K \geq 2$ colors, then without any loss of generality we can write

$$R = \begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_K \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \quad \text{with} \quad \sum_{j=1}^{K} \alpha_j = 1, \text{ and } \alpha_j \geq 0, \ \forall j, \tag{2.3.10}$$

by taking 1 as the central vertex. Taking $\theta = 1$, the matrix $\hat{R}$ is

$$\hat{R} = \frac{1}{K-1} \begin{pmatrix} 1-\alpha_1 & 1-\alpha_2 & \dots & 1-\alpha_K \\ 0 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \dots & 1 \end{pmatrix}, \tag{2.3.11}$$

which is clearly reducible. A directed graph associated with $\hat{R}$ for $K = 3$ is given below

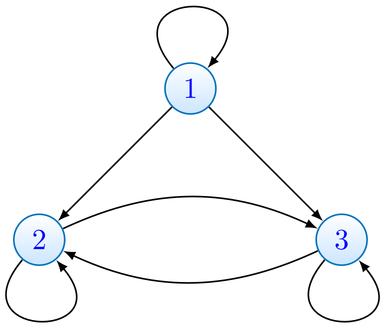


Figure 2.2: Graph of $\hat{R}$ when R is a star and 1 is the central vertex.

Clearly $\hat{R}$ is reducible as the center vertex 1 can not be reached from any other vertex of the graph. In the next theorem, we describe the limit of the urn configuration.

Theorem 2.3.3. *Let $\theta = 1$ and replacement matrix R be a star matrix as given in equation* (2.3.10) *and $R \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then,*

$$\frac{U_{n,1}}{n+1} \longrightarrow 1, \quad a.s. \tag{2.3.12}$$

Further, there exists a random variable $W \geq 0$, with $\mathbf{E}[W] > 0$, such that,

$$\frac{U_{n,j}}{n^{\gamma}} \longrightarrow \frac{\alpha_j}{K-1} W, \quad a.s. \ \forall j = 2, 3 \ldots, K, \tag{2.3.13}$$

where $\gamma = \dfrac{1-\alpha_1}{K-1} < 1$.

Remark 2.3.6. The case when $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\theta = 1$ has been studied in Section 1.3 of Chapter 1.

Proof of Theorem 2.3.3 : Since the matrix $\hat{R}$, as given in (2.3.11) is reducible without isolated blocks. Using Proposition 4.3 of [44] we get,

$$\frac{\hat{U}_{n,1}}{n+1} \to 0 \text{ and } \frac{\hat{U}_{n,j}}{n+1} \to \frac{1}{K-1}, \ \forall j \neq 1.$$

which implies

$$\frac{U_{n,1}}{n+1} \to 1 \text{ and } \frac{U_{n,j}}{n+1} \to 0, \ \forall j \neq 1.$$

Now recall that equation (2.1.7) provides the recursion:

$$\mathbf{E}\left[U_{n+1}\,\middle|\,\mathcal{F}_n\right] = U_n + \mathbf{E}\left[\chi_{n+1}\,\middle|\,\mathcal{F}_n\right] R = U_n + \frac{U_n}{n+1} AR.$$

Notice that, in this case, the matrix AR is given by

$$AR = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1-\gamma & \frac{\alpha_2}{K-1} & \frac{\alpha_3}{K-1} & \cdots & \frac{\alpha_K}{K-1} \\ 1-\gamma & \frac{\alpha_2}{K-1} & \frac{\alpha_3}{K-1} & \cdots & \frac{\alpha_K}{K-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-\gamma & \frac{\alpha_2}{K-1} & \frac{\alpha_3}{K-1} & \cdots & \frac{\alpha_K}{K-1} \end{pmatrix}, \tag{2.3.14}$$

where $\gamma = (1-\alpha_1)/(K-1)$. Thus, the eigenvalues of AR are $1, \gamma$ and $0, 0, \ldots, 0$ ($K-2$ times) and a eigenvector corresponding to the non-principal eigenvalue γ is

$$\xi = \Big(0, 1, 1, \ldots, 1\Big)'.$$

Therefore using again equation (2.1.7) we get

$$\mathbf{E}\left[U_{n+1}\xi\,\middle|\,\mathcal{F}_n\right] = U_n \left[I + \frac{AR}{n+1}\right] \xi = U_n\xi \left[1 + \frac{\gamma}{(n+1)}\right].$$

Let $\Pi_n(\gamma) = \prod_{i=1}^{n} \left[1 + \frac{\gamma}{i}\right]$ then, $W_n := U_n\xi / \Pi_n(\gamma)$ is a non-negative martingale and using Euler's product, for large n

$$\Pi_n(\gamma) \sim \frac{n^\gamma}{\Gamma(\gamma+1)}. \tag{2.3.15}$$

Equ:Approx We now show that this martingale is $\mathcal{L}^2$ bounded, which will then imply that

$$\frac{U_n\xi}{n^\gamma} \longrightarrow W \tag{2.3.16}$$

where W is a non-degenerate random variable. More precisely, W is nonzero with positive probability. We can write,

$$\mathbf{E}\left[W_{n+1}^2\,\middle|\,\mathcal{F}_n\right] = W_n^2 + \mathbf{E}\left[(W_{n+1} - W_n)^2\,\middle|\,\mathcal{F}_n\right]$$

and

$$\begin{aligned} W_{n+1} - W_n &= \frac{1}{\Pi_{n+1}(\gamma)} \left[U_{n+1}\xi - U_n\xi\left(1 + \frac{\gamma}{n+1}\right)\right] \\ &= \frac{1}{\Pi_{n+1}(\gamma)} \left[\chi_n R\xi - \frac{\gamma}{n+1} U_n\xi\right] \\ &= \frac{1}{\Pi_{n+1}(\gamma)} \left[(K-1)\chi_{n,1} - \frac{(n+1) - U_{n,1}}{n+1}\right] \end{aligned}$$

$$= \frac{K-1}{\Pi_{n+1}(\gamma)} \left[\chi_{n,1} - \mathbf{E}\left[\chi_{n,1} \,\middle|\, \mathcal{F}_n\right]\right]$$

Therefore,

$$\begin{aligned}
\mathbf{E}\left[W_{n+1}^2 \,\middle|\, \mathcal{F}_n\right] &= W_n^2 + \frac{(K-1)^2}{\Pi_{n+1}^2(\gamma)} \left(\mathbf{E}\left[\chi_{n,1} \,\middle|\, \mathcal{F}_n\right] - \mathbf{E}\left[\chi_{n,1} \,\middle|\, \mathcal{F}_n\right]^2\right) \\
&\leq W_n^2 + \frac{(K-1)^2}{\Pi_{n+1}^2(\gamma)} \mathbf{E}\left[\chi_{n,1} \,\middle|\, \mathcal{F}_n\right] \\
&= W_n^2 + \frac{1-\alpha_1}{(n+1)\Pi_{n+1}(\gamma)} \frac{U_n \xi}{\Pi_{n+1}(\gamma)} \\
&\leq W_n^2 + \frac{1-\alpha_1}{(n+1)\Pi_{n+1}(\gamma)} W_n \\
&\leq W_n^2 + \frac{1-\alpha_1}{2(n+1)\Pi_{n+1}(\gamma)} (1+W_n^2) \quad \left(\text{since } 2W_n \leq 1 + W_n^2\right) \\
&\leq W_n^2 + \frac{(1-\alpha_1)\Gamma(\gamma+1)}{2c_1(n+1)^{\gamma+1}} (1+W_n^2) \qquad (2.3.17)
\end{aligned}$$

The last inequality follows, as (for finite n) from equation (2.3.15) we get,

$$c_1 \frac{n^\gamma}{\Gamma(\gamma+1)} \leq \Pi_n(\gamma) \leq c_2 \frac{n^\gamma}{\Gamma(\gamma+1)}, \text{ for some } c_1, c_2 < \infty. \qquad (2.3.18)$$

Let $c := \frac{1}{2c_1}(1-\alpha_1)\Gamma(\gamma+1)$, then

$$\begin{aligned}
\mathbf{E}\left[W_{n+1}^2 + 1 \,\middle|\, \mathcal{F}_n\right] &\leq \left(1 + \frac{c}{(n+1)^{\gamma+1}}\right)(1+W_n^2) \\
&\leq (1+W_0^2) \prod_{j=1}^{n} \left(1 + \frac{c}{(j+1)^{\gamma+1}}\right) \\
&\leq (1+W_0^2) \exp\left(\sum_{j=1}^{n} \frac{c}{(j+1)^{\gamma+1}}\right) < \infty \quad (\text{since } \gamma > 0).
\end{aligned}$$

Thus W_n is $\mathcal{L}^2$-bounded and hence converges to a non-degenerate random variable say W. Now for a star matrix R (as given in equation (2.3.10)), the fundamental recursion (1.1.7) reduces to

$$U_{n+1,1} = U_{n,1} + \alpha_1 \chi_{n+1,1} + (1 - \chi_{n+1,1}) \qquad (2.3.19)$$

and

$$U_{n+1,h} = U_{n,h} + \alpha_h \chi_{n+1,1} \quad \forall\, h \neq 1. \qquad (2.3.20)$$

For $h \neq 1$ if $\alpha_h = 0$ then the $h-th$ column of the replacement matrix is a null column, that is color h is never reinforced and then

$$U_{n,h} = U_{0,h}$$

which is a constant initial configuration of color h. Now, suppose $h \neq 1$, is such that $\alpha_h > 0$. Then dividing both sides by α_h, we get

$$\frac{U_{n+1,h}}{\alpha_h} = \frac{U_{0,h}}{\alpha_h} + \sum_{j=1}^{n+1} \chi_j.$$

Since the above relation holds for every choice of $h > 0$, such that $\alpha_h > 0$, we get

$$\frac{U_{n+1,h}}{\alpha_h} - \frac{U_{n+1,l}}{\alpha_l} = \frac{U_{0,h}}{\alpha_h} - \frac{U_{0,l}}{\alpha_l} \tag{2.3.21}$$

for any $h, l \in \{2, 3, \cdots, K\}$ with $\alpha_h, \alpha_l > 0$. Multiplying the above equation by $\frac{\alpha_l}{1-\alpha_1}$ and taking sum over $l \neq 1$, we get

$$\frac{U_{n,h}}{\alpha_h} - \frac{1}{1-\alpha_1}\sum_{l\neq 1} U_{n,l} = \frac{U_{0,h}}{\alpha_h} - \frac{1}{1-\alpha_1}\sum_{l\neq 1} U_{0,l},$$

which can be written as,

$$\frac{U_{n,h}}{\alpha_h} - \frac{1}{K-1}U_n\xi = \frac{U_{0,h}}{\alpha_h} - \frac{1}{K-1}U_0\xi.$$

Now dividing both sides by n^γ,

$$\frac{1}{n^\gamma}\frac{U_{n,h}}{\alpha_h} - \frac{1}{K-1}\frac{U_n\xi}{n^\gamma} = \frac{1}{n^\gamma}\left[\frac{U_{0,h}}{\alpha_h} - \frac{1}{K-1}U_0\xi\right].$$

Note that the right hand side of the above expression goes to 0 as n tends to infinity. Therefore

$$\lim_{n\to\infty}\frac{1}{n^\gamma}\frac{U_{n,h}}{\alpha_h} - \frac{1}{K-1}\frac{U_n\xi}{n^\gamma} = 0$$

Using the limit from (2.3.16) we get,

$$\frac{U_{n,h}}{n^\gamma} \longrightarrow \frac{\alpha_h}{K-1}W.$$

□

Remark 2.3.7. When $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we get $\hat{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Notice that then $\hat{R}$ is the reinforcement rule for the classical Pólya urn scheme. Now using (1.1.7) we have

$$\mathbf{E}\left[U_{n+1}\,\middle|\,\mathcal{F}_n\right] = U_n + \frac{U_n}{n+1} = (n+2)\frac{U_n}{n+1},$$

which implies that each coordinate of the vector $\frac{U_n}{n+1}$, is a positive martingale and hence converges. Moreover, by exchangeability and arguments similar to the classical Pólya urn, we can easily show that,

$$\frac{U_{n,1}}{n+1} \longrightarrow Z \text{ a.s.},$$

where $Z \sim Beta(U_{0,1}, U_{0,2})$.

2.4 Asymptotics of the Colour Count Statistics

2.4.1 $\hat{R}$ is Irreducible.

Theorem 2.4.1. *Suppose $\hat{R}$ is irreducible then,*

$$\frac{N_{n,j}}{n} \longrightarrow \frac{1}{K\theta - 1}\left[\theta - \mu_j\right], \quad a.s.\ \forall\, 1 \leq j \leq K,$$

where $\mu = (\mu_1, \mu_1, \ldots, \mu_k)$ satisfies equation (2.3.2)*.*

Proof of Theorem 2.4.1 : Recall from equation (1.1.14) and (1.1.17),

$$N_{n,j} = \sum_{m=0}^{n-1} \mathbf{1}\left(Z_m = j\right), \ \ 1 \leq j \leq K.$$

and

$$U_{n+1} = U_0 + N_{n+1}R.$$

Therefore, we can write

$$\begin{aligned} N_n &= \sum_{i=1}^{n}\left(\chi_i - \mathbf{E}\left[\chi_i\,\middle|\,\mathcal{F}_{i-1}\right]\right) + \sum_{i=1}^{n}\mathbf{E}\left[\chi_i\,\middle|\,\mathcal{F}_{i-1}\right] \\ &= \sum_{i=1}^{n}\left(\chi_i - \mathbf{E}\left[\chi_i\,\middle|\,\mathcal{F}_{i-1}\right]\right) + \frac{1}{K\theta-1}\sum_{i=1}^{n}\left[\theta\mathbf{1} - \frac{U_{i-1}}{i}\right], \end{aligned} \tag{2.4.1}$$

Since $\left(\chi_i - \mathbf{E}\left[\chi_i\,\middle|\,\mathcal{F}_{i-1}\right]\right)_{i\geq 1}$ is a bounded martingale difference sequence, using Azuma's inequality (see [22]) we get

$$\frac{1}{n}\sum_{i=1}^{n}\left(\chi_i - \mathbf{E}\left[\chi_i\,\middle|\,\mathcal{F}_{i-1}\right]\right) \longrightarrow 0, \ \ a.s.. \tag{2.4.2}$$

Now using Theorem 2.3.1 and *Cesaro Lemma* (see [5]), we get

$$\frac{N_{n,j}}{n} \longrightarrow \frac{1}{K\theta - 1}\left[\theta - \mu_j\right], \quad a.s. \ \forall\, 1 \leq j \leq K.$$

□

Theorem 2.4.2. *Suppose $\hat{R}$ is irreducible, then there exists a* variance-co-variance *matrix $\tilde{\Sigma}$, such that*

$$\frac{N_n - \frac{n}{K\theta - 1}(\theta \mathbf{1} - \mu)}{\sigma_n} \Longrightarrow N\left(0, \tilde{\Sigma}\right),$$

where σ_n is given in equation (2.3.5) *and equation* (2.3.6). *Moreover,*

$$\Sigma = R^T \tilde{\Sigma} R, \tag{2.4.3}$$

where Σ is as in Theorem 2.3.2.

Remark 2.4.1. It is worth to note here that from definition (1.1.14), it follows that $\sum_{j=1}^{K} N_{n,j} = n+1$, thus $\tilde{\Sigma}$ is a *positive semi-definite* matrix. Further, from equation (2.4.3) it follows that rank $(\Sigma) \leq$ rank $\left(\tilde{\Sigma}\right)$ and equality holds, if and only if, the replacement matrix R is non-singular.

Proof of Theorem 2.4.2 : Notice that under our coupling N_n remains same for the two processes, namely, $(U_n)_{n\geq 0}$ and $\left(\hat{U}_n\right)_{n\geq 0}$. Thus applying Theorem 4.1 of [10] on the urn process $\left(\hat{U}_n\right)_{n\geq 0}$ we conclude that there exists a matrix $\tilde{\Sigma}$ such that,

$$\frac{N_n - n\mu A}{\sigma_n} \Longrightarrow N\left(0, \tilde{\Sigma}\right)$$

Finally the equation (2.4.3) follows from (1.1.17). This completes the proof. □

2.4.2 $\hat{R}$ is Reducible.

Recall that $\hat{R}$ has the form given in (2.3.10) when it is reducible.

Theorem 2.4.3. *Let R be a star matrix with 1 as a central vertex and $\theta = 1$, such that $R \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then*

$$\frac{N_{n,1}}{n} \longrightarrow 0, \quad a.s.$$

and,

$$\frac{N_{n,j}}{n} \longrightarrow \frac{1}{K-1}, \quad a.s. \ \forall\, 2 \leq j \leq K.$$

Remark 2.4.2. For $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\theta = 1$ using equation (1.1.17) and Remark (2.3.7) we get

$$\frac{N_{n,1}}{n} \longrightarrow 1 - Z \text{ a.s.},$$

where as before, $Z \sim Beta(U_{0,1}, U_{0,2})$.

Proof of Theorem 2.4.3 : The proof follows from equation (2.4.1) and (2.4.2). □

Theorem 2.4.4. *Let R be a star matrix with 1 as a central vertex and $\theta = 1$, such that $R \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then*

1. *if $\gamma = \dfrac{1-\alpha_1}{K-1} < 1/2$, then*

$$\frac{1}{\sqrt{n}} \left(\frac{n}{K-1} \mathbf{1} - N_{n,-} \right) \implies N \left(\mathbf{0}, \frac{1}{K-1} I - \frac{1}{(K-1)^2} J \right),$$

where $N_{n,-} = (N_{n,2}, \cdots, N_{n,K})$, and

$$\frac{N_{n,1}}{\sqrt{n}} \xrightarrow{P} 0.$$

2. *if $\gamma = \frac{1-\alpha_1}{K-1} > 1/2$, then*

$$\frac{1}{n^\gamma} \left(\frac{n}{K-1} - N_{n,j} \right) \xrightarrow{P} \frac{\alpha_j}{(K-1)(1-\alpha_1)} W, \quad \forall\, j \neq 1$$

and

$$\frac{N_{n,1}}{n^\gamma} \xrightarrow{P} \frac{1}{K-1} W$$

where W is as given in Theorem 2.3.3.

3. *if $\gamma = \frac{1-\alpha_1}{K-1} = 1/2$, then*

$$\frac{N_{n,-} - \dfrac{n}{K-1} \mathbf{1} + \dfrac{W}{(K-1)(1-\alpha_1)} \boldsymbol{\alpha}_-}{\sqrt{n}} \implies N \left(\mathbf{0}, \frac{1}{K-1} I - \frac{1}{(K-1)^2} J \right),$$

where $\boldsymbol{\alpha}_- = (\alpha_2, \alpha_3, \cdots, \alpha_K)$ and

$$\frac{N_{n,1}}{n^\gamma} \xrightarrow{P} \frac{1}{K-1} W.$$

Remark 2.4.3. Note that $\gamma < 1/2$, if and only if, $K \geq 4$ or $K = 3$ and $\alpha_1 > 0$ or $K = 2$ and $\alpha_1 > 1/2$.

Proof of Theorem 2.4.4 : Let $M_{n,j} := \sum_{i=1}^{n} \left(\chi_{i,j} - \mathbf{E}\left[\chi_{i,j} \,\middle|\, \mathcal{F}_{i-1}\right]\right)$ and $M_n = (M_{n,1}, M_{n,2}, \cdots, M_{n,K})$. Then $\{M_n, \mathcal{F}_n\}$ is a martingale. Define $X_i = (X_{i,1}, X_{i,2}, \cdots, X_{i,K})$ where

$$X_{i,j} := \frac{1}{\sqrt{n}} \left(\chi_{i,j} - \mathbf{E}\left[\chi_{i,j} \,\middle|\, \mathcal{F}_{i-1}\right]\right)$$

are the martingale differences and $(M_n)_{n\geq 1}$ is a k-dimensional bounded increment martingale.

Let $M_{n,-} := (M_{n,2}, \cdots, M_{n,K})$ and $X_{n,-} := (X_{n,2}, \cdots, X_{n,K})$. In this proof, we first provide a central limit theorem for $M_{n,-}$, and then for N_n. Observe that the (l, m)-th entry of the matrix $\mathbf{E}\left[X_{i,-}^T X_{i,-} \,\middle|\, \mathcal{F}_{i-1}\right]$ is

$$\begin{aligned}
\mathbf{E}\left[\left(X_{i,-}^n\right)^T X_{i,-}^n \,\middle|\, \mathcal{F}_{i-1}\right]_{(l,m)} &= \frac{1}{n}\mathbf{E}\left[\chi_{i,l}\chi_{i,m} \,\middle|\, \mathcal{F}_{i-1}\right] - \mathbf{E}\left[\chi_{i,l} \,\middle|\, \mathcal{F}_{i-1}\right]\mathbf{E}\left[\chi_{i,m} \,\middle|\, \mathcal{F}_{i-1}\right] \\
&= \begin{cases} \frac{1}{n}\mathbf{E}\left[\chi_{i,l} \,\middle|\, \mathcal{F}_{i-1}\right]\left(1 - \mathbf{E}\left[\chi_{i,l} \,\middle|\, \mathcal{F}_{i-1}\right]\right) & \text{if } l = m, \\ \frac{-1}{n}\mathbf{E}\left[\chi_{i,l} \,\middle|\, \mathcal{F}_{i-1}\right]\mathbf{E}\left[\chi_{i,m} \,\middle|\, \mathcal{F}_{i-1}\right] & \text{if } l \neq m \end{cases} \\
&= \begin{cases} \frac{1}{n(K-1)}\left(1 - \frac{U_{i-1,l}}{i}\right)\left(1 - \frac{1}{K-1}\left(1 - \frac{U_{i-1,j}}{i}\right)\right) & \text{if } l = m, \\ \frac{-1}{n(K-1)^2}\left(1 - \frac{U_{i-1,l}}{i}\right)\left(1 - \frac{U_{i-1,m}}{i}\right) & \text{if } l \neq m, \end{cases}
\end{aligned}$$

So, as $n \to \infty$, (using Theorem 2.3.3) we have

$$\sum_{i=1}^{n} \mathbf{E}\left[\left(X_{i,-}^n\right)^T X_{i,-}^n \,\middle|\, \mathcal{F}_{i-1}\right]_{(l,m)} \to \begin{cases} \frac{(K-2)}{(K-1)^2} & \text{if } l = m, \\ \frac{-1}{(K-1)^2} & \text{if } l \neq m, \end{cases}$$

Therefore,

$$\sum_{i=1}^{n} \mathbf{E}\left[\left(X_{i,-}^n\right)^T X_{i,-}^n \,\middle|\, \mathcal{F}_{i-1}\right] \to \frac{1}{K-1} I - \frac{1}{(K-1)^2} J,$$

and by the martingale central limit theorem [45], we get

$$\frac{1}{\sqrt{n}} M_{n,-} \implies N\left(0, \frac{1}{K-1} I - \frac{1}{(K-1)^2} J\right) \qquad (2.4.4)$$

Now for color 1, we have

$$\frac{1}{\sqrt{n}} M_{n,1} = \frac{-1}{\sqrt{n}} \sum_{j=1}^{K-1} M_{n,-}$$

which implies

$$\frac{1}{\sqrt{n}} M_{n,1} \stackrel{P}{\longrightarrow} 0.$$

We now prove the central limit theorem for N_n. By equation (2.4.1), we have

$$N_n = M_n + \sum_{i=1}^{n} \mathbf{E}\left[\chi_i \,\middle|\, \mathcal{F}_{i-1}\right]$$

Therefore,

$$\frac{n}{K-1}\mathbf{1} - N_{n,-} = -M_{n,-} + \frac{1}{K-1}\sum_{i=1}^{n} \frac{U_{i-1,-}}{i} \tag{2.4.5}$$

Form Theorem 2.3.3, we know that for each $j \neq 1$

$$\frac{U_{i-1,j}}{i^{\gamma}} \to \frac{\alpha_j}{K-1} W, \ a.s..$$

$$\begin{aligned}\sum_{i=1}^{n} \frac{U_{i-1,j}}{i} &\asymp \frac{\alpha_j}{K-1} W \sum_{i=1}^{n} i^{\gamma-1} \\ &\sim \frac{\alpha_j}{\gamma(K-1)} W n^{\gamma} = \frac{\alpha_j}{(1-\alpha_1)} W n^{\gamma}.\end{aligned}$$

Therefore,

$$\frac{1}{n^{\gamma}} \sum_{i=1}^{n} \frac{U_{i-1,j}}{i} \to \frac{\alpha_j}{(1-\alpha_1)} W \ a.s.. \tag{2.4.6}$$

Therefore for $\gamma < 1/2$, using equation (2.4.4), (2.4.5) and (2.4.6) we get

$$\frac{1}{\sqrt{n}}\left(\frac{n}{K-1}\mathbf{1} - N_{n,-}\right) \implies N\left(0, \frac{1}{K-1} I - \frac{1}{(K-1)^2} J\right),$$

and for $\gamma > 1/2$,

$$\frac{1}{n^{\gamma}}\left(\frac{n}{K-1} - N_{n,j}\right) \stackrel{P}{\longrightarrow} \frac{\alpha_j}{(K-1)(1-\alpha_1)} W \ \text{ for } j = 2, 3, \cdots, K$$

since then $M_{n,j}/n^{\gamma} \stackrel{P}{\longrightarrow} 0$. For $j = 1$, we have

$$\begin{aligned}N_{n,1} &= n - \sum_{j=2}^{K} N_{n,j} \\ &= \sum_{j=2}^{K}\left(\frac{n}{K-1} - N_{n,j}\right)\end{aligned}$$

Therefore for $\gamma < 1/2$ we have

$$\frac{N_{n,1}}{\sqrt{n}} \xrightarrow{P} 0.$$

and for $\gamma > 1/2$, we have

$$\frac{1}{n^{\gamma}} N_{n,1} = \sum_{j=2}^{K} \frac{1}{n^{\gamma}} \left(\frac{n}{K-1} - N_{n,j} \right) \xrightarrow{P} \frac{1}{(K-1)(1-\alpha_1)} W \sum_{j=2}^{K} \alpha_j = \frac{1}{K-1} W.$$

From equation (2.4.5) we have

$$N_{n,-} - \frac{n}{K-1}\mathbf{1} + \frac{1}{K-1} \sum_{i=1}^{n} \frac{U_{i-1,-}}{i} = M_{n,-}$$

$$\frac{N_{n,-} - \frac{n}{K-1}\mathbf{1} + \frac{1}{K-1} \sum_{i=1}^{n} \frac{U_{i-1,-}}{i}}{\sqrt{n}} = \frac{M_{n,-}}{\sqrt{n}}$$

therefore for $\gamma = 1/2$, we get

$$\frac{N_{n,-} - \frac{n}{K-1}\mathbf{1} + \frac{W}{(K-1)(1-\alpha_1)}\boldsymbol{\alpha}_-}{\sqrt{n}} \Longrightarrow N\left(\mathbf{0}, \frac{1}{K-1} I - \frac{1}{(K-1)^2} J\right)$$

where $\boldsymbol{\alpha}_- = (\alpha_2, \alpha_3, \cdots, \alpha_K)$ and

$$\begin{aligned} N_{n,1} &= \sum_{j=2}^{K} \left(\frac{n}{K-1} - N_{n,j} \right) \\ &= \frac{1}{K-1} W + \sum_{j=2}^{K} \left(\frac{n}{K-1} - N_{n,j} - \frac{\alpha_j}{(K-1)(1-\alpha_1)} W \right) \end{aligned}$$

$\Longrightarrow$

$$\frac{N_{n,1}}{\sqrt{n}} \xrightarrow{P} \frac{1}{K-1} W.$$

□

Chapter 3

Negatively Reinforced Urn Model with Lipschitz Weight Function [1]

3.1 Introduction

In this chapter, we consider a generalization of linear negatively reinforced urn models to negatively reinforced urn models with non-increasing Lipschitz weight function. In case of the classical urn models, one such generalization to general *increasing* weight functions with a random replacement rule have been studied by Laurelle and Pages [55]. For such urn models they obtained the almost sure convergence and central limit theorem results of the random configuration of the urn. The main tool used to study such models is *stochastic approximation method,* which is a powerful tool to study recursive algorithms.

In this chapter, we investigate a generalization of negatively reinforced urn models for Lipschitz *non-increasing* weight functions. We first show that for a Lipschitz weight function and sufficiently large number of colours, the colour proportions and proportion of colour counts converge almost surely to a constant vector for any choice of replacement matrix.

As mentioned in Chapter 1, negatively reinforced models have applications in the load balancing problems in a resource constraint system and in such load balance problems, uniform is the desirable limiting distribution. Later in this chapter we will notice that this can be achieved only if we choose a doubly stochastic replacement matrix. Therefore, in this chapter

[1] This chapter is based on the paper entitled "*Negatively Reinforced Balanced Urn Schemes*" [13].

we will mainly focus on the doubly stochastic replacement matrices, and show the almost sure convergence of the random urn proportions to the uniform distribution holds under some very mild assumptions on the weight function. We will also obtain central limit theorems for the case when the replacement matrix is a doubly stochastic matrix.

Recall from Section 1.1 of Chapter 1, that the negatively reinforced urn model is defined as a stochastic process $(U_n)_{n\geq 0}$ which satisfies the fundamental recursion as given in equation (1.1.7), that is

$$U_{n+1} = U_n + \chi_{n+1} R \tag{3.1.1}$$

where $\chi_{n+1} := (\mathbf{1}(Zn = j))_{1\leq j\leq K}$, with a balanced replacement matrix R, such that for a given weight function $w : [0,1] \to \mathbb{R}^+$,

$$\mathbf{E}\left[\chi_{n+1}\Big|, \mathcal{F}_n\right] = \frac{\mathbf{w}(Y_n)}{S_w(Y_n)} \tag{3.1.2}$$

where $Y_n = \dfrac{U_n}{n+1}$, $\mathbf{w}(Y_n) = (w(Y_{n,1}), \cdots, w(Y_{n,K}))$ and $S_w(Y_n) := \sum_{j=1}^K w(Y_{n,j})$.

Since it is not possible to use simple martingale technique in case of non-linear weight functions, we will be using the stochastic approximation method, as introduced in the work of Kushner and Clark [54], Benaïm [20] and Borkar [23], to obtain the asymptotics for the urn configuration and the colour count statistics for negatively reinforced urn models. In Appendix A we present a detailed review of the stochastic aproximation theory. In the next section, we first define the stochastic approximation algorithm and then we show that the urn configuration vector $(Y_n)_{n\geq 0}$ and proportion of colour counts $\left(\tilde{Y}_n\right)_{n\geq 0}$ can also be written as a stochastic approximation algorithm.

3.2 Stochastic Approximation and Urn

A stochastic approximation algorithm $(Y_n)_{n\geq 0}$ is a stochastic process in $\mathbb{R}^d$, as defined in equation (A.1.1) in Appendix A, is given by

$$Y_{n+1} = Y_n + \gamma_n h(Y_n) + \gamma_n M_{n+1} \quad \text{for } n \geq 0, \tag{3.2.1}$$

for $h : \mathbb{R}^d \to \mathbb{R}^d$, such that

(i) γ_n is a sequence of positive real numbers such that

$$\sum_{n\geq 1} \gamma_n = \infty \text{ and } \sum_{n\geq 1} \gamma_n^2 < \infty$$

(ii) $(M_n)_{n\geq 1}$ is a square integrable martingale difference sequence with respect to the filtration $\mathcal{F}_n = \sigma\{Y_m, M_m,\ m \leq n\}$,

$$\sup_{n\geq 0} E\left[\|M_{n+1}\|^2\right] < \infty. \tag{3.2.2}$$

As discussed in Appendix A, under certain assumptions, asymptotics of Y_n can be obtained using the stochastic approximation theory, which essentially relates the recursion in (3.2.6) with the ODE

$$\dot{y} = h(y). \tag{3.2.3}$$

3.2.1 Random Urn Configuration

Observe that, for the urn model defined in equation (3.1.1), we have

$$E\left[U_{n+1} - U_n | \mathcal{F}_n\right] = \frac{\mathbf{w}(Y_n)}{S_w(Y_n)} R \tag{3.2.4}$$

Therefore the recurrence relation (3.1.1) can be written as

$$\begin{aligned} U_{n+1} &= U_n + E[\chi_{n+1}|\mathcal{F}_n]R + \left[\chi_{n+1} - E[\chi_{n+1}|\mathcal{F}_n]\right]R \\ &= U_n + \frac{\mathbf{w}(Y_n)}{S_w(Y_n)} R + M_{n+1} R \end{aligned} \tag{3.2.5}$$

where $M_{n+1} = \chi_{n+1} - E\left[\chi_{n+1}|\mathcal{F}_n\right]$ is an $\mathcal{F}_n$ martingale difference. Now observe,

$$\begin{aligned} \frac{U_{n+1}}{n+2} &= \frac{U_n}{n+2} + \frac{1}{n+2}\frac{\mathbf{w}(Y_n)}{S_w(Y_n)} R + \frac{1}{n+2} M_{n+1} R \\ \implies Y_{n+1} &= Y_n \frac{n+1}{n+2} + \frac{1}{n+2}\frac{\mathbf{w}(Y_n)}{S_w(Y_n)} R + \frac{1}{n+2} M_{n+1} R \\ \implies Y_{n+1} &= Y_n + \frac{1}{n+2}\left(\frac{\mathbf{w}(Y_n)}{S_w(Y_n)} R - Y_n\right) + \frac{1}{n+2} M_{n+1} R \end{aligned}$$

which is exactly of the form given in the equation (3.2.1), that is the urn configuration Y_n can be written as a K-dimensional stochastic approximation algorithm

$$Y_{n+1} = Y_n + \gamma_n h(Y_n) + \gamma_n M_{n+1} R \tag{3.2.6}$$

where $\gamma_n = \dfrac{1}{n+2}$, and $h : \mathbb{R}^K \to \mathbb{R}^K$ is given by

$$h(y) = \frac{\mathbf{w}(y)}{S_w(y)} R - y. \tag{3.2.7}$$

where we extend the w continuously to whole of $\mathbb{R}$, by making it a constant function outside the interval $[0, 1]$, that is, $w(y) = w(0)$ for $y \leq 0$ and $w(y) = w(1)$ for $y \geq 1$. Also note that $\gamma_n \sim \mathcal{O}\left(n^{-1}\right)$ satisfies the required conditions given in (ii) and $(M_{n+1}R)_{n\geq 1}$ is a martingale difference sequence that is

$$E\left[M_{n+1}R|\mathcal{F}_n\right] = 0 \tag{3.2.8}$$

and by using cauchy Schwartz ineuality we get

$$\|M_{n+1}R\|^2 = \sum_{i=1}^{K} \left(\sum_{j=1}^{K} M_{n+1,j} R_{j,i} \right)^2 \tag{3.2.9}$$

$$\leq \sum_{i=1}^{K} \left(\sum_{j=1}^{K} R_{j,i}^2 \right) \left(\sum_{j=1}^{K} M_{n+1,j}^2 \right) \tag{3.2.10}$$

Now since R is a stochastic matrix, $R_{j,i} \leq 1$ and therefore we get

$$\|M_{n+1}R\|^2 \leq K \sum_{j=1}^{K} |\chi_{n+1,j} - E[\chi_{n+1,j}|\mathcal{F}_n]|^2 \tag{3.2.11}$$

$$\leq K \sum_{j=1}^{K} |\chi_{n+1,j}|^2 + E[\chi_{n+1,j}|\mathcal{F}_n]^2 \tag{3.2.12}$$

Now since $\chi_{n+1,j}^2 = \chi_{n+1,j}$ (as it only takes value 0 or 1) and $\sum_{j=1}^{K} \chi_{n+1,j} = 1$, therefore we get

$$\|M_{n+1}R\|^2 \leq K \left(1 + \sum_{j=1}^{K} E[\chi_{n+1,j}|\mathcal{F}_n]^2 \right) \tag{3.2.13}$$

$$\leq K(1+K). \tag{3.2.14}$$

Thus $(M_{n+1}R)_{n\geq 0}$ also satisfies the conditions given in equation (3.2.2). Therefore, the ODE associated to (3.2.6) is

$$\dot{y} = h(y) \tag{3.2.15}$$

where h is given in equation (3.2.7).

3.2.2 Colour Count Statistics

Recall that the colour count statistics is given by, $N_n = \sum_{j=1}^{n} \chi_j$ and the colour count proportions are $\tilde{Y}_n := \dfrac{N_n}{n}$, we can write

$$\begin{aligned}
N_{n+1} &= N_n + \chi_{n+1} \\
&= N_n + E\left[\chi_{n+1}|\mathcal{F}_n\right] + \left(\chi_{n+1} - E\left[\chi_{n+1}|\mathcal{F}_n\right]\right) \\
&= N_n + \frac{\mathbf{w}(Y_n)}{S_w(Y_n)} + M_{n+1} \\
\frac{N_{n+1}}{n+1} &= \frac{N_n}{n} + \frac{1}{n+1}\left[\frac{\mathbf{w}(Y_n)}{S_w(Y_n)} - \frac{N_n}{n}\right] + \frac{1}{n+1}M_{n+1} \\
\implies \qquad \tilde{Y}_{n+1} &= \tilde{Y}_n + \frac{1}{n+1}\left[\frac{\mathbf{w}\left(Y_n\right)}{S_w\left(Y_n\right)} - \tilde{Y}_n\right] + \frac{1}{n+1}M_{n+1}
\end{aligned} \tag{3.2.16}$$

Recall from equation (1.1.17) that the relation between U_n and N_n is given by

$$U_n = U_0 + N_n R$$

$$\implies Y_n = \frac{1}{n+1}Y_0 + \frac{n}{n+1}\tilde{Y}_n R =: \tilde{Y}_n R + \delta_n \tag{3.2.17}$$

for

$$\delta_n = \frac{1}{n+1}Y_0 - \frac{1}{n+1}\tilde{Y}_n R \tag{3.2.18}$$

Therefore we can rewrite equation (3.2.16) as

$$\tilde{Y}_{n+1} = \tilde{Y}_n + \frac{1}{n+1}\left[\frac{\mathbf{w}\left(\tilde{Y}_n R + \delta_n\right)}{S_w\left(\tilde{Y}_n R + \delta_n\right)} - \tilde{Y}_n\right] + \frac{1}{n+1}M_{n+1} \tag{3.2.19}$$

$$= \tilde{Y}_n + \frac{1}{n+1}\left[\frac{\mathbf{w}\left(\tilde{Y}_n R\right)}{S_w\left(\tilde{Y}_n R\right)} - \tilde{Y}_n\right] + \frac{1}{n+1}\epsilon_n + \frac{1}{n+1}M_{n+1} \tag{3.2.20}$$

where

$$\epsilon_n = \frac{\mathbf{w}\left(\tilde{Y}_n R + \delta_n\right)}{S_w\left(\tilde{Y}_n R + \delta_n\right)} - \frac{\mathbf{w}\left(\tilde{Y}_n R\right)}{S_w\left(\tilde{Y}_n R\right)}.$$

Therefore $\tilde{Y}_n$ can also be written as a stochastic approximation recursion as given in equation (3.1.1). Since $\delta_n \to 0$, $\epsilon_n \to 0$, as $n \to \infty$, the ODE associated to (3.2.20) is

$$\dot{\tilde{y}} = \tilde{h}\left(\tilde{y}\right) \tag{3.2.21}$$

where $\tilde{h} : \mathbb{R}^K \to \mathbb{R}^K$ is such that

$$\tilde{h}\left(\tilde{y}\right) = \frac{\mathbf{w}\left(\tilde{y}R\right)}{S_w\left(\mathbf{w}(\tilde{y}R)\right)} - \tilde{y}. \tag{3.2.22}$$

3.3 Almost Sure Convergence

To start with, we need the ODE in equation (3.2.3) and (3.2.21) to have a unique solution. A sufficient condition for the ODE to have a unique solution, is when h and $\tilde{h}$ are Lipschitz functions. We will assume throughout this chapter that the function w is continuously differentiable, which implies that the function h and $\tilde{h}$ are both Lipschitz and this ensures that the associated ODEs have unique solution for any initial vector Y_0. To present our main results we will need the following definitions.

Definition 3.3.1. For the stochastic algorithm defined in (3.2.1), a point y^* is called an equilibrium point if $h(y^*) = 0$.

Note that, for the h function given in equation (3.2.7), y^* is an equilibrium point if

$$h(y^*) = 0 \iff \mathbf{w}(y^*)R = S_w(y^*)y^*. \tag{3.3.1}$$

The equilibrium points of h are important as they are possible limit points for the solution of the ODE (3.2.3). In case of a linear weight function w, where h is of the form

$$h(y) = y\left[AR - I\right]$$

there exists a unique equilibrium point assuming that the stochastic matrix AR is irreducible, that is AR has a unique stationary distribution. whereas, for a nonlinear weight function w or h the unique equilibrium point is guaranteed assuming that the function $F : \mathbb{R}^K \to \mathbb{R}^K$ defined as

$$F(y) := \frac{\mathbf{w}(y)}{S_w(y)}R, \tag{3.3.2}$$

is a contraction map. We now present the results depending on whether F is a contraction.

3.3.1 F is a Contraction

Theorem 3.3.1. *Suppose w is a non-increasing weight function and F is a contraction map then*

$$Y_n \longrightarrow y^* \;\; a.s., \;\; \textit{and} \;\; \tilde{Y}_n \longrightarrow \tilde{y}^* \;\;\; a.s. \tag{3.3.3}$$

where y^* is the unique fixed point of F and

$$\tilde{y}^* = \frac{w(y^*)}{S_w(y^*)}. \tag{3.3.4}$$

In particular, convergence in (3.3.3) *holds, whenever non-increasing function w is a Lip(M) function and $\sqrt{K} > \dfrac{2M}{w(1)}$.*

Corollary 3.3.1. *From equation* (3.2.17) *we get*

$$y^* = \tilde{y}^* R,$$

or

$$\tilde{y}^* = \frac{w(y^*)}{S_w(y^*)}$$

where y^* and $\tilde{y}^*$ are given in Theorem 3.3.1.

In the next Proposition, we obtain sufficient conditions under which F is a contraction map.

Proposition 3.3.1. *Suppose w is a non-increasing $Lip(M)$ function then F is a contraction whenever*

(i) $w(1) > 0$ *and* $\sqrt{K} > \dfrac{2M}{w(1)}$

or

(ii) w *is a convex function and* $\sqrt{K}w(1/K) > 2M$.

Remark 3.3.1. If w is a non-increasing convex weight function and $w(0) < \infty$, then F is a contraction whenever $\sqrt{K} > \dfrac{4M}{w(0)}$, for K sufficiently large such that $w(1/K) > w(0)/2$.

Proof:

$$\begin{aligned}
\|F(x) - F(y)\| &= \left\| \frac{\mathbf{w}(x)}{S_w(x)} - \frac{\mathbf{w}(y)}{S_w(y)} \right\| \\
&= \left\| \frac{\mathbf{w}(x)S_w(y) - \mathbf{w}(y)S_w(x)}{S_w(x)S_w(y)} \right\| \\
&= \left\| \frac{\left(S_w(y) - S_w(x)\right)\mathbf{w}(x) - S_w(x)\left(\mathbf{w}(y) - \mathbf{w}(x)\right)}{S_w(x)S_w(y)} \right\| \\
&\leq \frac{|S_w(y) - S_w(x)| \ \|\mathbf{w}(x)\| \ + \ S_w(x) \ \|\mathbf{w}(y) - \mathbf{w}(x)\|}{S_w(x)S_w(y)}
\end{aligned}$$

Note that

$$\|\mathbf{w}(x)\|^2 = \sum_{i=1}^{K} |w(x_i)|^2 \leq \left(\sum_{i=1}^{K} w(x_i)\right)^2 = S_w(x)^2$$

$$\implies \|\mathbf{w}(x)\| \leq S_w(x). \tag{3.3.5}$$

Therefore,

$$\|F(x) - F(y)\| \leq \frac{|S_w(y) - S_w(x)| + \|\left(\mathbf{w}(x) - \mathbf{w}(y)\right)\|}{S_w(y)} \tag{3.3.6}$$

Now since w is a $Lip(M)$ function we get

$$\begin{aligned}
\|\mathbf{w}(x) - \mathbf{w}(y)\|^2 &= \sum_{i=1}^{K} |w(x_i) - w(y_i)|^2 \\
&\leq M^2 \sum_{i=1}^{K} |x_i - y_i|^2 = M^2\|x - y\|^2
\end{aligned} \tag{3.3.7}$$

and

$$\begin{aligned}
|S_w(y) - S_w(x)| &= \left| \sum_{j=1}^{K} w(y_j) - w(x_j) \right| \\
&\leq \sum_{j=1}^{K} |w(y_j) - w(x_j)| \\
&\leq M \sum_{j=1}^{K} |y_j - x_j| = M\|y - x\|_1
\end{aligned}$$

$$\leq M\sqrt{K}\|x-y\| \tag{3.3.8}$$

The last inequality follows by Cauchy-Schwartz inequality. Finally from equations (3.3.7), (3.3.8) and (3.3.6), we get

$$\|F(x)-F(y)\| \leq \frac{M\left(1+\sqrt{K}\right)}{S_w(y)}\|x-y\| \tag{3.3.9}$$

Case (i): If $w(1)>0$, then we can write $S_w(y)\geq Kw(1)$ and therefore from equation (3.3.9) we get

$$\|F(x)-F(y)\| \leq \frac{M\left(1+\sqrt{K}\right)}{Kw(1)}\|x-y\| \leq \frac{2M}{\sqrt{K}w(1)}\|x-y\| \tag{3.3.10}$$

Thus F is a contraction if $\sqrt{K} > \dfrac{2M}{w(1)}$.
Case (ii): Assuming that w is a convex function then

$$S_w(y) \geq Kw(1/K),\ \forall y\ \in \Delta_K.$$

Therefore from equation (3.3.9) we get

$$\|F(x)-F(y)\| \leq \frac{M\left(1+\sqrt{K}\right)}{Kw(1/K)}\|x-y\| \leq \frac{2M}{\sqrt{K}w(1/K)}\|x-y\| \tag{3.3.11}$$

Thus F is a contraction if $\sqrt{K}w(1/K) > 2M$. □

Proof of Theorem 3.3.1 : Suppose F is a contraction, then there exists a unique y^* such that $F(y^*)=y^*$, that is a unique fixed point of F. Then

$$h(y^*)=0.$$

that is y^* is also a unique equilibrium. Now using Theorem 2. and Corollary 3. from [23] (page 126) we get

$$Y_n \to y^*, \quad \text{as } n\to\infty.$$

Now if $\tilde{y}^*$ is an equilibrium point of the ODE in equation (3.2.21) that is $\tilde{y}^*$ satisfies

$$\tilde{y}^* = \frac{\mathbf{w}(\tilde{y}^*R)}{S_w(\tilde{y}^*R)}$$

then

$$\tilde{Y}_n \to \tilde{y}^*, \quad \text{as } n\to\infty.$$

□

3.3.2 F is not a Contraction

In the case when F is not a contraction, we will only consider doubly stochastic replacement matrices. We start with the following observation.

Proposition 3.3.2. *The uniform vector* $\frac{1}{K}\mathbf{1}$ *is an equilibrium point of the ODE in equation* (3.2.3)*, if and only if,* R *is a doubly stochastic matrix.*

Proof: Note that,

$$h\left(\frac{1}{K}\mathbf{1}\right) = 0 \iff \frac{\mathbf{w}\left(\frac{1}{K}\right)}{S_w\left(\frac{1}{K}\mathbf{1}\right)}R = \frac{1}{K}\mathbf{1} \tag{3.3.12}$$

$$\iff \frac{1}{K}\mathbf{1}R = \frac{1}{K}\mathbf{1} \tag{3.3.13}$$

Thus, uniform is an equilibrium point, if and only if, R is a doubly stochastic matrix. □

Assuming that R is doubly stochastic, $\frac{1}{K}\mathbf{1}$ is an equilibrium point for both the ODEs given in equation (3.2.3) and (3.2.21), that is

$$h\left(\frac{1}{K}\mathbf{1}\right) = 0 \quad \text{and} \quad \tilde{h}\left(\frac{1}{K}\mathbf{1}\right) = 0 \tag{3.3.14}$$

where h and $\tilde{h}$ are defined in equation (3.2.7) and (3.2.22).

Definition 3.3.2. An equilibrium point y^* is called *stable (or attractor)*, if all the eigenvalues of the Jacobian matrix of h at y^*, that is $\left.\frac{\partial h(y)}{\partial y}\right|_{y=y^*}$ have negative real parts, otherwise it is called *unstable*.

In the next Proposition, we give a sufficient condition under which the unique equilibrium point $\frac{1}{K}\mathbf{1}$ is stable for both the ODEs.

Proposition 3.3.3. *Suppose* w *is a non-increasing function on* $[0, 1]$*, and* R *is doubly stochastic matrix then,* $\frac{1}{K}\mathbf{1}$ *is a stable equilibrium, if for every eigenvalue* λ *of* R

$$\Re(\lambda) > \frac{Kw\left(\frac{1}{K}\right)}{w'\left(\frac{1}{K}\right)}. \tag{3.3.15}$$

Proof: The Jacobian matrix of h is given by

$$\frac{\partial h(y)}{\partial y} = \frac{\partial \mathbf{w}(y)/S_w(y))}{\partial y} R - I \tag{3.3.16}$$

where,

$$\frac{\partial}{\partial y}\frac{\mathbf{w}(y)}{S_w(y)} = \begin{bmatrix} \frac{w'(y_1)}{S_w(y)} - \frac{w(y_1)w'(y_1)}{S_w(y)^2} & -\frac{w(y_1)w'(y_2)}{S_w(y)^2} & \cdots & -\frac{w(y_1)w'(y_K)}{S_w(y)^2} \\ -\frac{w(y_2)w'(y_1)}{S_w(y)^2} & \frac{w'(y_2)}{S_w(y)} - \frac{w(y_2)w'(y_2)}{S_w(y)^2} & \cdots & -\frac{w(y_2)w'(y_K)}{S_w(y)^2} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{w(y_K)w'(y_1)}{S_w(y)^2} & -\frac{w(y_K)w'(y_2)}{S_w(y)^2} & \cdots & \frac{w'(y_K)}{S_w(y)} - \frac{w(y_K)w'(y_K)}{S_w(y)^2} \end{bmatrix} \tag{3.3.17}$$

That is,

$$\frac{\partial \mathbf{w}(y)/S_w(y)}{\partial y} = diag\left(\frac{\mathbf{w}'(y)}{S_w(y)}\right) + \left(\left(\frac{-w(y_i)w'(y_j)}{S_w(y)^2}\right)\right)_{i,j=1,2,\cdots,K}. \tag{3.3.18}$$

Therefore

$$\begin{aligned} \frac{\partial h(y)}{\partial y}\Big|_{y=\frac{1}{K}\mathbf{1}} &= \left(bI - \frac{b}{K}J\right) R - I \\ &= bR - \frac{b}{K}J - I \end{aligned} \tag{3.3.19}$$

where

$$b := \frac{w'\left(\frac{1}{K}\right)}{Kw\left(\frac{1}{K}\right)}, \tag{3.3.20}$$

Note that $b \leq 0$ as w is a non-increasing function. By Perron Frobenius Theorem, the stochastic matrix R has maximal eigenvalue 1. That is the absolute real part of all eigenvalue of a stochastic matrix R is less than 1, so without loss we assume $1 > \Re(\lambda_1) \geq \Re(\lambda_2) \geq \cdots \geq \Re(\lambda_s) \geq -1$. Note that the right eigenvector corresponding to the maximal eigenvalue 1 of R is $\mathbf{1}^T$ and

$$Dh\left(\frac{1}{K}\mathbf{1}\right)\mathbf{1}^T = \left(bI - \frac{b}{K}J\right) R\mathbf{1}^T - \mathbf{1}^T = -\mathbf{1}^T \tag{3.3.21}$$

Thus -1 is an eigenvalue of $Dh\left(\frac{1}{K}\mathbf{1}\right)$. Now, for an eigenvalue $\lambda_i(\neq 1)$ of R, and the corresponding right eigenvector v_i^T which is orthogonal to $\mathbf{1}^T$, we have

$$Dh\left(\frac{1}{K}\mathbf{1}\right)v_i^T = \left(bI - \frac{b}{K}J\right)Rv_i^T - v_i^T = (b\lambda_i - 1)v_i^T \tag{3.3.22}$$

Therefore the Jacobian matrix $Dh\left(\frac{1}{K}\mathbf{1}\right)$ has eigenvalues $b\lambda_i - 1$ for every $i = 1, \cdots, s$. Thus the equilibrium point $\frac{1}{K}\mathbf{1}$ is stable, if and only if,

$$\Re(b\lambda_i - 1) < 0, \ \ \forall\, i = 1, 2, \cdots, s. \tag{3.3.23}$$

$$\iff \Re(\lambda_s) > \frac{1}{b} = \frac{Kw\left(\frac{1}{K}\right)}{w'\left(\frac{1}{K}\right)}. \tag{3.3.24}$$

This completes the proof. □

Now note that

$$\frac{\partial \tilde{h}(\tilde{y})}{\partial \tilde{y}} = \frac{\partial}{\partial y}\frac{\mathbf{w}\,(y)}{S_w\,(y)}R - I = \frac{\partial h(y)}{\partial y} \tag{3.3.25}$$

and therefore, the condition for stability in equation (3.3.15) is same for both the ODEs.

Remark 3.3.2. Since $\Re(\lambda_j) \geq -1$ for every $j = 1, \cdots, s$, another sufficient condition for the stability is

$$Kw\left(\frac{1}{K}\right) > -w'\left(\frac{1}{K}\right). \tag{3.3.26}$$

Now assuming that $w(0), w'(0+) < \infty$, equation (3.3.15) or (3.3.26) hold for K sufficiently large.

Clearly from equation (3.3.26), $\frac{1}{K}\mathbf{1}$ is a stable point whenever all the eigenvalues of a doubly stochastic matrix R have positive real part. In particular $\frac{1}{K}\mathbf{1}$ is stable for any choice of weight function when the replacement matrix is of Pólya type, that is when $R = I$. Moreover, for a doubly stochastic matrix R, $\frac{1}{k}\mathbf{1}$ is an equilibrium point for both the ODEs given in equation (3.2.3) and (3.2.21), that is

$$h\left(\frac{1}{k}\mathbf{1}\right) = 0 \quad \text{and} \quad \tilde{h}\left(\frac{1}{k}\mathbf{1}\right) = 0 \tag{3.3.27}$$

where h and $\tilde{h}$ are defined in equation (3.2.7) and (3.2.22). In the next Theorem we show that the vector of urn proportions and colour count proportions converge to uniform vector $\frac{1}{K}\mathbf{1}$, when the replacement matrix R is of the form $R = \frac{1}{2}[I + \tilde{R}]$, for a non-negative doubly stochastic matrix $\tilde{R}$.

Theorem 3.3.2. *Suppose w is a non-increasing weight function and R is a doubly stochastic matrix of the form $R = \dfrac{1}{2}\left[I + \tilde{R}\right]$ for a doubly stochastic matrix $\tilde{R}$, then*

$$Y_n \longrightarrow \frac{1}{k}\mathbf{1} \;\; a.s. \quad \text{and} \quad \tilde{Y}_n \longrightarrow \frac{1}{k}\mathbf{1} \;\; a.s. \tag{3.3.28}$$

Remark 3.3.3. Note that, for this special form of doubly stochastic replacement matrix all its diagonal entries are at least $1/2$, therefore the color chosen according to non-increasing weight function indeed gets reinforced more than every other colour. In particular, this ensures negative reinforcement both in the selection step and reinforcement step.

We will also observe later in section 3.5 that the above almost sure convergence does not hold for any doubly stochastic replacement matrix. In particular, we show that for certain non-linear weight function and a particular doubly stochastic matrix, which is not of the form $\dfrac{1}{2}\left[I + \tilde{R}\right]$, the vector of urn proportions does not converge to $\frac{1}{k}\mathbf{1}$.

Proof of Theorem 3.3.2 : We first note that if R is a doubly stochastic matrix of the form $\frac{1}{2}\left[I + \tilde{R}\right]$ then the condition for stability in Proposition 3.3.3 is satisfied, thus $\frac{1}{k}\mathbf{1}$ is a stable equilibrium point. Now we show that for any initial vector y_0, the unique solution of the ODE $\dot{y} = h(y)$ converges to $\frac{1}{k}\mathbf{1}$.
Let y_0 be the initial configuration vector, such that $y_0 \neq \frac{1}{k}\mathbf{1}$, and ϕ be the unique solution of the ODE $\dot{y} = h(y)$, then $\phi(t) = \left(\phi^1(t), \phi^2(t), \cdots, \phi^k(t)\right)$ satisfies

$$\frac{d\phi^i(t)}{dt} = h(\phi(t))_i = \frac{w\left(\phi^i(t)\right)}{S_w\left(\phi(t)\right)} - \phi^i(t), \quad \text{for every } i = 1, 2, \cdots, k, \tag{3.3.29}$$

where $S_w(\phi(t)) = \sum_{j=1}^{k} w\left(\phi^j(t)\right)$. Now to show that $\phi(t) \to 1/k\mathbf{1}$ as $t \to \infty$, it is enough to show that $\min_j \phi^j(t) \to \dfrac{1}{k}$. Now define $M : \mathbb{R}^+ \to \{1, 2, \cdots, k\}$ and $m : \mathbb{R}^+ \to \{1, 2, \cdots, k\}$ such that

$$M(t) = arg \max_j \phi^j(t). \tag{3.3.30}$$

and

$$m(t) = arg \min_j \phi^j(t). \tag{3.3.31}$$

Then $\phi^{M(t)}(t) = \max_{1 \leq j \leq k} \phi^j(t) \in \left[\dfrac{1}{k}, 1\right]$, and $\phi^{m(t)}(t) = \min_{1 \leq j \leq k} \phi^j(t) \in \left[0, \dfrac{1}{k}\right]$, therefore $\phi^{m(t)}(t) - \phi^{M(t)}(t) \leq 0$ and we now show that it is increasing in t.

$$\frac{d}{dt}\left(\phi^{m(t)}(t) - \phi^{M(t)}(t)\right)$$

$$= \frac{\sum_{j=1}^{k} w\left(\phi^j(t)\right) R_{j,m(t)}}{S_w(\phi)} - \frac{\sum_{j=1}^{k} w\left(\phi^j(t)\right) R_{j,M(t)}}{S_w(\phi)} - \left(\phi^{m(t)}(t) - \phi^{M(t)}(t)\right) \quad (3.3.32)$$

The first term in the above expression is:

$$\begin{aligned}
&\frac{\sum_{j=1}^{k} w\left(\phi^j(t)\right) R_{j,m(t)}}{S_w(\phi)} - \frac{\sum_{j=1}^{k} w\left(\phi^j(t)\right) R_{j,M(t)}}{S_w(\phi)} \\
&= \frac{w\left(\phi^{m(t)}(t)\right) - w\left(\phi^{M(t)}(t)\right)}{2S_w(\phi)} + \frac{\sum_{j=1}^{k} w\left(\phi^j(t)\right) \tilde{R}_{j,m(t)} - w\left(\phi^j(t)\right) \tilde{R}_{j,M(t)}}{2S_w(\phi)} \\
&\geq \frac{w\left(\phi^{m(t)}(t)\right) - w\left(\phi^{M(t)}(t)\right)}{2S_w(\phi)} + \frac{w\left(\phi^{M(t)}(t)\right) - w\left(\phi^{m(t)}(t)\right)}{2S_w(\phi)} \\
&= 0. \qquad (3.3.33)
\end{aligned}$$

(The above inequality holds, since $\tilde{R}$ is a doubly stochastic and w is non-increasing). Therefore we get $\phi^{m(t)}(t) - \phi^{M(t)}(t)$ is increasing in t. Now combining (3.3.32) and (3.3.33) we get

$$\frac{d}{dt}\left(\phi^{m(t)}(t) - \phi^{M(t)}(t)\right) \geq \phi^{M(t)}(t) - \phi^{m(t)}(t) \geq 0. \quad (3.3.34)$$

Now, integrating equation (3.3.34) in the interval $[0, s]$, we get

$$\left(\phi^{m(s)}(s) - \phi^{M(s)}(s)\right) - \left(\phi^{m(0)}(0) - \phi^{M(0)}(0)\right) \geq \int_0^s \left(\phi^{M(t)}(t) - \phi^{m(t)}(t)\right) dt \quad (3.3.35)$$

Notice that the first term on the L.H.S. that is, $\left(\phi^{m(s)}(s) - \phi^{M(s)}(s)\right) \leq 0$ for every s and therefore we get

$$\int_0^s \left(\phi^{M(t)}(t) - \phi^{m(t)}(t)\right) dt \leq -\left(\phi^{m(0)}(0) - \phi^{M(0)}(0)\right) \quad (3.3.36)$$

The R.H.S. in the above expression is finite and independent of s, therefore we get

$$\int_0^\infty \left(\phi^{M(t)}(t) - \phi^{m(t)}(t)\right) dt < \infty$$

Now since the integrand in the above equation is positive and decreasing, we get

$$\phi^{M(t)}(t) - \phi^{m(t)}(t) \to 0, \quad \text{as } t \to \infty$$

Further it implies that, for every $j = 1, 2, \cdots, k$

$$\phi^j(t) \to \frac{1}{k}, \quad \text{as } t \to \infty.$$

This completes the proof. □

3.4 Scaling Limits

In this section, we will obtain the central limit theorems for Y_n and $\tilde{Y}_n$. Throughout this section we will consider the following two assumptions

(A1) w is a differentiable function with bounded derivatives in $[0, 1]$.

(A2) Y_n converges almost surely to the uniform vector $\frac{1}{K}\mathbf{1}$.

We will again use the stochastic approximation method to obtain central limit theorems. The rate of convergence of the discrete stochastic approximation process depends on the eigenvalues of the Jacobian matrix when evaluated at the limiting vector. For the ODE associated with Y_n, the Jacobian matrix of h at the equilibrium point $\frac{1}{K}\mathbf{1}$ as evaluated in equation (3.3.19) is given by

$$Dh\left(\frac{1}{K}\mathbf{1}\right) := \frac{\partial h(y)}{\partial y}\Big|_{y=\frac{1}{K}\mathbf{1}} = bR - \frac{b}{K}J - I \tag{3.4.1}$$

where,

$$b = \frac{w'\left(\frac{1}{K}\right)}{Kw\left(\frac{1}{K}\right)} \tag{3.4.2}$$

and if R has $s+1$ distinct eigenvalues $1, \lambda_1, \cdots, \lambda_s$, such that $1 > \Re(\lambda_1) \geq \Re(\lambda_2) \geq \cdots \geq \Re(\lambda_s)$, then the Jacobian matrix $Dh\left(\frac{1}{K}\mathbf{1}\right)$ has eigenvalues 1, $b\lambda_i - 1$ for $i = 1, \cdots, s$. Now define

$$\rho := \max\{0, 1 - b\Re(\lambda_s)\} \tag{3.4.3}$$

First, we note that the $Dh\left(\dfrac{1}{K}\mathbf{1}\right)$ is a diagonal matrix, if and only if,

$$bR_{i,j} - b/K = 0 \quad \forall i \neq j \quad \Longleftrightarrow \quad R = \frac{1}{K}J.$$

In fact, for this choice of R, we have

$$U_{n,i} = U_{0,i} + \frac{n}{K}, \quad \forall\, i = 1, 2, \cdots, K$$

which is a deterministic recursion, leading to the solution

$$Y_{n,i} = \frac{U_{0,i}}{n+1} + \frac{n}{(n+1)K} \tag{3.4.4}$$

Thus for any weight function w, if $Dh\left(\frac{1}{K}\mathbf{1}\right)$ is a diagonal matrix then we have

$$Y_{n+1,i} \to \frac{1}{K} \quad \text{as } n \to \infty, \ \ \forall\, i = 1, 2, \cdots, K. \tag{3.4.5}$$

We now present the the CLT results obtained for general Jacobian matrix $Dh\left(\frac{1}{K}\mathbf{1}\right)$ in the next three subsections, depending upon the value of ρ.

3.4.1 The case $\rho > 1/2$

Theorem 3.4.1. *Suppose w is a non-increasing function and R is a doubly stochastic matrix such that $\rho > 1/2$, then under assumptions **(A1)** and **(A2)**,*

$$\sqrt{n}\left(Y_n - \frac{1}{K}\mathbf{1}\right) \implies N(0, \Sigma_1) \tag{3.4.6}$$

and

$$\sqrt{n}\left(\tilde{Y}_n - \frac{1}{K}\mathbf{1}\right) \implies N\left(0, \tilde{\Sigma}_1\right) \tag{3.4.7}$$

with

$$\tilde{\Sigma}_1 = \frac{1}{K}\left[\Lambda_1 - \frac{1}{K(1-2b)}J\right] \quad \text{and} \quad \Sigma_1 = R^T\tilde{\Sigma}_1 R \tag{3.4.8}$$

where Λ_1 is the unique solution of the Sylvester's equation (see [21])

$$A\Lambda_1 - \Lambda_1 A^T = I \tag{3.4.9}$$

for $A = \frac{1}{2}I - bR^T$. In particular, if R is a normal matrix then

$$\Sigma_1 = \frac{1}{K}\left[R^T\left(I - b(R^T + R)\right)^{-1} R - \frac{1}{K(1-2b)}J\right] \tag{3.4.10}$$

where b is defined in equation (3.4.2).

Remark 3.4.1. Note that for a Pólya type urn, that is when $R = I$, assumption **(A2)** holds and $\rho = 1 - b > \frac{1}{2}$, therefore under assumption **(A1)** Theorem 3.4.1 holds with

$$\Sigma_1 = \frac{1}{K(1-2b)}\left[I - \frac{1}{K}J\right] = \frac{1}{1-2b}\Gamma, \tag{3.4.11}$$

where $\Gamma = \frac{1}{K}I - \frac{1}{K^2}J$.

Proof of Theorem 3.4.1 : Since $\rho > 1/2$, by Theorem A.3.1 in Appendix A we get

$$\sqrt{n}\left(Y_n - \frac{1}{K}\mathbf{1}\right) \implies N(0, \Sigma_1)$$

where

$$\Sigma_1 = \int_0^\infty \left(e^{uH}\right)^T \Gamma_1 \left(e^{uH}\right) du,$$

and

$$H = \frac{\partial h}{\partial y}\Big|_{y=\frac{1}{K}\mathbf{1}} + \frac{1}{2}I = bR - \frac{b}{K}J - \frac{1}{2}I$$

and

$$\begin{aligned}
\Gamma_1 &= \lim_{n\to\infty} R^T E\left[M_{n+1}^T M_{n+1}\Big|\mathcal{F}_n\right] R \\
&= \lim_{n\to\infty} R^T E\left[\left(\chi_{n+1} - \frac{\mathbf{w}(Y_n)}{S_w(Y_n)}\right)^T \left(\chi_{n+1} - \frac{\mathbf{w}(Y_n)}{S_w(Y_n)}\right)\Big|\mathcal{F}_n\right] R \\
&= \lim_{n\to\infty} R^T \left[E\left[\chi_{n+1}^T \chi_{n+1}\Big|\mathcal{F}_n\right] - \frac{\mathbf{w}(Y_n)^T \mathbf{w}(Y_n)}{S_w(Y_n)^2}\right] R \\
&= R^T \left[\frac{1}{K}I - \frac{1}{K^2}J\right] R.
\end{aligned}$$

Now observe that $JR = RJ = J$, because R is doubly stochastic. Therefore

$$e^{uH} = e^{buR - \frac{bu}{K}JR - \frac{u}{2}I} = e^{buR - \frac{bu}{K}J - \frac{u}{2}I}$$

Again since R commutes with J and I, we can write

$$\begin{aligned}
e^{uH} &= e^{buR} e^{-(bu/K)J} e^{-(u/2)I} \\
&= e^{-u/2}\left[\sum_{j=0}^{\infty} \frac{\left(\frac{-bu}{K}J\right)^j}{j!}\right] e^{buR} \\
&= e^{-u/2}\left[I + \sum_{j=1}^{\infty}\left(\frac{-bu}{K}\right)^j \frac{K^{j-1}J}{j!}\right] e^{buR} \\
&= e^{-u/2}\left[I + \frac{e^{-bu}-1}{K}J\right] e^{buR}
\end{aligned} \tag{3.4.12}$$

Now

$$e^{uH^T}\Gamma_1 = e^{-u/2} e^{buR^T}\left[I + \frac{e^{-bu}-1}{K}J\right] R^T \left[\frac{1}{K}I - \frac{1}{K^2}J\right] R$$

$$
\begin{aligned}
&= e^{-u/2} e^{buR^T}\left[R^T + \frac{e^{-bu}-1}{K}J\right]\left[\frac{1}{K}R - \frac{1}{K^2}J\right] \\
&= e^{-u/2} e^{buR^T}\left[\frac{1}{K}R^T R - \frac{1}{K^2}J\right] \qquad (3.4.13)
\end{aligned}
$$

$$
\begin{aligned}
e^{uH^T}\Gamma_1 e^{uH} &= e^{-u} e^{buR^T}\left[\frac{1}{K}R^T R - \frac{1}{K^2}J\right]\left[I + \frac{e^{-bu}-1}{K}J\right] e^{buR} \\
&= e^{-u} e^{buR^T}\left[\frac{1}{K}R^T R - \frac{1}{K^2}J\right] e^{buR} \\
&= e^{-u}\left[\frac{1}{K}e^{buR^T} R^T R e^{buR} - \frac{e^{2bu}}{K^2}J\right] \\
&= e^{-u} R^T\left[\frac{1}{K}e^{buR^T} e^{buR} - \frac{e^{2bu}}{K^2}J\right] R \qquad (3.4.14)
\end{aligned}
$$

The last step follows as R and e^{buR} commute. Now we can rewrite the last expression as

$$
\begin{aligned}
&= \frac{1}{K}R^T\left[e^{-u} e^{buR^T} e^{buR} - \frac{e^{-u(1-2b)}}{K}J\right] R \\
&= \frac{1}{K}R^T\left[e^{-\frac{u}{2}(I-2bR^T)} e^{\frac{u}{2}(2bR-I)} - \frac{e^{-u(1-2b)}}{K}J\right] R \qquad (3.4.15)
\end{aligned}
$$

Thus,

$$
\int_0^\infty e^{uH^T}\Gamma_1 e^{uH} du = \frac{1}{K}R^T\left[\Lambda_1 - \frac{1}{K(1-2b)}J\right] R \qquad (3.4.16)
$$

where

$$
\Lambda_1 = \int_0^\infty e^{-u(1/2I - bR^T)} e^{u(bR-1/2I)} du \qquad (3.4.17)
$$

which satisfies the Sylvesters equation :

$$
A\Lambda_1 - \Lambda_1 B = I. \qquad (3.4.18)
$$

for $A = \frac{1}{2}I - bR^T$ and $B = bR - \frac{1}{2}I = -A^T$. Now if R is a normal matrix then

$$
\Lambda_1 = (A-B)^{-1} = (I - b(R+R^T))^{-1}
$$

satisfies the Sylvesters equation, if and only if

$$
\begin{aligned}
& I = A(A-B)^{-1} - (A-B)^{-1}B \\
\iff & A - B = (A-B)A(A-B)^{-1} - B \\
\iff & A = (A-B)A(A-B)^{-1} \\
\iff & A(A-B) = (A-B)A
\end{aligned}
$$

$$\iff AB = BA$$
$$\iff AA^T = A^T A$$
$$\iff R^T R = RR^T.$$

Therefore for a normal matrix R

$$\Sigma_1 = \frac{1}{K} R^T \left[(I - b(R + R^T))^{-1} - \frac{1}{K(1-2b)} J \right] R.$$

Similarly, if $\rho > 1/2$ then by Theorem A.3.1 in Appendix A we get

$$\sqrt{n} \left(\tilde{Y}_n - \frac{1}{K} \mathbf{1} \right) \implies N(0, \tilde{\Sigma}_1)$$

where

$$\tilde{\Sigma}_1 = \int_0^\infty \left(e^{uH} \right)^T \tilde{\Gamma}_1 \left(e^{uH} \right) du,$$

for

$$H = \frac{\partial \tilde{h}}{\partial y} \Big|_{y = \frac{1}{K} \mathbf{1}} + \frac{1}{2} I = bR - \frac{b}{K} J - \frac{1}{2} I$$

and

$$\tilde{\Gamma}_1 = \lim_{n \to \infty} E \left[M_{n+1}^T M_{n+1} \Big| \mathcal{F}_n \right] = \left[\frac{1}{K} I - \frac{1}{K^2} J \right].$$

Now similar to the expression obtained in equation (3.4.14) we get

$$e^{uH^T} \tilde{\Gamma}_1 e^{uH} = e^{-u} \left[\frac{1}{K} e^{buR^T} e^{buR} - \frac{e^{2bu}}{K^2} J \right] \tag{3.4.19}$$

and therefore,

$$\tilde{\Sigma}_1 = \int_0^\infty e^{uH^T} \tilde{\Gamma}_1 e^{uH} du = \frac{1}{K} \left[\Lambda_1 - \frac{1}{K(1-2b)} J \right]. \tag{3.4.20}$$

where Λ_1 satisfies the Sylvesters equation (3.4.18). □

3.4.2 The case $\rho = 1/2$

Note that

$$\rho = \frac{1}{2} \iff \Re(\lambda_s) = \frac{Kw\left(\frac{1}{K}\right)}{2w'\left(\frac{1}{K}\right)}$$

and since $\Re(\lambda_s) \geq -1$ thus, $\rho = \dfrac{1}{2}$ case is possible only when $Kw\left(\frac{1}{K}\right) \leq -2w'\left(\frac{1}{K}\right)$. Let $\nu := \max_{1 \leq i \leq s} \{\nu_i : \Re(\lambda_i) = \Re(\lambda_s)\}$, where ν_i is the multiplicity of eigenvalue λ_i.

Theorem 3.4.2. *Let w be a non-increasing, twice differentiable weight function such that $\rho = 1/2$, then under assumption **(A2)**,*

$$\frac{\sqrt{n}}{(\log n)^{\nu-1/2}}\left(Y_n - \frac{1}{K}\mathbf{1}\right) \implies N\left(0, \Sigma_2\right) \tag{3.4.21}$$

and

$$\frac{\sqrt{n}}{(\log n)^{\nu-1/2}}\left(\tilde{Y}_n - \frac{1}{K}\mathbf{1}\right) \implies N\left(0, \tilde{\Sigma}_2\right) \tag{3.4.22}$$

where

$$\tilde{\Sigma}_2 = \frac{1}{K}\Lambda_2, \quad \textit{and} \quad \Sigma_2 = R^T\tilde{\Sigma}_2 R, \tag{3.4.23}$$

and

$$\Lambda_2 = \lim_{n\to\infty}\frac{1}{(\log n)^{2\nu-1}}\int_0^{\log n} e^{-u}e^{buR^T}e^{buR}du. \tag{3.4.24}$$

Proof: Suppose the Jacobian $Dh(\frac{1}{K}\mathbf{1})$ is not a diagonal matrix and $\rho = 1/2$, then we need to verify the the following two assumptions

1. $$\frac{1}{n}\sum_{m=1}^{n} E\left[\|M_mR\|^2 I\{\|M_mR\| \geq \epsilon\sqrt{n}\}\Big|\mathcal{F}_{m-1}\right] \to 0. \tag{3.4.25}$$

   a.s. or in L^1, for all $\epsilon > 0$.

2. For some $\epsilon > 0$, as $y \to \frac{1}{K}\mathbf{1}$

$$h(y) = h\left(\frac{1}{K}\mathbf{1}\right) + \left(y - \frac{1}{K}\mathbf{1}\right)Dh\left(\frac{1}{K}\mathbf{1}\right) + o\left(\|y - \frac{1}{K}\mathbf{1}\|^{1+\epsilon}\right) \tag{3.4.26}$$

The Linderberg condition in equation (3.4.25) holds, since from equation (3.2.14) we have

$$\|M_mR\|^2 \leq K(K+1) \quad \text{for all } m.$$

and for $\sqrt{n} > \frac{K(K+1)}{\epsilon}$, $I\{\|M_mR\| \geq \epsilon\sqrt{n}\} = 0$ for all m. The second condition (3.4.26) is also satisfied as h is twice differentiable. Thus for $\rho = 1/2$, by Theorem A.3.1 in Appendix A we get

$$\frac{\sqrt{n}}{\log n^{\nu-1/2}}\left(Y_n - \frac{1}{K}\mathbf{1}\right) \implies N\left(0, \Sigma_2\right)$$

where

$$\Sigma_2 = \lim_{n\to\infty}\frac{1}{(\log n)^{2\nu-1}}\int_0^{\log n}\exp(uH^T)\Gamma\exp(uH)du.$$

Now from equation (3.4.15), we get

$$\Sigma_2 = \frac{1}{K} R^T \left[\Lambda_2 - \lim_{n\to\infty} \frac{1-n^{-(1-2b)}}{(\log n)^{2\nu-1}} J \right] R$$
$$= \frac{1}{K} R^T \Lambda_2 R$$

where

$$\Lambda_2 = \lim_{n\to\infty} \frac{1}{(\log n)^{2\nu-1}} \int_0^{\log n} \exp\left(-u(1/2I - bR^T)\right) \exp\left(u(bR - 1/2I)\right) du. \tag{3.4.27}$$

Similarly for $\tilde{Y}_n$ the required Linderberg condition holds that is

$$\frac{1}{n} \sum_{m=1}^{n} E\left[\|M_m\|^2 I\{\|M_m\| \geq \epsilon\sqrt{n}\} \middle| \mathcal{F}_{m-1} \right] \to 0. \tag{3.4.28}$$

and therefore by Theorem A.3.1 in Appendix A we get

$$\tilde{\Sigma}_2 = \frac{1}{K} \left[\Lambda_2 - \lim_{n\to\infty} \frac{1-n^{-(1-2b)}}{(\log n)^{2\nu-1}} J \right] = \frac{1}{K} \Lambda_2$$

where Λ_2 is as given in equation (3.4.27).

□

3.4.3 The case $\rho < 1/2$

Note that

$$\rho < \frac{1}{2} \iff \Re(\lambda_s) < \frac{Kw\left(\frac{1}{K}\right)}{2w'\left(\frac{1}{K}\right)}$$

and thus, $\rho < \frac{1}{2}$ case is not possible whenever $Kw\left(\frac{1}{K}\right) > -2w'\left(\frac{1}{K}\right)$, which is true for sufficiently large K assuming that $w(0)$ and $w'(0)$ are both finite. Therefore for a negatively reinforced urn scheme, $\rho < \frac{1}{2}$ is a rare case, and in this case we have the following convergence result.

Theorem 3.4.3. *Let w be a non-increasing weight function which is twice differentiable and R be a doubly stochastic matrix, such that $0 < \rho < 1/2$, then under assumptions **(A1)** and **(A2)**, there are complex random variables $\xi_1, \cdots, \xi_s$ such that*

$$\frac{n^\rho}{(\log n)^{\nu-1}} \left(Y_n - \frac{1}{K}\mathbf{1} \right) - X_n \xrightarrow{a.s.} 0 \tag{3.4.29}$$

where

$$X_n = \sum_{i:\Re(\lambda_i)=(1-\rho)/b} e^{-i(1-bIm(\lambda_i)\log n}\xi_i v_i$$

and v_i is the left eigenvector of $Dh\left(\frac{1}{K}\mathbf{1}\right)$ with respect to the eigenvalue $b\lambda_i - 1$.

Proof: Proof of this theorem follows from Theorem A.3.1 in Appendix A. □

3.5 Examples [2]

In this section, we look at three different decreasing weight functions including linear, power law and exponential. We also describe our results for these three specific weight functions by simulations, not just to support our convergence results but also to observe the time required for the desired accuracy of convergence.

1. **Linear weight function:** Let

$$w_\theta(y) = \theta - y; \quad \theta \geq 1$$

$y \in [0,1]$, the stochastic approximation algorithm in equation (3.2.6) then holds with

$$h(y) = y\,(AR - I) \tag{3.5.1}$$

for

$$A = \frac{1}{K\theta - 1}\left(\theta J - I\right).$$

Notice that, an equilibrium point of the associated ODE is also a stationary distribution of AR. Thus if AR is irreducible and y^* is its unique stationary distribution. then since all the eigenvalues of the Jacobian matrix $AR - I$ have negative real part, we get

$$Y_n \longrightarrow y^* \;\; a.s..$$

The above almost sure convergence result was also proved in Chapter 2. If y^* is unique stationary distribution of AR then $\nu = y^*A$ is the unique stationary distribution of RA.

[2] All the simulations in this section are done using R. See `https://goo.gl/duLnQx` for the R-code used.

In fact in Chapter 2, a necessary and sufficient condition for $\hat{R} = RA$ to be irreducible is given and also convergence for the case when RA is reducible is obtained.

Simulations for convergence: We present simulation results for Pólya type urn model, that is when $R = I$ and observe the convergence to uniform vector $\frac{1}{K}\mathbf{1}$ as $n \to \infty$. To verify the convergence, we define

$$E_n(K) := \sup_{U_0 \in \{\alpha_1, \cdots, \alpha_m\}} \left\| Y_n - \frac{1}{K}\mathbf{1} \right\|_{TV} \tag{3.5.2}$$

where $\alpha_1, \cdots, \alpha_m$ are randomly chosen initial configurations from Dirichlet distribution with parameter $\frac{1}{K}\mathbf{1}$, that is m initial configurations chosen uniformly at random from the simplex $\{y \in \mathbb{R}^K : \sum_{i=1}^K y_i = 1\}$, and $\| \cdot \|_{TV}$ denotes the total variation norm. In the graphs below, we plot the $\{E_n(K)\}_{1 \le n \le 10000}$ with $m = 100$, for different number of colours $K = 2$ and 3, and observe that in both these cases the total variation distance goes to 0, as n increases, irrespective of the starting configuration.

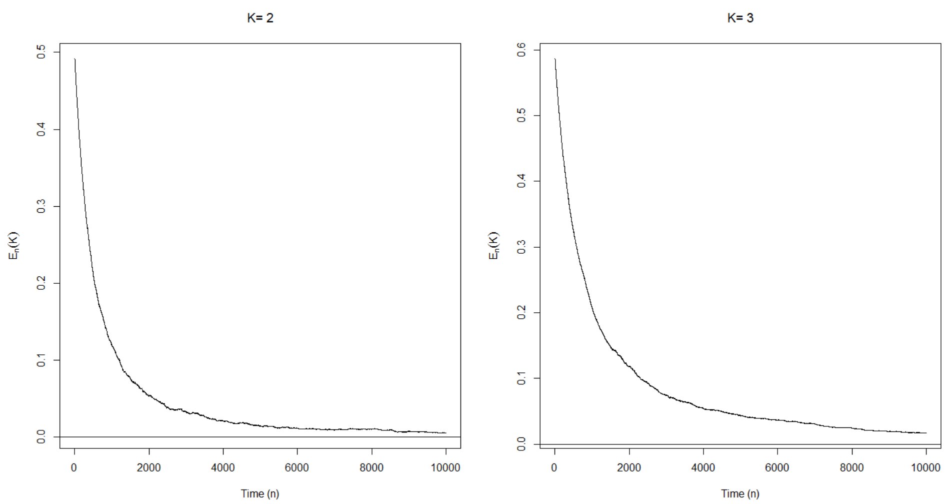


Figure 3.1: Convergence of the urn configuration to uniform for a linear weight function with $\theta = 1$.

For the central limit theorem, consider a doubly stochastic matrix R. In this case, the constant b, as defined in equation (3.4.2) is

$$b = -\frac{1}{K\theta - 1}$$

and ρ as defined in equation (3.4.3) is

$$\rho = 1 + \frac{\Re(\lambda_s)}{K\theta - 1}.$$

Now we separately consider $K = 2$ and $K \geq 3$, in order to identify the possible values of ρ.

Consider $K = 2$, and let $R = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$, where $p \in [0, 1]$. Then R has eigenvalues 1 and $2p - 1$ and

$$\rho = 1 + \frac{2p-1}{2\theta - 1}.$$

$$\rho \geq \frac{1}{2} \iff 2p - 1 \geq \frac{1 - 2\theta}{2} \tag{3.5.3}$$

Therefore, by Theorem 3.4.1 and Theorem 3.4.2, we get

$$\sigma_n \left(Y_{n,1} - \frac{1}{2} \right) \implies N\left(0, \sigma^2\right) \tag{3.5.4}$$

where

$$\sigma_n = \begin{cases} \sqrt{\frac{n}{\log n}} & \text{if the eigenvalues of } R \text{ are 1 and } \lambda = \frac{1-2\theta}{2} \text{ with } \theta \in \left[1, \frac{3}{2}\right]; \\ \\ \sqrt{n} & \text{if the eigenvalues of } R \text{ are 1 and } \lambda > \frac{1-2\theta}{2} \text{ with } \theta \in \left[1, \frac{3}{2}\right]; or\ \theta > \frac{3}{2}. \end{cases}$$

and using Theorem 3.4.3, we know that $n^{\rho}(Y_n - \frac{1}{K})$ convergence to a random variable, and this covers the case when $K = 2$, $\theta \leq \frac{3}{2}$, and eigenvalue of R is $< \frac{1-2\theta}{2}$, as mentioned in Remark 2.3.5 of Chapter 2.

Simulations for asymptotic normality: For the Pólya type urn with 2 colors, the histogram plot and the Q-Q plot obtained through simulations are given below:

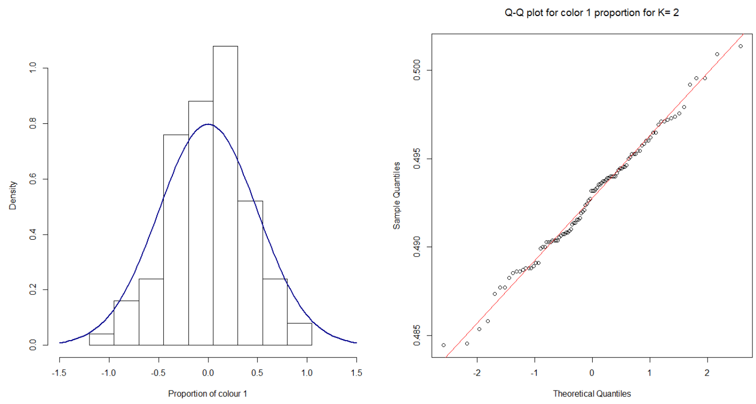


Figure 3.2: Asymptotic normality of the color 1 proportion when $\theta = 1$.

Now, in the figure below, we observe that ρ is equal to $\frac{1}{2}$ on the highlighted straight line, and below this line ρ is less than $\frac{1}{2}$. Note that for a large range of minimum eigenvalue of R and parameter θ, ρ is greater than $\frac{1}{2}$ for which the asymptotic normality holds with scaling factor $\sqrt{n}$.

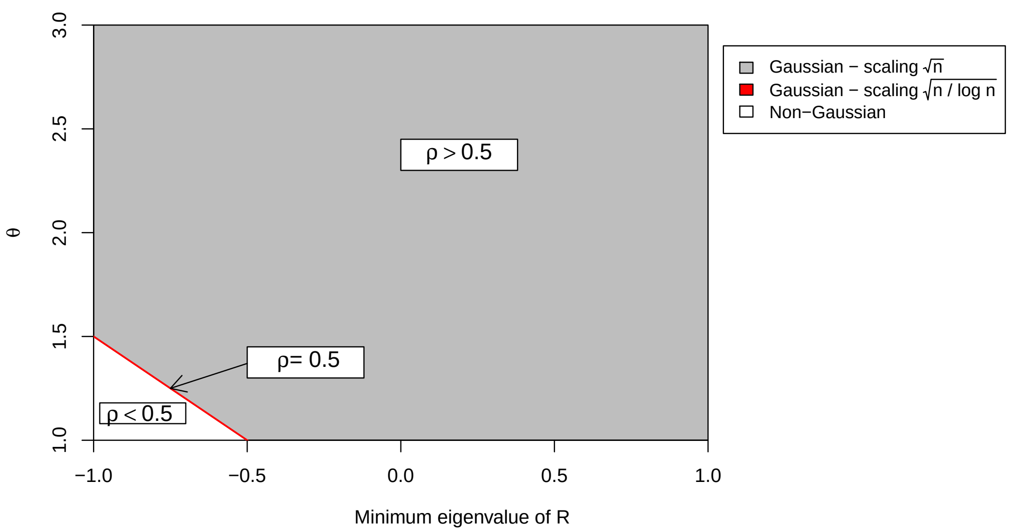


Figure 3.3: Range of ρ for given θ and minimum eigenvalue of R

Now for $K \geq 3$, using the fact that $\Re(\lambda_s) \geq -1$, we get

$$\rho \geq 1 - \frac{1}{K\theta - 1}$$

and therefore for $K \geq 3$, $\rho \geq 1/2$. Thus, there is no non-Gaussian limiting behavior. In fact in this case we always have Gaussian limit with $\sqrt{n}$ scaling, except when $\rho = \frac{1}{2}$, which can only happen when $K = 3$ and then

$$\rho = 1/2 \quad \Longleftrightarrow \quad \Re(\lambda_s) = -\frac{3\theta - 1}{2}.$$

Which is possible only when $\theta = 1$ and $\Re(\lambda_s) = -1$, and for a 3×3 stochastic matrix, there can only be at most one eigenvalue with real part equal to -1.

The above result for $K \geq 2$ and $\rho \geq \frac{1}{2}$ has already been obtained in Chapter 2. In fact central limit theorem for a general class of replacement matrices is given in Chapter 2.

2. **Inverse power law weight function:** Let

$$w(x) = (\theta + x)^{-\alpha}, \text{ for } \theta, \alpha > 0 \tag{3.5.5}$$

and R be a doubly stochastic matrix.

Then $b = -\frac{\alpha}{K\theta + 1}$ and therefore by Proposition 3.3.3, $\frac{1}{K}\mathbf{1}$ is a stable equilibrium point if

$$\Re(\lambda_s) > -\frac{K\theta + 1}{\alpha}. \tag{3.5.6}$$

In particular, the above condition for stability holds if $R = I$ or if $\alpha < K\theta + 1$. Now,

$$\rho = 1 + \frac{\alpha}{K\theta + 1}\Re(\lambda_s)$$

and then

$$\rho \geq 1/2 \iff \Re(\lambda_s) \geq -\frac{K\theta + 1}{2\alpha}.$$

Therefore, the scaling for central limit theorems depend on the values of α and θ.

In particular, for $\alpha = 1$ the condition for stability in equation (3.5.6) holds for $K \geq 2$ and thus by Theorem 3.3.2 we get

$$Y_n \longrightarrow \frac{1}{K}\mathbf{1} \;\; a.s.$$

Simulations for convergence: In the graphs below, we plot the $\{E_n(K)\}_{1\leq n\leq 10000}$ with $m = 100$, as defined in equation (3.5.2), for $K = 2$ and $K = 3$, and observe that the total variation distance goes to 0, as n increases, irrespective of the starting configuration.

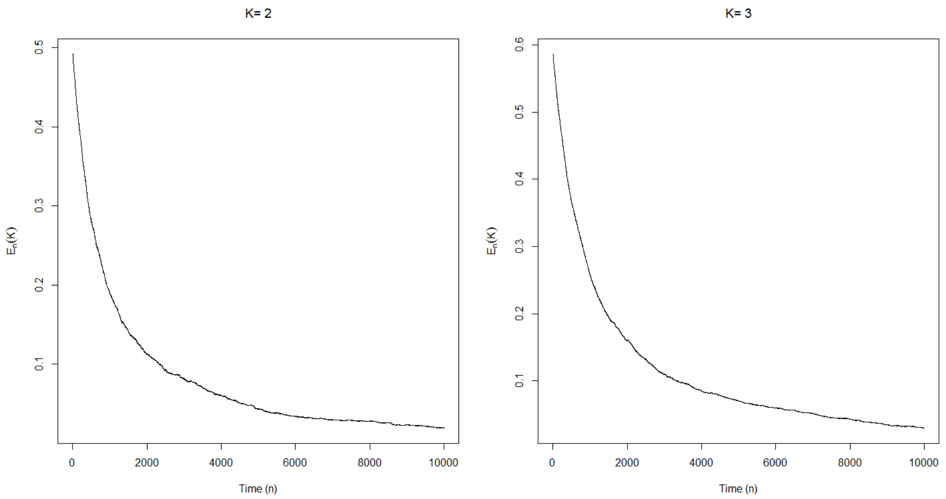


Figure 3.4: Convergence of the total variation distance to 0 when $\theta = 1, \alpha = 1$.

For the central limit theorem, as shown in the figure below ρ takes value more than $\frac{1}{2}$ in the shaded region and thus for given θ and R, the central limit theorems hold accordingly.

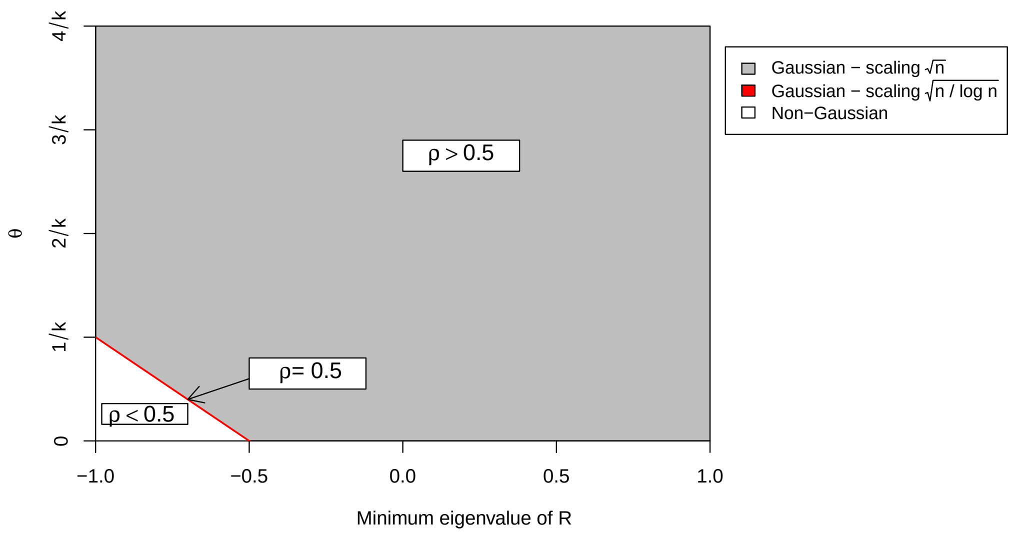


Figure 3.5: Range of ρ for given θ (with $\alpha = 1$) and minimum eigenvalue of R

Note that, above a critical value for θ, we observe asymptotic normality with the scaling factor of $\sqrt{n}$ for any choice of replacement matrix R. Further, the region for $\rho < \frac{1}{2}$ decreases as we increase the number of colours K.

Simulations for asymptotic normality: For the Pólya type urn with $K = 2$ and $K = 3$, the histogram plot and the Q-Q plot obtained through simulations are given below:

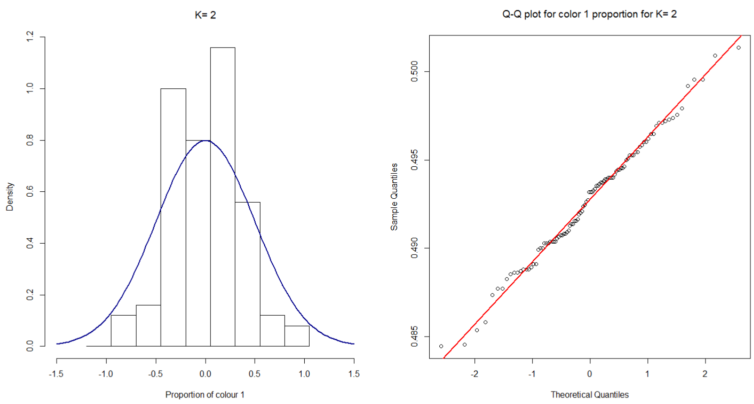


Figure 3.6: Asymptotic normality of the color 1 proportion when $\theta = 1, \alpha = 1$ and $K = 2$.

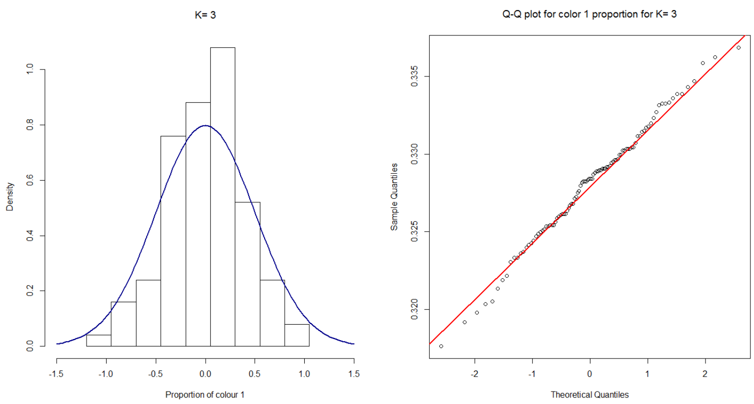


Figure 3.7: Asymptotic normality of the color 1 proportion when $\theta = 1, \alpha = 1$ and $K = 3$.

Remark 3.5.1. The inverse power law weight function with $\theta = \alpha = 1$ is $w(x) = 1/(1 + x)$, and we observe in Figure 3.4, that in this case the time required for the colour proportions to converge to uniform is much longer, as compared to the convergence time for a linear weight function. We observe the similar pattern for the inverse weight $w(x) = 1/x$, for which we present simulation results in the next chapter.

3. **Exponential weight function** Let

$$w(x) = \exp\left(-\frac{x}{\theta}\right), \text{ for } \theta > 0$$

then

$$b = -\frac{1}{K\theta} \quad \text{and } \rho = 1 + \frac{\Re(\lambda_s)}{K\theta}$$

and thus $\frac{1}{K}\mathbf{1}$ is a stable equilibrium for a doubly stochastic matrix if

$$\Re(\lambda_i) > -K\theta; \quad \text{for } i = 1, 2, \cdots, s.$$

and

$$\rho \geq 1/2 \iff \Re(\lambda_s) \geq -\frac{K\theta}{2}$$

As shown in the following graph, $\rho > \frac{1}{2}$ in the shaded region and equal to $\frac{1}{2}$ on the straight line.

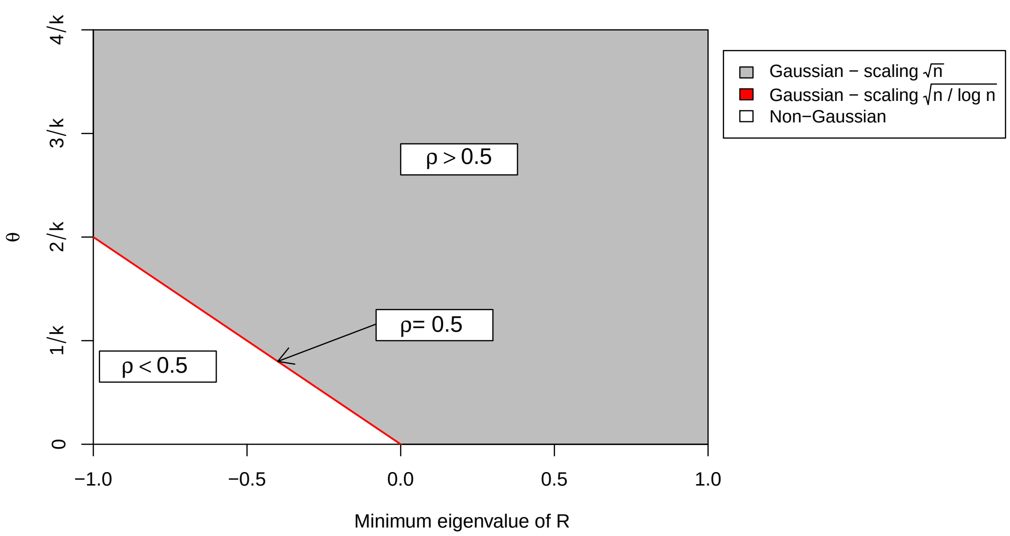


Figure 3.8: Range of ρ for given θ and minimum eigenvalue of R

Simulations for convergence: In the graphs below, we plot the $\{E_n(K)\}_{1 \leq n \leq 10000}$ as defined in equation (3.5.2), with $m = 100$, for a exponential weight function with $\theta = 1$. We observe that the total variation distance goes to 0, as n increases, irrespective of the starting configuration.

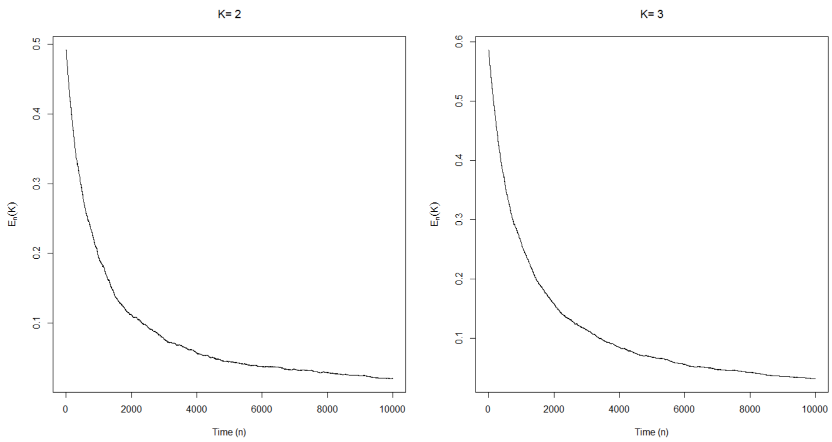


Figure 3.9: Convergence of the total variation distance to 0 when $\theta = 1$ and $R = I$.

Simulations for asymptotic normality: For the Pólya type urn with $K = 2$ and $K = 3$, the histogram plot and the Q-Q plot obtained through simulations are given below:

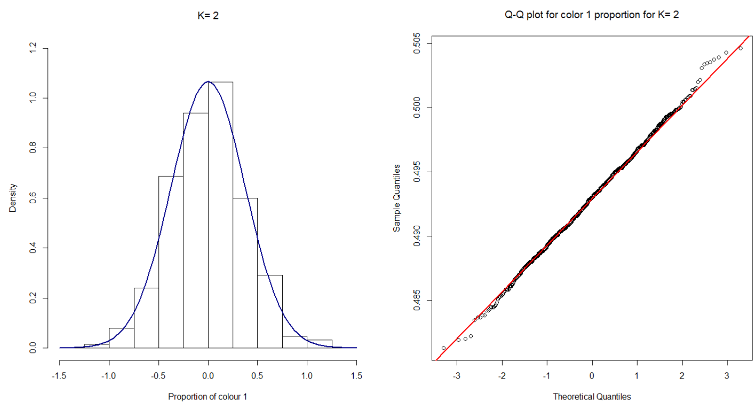


Figure 3.10: Asymptotic normality of the color 1 proportion when $\theta = 1$ and $K = 2$.

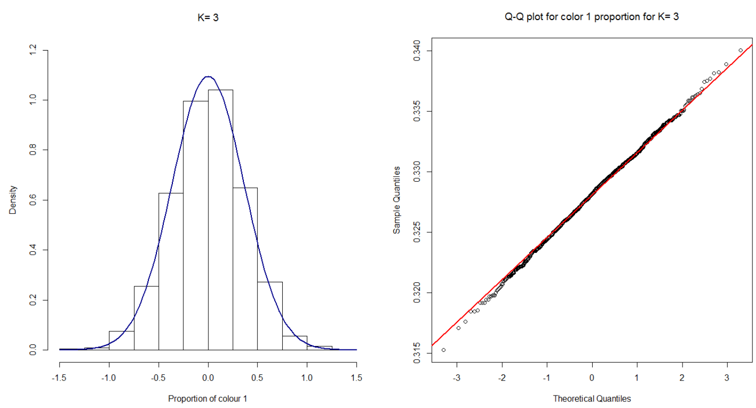


Figure 3.11: Asymptotic normality of the color 1 proportion when $\theta = 1$ and $K = 3$.

Remark 3.5.2. We observe in Figure 3.9 that the total variation distance decreases to 0 much faster when compared to other two weight functions. The reason behind this observation is that the colour with minimum proportion gets reinforced with very high probability when the weight function is exponentially decreasing.

Chapter 4

Inverse Weight Function

4.1 Model

In this chapter, we present simulation study for the inverse weight function, that is, $w(x) = \frac{1}{x}$, for $x \in (0, 1]$. For this choice of weight function, the technique used in Chapter 3 does not hold. This is because, for the asymptotic results obtained using the stochastic approximation theory, we need the approximated ODE given in equation (3.2.3) to have an unique solution. This is guaranteed under the sufficient condition that the function h is Lipschitz. However, the inverse function is not Lipschitz on $(0, 1]$. Note that, the inverse function defined defined on $[\epsilon, 1]$, for $\epsilon > 0$ is Lipschitz with all bounded derivatives. Therefore, if the initial distribution is such that for some $\epsilon > 0$, $U_{o,i} > \epsilon\ \forall i$, then we expect that the results stated in Chapter 3 should also hold for inverse weight function. However, it seems technically difficult and we have been unable to prove it. In the next section, we present simulation results as evidence of almost sure convergence and asymptotic normality.

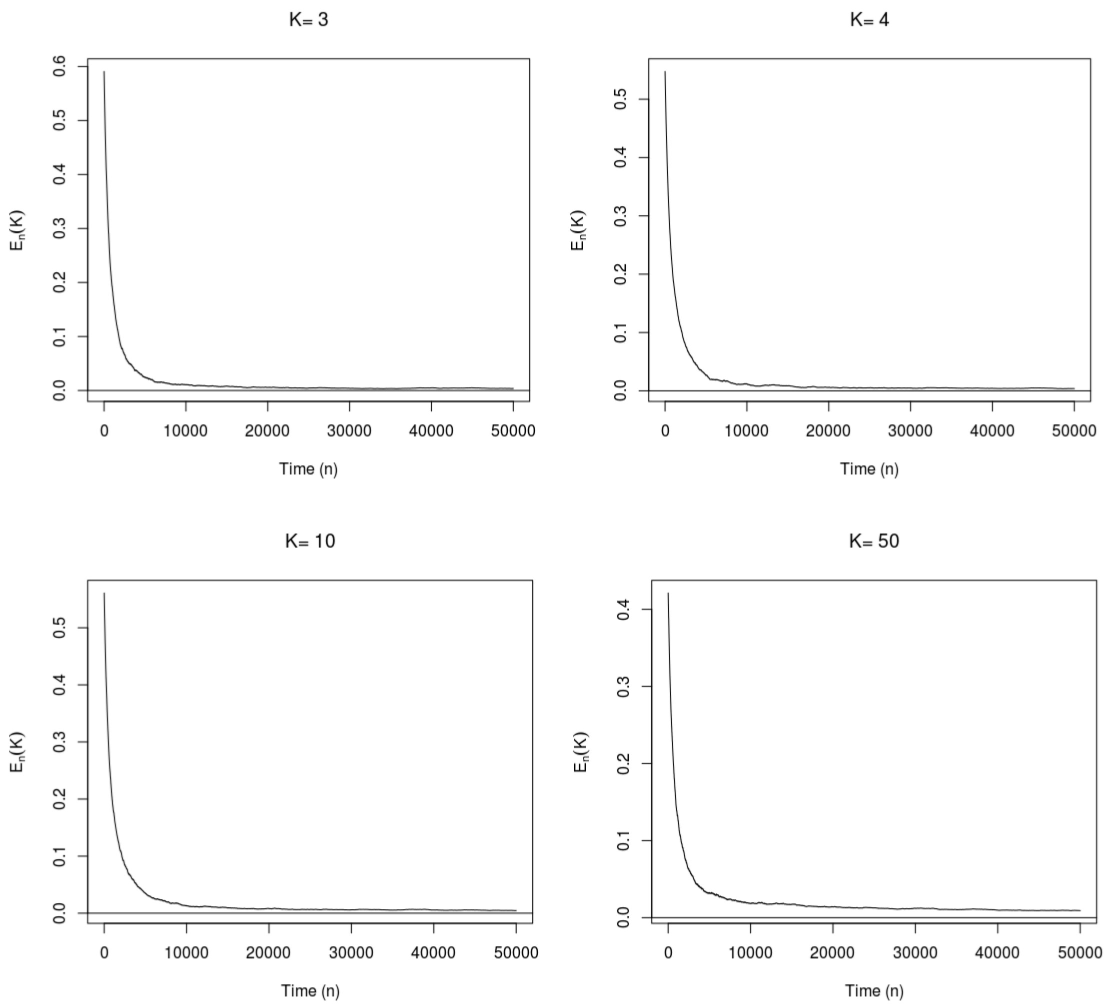
4.2 Simulation Study [1]

4.2.1 Convergence to Uniform

In this section, we present simulation results for the negatively reinforced urn model with inverse weight function and observe the convergence to uniform $\frac{1}{K}\mathbf{1}$ as $n \to \infty$. To verify the convergence, we use the function defined in equation (3.5.2) of Chapter 3 In the graphs

[1] All the simulations in this section are done using R. See `https://bit.ly/2GFuXI4` for the R-code used.

below, we plot the $\{E_n(K)\}_{1\leq n\leq 50000}$ with $m = 100$, for different number of colours $K \in \{3, 4, 10, 50, 200, 500\}$, and observe that in all these cases the total variation distance goes to 0, as n increases, irrespective of the starting configuration.



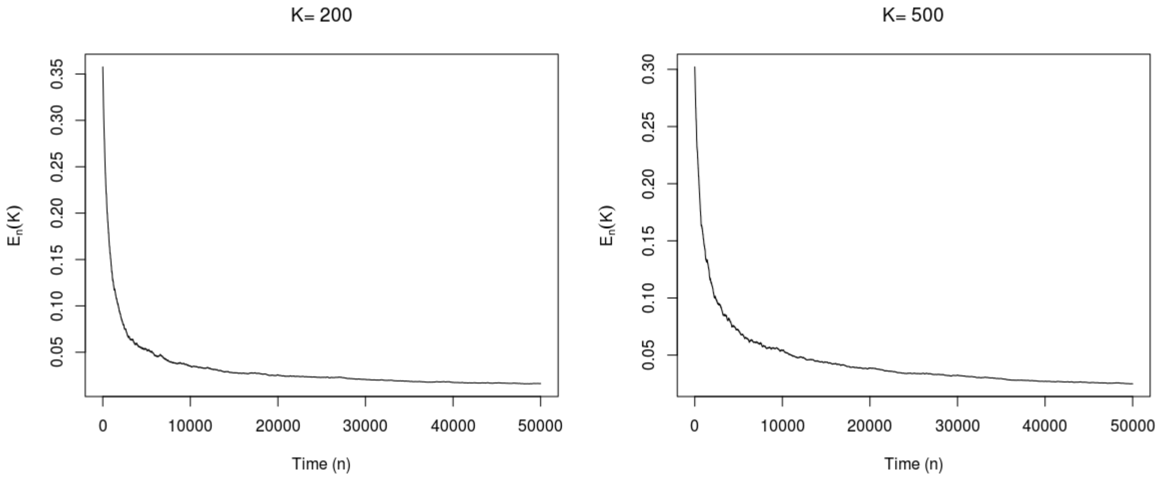
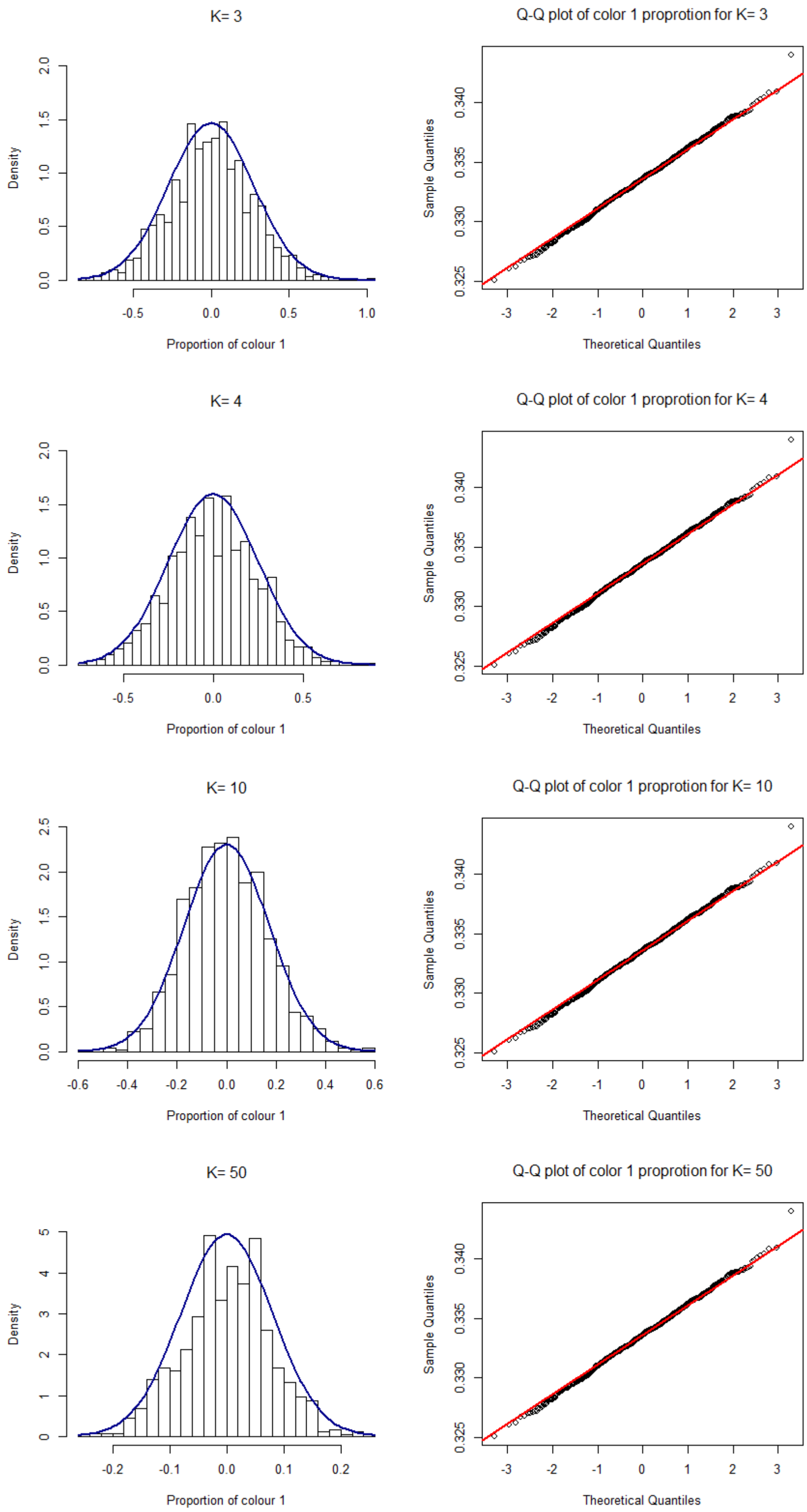


Figure 4.1: Convergence of the urn configuration to uniform for a negatively reinforced urn model with inverse weight function

4.2.2 Asymptotic Normality

We now give the simulation results for the asymptotic normality with the inverse weight function with $K \in \{3, 4, 10, 50, 200, 500\}$. To check the asymptotic normality, we present histograms and normal Q-Q plots. In a normal Q-Q plot, quantiles of the data generated are plotted against the quantiles of a normal distribution, and a straight line plot suggest that the data generated follows a normal distribution. Here we present histograms and normal Q-Q plots for 1000 iterations of the normalised proportion of colour 1, that is $\sqrt{N}\left(Y_{N,1} - \frac{1}{K}\right)$, for $N = 10000$, where initial distribution is chosen uniformly at random from Dirichlet $\left(\frac{1}{K}\mathbf{1}\right)$.

Note that for the inverse weight function the constant b (as defined in equation (3.4.2)) is equal to -1. Therefore from equation (3.4.11), we expect the limiting variance for each of the colour proportions to be equal to $\frac{1}{3}\left(\frac{1}{K} - \frac{1}{K^2}\right)$. To verify this, we overlay each of the histograms with the density curve of normal with mean 0 and variance $\frac{1}{3}\left(\frac{1}{K} - \frac{1}{K^2}\right)$.


K= 3
Density
Proportion of colour 1
Q-Q plot of color 1 proprotion for K= 3
Sample Quantiles
Theoretical Quantiles
K= 4
Density
Proportion of colour 1
Q-Q plot of color 1 proprotion for K= 4
Sample Quantiles
Theoretical Quantiles
K= 10
Density
Proportion of colour 1
Q-Q plot of color 1 proprotion for K= 10
Sample Quantiles
Theoretical Quantiles
K= 50
Density
Proportion of colour 1
Q-Q plot of color 1 proprotion for K= 50
Sample Quantiles
Theoretical Quantiles

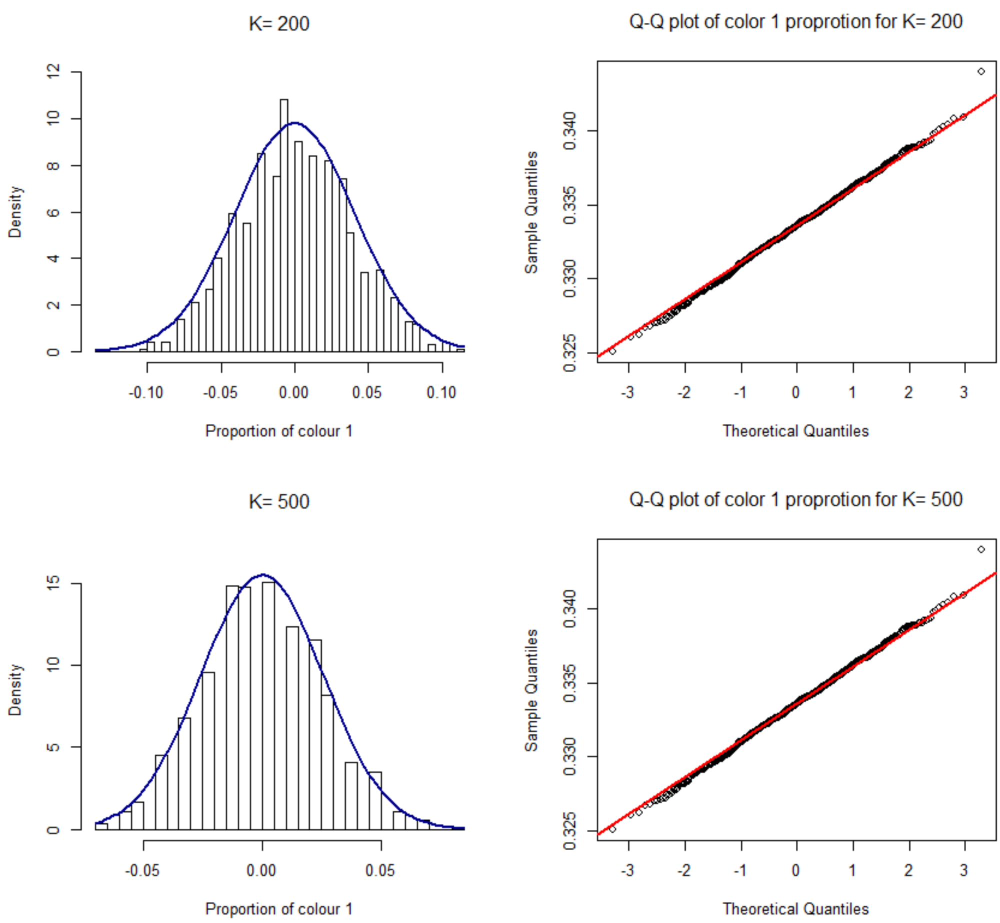


Figure 4.2: Histogram showing the asymptotic normality of colour 1 proportions in 5000 iterations.

From the above graphs, we can observe that for inverse weight function also the asymptotic normality of colour proportions should hold.

4.3 A Conjecture

As we observed in the simulations, that for a Pólya type replacement matrix that is for $R = I$ with a inverse weight function, that all the urn proportions converge to the constant $\frac{1}{K}$ also by observing the histogram and Q-Q plot, as a conclusion, we make the following conjecture:

Conjecture 4.3.1. *Suppose* $w(x) = \frac{1}{x}$, *for* $x \in (0, 1]$ *and the initial configuration* U_0 *is such that* $U_{0,i} > 0 \; \forall i$, *then*

$$Y_{n,i} \longrightarrow \frac{1}{K}\mathbf{1} \;\; a.s.. \tag{4.3.1}$$

Further,

$$\sqrt{n}\left(Y_n - \frac{1}{K}\mathbf{1}\right) \implies \mathcal{N}(0, \Sigma_1), \tag{4.3.2}$$

where Σ_1 is as given in Theorem 3.4.1 with $b = -1$.

Next we look at the dependence of initial configuration on the time required for the convergence to uniform.

4.4 Convergence Time [2]

As observed in the previous section that the urn proportions must converge to uniform for inverse weight function, for any choice of initial urn configuration vector U_0, but naturally the time required for the convergence depends on the choice of initial vector. For example, it is easy to observe that if the initial configuration is closer to uniform, then the configuration will remain closer to the uniform, on the other hand if the initial configuration is such that it is near a boundary point in the simplex $\left\{y \in \mathbb{R}^K : \sum_{i=1}^K y_i = 1\right\}$, then the time required for the urn configuration to reach in a neighbourhood of uniform will be very large.

More precisely, we define the time required for convergence to uniform by the minimum time after which the the total variation distance between urn proportions and the uniform vector is less than a given $\epsilon(> 0)$. That is, for a given $\epsilon > 0$, we define

$$T(U_0, \epsilon) := \inf\left\{n : \left\|Y_n - \frac{1}{K}\mathbf{1}\right\|_{TV} < \epsilon\right\}$$

Consider negative reinforcement urn model with $K = 3$ and U_0 takes different values in the simplex (of dimension 2) and $\epsilon = .001$. Then for a graphical representation of the dependence $T(U_0, \epsilon)$ on U_0, we divide the time required in 9 different classes and assign 9 different colours from light green to dark red, where green is for the smallest and red is for the largest value of $T(U_0, \epsilon)$.

[2]The simulations in this section are done using R. See `https://bit.ly/2GGydyI` for the R-code used.

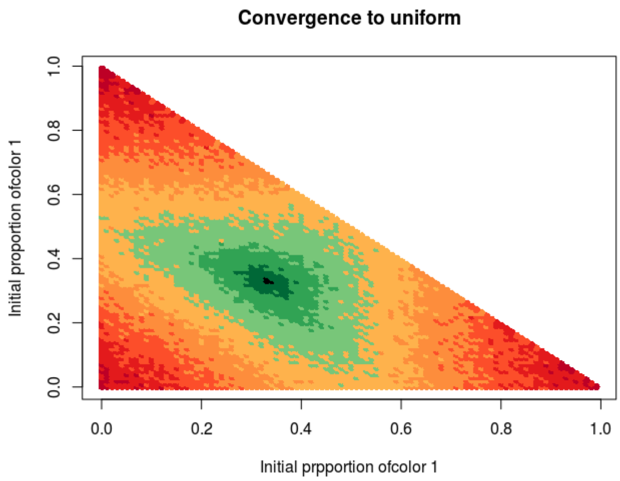


Figure 4.3: Rate of convergence to uniform for a negatively reinforced urn model with inverse weight function and 3 colours.

From the above graph we can observe that if the initial configuration is closer to the uniform vector $\frac{1}{K}\mathbf{1}$, then the total variation distance between vector of urn proportions and uniform vector becomes less than ϵ much faster than the points which are near the boundary. In fact this also explains the difficulty in studying the urn model with inverse weight function if defined on the interval $(0, 1]$, since near the boundary points of the simplex $\{y \in \mathbb{R}^K : \sum_{i=1}^K y_i = 1\}$, the time required for convergence is very large.

Chapter 5

Choice of Two in Weighted Negative Reinforcement [1]

5.1 Introduction

In this Chapter, we study an algorithm which can be used for implementation of a negatively reinforced urn scheme, defined in Chapter 1. Here instead of assigning probabilities proportional to a decreasing weight function to the colour proportion of all K colours, we choose $2 \leq d \leq K$ colours uniformly at random from the set of K colours and then reinforce one of the chosen d colours according to probabilities proportional to a decreasing weight function. Later in this Chapter we will show that $d = 2$, that is two choices not only enforces negative reinforcement, it is also optimal in some sense. This phenomenon is mostly referred as *two-choice paradigm* [8].

The two choice paradigm has been validated in the theory of balls and bins models, in the work by [8]. In ball and bins models, n balls are sequentially placed into n bins according to a given rule. It is shown in [8] that if the balls are thrown in a bin, which is selected uniformly at random then the maximum load is approximately $\frac{\log n}{\log \log n}$ with high probability. Further in [8] they showed that if the balls are placed sequentially in the least loaded of randomly chosen $d \geq 2$ bins, then the maximum load is $\frac{\log \log n}{\log d} + \Theta(1)$ with high probability. In particular, this implies that there is considerable reduction in the maximum load if we choose $d = 2$ over $d = 1$, and it further reduces only by a constant factor. In this Chapter, we show that such phenomenon

[1] This chapter is based on the paper entitled "*Choice of Two in Weighted Negative Reinforcement*" [12].

holds for urn models as well. That is, choice of two is some what optimal. Recall that, in the classical Pólya urn model, the probability of reinforcing a colour is proportional to the number of balls of that colour in the urn, and the limiting configuration in this case depends on the initial configuration. In fact, irrespective of the initial configuration the urn configuration for such models will converge to a state of disbalance with probability one. The negatively reinforced urn models discussed in the previous Chapters for load balancing problems, aims to achieve a balanced configuration asymptotically. In Chapters 2 and 3, it was shown that for the Pólya type negatively reinforced urn models, uniform vector is the limiting configuration.

Some authors have also studied the asymptotics of multiple drawing urn models [34, 29, 56, 53], where $m \geq 1$ many balls are drawn from the urn at every time $n \geq 1$ and reinforcement is done according to the observed balls. Our work in this chapter is different form multiple drawing in an urn, as studied in [34, 29, 53, 56], as we reinforce only according to one colour which is chosen according to decreasing weight function.

In this Chapter, we introduce the *weighted negatively reinforced Pólya urn models* with two or more choices and investigate their asymptotics. The dynamics of this model can be briefly described as follows:

> At every time $n \geq 1$, sample d ($\geq$ 2) colours without replacement form the set of colours and choose a colour to reinforce according to a non-increasing weight function.

We show that the urn configuration for this model converges to a uniform vector almost surely, moreover the choice of two balls makes a significant difference over reinforcing random colour and further we observe that there are only minor improvements (only by a constant factor) for more than two choices. Here, by an improved model we mean that the limiting variance of the urn configuration is less than the other, in the sense that the difference is a negative semi-definite matrix. In the next section we first describe the model in detail.

5.2 Weighted Negative Reinforcement with d Choices

Recall that $U_n = (U_{n,1}, \cdots, U_{n,K})$ denotes the configuration of the urn at time n. For a fixed $d \in \{2, 3, \cdots, K\}$, we update the urn at time $n+1$ as follows:

Step 1: `Select` $d \geq 2$ `colours from the set of colours` $\{1, 2, \cdots, K\}$`, uniformly at random without replacement`

Step 2: `Reinforce a colour out of the` d `chosen colours, according to a non-increasing weight function in` **`Step 1`**`.`

Since we only consider Pólya type replacement scheme, that is, only one ball is added into the urn at every time, the dynamics of the urn configuration can be written as

$$U_{n+1} = U_n + \chi_{n+1}. \tag{5.2.1}$$

Suppose $w : [0,1] \to \mathbb{R}^+$ be a non-increasing weight function and the d randomly chosen colours in **Step 1** are $Z_{n+1,1}, Z_{n+1,2}, \cdots, Z_{n+1,d}$, then at time $n+1$ the conditional distribution of χ_{n+1} for weighted negative reinforcement with d choices, is given by

$$\chi_{n+1} = \begin{cases} \mathbf{u}_{Z_{n+1,1}}, & \text{w.p. } \dfrac{w\left(Y_{n,Z_{n+1,1}}\right)}{\sum_{j=1}^{d} w\left(Y_{n,Z_{n+1,j}}\right)} \\ \mathbf{u}_{Z_{n+1,2}}, & \text{w.p. } \dfrac{w\left(Y_{n,Z_{n+1,2}}\right)}{\sum_{j=1}^{d} w\left(Y_{n,Z_{n+1,j}}\right)} \\ \vdots & \\ \mathbf{u}_{Z_{n+1,d}}, & \text{w.p. } \dfrac{w\left(Y_{n,Z_{n+1,d}}\right)}{\sum_{j=1}^{d} w\left(Y_{n,Z_{n+1,j}}\right)} \end{cases} \tag{5.2.2}$$

where $Y_n = \dfrac{U_n}{n+1}$. Therefore, the conditional distribution of χ_{n+1} defined in (5.2.2) is given by

$$\mathbb{P}\left(\chi_{n+1} = \mathbf{u}_i \middle| \mathcal{F}_n\right) = E\left[\mathbb{P}\left(\chi_{n+1} = \mathbf{u}_i \middle| \mathcal{F}_n\right) \middle| Z_{n+1,1}, Z_{n+1,2}, \cdots Z_{n+1,d}\right] \tag{5.2.3}$$

Let $S^{(d)}$ denotes the set of all subsets of $\{1, 2, \cdots, K\}$ which are of size d then, we can write

$$\mathbb{P}\left(\chi_{n+1} = \mathbf{u}_i \middle| \mathcal{F}_n\right) = \frac{1}{\binom{K}{d}} \sum_{s \in S^{(d)}: i \in s} \frac{w(Y_{n,i})}{\sum_{l \in s} w\left(Y_{n,l}\right)} \tag{5.2.4}$$

$$= \frac{d}{K} \frac{1}{\binom{K-1}{d-1}} \sum_{s \in S^{(d)}: i \in s} \frac{w(Y_{n,i})}{w(Y_{n,i}) + \sum_{l(\neq i) \in s} w\left(Y_{n,j}\right)} \tag{5.2.5}$$

$$= \frac{d}{K} \mathbb{E}\left[\frac{w(Y_{n,i})}{w(Y_{n,i}) + w(Y_{n,J_1}) + w(Y_{n,J_2}) + \cdots + w(Y_{n,J_{d-1}})}\right] \tag{5.2.6}$$

where $\{J_1, J_2, \cdots, J_{d-1}\}$ is a set of $d-1$ elements, sampled uniformly at random without replacement from the set $\{1, 2, \cdots, K\} \setminus \{i\}$. Recall that we extend w continuously to whole of $\mathbb{R}$, by making it a constant function outside the interval $[0, 1]$, that is, $w(y) = w(0)$ for $y \leq 0$ and $w(y) = w(1)$ for $y \geq 1$.

Proposition 5.2.1. *The process $(Y_n)_{n \geq 0}$ can be written as a stochastic approximation algorithm given by*

$$Y_{n+1} = Y_n + \gamma_n h(Y_n) + \gamma_n M_{n+1} \tag{5.2.7}$$

where $\gamma_n = \dfrac{1}{n+2} \sim \mathcal{O}(n^{-1})$, $\quad h : \mathbb{R}^K \to \mathbb{R}^K$ is such that

$$h(y)_i = \frac{d}{K} \mathbb{E}\left[\frac{w(y_i)}{w(y_i) + w(y_{J_1}) + w(y_{J_2}) + \cdots + w(y_{J_{d-1}})}\right] - y_i, \text{ for } i = 1, 2, \cdots, K \tag{5.2.8}$$

where $\{J_1, J_2, \cdots, J_{d-1}\}$ is a sample of size $d-1$ drawn uniformly at random without replacement from the set $\{1, 2, \cdots, K\} \setminus \{i\}$, and $M_{n+1} = \chi_{n+1} - E\left[\chi_{n+1} | \mathcal{F}_n\right]$ is a bounded martingale difference sequence.

Proof: We can re-write the fundamental recursion in equation (5.2.1) as

$$U_{n+1} = U_n + \mathbf{E}\left[\chi_{n+1} \middle| \mathcal{F}_n\right] + \chi_{n+1} - \mathbf{E}\left[\chi_{n+1} \middle| \mathcal{F}_n\right]$$

which implies

$$Y_{n+1} = Y_n + \frac{1}{n+2}\left[\mathbf{E}\left[\chi_{n+1} \middle| \mathcal{F}_n\right] - Y_n\right] + \frac{1}{n+2} M_{n+1}$$

where M_{n+1} is the martingale difference $\chi_{n+1} - \mathbf{E}\left[\chi_{n+1} \middle| \mathcal{F}_n\right]$. Now using 5.2.6, we get

$$Y_{n+1} = Y_n + \frac{1}{n+2} h\left(Y_n\right) + \frac{1}{n+2} M_{n+1}$$

where

$$h\left(y_1, \cdots, y_K\right)_i = \frac{d}{K}\mathbb{E}\left[\frac{w(y_i)}{w(y_i)+w(y_{J_1})+w(y_{J_2})+\cdots+w(y_{J_{d-1}})}\right] - y_i. \tag{5.2.9}$$

Therefore Y_n satisfies (5.2.7). □

Therefore the stochastic approximation algorithm in equation (5.2.7) can be approximated by the ODE:

$$\dot{y} = h(y) \tag{5.2.10}$$

where h is as given in equation (5.2.8).

Proposition 5.2.2. *The dynamics $\dot{y} = h(y)$, where h is as given as in* (5.2.8)*, is a cooperative system of differential equations, that is*

$$\frac{\partial h_i}{\partial y_j} > 0, \quad \forall i \neq j.$$

Proof: Note that for $i \neq j$

$$\begin{aligned}
\frac{\partial h_i}{\partial y_j} &= \frac{1}{\binom{K}{d}}\frac{\partial}{\partial y_j}\sum_{s\in S^{(d)}:i\in s}\frac{w(y_i)}{\sum_{l\in s} w\left(y_l\right)} \\
&= \frac{1}{\binom{K}{d}}\sum_{s\in S^{(d)}:i,j\in s}\frac{\partial}{\partial y_j}\frac{w(y_i)}{\sum_{l\in s} w\left(y_l\right)} \\
&= \frac{1}{\binom{K}{d}}\sum_{s\in S^{(d)}:i,j\in s}\frac{-w(y_i)w'(y_j)}{\left(\sum_{l\in s} w\left(y_l\right)\right)^2}
\end{aligned}$$

Since w is non-increasing, $w'(y) < 0$, we get $\dfrac{\partial h_i}{\partial y_j} > 0$, for every $i \neq j$. Thus $\dot{y} = h(y)$ is a cooperative system of differential equations. □

Lemma 5.2.1. *Suppose w is a non-increasing function, then $\frac{1}{K}\mathbf{1}$ is the unique equilibrium point of ODE in equation* (5.2.10)*.*

Proof: Note that, for $i \in \{1, 2\cdots, K\}$,

$$h\left(\frac{1}{K}\mathbf{1}\right)_i = \frac{1}{\binom{K}{d}}\sum_{s\in S^{(d)}:i\in s}\frac{1}{d} - \frac{1}{K} \tag{5.2.11}$$

$$= \binom{K}{d}^{-1}\binom{K-1}{d-1}\frac{1}{d} - \frac{1}{K} = 0 \tag{5.2.12}$$

Therefore $\dfrac{1}{K}\mathbf{1}$ is an equilibrium point. Now to prove the uniqueness, suppose $h(y) = 0$ for some y then from equation (5.2.8), for all i we get

$$\frac{d}{K}\mathbb{E}\left[\frac{w(y_i)}{w(y_i) + \sum_{l=1}^{d-1} w(y_{J_l})}\right] = y_i$$
$$\frac{d}{K}\mathbb{E}\left[\frac{w(y_i)}{w(y_i) + \sum_{l=1}^{d-1} w(y_{J_l})}\right] - \frac{1}{K} = y_i - \frac{1}{K}$$

$$\frac{1}{K}\mathbb{E}\left[\frac{\sum_{l=1}^{d-1}\left(w(y_i) - w(y_{J_l})\right)}{w(y_i) + \sum_{l=1}^{d-1} w(y_{J_l})}\right] = y_i - \frac{1}{K}$$

Without loss we can assume that y is such that $y_1 \leq y_2 \leq \cdots \leq y_K$ then, since w is a non-increasing function $w(y_1) \geq w(y_2) \geq \cdots \geq w(y_K)$. Therefore,

$$\frac{1}{K}\mathbb{E}\left[\frac{\sum_{l=1}^{d-1}\left(w(y_1) - w(y_{J_l})\right)}{w(y_i) + \sum_{l=1}^{d-1} w(y_{J_l})}\right] \geq 0$$

and then

$$\begin{aligned} y_1 - \frac{1}{K} &\geq 0 \\ \Longrightarrow y_i &\geq \frac{1}{K} \quad \forall i \\ \Longrightarrow y_i &= \frac{1}{K} \quad \forall i. \end{aligned}$$

Thus $\frac{1}{K}\mathbf{1}$ is the unique equilibrium.

□

Proposition 5.2.3. *Let w be a non-increasing function, then $\frac{1}{K}\mathbf{1}$ is a stable equilibrium.*

Proof: For h defined in equation (5.2.8), for $i \neq j$ we get form equation

$$\begin{aligned} \frac{\partial h_i}{\partial y_j}\Big|_{y=\frac{1}{K}} &= \frac{1}{\binom{K}{d}} \sum_{s\in S^{(d)}:i,j\in s} \frac{-w'(1/K)}{d^2 w(1/K)} \\ &= \frac{1}{\binom{K}{d}} \binom{K-2}{d-2} \frac{-w'(1/K)}{d^2 w(1/K)} \\ &= \frac{-(d-1)w'(1/K)}{dK(K-1)w(1/K)} \\ &= \frac{-b(d-1)}{d(K-1)} \end{aligned} \tag{5.2.13}$$

where b is as defined in Chapter 3, equation 3.4.2 that is

$$b = \frac{w'\left(\frac{1}{K}\right)}{Kw\left(\frac{1}{K}\right)} \tag{5.2.14}$$

and

$$\begin{aligned}
\frac{\partial h_i}{\partial y_i} &= \frac{1}{\binom{K}{d}} \sum_{s \in S^{(d)}: i \in s} \frac{\partial}{\partial y_i} \frac{w(y_i)}{\sum_{l \in s} w(y_l)} - 1 \\
&= \frac{1}{\binom{K}{d}} \sum_{s \in S^{(d)}: i \in s} \frac{\partial}{\partial y_i} \left(1 - \frac{\sum_{l(\neq i) \in s} w(y_l)}{\sum_{l \in s} w(y_l)} \right) - 1 \\
&= \frac{1}{\binom{K}{d}} \sum_{s \in S^{(d)}: i \in s} \frac{w'(y_i) \sum_{l(\neq i) \in s} w(y_l)}{\left(\sum_{l \in s} w(y_l)\right)^2} - 1
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial h_i}{\partial y_i}\right|_{y=\frac{1}{K}} &= \frac{1}{\binom{K}{d}} \sum_{s \in S^{(d)}: i \in s} \frac{(d-1)w'(1/K)}{d^2 w(1/K)} - 1 \\
&= \binom{K}{d}^{-1} \binom{K-1}{d-1} \frac{(d-1)w'(1/K)}{d^2 w(1/K)} - 1 \\
&= \frac{(d-1)w'(1/K)}{dKw(1/K)} - 1 \\
&= \frac{b(d-1)}{d} - 1
\end{aligned} \tag{5.2.15}$$

Therefore,

$$\left.\frac{\partial h}{\partial y}\right|_{y=\frac{1}{K}} = \frac{-b(d-1)}{d(K-1)}[J - I] + \left(\frac{b(d-1)}{d} - 1 \right) I \tag{5.2.16}$$

$$= \frac{-b(d-1)}{d(K-1)} J + \frac{b(d-1)K}{d(K-1)} I - I \tag{5.2.17}$$

$$= \frac{b(d-1)}{d(K-1)} [KI - J] - I \tag{5.2.18}$$

Note that the eigenvalues of $\left.\frac{\partial h}{\partial y}\right|_{y=\frac{1}{K}}$ are -1, $\frac{bK(d-1)}{d(K-1)} - 1$ ($K-1$ many). Since $b < 0$, all the eigenvalues are real and negative, therefore $\frac{1}{K}\mathbf{1}$ is a stable equilibrium by Remark (A.2.1) in Appendix A □

Theorem 5.2.1. *Suppose w is a non-increasing function, then for a weighted negatively rein-*

forced urn model with d choices

$$\frac{U_n}{n+1} \xrightarrow{a.s.} \frac{1}{K}\mathbf{1}. \tag{5.2.19}$$

Proof: We showed that the ODE associated is a cooperative system and for the cooperative systems, the limit is always inside the set of equilibrium points (see Theorem A.4.1 and Theorem (A.4.2) in the Appendix A). Therefore the result holds by using Lemma 5.2.1. □

To obtain the central limit theorem results, note that all the eigenvalues of $Dh\left(\frac{1}{K}\mathbf{1}\right)$ are real and $\rho = 1 - \dfrac{bK(d-1)}{d(K-1)} > \dfrac{1}{2}$, since $b < 0$. Therefore by Theorem A.3.1 in Appendix A, we get the following central limit theorem.

Theorem 5.2.2. *Suppose w is a non-increasing differentiable function, then*

$$\sqrt{n}\left(Y_n - \frac{1}{K}\mathbf{1}\right) \implies N\left(0, (1-C_{w,d})^{-1}\Gamma\right) \tag{5.2.20}$$

where $C_{w,d} = \dfrac{2b(d-1)K}{d(K-1)}$ *and* $\Gamma = \dfrac{1}{K}I - \dfrac{1}{K^2}J.$

Proof: The asymptotic normality follows by Theorem A.3.1 from Appendix A, with limiting variance matrix

$$\Sigma = \int_0^\infty \left(e^{Hu}\right)^T \Gamma e^{Hu} du,$$

where $H = Dh\left(\frac{1}{K}\mathbf{1}\right) + \frac{1}{2}I$ and

$$\Gamma = \lim_{n\to\infty} \mathbf{E}\left[M_{n+1}^T M_{n+1} | \mathcal{F}_n\right]. \tag{5.2.21}$$

Note that

$$\Gamma = \lim_{n\to\infty} \mathbf{E}\left[\chi_{n+1}^T \chi_{n+1} | \mathcal{F}_n\right] - \mathbf{E}\left[\chi_{n+1}^T | \mathcal{F}_n\right] \mathbf{E}\left[\chi_{n+1} | \mathcal{F}_n\right] \tag{5.2.22}$$

$$= \frac{1}{K}I - \frac{1}{K^2}J. \tag{5.2.23}$$

Now using (5.2.18) we get

$$H = \frac{b(d-1)}{d(K-1)}[KI - J] - \frac{1}{2}I \tag{5.2.24}$$

which is a symmetric matrix. Therefore the limiting variance covariance matrix is given by

$$\Sigma = \int_0^\infty \left(e^{uH}\right)^T \Gamma e^{uH} du$$

$$= \Gamma \int_0^\infty e^{uH} e^{uH} du \tag{5.2.25}$$

$$= \Gamma \int_0^\infty e^{-u(-2H)} du \tag{5.2.26}$$

$$= \Gamma(-2H)^{-1} \tag{5.2.27}$$

Now let $C_{w,d} = \dfrac{2b(d-1)K}{d(K-1)}$ then from equation (5.2.24) we have

$$\begin{aligned} 2H &= \left(\frac{2b(d-1)K}{d(K-1)} - 1\right) I - \frac{2b(d-1)}{d(K-1)} J \\ &= (C_{w,d} - 1)I - \frac{C_{w,d}}{K} J \\ (-2H)^{-1} &= (1 - C_{w,d})^{-1} \left(I - \frac{C_{w,d}}{K} J\right) \end{aligned}$$

Note that $(1 - C_{w,d})^{-1}$ is always defined, since $C_{w,d} < 0$. Now

$$\Sigma = (1 - C_{w,d})^{-1}\Gamma(I - \frac{C_{w,d}}{K} J) \tag{5.2.28}$$

$$= (1 - C_{w,d})^{-1}\Gamma \tag{5.2.29}$$

□

5.3 Power of Two Choices in Weighted Negative Reinforcement

To show that the two choice paradigm holds, we first compare the model of weighted negative reinforcement with two choice, with the weighted negative reinforcement model with just one choice.

5.3.1 Comparison with The Weighted Negative Reinforcement with One Choice

The weighted negatively reinforced model with one choice can be defined as the random reinforcement model with K colours, as a K dimensional Markov process $(V_n)_{n \geq 0}$ such that, at every time n a colour is chosen uniformly at random. Then for every $i = 1, \cdots, K$

$$V_{n+1,i} = \begin{cases} V_{n,i} + 1 & \text{w.p. } \frac{1}{K} \\ V_{n,i} & \text{w.p.} 1 - \frac{1}{K} \end{cases}$$

We can also write

$$V_{n+1} = V_n + \chi'_{n+1} \tag{5.3.1}$$

where

$$P\left(\chi'_{n+1} = \mathbf{u}_i\right) = \frac{1}{K}, \quad \forall i = 1, \cdots, K.$$

Theorem 5.3.1. *Consider a random reinforcement model $(V_n)_{n\geq 0}$, then as $n \to \infty$*

$$\frac{V_n}{n+1} \xrightarrow{a.s.} \frac{1}{K}\mathbf{1}, \tag{5.3.2}$$

and

$$\sqrt{n}\left(\frac{V_n}{n+1} - \frac{1}{K}\right) \implies N(0, \Gamma). \tag{5.3.3}$$

Proof: Note that $(\chi'_n)_{n\geq 1}$ are independent and identically distributed sequence of random variables. In fact for every $i = 1, \cdots, K$, $(\chi'_{n,i})_{n\geq 1}$ are identically distributed $Ber\left(\frac{1}{K}\right)$ variables and

$$V_{n,i} = V_{0,i} + \sum_{m=1}^{n} \chi'_{m,i}$$

and therefore by SLLN

$$\frac{1}{n}\sum_{m=1}^{n} \chi'_m \to \frac{1}{K}\mathbf{1} \quad a.s.$$

which implies

$$\frac{V_n}{n+1} \to \frac{1}{K}\mathbf{1} \quad a.s. \tag{5.3.4}$$

Now since $Var(\chi'_{n,i}) = \frac{1}{K} - \frac{1}{K^2}$ and $Cov(\chi'_{n,i}, \chi'_{n,j}) = \frac{-1}{K^2}$. Therefore by Lindeberg-Levy central limit theorem we get

$$\sqrt{n}\left(\frac{V_n}{n+1} - \frac{1}{K}\mathbf{1}\right) \to N(0, \Gamma) \tag{5.3.5}$$

where $\Gamma = \frac{1}{K}I - \frac{1}{K^2}J$. □

The following theorem gives a comparison between the model with two choices and random reinforcement model using the limiting variance-covariance matrices obtained in corresponding central limit theorems, Theorem 5.2.2 and Theorem 5.3.1.

Proposition 5.3.1. *Let Σ be the limiting variance matrix for the process $(U_n)_{n\geq 0}$ with two choices, and Γ be the limiting variance matrix for the process $(V_n)_{n\geq 0}$, then $\Sigma < \Gamma$ in the sense that $(\Sigma - \Gamma)$ is a negative semi-definite matrix.*

Proof: From Theorem 5.2.2 we have

$$\Sigma = (1 - C_{w,2})^{-1}\Gamma = \left(1 - \frac{bK}{K-1}\right)^{-1}\Gamma$$

and since $b < 0$, $\left(1 - \frac{bK}{K-1}\right)^{-1} < 1$, as w is a non-increasing weight function. Thus $\Sigma - \Gamma$ is a negative semi-definite matrix, as Γ is a non negative semi-definite matrix. □

5.3.2 Comparison with The Weighted Negative Reinforcement with $d \geq 3$ Choices

We again compare the two models, by comparing their limiting variance covariance matrices. Suppose Σ_d is the limiting variance matrix for the urn process U_n with d choices then for $d \geq 3$, $\Sigma_d < \Sigma_2$, but note that the constant $C_{w,d}$ in the definition of Σ_d changes only by a constant factor when compared to $C_{w,2}$.

Thus we can conclude that making two choices for weighted negative reinforcement is an optimal number of choice.

Appendix A

Review of Stochastic Approximation

A.1 Introduction

In this chapter we review some of the principle ideas of the *stochastic approximations*. A stochastic approximation algorithm as given in equation (3.2.1) generates a discrete time stochastic processes $(Y_n)_{n\geq 0}$ given by

$$Y_{n+1} = Y_n + \gamma_n \left[h\left(Y_n\right) + M_{n+1}\right],\ n \geq 0. \tag{A.1.1}$$

The theory of stochastic approximation started from the works of Robbins and Monro [69] and Kiefer and Wolfowitz [50] and has been extensively used in problems of signal processing, adaptive control [60] communication networks and artificial intelligence. Using the method of stochastic approximation one can track the long term behavior of such system, which are not deterministic and involve random changes at every time step. To analyze the long term behavior of (A.1.1) the main idea in stochastic approximation theory is to consider it as an approximation to the solution of the ordinary differential equation (ODE)

$$\dot{\mathbf{y}}(t) = h\left(\mathbf{y}(t)\right), t \geq 0. \tag{A.1.2}$$

This approach can be considered if the step sizes γ_n are small and the noise part vanishes out. The almost sure dynamics of stochastic approximation with decreasing step sizes was studied by Benaïm [20]. To formalize the relation between the two, we will make the following assumptions:

$(\mathbf{A1})$ The map $h : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz.

$(\mathbf{A2})$ Step sizes $(\gamma_n)_{n\geq 1}$ is a sequence of non-random positive real numbers satisfying

$$\sum_{n=1}^{\infty} \gamma_n = \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} \gamma_n^2 < \infty.$$

$(\mathbf{A3})$ $(M_n)_{n\geq 1}$ is a square integrable martingale difference sequence with respect to the natural filtration $\mathcal{F}_n = \sigma\left(Y_m, M_m,\ m \leq n\right)$, so that

$$\sup_{n\geq 0} \mathbf{E}\left[\|M_{n+1}\|^2\right] < \infty\ a.s.$$

$(\mathbf{A4})$ The iterates of (A.1.1) remain bounded almost surely, that is

$$\sup_{n} \|Y_n\| < \infty. \tag{A.1.3}$$

Assumption $(\mathbf{A1})$ ensures that the ODE (A.1.2) has a unique solution for any initial vector Y_0. Let $\Phi_t(y_0)$ be the unique solution of the ODE given in (A.1.2) when $\mathbf{y}_0 = y_0$. The set $\left(\Phi_t : \mathbb{R}^d \to \mathbb{R}^d\right)_{t\geq 0}$ is called the *flow* of the system which satisfies the following conditions

1. $\Phi_0 = Identity$

2. $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for every $t, s \geq 0$.

The relation of the flow to the ordinary differential equations is that the unique solution of the ODE when $Y(0) = y$ defines a unique flow $\Phi_t(y)_{t\geq 0}$, that is it satisfies

$$\frac{d\Phi_t(y)}{dt} = h\left(\Phi_t(y)\right). \tag{A.1.4}$$

Definition A.1.1. Let Φ_t be a flow, then for a continuous function $Y : \mathbb{R}^+ \to \mathbb{R}^d$,

$$d_{\Phi,t,T}\left(Y\right) := \sup_{0\leq s\leq T} d\left(Y(t+s), \Phi_s(Y(t))\right)$$

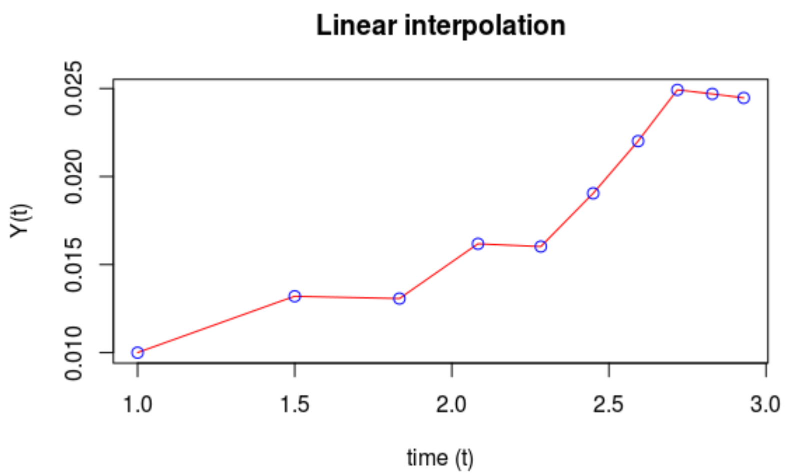
for some $T > 0$, denotes the divergence over the time interval $[t, t+T]$ between Y and the flow Φ started at $Y(t)$. The function Y is called an *asymptotic pseudotrajectory* for Φ if

$$\lim_{t\to\infty} d_{\Phi,t,T}\left(Y\right) = 0 \tag{A.1.5}$$

The next theorem by Benaïm [20] gives the relationship between the mean limit ODE and the discrete process Y_n. For this purpose, define a continuous time version of the discrete process Y_n as its linear interpolation, given by

$$Y(t_n) = Y_n,\ Y(t) = \frac{(t_{n+1} - t)}{\gamma_n} Y(t_n) + \frac{(t - t_n)}{\gamma_n} Y(t_{n+1}) \quad \text{for } t \in (t_n, t_{n+1}) \tag{A.1.6}$$

where $t_n = \sum_{i=0}^{n-1} \gamma_i$. The graph below gives the linear interpolation of a simulated process for 10 time points.



The next result shows that the stochastic approximations are asymptotic pseudotrajectories.

Theorem A.1.1 (Proposition 4.1 of Benaïm [20])**.** *The interpolated process* $(Y(t))_{t\geq 0}$ *is almost surely an asymptotic pseudotrajectory of the flow* Φ *induced by the ODE A.1.2, that is*

$$\lim_{t\to\infty} \sup_{0\leq s\leq T} d\left(Y\left(t+s\right), \Phi_s(Y(t))\right) = 0. \tag{A.1.7}$$

Theorem A.1.2 (Proposition 3.2 of Benaïm [20])**.** *Let* $Y : \mathbb{R}^+ \to \mathbb{R}^d$ *be a continuous function whose image has compact closure in* $\mathbb{R}^d$*.*

(1) Y *is asymptotic pseudotrajectory of* Φ

(2) Y *is uniformly continuous and every limit point of* $(Y(t+s) : s \geq 0)$ *is a fixed point of* Φ

(3) The sequence $(Y(t+s) : s \geq 0)_{t\geq 0}$ *is relatively compact in* $C(\mathbb{R}^+, \mathbb{R}^d)$*.*

Then (1) and (2) are equivalent and imply (3).

Proof: Let K be the compact closure of $Y(t)$. Suppose (1) holds, then by continuity of Φ and compactness of K, for an $\epsilon > 0$ there exists $a > 0$ such that

$$d\left(\Phi(y), y\right) < \frac{\epsilon}{2} \quad \text{for } |s| \leq a$$

uniformly in $x \in K$. Therefore

$$d\left(\Phi_s(Y(t)), Y(t)\right) < \frac{\epsilon}{2} \quad \forall t \geq 0, |s| \leq a.$$

Since Y is an asymptotic pseudotrajectory of Φ, there exists $t_0 > 0$ such that

$$d\left(\Phi_s(Y(t)), Y(t+s)\right) < \frac{\epsilon}{2} \quad \forall t \geq t_0, |s| \leq a$$

Thus

$$d\left(Y(t+s), Y(t)\right) < \epsilon \;\; \forall t \geq t_0, |s| \leq a$$

This proves the uniform continuity of Y, where as Lemma 3.1 of Benaïm [20] implies that any limit point of $\left(Y(t+s)\right)_{s\geq 0}$ is a fixed point of Φ. This completes the proof.
Now suppose that (2) holds. Since $(Y(t))$ is relatively compact and Y is uniformly continuous, $\left(Y(t+s)\right)_{s\geq 0}$ is equicontinuous and for each $s \geq 0$, $\left(Y(t+s)\right)_{t\geq 0}$. is relatively compact in M. Hence by Arzela-Ascoli Theorem $\left(Y(t+s)\right)_{s\geq 0}$ is relatively compact. Thus (2) implies (3). Now, if $\{Y(t+s)\}_{s\geq 0}$ is relatively compact, then by (2) we get

$$\lim_{t\to\infty} d(Y(t+s), \Phi_s(Y(t))) \to 0$$

which implies (1). Therefore (2) also implies (1) and this completes the proof. □

Proof of Theorem A.1.1: Let Y, and $\overline{Y}$ denote the linear and piecewise constant interpolation of Y_n, defined as

$$Y(t_n) = Y_n, \; Y(t) = \frac{(t_{n+1} - t)}{\gamma_n} Y(t_n) + \frac{(t - t_n)}{\gamma_n} Y(t_{n+1}); \quad \text{for } \; t \in (t_n, t_{n+1})$$

and

$$\overline{Y}(t) = Y_n; \;\; \text{for } t \in [t_n, t_{n+1})$$

where $t_n = \sum_{i=0}^{n-1} \gamma_i$. Similarly define linear interpolation of M_n as

$$M(t_n) = \sum_{i=0}^{n-1} \gamma_i M_i$$

$$M(t) = \frac{(t_{n+1} - t)}{\gamma_n} M(t_n) + \frac{(t - t_n)}{\gamma_n} M(t_{n+1}) \quad \text{for } \; t \in (t_n, t_{n+1}).$$

and piecewise interpolation as

$$\overline{M}(t) = M_n \ \text{ for } t \in [t_n, t_{n+1})$$

Then $Y(t)$ satisfies

$$Y(t) - Y(0) = \int_0^t h(\overline{Y}(u)) + \overline{M}(u)du \tag{A.1.8}$$

Now by continuity of h and assumption $(\mathbf{A4})$ there exists $C > 0$ such that $\|h(Y(t))\| \leq C$ for all $t \geq 0$. Therefore, for all $T > 0$ we get

$$\limsup_{t\to\infty} \sup_{0\leq\delta\leq T} \|Y(t+\delta) - Y(t)\| \leq CT + \limsup_{t\to\infty} \sup_{0\leq\delta\leq T} \| \int_t^{t+\delta} \overline{M}(u)du.\| \tag{A.1.9}$$

Under assumption (A2) and (A3), by Proposition 4.2 of Benaïm [20], we get

$$\lim_{t\to\infty} \sup_{0\leq\delta\leq T} \| \int_t^{t+\delta} \overline{M}(u)du\| = 0. \tag{A.1.10}$$

Thus we get

$$\limsup_{t\to\infty} \sup_{0\leq\delta\leq T} \|Y(t+\delta) - Y(t)\| \leq CT. \tag{A.1.11}$$

Hence $Y(t)$ is uniformly continuous. Define the translation flow Θ^t for $t \geq 0$ by

$$\Theta^t(Y)(s) = Y(t+s) \ \text{ for } s \geq 0.$$

Now observe,

$$\begin{aligned}
\Theta^t(Y)(s) &= Y(t) + \int_t^{t+s} h(\overline{Y}(u)) + \overline{M}(u)du \\
&= Y(t) + \int_t^{t+s} h(Y(u))du + \int_t^{t+s} h(\overline{Y}(u)) - h(Y(u))du + \int_t^{t+s} \overline{M}(u)du \\
&= L_h(\Theta^t(Y)) + A_t + B_t
\end{aligned}$$

where

$$L_h(Y)(s) = Y(0) + \int_0^s h(Y(u))du$$

$$A_t = \int_t^{t+s} h\left(\overline{Y}(u)\right) - h\left(Y(u)\right)du \ \text{ and } \ B_t = \int_t^{t+s} \overline{M}(u)du.$$

By equation (A.1.10) we get

$$\lim_{t\to\infty} B_t \to 0 \ \ a.s.$$

For any $T > 0$ and $t \in [t_n, t_{n+1})$ for $u \in [t, t+T]$ then (A.1.9) implies that

$$\begin{aligned}\|Y(u) - \overline{Y}(u)\| &= \| \int_{t_n}^{u} h(\overline{Y}(u)) + \overline{M}(u)du \; \| \\ &\leq C\gamma_n + \| \int_{t_n}^{u} \overline{M}(u)du\|\end{aligned}$$

and now by assumptions $(\mathbf{A3})$ and $(\mathbf{A4})$,

$$\sup_{t\leq u\leq t+T} \|Y(u) - \overline{Y}(u)\| \to 0$$

Let Y^* be a limit point of $\left(\Theta^t(Y)(s)\right)_{s\geq 0}$ then

$$Y^* = Y(0) + L_h(Y^*)$$

Since Φ is the flow it satisfies

$$\Phi(s) = \Phi(0) + \int_0^s h(\Phi(u))du$$

Thus by uniqueness of integral curves we get

$$Y^* = \Phi(Y^*)$$

Now by Theorem A.1.2 it follows that $(Y(t))_{t\geq 0}$ is an asymptotic pseudotrajectory of Φ. □

A.2 Convergence of Stochastic Approximation Process

For the limit set of the asymptotic pseudotrajectory $Y(t)$, let M be a subset of $\mathbb{R}^d$. Then M is invariant for the flow Φ_t if

$$\Phi_t(M) = M.$$

A compact invariant set M is said to be *internally chain transitive*, if for any $x, y \in \mathbb{R}^d$ and $\epsilon > 0, T > 0$, there exists $n \geq 1$ and points $Y_0 = x, Y_1, \cdots, Y_n = y$ in M, such that the solution of the ODE initiated at Y_i, that is $\Phi_t(Y_i)$ meets with the ϵ- neighbourhood of Y_{i+1} after time T. For a function $Y(t) : \mathbb{R}^+ \to \mathbb{R}$ define a limit set $L(Y(t))$ be the set of limit points, that is $p \in L(Y(t))$ if there exists a subsequence $(Y(t_k))_{k\geq 1}$ such that $\lim_{k\to\infty} Y(t_k) \to p$, that is

$$L\left(Y(t)\right) = \cap_{t\geq 0}\overline{Y[t,\infty)}$$

Theorem A.2.1 (Theorem 5.7 of Benaïm [20]). *The limit set $L(Y(t))$ of $Y(t)$ is almost surely an internally chain transitive set for the flow of the mean limit ODE A.1.2.*

A point $\mathbf{y}^*$ is called an equilibrium point if $\Phi_t(\mathbf{y}^*) = \mathbf{y}^*$ for all t, and when Φ is the flow induced by h, the equilibrium coincide with the zeros of h. If the system A.1.2 possesses an equilibrium point $\mathbf{y}^*$, that is $h(\mathbf{y}^*) = 0$, then $Y(t) = \mathbf{y}^*$ for all t, is a solution. These are the simplest kind of solution.

Corollary A.2.1. *If the only internally chain transitive invariant set are isolated equilibrium points of h then Y_n converges a.s. to the set of equilibrium points of h.*

A set $A \subset M$ is called an *attractor* if

1. A is non-empty compact and invariant, and
2. there exists an open neighborhood O of A with the following property:

$$d(\Phi(x,t), A) \to 0 \text{ as } t \to \infty \quad \text{uniformly in } x \in O.$$

The invariant property of set A implies that every trajectory initiated in A remains in A, and the second property here implies that every trajectory initiated in O converges to A. Such a set uniformly attracts a neighborhood of itself. Indeed, if the dynamics starts inside of the attractor, it will stay there. The *basin* of A is a positively invariant set containing all points x such that $d(\Phi(x,t), A) \to 0$ as $t \to \infty$.

A global attractor is an attractor whose basin is the whole space. An equilibrium point which is an attractor is called *asymptotically stable*. More precise definition is given below.

Definition A.2.1. The equilibrium point $\mathbf{y}^*$ is said to stable if given $\epsilon > 0$ there is a $\delta > 0$ such that $\|\Phi(t, y) - \mathbf{y}^*\| < \epsilon$ for all $t > 0$ and for all y such that $\|y - \mathbf{y}^*\| < \delta$. If δ is chosen so that $\mathbf{y}^*$ is not only stable but also $\Phi(t, y) \to \mathbf{y}^*$ as $t \to \infty$, then $\mathbf{y}^*$ is said to be asymptotically stable. If $\mathbf{y}^*$ is not stable then it is said to be unstable.

Theorem A.2.2 (Theorem 6.10 of Benaïm [20]). *Let A be an attractor with basin W and $B \subset W$ be a compact subset. If $Y(t_k) \in B$ for some sequence $t_k \to \infty$, then*

$$L(Y) = \cap_{t \geq 0} \overline{Y([t, \infty)} \subset A.$$

Since $\mathbb{R}^d$ is locally compact (by Heine Borel Theorem), from Theorem 7.3 in [20], we have the following theorem:

Theorem A.2.3 (Convergence with positive probability toward an attractor)**.** *Let $A \subset \mathbb{R}^d$ be an attractor for Φ with basin of attraction $B(A)$, such that there exists a point $p \in B(A)$ such that for every $t > 0$ and every open neighborhood U of p*

$$P(\exists s \geq t : Y(s) \in U) > 0$$

then

$$P(L(Y) \subset A) > 0$$

Now we consider a special form of recursion, which is useful in the context of urn models. Suppose the recursion can be written as

$$Y_{n+1} = Y_n + \gamma_n[F(Y_n) - Y_n] + \gamma_n M_{n+1} \tag{A.2.1}$$

that is $h(y) = F(y) - y$, for a continuous function F. Let $E = \{p : F(p) = p\}$ be the set of fixed points of F or the set of equilibrium points, then for $K = 2$ we have

Theorem A.2.4 ([66] Corollary 2.7)**.** *If F is a continuous function then*

$$P\left(\lim_{n\to\infty} Y_n \in E\right) = 1 \tag{A.2.2}$$

The above Theorem only says that for a 2-dimensional stochastic approximation the possible limit points are inside the set of fixed points. In 1980 *Hill, Lane and Sudderth* [46] first considered the two colour case, that is $K = 2$, and showed that not all fixed point of F can be the limit point of Y_n satisfying equation (A.1.1). Later in 1986 *Arthur, Ermoliev and Kaniovski* [6] generalized the results to higher dimensional situations by classifying fixed points into stable and unstable points.

Theorem A.2.5 ([6])**.** *If p is a stable fixed point of F then*

$$P(Y_n \to p) > 0 \tag{A.2.3}$$

if p is an unstable fixed point of F and is not a vertex of a d-simplex then

$$P(Y_n \to p) = 0. \tag{A.2.4}$$

The next theorem states that if there is a unique equilibrium point and the function F is a contraction then the random process Y_n converges to the equilibrium point almost surely.

Theorem A.2.6 ([23] Section 10.3 Theorem 2.)**.** *Suppose F is a contraction function that is*

$$\|F(x) - F(y)\| < \|x - y\| \tag{A.2.5}$$

and if y^* is its unique fixed point then, $V(y,t) = \|y(t) - y^*\|$ (with the same norm as in equation (A.2.5)*) is a strictly decreasing function for any non-constant solution $y(t)$ of the ODE given in equation* (A.1.2)*. Moreover y^* is the unique globally asymptotically stable equilibrium point.*

In the next two sections we look at the linear ODE and linearization of the nonlinear ODE respectively.

A.2.1 Linear Differential Equation

Let A be a $d \times d$ matrix and

$$\dot{\mathbf{y}} = A\mathbf{y} \tag{A.2.6}$$

be a linear differential equation in $\mathbb{R}^d$. The solution of (A.2.6) can be written as $e^{At}Y(0)$, where e^{At} is given by

$$e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots$$

The components $\Phi_i(t)$ of the solution of the linear ODE (A.2.6) are linear combinations of the following functions

1. $e^{\lambda t}$ whenever λ is a real eigenvalue of A;
2. $e^{at}\cos bt$ and $e^{at}\sin bt$, that is real and imaginary part of $e^{\mu t}$, whenever $\mu = a + ib$ is a complex eigenvalue of A,
3. $t^j e^{\lambda t}$, or $t^j e^{at}\cos bt$ and $t^j e^{at}\sin bt$ for $0 \leq j < m$, if the eigenvalue λ or μ with multiplicity m

where the coefficients are the corresponding eigenvectors. The oscillatory part introduced by the complex eigenvalue μ can be neglected if and only if $a < 0$. An equilibrium point is called hyperbolic if no eigenvalue of A has real part 0.

Remark A.2.1. The sufficient condition for an equilibrium of a linear ODE to be asymptotically stable is that the real part of all the eigenvalues of matrix A are negative. Since then the exponential part of the solution converges to 0 as $t \to \infty$.

For the non-convergence phenomenon of the stochastic approximation the results are available for linearly unstable points, which requires one of the eigenvalue of the Jacobian matrix of h has positive real part.

Theorem A.2.7. *Let Y_n be a stochastic approximation process and let p be a linearly unstable equilibrium for the flow induced by the ODE* (A.1.2) *then*

$$P\left(\lim_{n\to\infty} d(Y_n, p) = 0\right) = 0 \tag{A.2.7}$$

A.3 Central Limit Theorems

We now state the central limit theorem results known in stochastic approximation theory, assuming that the process $(Y_n)_{n\geq 1}$ converges almost surely to y^*. The eigenvalues of the Jacobian matrix of h at the convergence point y^* determine the scaling order for the CLT results. Let $Dh(y^*)$ be the Jacobian matrix of h at $y = y^*$ and $\lambda_1, \cdots, \lambda_s$ be s distinct the eigenvalues of $Dh(y^*)$. Suppose the Jordan canonical decomposition of $Dh(y^*)$ is given by

$$T^{-1}Dh(y^*)T = diag(J_1, J_2, \cdots, J_s)$$

for a $d \times d$ invertible matrix T such that each J_i is of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & 0 & \lambda_i & \ddots & \vdots \\ 0 & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & \lambda_i \end{bmatrix}.$$

Define $\rho := -\max_{1\leq i\leq s}\{0, \Re(\lambda_i)\}$ and $\nu = \max_{1\leq i\leq s}\{\nu_i : \Re(\lambda_i) = -\rho\}$, where $\nu_i := dim(J_i)$.

Theorem A.3.1 ([20, 82])**.** *Suppose y^* is a stable equilibrium point of h, such that $Y_n \to y^*$ a.s.*

and

$$E\left[M_{n+1}^T M_{n+1}\middle|\mathcal{F}_n\right] \to \mathbf{\Gamma} \quad a.s. \text{ or in } L_1$$

where $\mathbf{\Gamma}$ *is deterministic symmetric positive semidefinite matrix, and the Lindeberg condition is satisfied, that is for every* $\epsilon > 0$

$$\frac{1}{n}\sum_{m=1}^{n} E\left[\|M_m\|^2 \mathbf{I}\{\|M_m\| \geq \epsilon\sqrt{n}\}\middle|\mathcal{F}_{m-1}\right] \to 0 \;\; a.s..$$

Then

1. *If* $\rho > \frac{1}{2}$, *and*

$$h(y) = h(y^*) + (y - y^*)Dh(y^*) + o(\|y - y^*\|)$$

as $y \to y^*$, *then*

$$\sqrt{n}\,(Y_n - y^*) \;\Rightarrow\; N\,(0, \Sigma)$$

where

$$\Sigma = \int_0^\infty \left(e^{\left(Dh(y^*)+\frac{1}{2}I\right)u}\right)^T \mathbf{\Gamma} e^{\left(Dh(y^*)+\frac{1}{2}I\right)u} du \tag{A.3.1}$$

2. *If* $\rho = \frac{1}{2}$ *and for some* $\delta > 0$

$$h(y) = h(y^*) + (y - y^*)Dh(y^*) + o(\|y - y^*\|^{1+\delta})$$

as $y \to y^*$, *then*

$$\frac{\sqrt{n}}{\log n^{\nu-\frac{1}{2}}}\,(Y_n - y^*) \;\Longrightarrow\; N\left(0, \tilde{\Sigma}\right).$$

where

$$\tilde{\Sigma} = \lim_{n\to\infty} \frac{1}{(\log n)^{2\nu-1}} \int_0^{\log n} \left(e^{\left(Dh(y^*)+\frac{1}{2}I\right)u}\right)^T \mathbf{\Gamma} e^{\left(Dh(y^*)+\frac{1}{2}I\right)u} du$$

3. *For* $0 < \rho < \frac{1}{2}$, *if*

$$h(y) = h(y^*) + (y - y^*)Dh(y^*) + o(\|y - y^*\|^{1+\delta}) \quad \text{as } y \to y^*$$

for some $\delta > 0$ *and*

$$E\left[(M_m)^T M_m\middle|\mathcal{F}_{m-1}\right] = \mathcal{O}(n) \quad a.s. \text{ or in } L_1.$$

Then there are complex random variables $\xi_1, \cdots, \xi_s$ *such that*

$$n^\rho\,(Y_n - y^*) - X_n \;\overset{a.s.}{\longrightarrow}\; 0$$

where X_n is random vector defined as

$$X_n = \sum_{i:\Re(\lambda_i)=\rho,\nu_i=\nu} n^{-iIm(\lambda_i)}\xi_i v_i$$
$$= \sum_{i:\Re(\lambda_i)=\rho,\nu_i=\nu} e^{-iIm(\lambda_i)\log n}\xi_i v_i$$

where v_i is the right eigenvector of $Dh(y^*)$ with respect to the eigenvalue λ_i.

A.4 Monotone Dynamical Systems

The system of ordinary differential equation (A.1.2) models a competing system if

$$\frac{\partial h_i}{\partial \mathbf{y}_j} \leq 0, \text{ for } i \neq j \tag{A.4.1}$$

and it models a cooperative sysmtem if

$$\frac{\partial h_i}{\partial \mathbf{y}_j} \geq 0, \text{ for } i \neq j$$

and it is called *monotone* if it is either competing or cooperative system. For example, *Kolmogorov model* of d cooperating species is given by

$$\dot{\mathbf{y}}_i = \mathbf{y}_i F_i(\mathbf{y}) \quad \text{for } i = 1, 2, \cdots, d. \tag{A.4.2}$$

where each $\mathbf{y}_i \geq 0$ and

$$\frac{\partial F_i}{\partial \mathbf{y}_j} \geq 0, \text{ for } i \neq j.$$

As shown by Hirsh and Smith [47], the long term behavior of the monotone dynamical systems is severely limited, in particular, it is shown that if the flow of the ODEs are bounded then the flow induced by every initial state converges to an equilibrium. In particular, in case of a unique equilibrium the process converges to the equilibrium almost surely. To formalize these results we need the following definitions:

Definition A.4.1 (Ordered space)**.** A metric space X is called an ordered space if it is endowed with a metric d and an order relation $R \subset X \times X$, such that $(x, y) \in R$ if and only if $x \leq y$

where $\leq$ is partial order.

Definition A.4.2 (Monotone semiflow)**.** A map $f : X_1 \to X_2$ between ordered spaces is *monotone* if

$$x \leq y \implies f(x) \leq f(y)$$

strictly monotone if

$$x < y \implies f(x) < f(y)$$

Suppose the solution of the ODE is defined, then let $\Phi = \{\Phi_t : \mathbb{R}^+ \to \mathbb{R}^n\}$ denote the resulting semiflow such that the solution with initial value y is given by $y(t) = \Phi_t(y)$. Further if the dynamical system is cooperative then Φ_t preserves the order, that is for $t \geq 0$

$$x \leq y \implies \Phi_t(x) \leq \Phi_t(y)$$

In other words Φ_t is monotone.

A.4.1 Basic Results

Theorem A.4.1 (Hirsch and Smith [47]: Convergence Criterion)**.** *Suppose the flow Φ is monotone and if the set $\{T > 0 : \Phi_T(x) \geq x\}$ is open and nonempty. then the trajectory initiated at a point x converges to an equilibrium.*

The condition mentioned in the above theorem is not very easy to verify since the flow is not always available for a ODE with arbitrary function h. The next theorem states the convergence of the empirical measure. The empirical occupation measure of the process Y_n is the random measure defined by

$$\tau_n(A) = \frac{1}{n+1} \sum_{i=0}^{n} \mathbf{1}_A(Y_i)$$

for every Borel set $A \subset \mathbb{R}^d$

Theorem A.4.2 (Pemantle Theorem 4.3)**.** *If h is cooperative and the Jacobian of h is irreducible then as the step size $\gamma_n \to 0$, the support of the empirical measure of the stochastic approximation process converge in probability to the set of equilibria.*

If in addition either h is real analytic or has only finitely many stable equilibria, then the support of the empirical distribution converges to an asymptotically stable equilibrium.

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