Recursive Distributional Equations and Recursive Tree Processes : Lecture - III

Antar Bandyopadhyay

(Joint work with Professor David J. Aldous)

[Work done at UC, Berkeley and IMA, Minneapolis]

Mini-Workshop on Recursive Distributional Equations
J.W. Goethe Universitt Frankfurt a.M., Germany

Department of Mathematics
Chalmers University of Technology
Gothenburg, Sweden

March 9, 2005

Tree-Structured Coupling From the Past

Finite RTF :

Given a RDE

$$X \stackrel{d}{=} g\left(\xi; \left(X_j, 1 \leq j \leq^* N\right)\right) \quad \text{on} \quad S,$$

we can define a RTF

$$\left(\mathbb{T}_\infty; \left(\xi_{\mathbf{i}}, N_{\mathbf{i}}\right)_{\mathbf{i} \in \mathcal{V}}; g\right).$$

Remark : Associated with a RTF there is a Galton-Watson branching process tree rooted at $\emptyset$ defined only through $\{N_{\mathbf{i}} \,|\, \mathbf{i} \in \mathcal{V}\}$, call it $\mathcal{T}$. Essentially any associated invariant RTP lives on $\mathcal{T}$.

Proposition 1 *If $\mathcal{T}$ is almost surely finite (equivalently $\mathbf{E}[N] \leq 1$ and $\mathbf{P}(N = 1) < 1$) then the associated RDE has unique solution and the RTP is endogenous.*

Domain of the Function g

- The innovation ξ takes values in some measurable space $(\Theta, \mathfrak{F})$.

- Recall our sample space is S.

- The function g which takes values in S, is defined on the space

$$\Theta^* := \Theta \times \bigcup_{0 \leq d \leq \infty} S^d.$$

Here S^∞ is the usual infinite product space and $S^0 := \{\Delta\}$ where Δ is some "known object" !

Proof of Proposition 1 :

- Let $\mathfrak{I}$ be the set of all finite rooted trees with vertex weights.
- We define a function $h : \mathfrak{I} \to S$ as follows
  - ▶ Let $\mathcal{T} \in \mathfrak{I}$ with weights $(w_{\mathbf{i}})$.
  - ▶ If a vertex $\mathbf{i}$ is a leaf then define

$$x_{\mathbf{i}} := g\left(w_{\mathbf{i}}; \varDelta\right).$$

  - ▶ For an internal vertex $\mathbf{i}$ with $n_{\mathbf{i}} \geq 1$ children recursively define

$$x_{\mathbf{i}} := g\left(w_{\mathbf{i}}; \left(x_{\mathbf{i}j}, 1 \leq j \leq n_{\mathbf{i}}\right)\right)$$

  - ▶ Take $h\left(\mathcal{T}\right) = x_{\emptyset}$, where $\emptyset$ is the root of $\mathcal{T}$.

Continuing ...

- GW tree $\mathcal{T}$ with the node weights $(\xi_{\mathbf{i}})$ is an element of $\mathfrak{I}$, let $X_\emptyset$ be the h value of it.

- For a vertex $\mathbf{i}$ of $\mathcal{T}$ let $\mathcal{T}_{\mathbf{i}}$ be the family tree generated by $\mathbf{i}$. Then $\mathcal{T}_{\mathbf{i}}$ with the node weights is also an element of $\mathfrak{I}$, let $X_{\mathbf{i}}$ be it's h value.

- It follows from definition of h that $(X_{\mathbf{i}})$ is a RTP with some marginal. Thus the RDE has a solution and it is endogenous.

- Finally if μ is a solution of the RDE, let $(Y_{\mathbf{i}})$ be invariant RTP with marginal μ. From definition for a leaf $\mathbf{i}$ we must have $Y_{\mathbf{i}} = X_{\mathbf{i}}$ a.s. Now since the tree is a.s. finite so by recursion we get

  $$Y_\emptyset = X_\emptyset \text{ a.s.}$$

  This proves the uniqueness.

Pointwise Monotone g-function :

Proposition 2 *Suppose $S = \mathbb{R}^+$. Assume the following properties for g*

(i) For each fixed θ and $1 \leq n \leq \infty$,

$$g\left(\theta; \left(x_j, 1 \leq j \leq^* n\right)\right) \leq g\left(\theta; \left(y_j, 1 \leq j \leq^* n\right)\right),$$

whenever $x_j \leq y_j$ for all $1 \leq j \leq^* n$.

(ii) For each fixed θ, the map $\mathbf{x} \mapsto g\left(\theta; \mathbf{x}\right)$ is continuous with respect to increasing limits.

Suppose further that, for the operator T, the sequence $\left(T^n\left(\delta_0\right)\right)_{n \geq 0}$ is tight. Then

$$T^n\left(\delta_0\right) \stackrel{d}{\longrightarrow} \mu$$

where μ is a solution of the RDE and the invariant RTP with marginal μ is endogenous.

Remarks :

- In many applications the function g will naturally satisfy the assumptions of the proposition.
- The tightness of the sequence $(T^n(\delta_0))_{n\geq 0}$ is equivalent to having a solution supported on $\mathbb{R}^+$.
- μ when exists has the property that $\mu \preccurlyeq \nu$ for any other solution ν.
- In context of *interacting particle system* μ parallels the concept of *lower invariant measure*.
- The theorem do not provide any obvious uniqueness criterion !

Proof of Proposition 2 :

- Notice that under assumption (i), $\left(T^n\left(\delta_0\right)\right)_{n\geq 0}$ is increasing in stochastic-ordering.

- Tightness of $\left(T^n\left(\delta_0\right)\right)_{n\geq 0}$ implies $T^n\left(\delta_0\right) \stackrel{d}{\longrightarrow} \mu$.

- Moreover by (ii), μ is a fixed-point of T.

- Let $\left(X_{\mathbf{i}}\right)_{\mathbf{i}\in\mathcal{V}}$ be the RTP with marginal μ.

- For each $d \geq 0$ we define a d-*depth RTP*, say $\left(X_{\mathbf{i}}^{(d)}\right)_{|\mathbf{i}|\leq d}$, which satisfy the recursion with same innovations but has $X_{\mathbf{i}}^{(d)} = 0$ when $|\mathbf{i}| = d$.

- Then by assumption (i) we get

$$0 \leq X_{\emptyset}^{(1)} \leq X_{\emptyset}^{(2)} \leq \cdots \leq X_{\emptyset} \text{ a.s.}$$

- On the other hand from definition $X_{\emptyset}^{(d)} \sim T^d\left(\delta_0\right)$.

- Thus $X_{\emptyset}^{(d)} \uparrow X_{\emptyset}$ a.s. which proves the endogeny.

Examples of "Max"-type RDEs

Discounted Tree Sums

Consider the following RDE

$$X \stackrel{d}{=} \eta + \max_{1 \leq j \leq^* N} \xi_j X_j \quad \text{on} \quad S = \mathbb{R}^+,$$

where $(\eta; (\xi_j, 1 \leq j \leq^* N))$ has a given law and is independent of $(X_j)_{j \geq 1}$ which are i.i.d. copies of X. We will consider the general case where $N \leq \infty$ and possibly random.

A Brief History :

- There are many interesting examples studied by various authors [Athreya (1985), Devroye (2001), Durrett and Limic (2002)] falling under this general RDE.

- Special case of non-random finite N has been studied by Rachev and Rüschendorf (1998).

- The *homogeneous* case, that is by taking $\eta = 0$ a.s. was considered independently by Jagers and Rösler (2004).

- In a recent work Neininger and Rüschendorf (2005) considered a more general form with multiple η's but non-random N.

A "Story" and A Potential Solution

- For simplicity let us assume $N = 2$ a.s.
- $(\eta_{\mathbf{i}}; (\xi_{\mathbf{i}1}, \xi_{\mathbf{i}2}))_{\mathbf{i} \in \mathcal{V}}$ be the i.i.d. innovation process.
- Think $\eta_{\mathbf{i}}$ to be the weight of vertex $\mathbf{i}$.
- Think $\xi_{\mathbf{i}1}$ and $\xi_{\mathbf{i}2}$ as edge-weights for the two edges coming out of the vertex $\mathbf{i}$.
- For the path $(\emptyset = v_0, v_1, \ldots, v_d)$ from the root $\emptyset$ to v_d define the *influence* of vertex v_d at the root as

$$\eta_{v_d} \prod_{k=1}^{d} \xi_{(v_{k-1}, v_k)}.$$

- For an infinite path $\pi := (\emptyset = v_0, v_1, v_2, \ldots)$ the total influence is

$$\sum_{d=0}^{\infty} \eta_{v_d} \prod_{k=1}^{d} \xi_{(v_{k-1}, v_k)}.$$

- Let X be the maximal influence of any infinite path, that is,

$$X = \sup_{(\emptyset = v_0, v_1, v_2, \ldots)} \sum_{d=0}^{\infty} \eta_{v_d} \prod_{k=1}^{d} \xi_{(v_{k-1}, v_k)}.$$

- If $X < \infty$ a.s. then it is a solution of the RDE.

A Contraction Argument

Theorem 1 *Suppose that* $0 \leq \xi_j < 1$ *for* $j \geq 1$ *and* $\eta \geq 0$ *has all moments finite. Suppose for some* $1 \leq p < \infty$ $c(p) := \mathbf{E}\left[\sum_{j=1}^{N} \eta_j\right] < \infty$ *then*

(a) If T *is the associated operator for the RDE then*

$$T^n(\delta_0) \xrightarrow{d} X < \infty \text{ a.s.}$$

and X *has all moments finite.*

(b) There is a $p_0 \geq 1$ *such that* $c(p_0) < 1$ *and* X *is the unique solution amongst possible solutions with finite* ${p_0}^{th}$ *moment.*

Remarks :

- Here T becomes a contraction under (standard) Wasserstein metric with contraction coefficient $c(p)$.

- We note the function g here is "nice" in the sense that it satisfies both conditions of Proposition 2.

- The contractive property proves that the sequence $\left(T^n\left(\delta_0\right)\right)_{n\geq 0}$ is tight, which is equivalent of proving $X < \infty$ a.s.

- We only consider the case $0 \leq \xi_j < 1$, but it makes sense to relax this assumption. All our examples satisfy this assumption though.

Three Interesting Examples :

Example 1 (Discounted BRW) : [Athreya (1985)]

$$X \stackrel{d}{=} \eta + c \max(X_1, X_2) \quad \text{on} \quad \mathbb{R}^+,$$

where (X_1, X_2) are i.i.d. copies of X and are independent of η and $0 < c < 1$ is a constant.

- Instead of binary branching one can also consider a random branching with distribution N.
- Here $c(p) = \mathbf{E}[N] \times c^p$, so by our theorem it has a solution with all moments finite if $\mathbf{E}[N] < \infty$ and if we assume all moments of η are finite. Moreover if c is "small" then this is unique amongst possible solutions with finite mean.
- One interpretation is as *inhomogeneous* percolation on the planted (binary) tree (the root has degree one), where an edge at depth d has traversal time distributed as $c^d \eta$. Then X is the time for the entire tree to be traversed.

Example 2 (FIND Algorithm) : [Devroye (2001)]

$$X \stackrel{d}{=} 1 + \max(UX_1, (1-U)X_2) \quad \text{on} \quad \mathbb{R}^+,$$

where (X_1, X_2) are i.i.d. copies of X and are independent of $U \sim$ Uniform $(0,1)$.

- Arise in the context of the probabilistic worst-case analysis of Hoare's FIND algorithm.
- Here $c(p) = \frac{2}{1+p}$ and so this also has a solution with all moments finite. This solution is unique amongst possible solutions with finite $(1+\varepsilon)^{\text{th}}$ moment, for some $\varepsilon > 0$.
- Devroye (2001) proved that any solution has all moments finite and hence proving uniqueness.

Example 3 : [Durrett and Limic (2002)]

$$X \stackrel{d}{=} \eta + \max_{j \geq 1} e^{-\xi_j} X_j \quad \text{on} \quad \mathbb{R}^+,$$

where $(X_j)_{j\geq 1}$ are i.i.d. copies of X and are independent of $\left(\eta; (\xi_j)_{j\geq 1}\right)$. Here $(\xi_j)_{j\geq 1}$ are points of a Poisson point process with rate 1 on $(0, \infty)$ and independent of $\eta \sim$ Exponential (1).

Remarks :

- The model studied by Durrett and Limic (2002) gives this equation through the following story.

- Consider the following Markov process on countable subsets of $[0, \infty)$
  - ▶ Each individual at position x lives for an independent Exponential (e^x) lifetime.
  - ▶ After which it dies and instantaneously gives birth to infinitely many offsprings which are placed at positions $(x + \xi_j)_{j \geq 1}$ where $(\xi_j)_{j \geq 1}$ are points of a Poisson process of rate 1 on $(0, \infty)$.
  - ▶ Then X is the extinction time for the process started with a single individual at position 0.

- $c(p) = \frac{1}{p}$, so the general result proves that there is a solution with all moments finite which is also endogenous.

- Interesting enough in this case there are many other solutions, which of course do not have finite expectation !

Homogeneous Equation and Uniqueness

Proposition 3 *Consider the homogeneous RDE*

$$X \stackrel{d}{=} \max_{1 \leq j \leq^* N} \xi_j X_j \quad \textit{on} \;\; \mathbb{R}^+,$$

where $(X_j)_{j \geq 1}$ *are i.i.d. copies of* X *and are independent of* $(\xi_j)_{j \geq 1}$. *Suppose dist*(Y) *is a non-zero solution of it. Let* T *be the operator associated with the original (inhomogeneous) RDE and we assume that* $X < \infty$ *a.s. Then for each* $0 \leq a < \infty$

$$T^n\left(\textit{dist}\left(aY\right)\right) \stackrel{d}{\longrightarrow} \mu_a$$

where each μ_a *are fixed-point of* T, $\mu_0 = \textit{dist}\,(X)$ *and* $\mu_a \preccurlyeq \mu_b$ *if* $0 \leq a \leq b < \infty$.

Remarks :

- Under some technical conditions one can also prove that μ_a's are all distinct.

- In particular for Example 3 one can explicitly find all the solutions of the homogeneous equation, which then give other solutions of the original equation.

- If we take log-transformation of the homogeneous equation then we get the following RDE

$$\widehat{X} \stackrel{d}{=} \max_{1 \leq j \leq^* N} \left(\widehat{\xi_j} + \widehat{X_j} \right).$$

  This equation relates to (possible) appropriately centered limit of the right-most position of a BRW. In general it is non-trivial to solve.

- We conjecture that the solution μ_a is not endogenous if $a > 0$.

Mean-Field Combinatorial Optimization Problems

Mean-Field Model of Distance

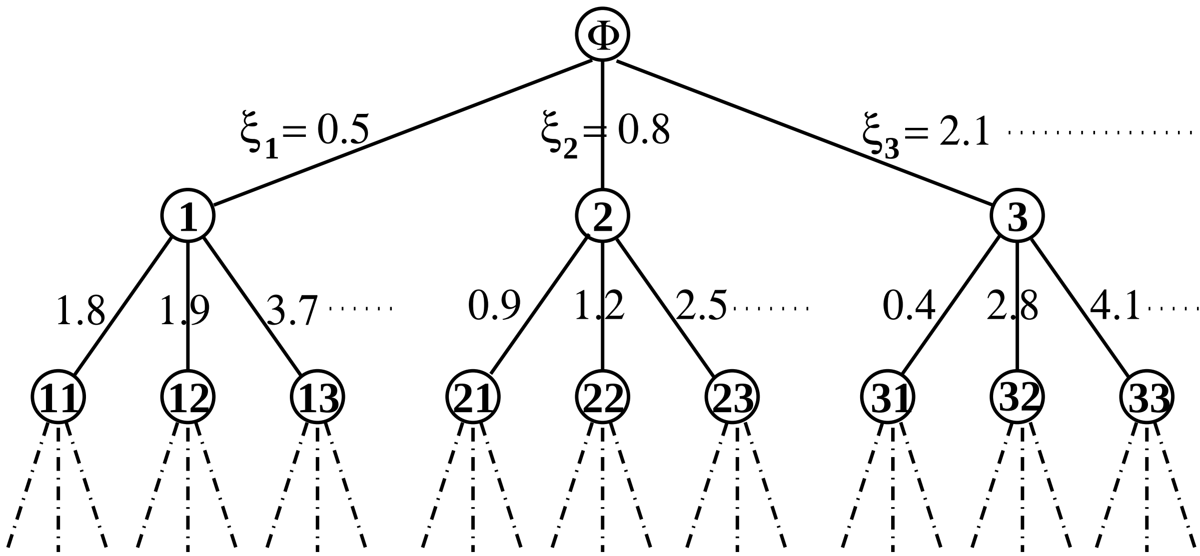
- We have n points.
- $\binom{n}{2}$ inter point distances are i.i.d. random variables.

  Note : This is different than the *Uniform-Euclidean* model, which is to take n uniformly distributed points inside a closed box in some Euclidean space.

- Think it as the complete graph K_n with i.i.d. edge lengths.
- The model with i.i.d. Exponential edge lengths is of particular interest.

Question : What happens to this model as n becomes large ?

PWIT (**P**oisson **W**eighted **I**nfinite **T**ree)



- It has the skeleton $\mathbb{T}_\infty$ rooted at $\emptyset$.

- For the edges $(\mathbf{i}, \mathbf{i}j)_{j \geq 1}$ coming out of the vertex $\mathbf{i}$, the weights are points of Poisson point process of rate 1 on $(0, \infty)$ written as $\left(\xi_{\mathbf{i}j}\right)_{j \geq 1}$. The processes are independent as $\mathbf{i}$ varies.

Local Weak Limit

- Considered the *rooted* graph G_n with edge lengths, obtained from K_n with i.i.d. Exponential $\left(\frac{1}{n}\right)$ edge lengths and selecting a vertex uniformly at random as the root.

- Then G_n converges in the sense of *local weak convergence* to the **PWIT** . [Aldous (1992, 2001), Aldous and Steele (2003)]

- The main reason is the following simple observation

  Suppose $\left(\xi^n_{(1)} < \xi^n_{(2)} < \cdots < \xi^n_{(n)}\right)$ is a ordered statistics from n i.i.d. Exponential random variables with mean n then

$$\left(\xi^n_{(i)}\right)_{i=1}^{n} \xrightarrow{d} \text{Poisson Process}\,(1)\,.$$

Random Assignment Problem

Problem : Suppose n is even, on K_n find the *complete matching* which has minimum total length.

More preciously the objective function is defined as

$$M_n := \inf \left\{ \text{length}\,(\mathcal{M}) \;\middle|\; \mathcal{M} \text{ is a matching} \right\}$$

where length of a matching $\mathcal{M}$ is the sum of the lengths of the edges in the matching.

Remark : Typically it is stated in the setting of bipartite graph but from asymptotic point of view they give same answer !

Theorem 2 (Aldous (2001)) *For i.i.d. Exponential edge lengths with mean* n

$$\frac{2}{n}\,\mathbf{E}\left[M_n\right] \to \frac{\pi^2}{6}.$$

Remark : Mézard and Parisi "proved" this result in 1987 using *non-rigorous* statistical physics argument.

A Program

(How to Prove Such a Theorem ?)

- We know that the limiting structure is **PWIT** .
- Suppose we have a *corresponding* optimization problem on the infinite-size model.
- Suppose we can solve that optimization problem and can compute the optimal solution.
- Suppose we can prove convergence of the finite problem to the infinite problem.
- All this will give the result !!!

Optimal Matching Problem on PWIT

Problem : Find a complete matching on **PWIT** which is *invariant* and minimizes *average* edge length.

Remarks :

- Here *invariant* means, intuitively, that in defining the complete matching on the **PWIT** , the root $\emptyset$ should not play any special role.

- Once a matching is *invariant* by *average* edge length we will mean the expected length of the edge which matches the root $\emptyset$ with one of its children.

540° argument

- **Step 1 :** For each vertex $\mathbf{i}$ of the **PWIT**, let $\mathbf{T}^{\mathbf{i}}$ be the infinite tree rooted at $\mathbf{i}$ containing only its descendents. We define the quantities
  - $W_{\mathbf{i}} :=$ *Total* weight of optimal matching on $\mathbf{T}^{\mathbf{i}}$.
  - $\widetilde{W}_{\mathbf{i}} :=$ *Total* weight of optimal matching on $\mathbf{T}^{\mathbf{i}} \setminus \{\mathbf{i}\}$.
  - $X_{\mathbf{i}} := W_{\mathbf{i}} - \widetilde{W}_{\mathbf{i}}$. **Note :** $X_{\mathbf{i}} = \infty - \infty$!
- **Step 2 :** Assuming these quantities make sense one can write the following *recursion*

$$X_{\emptyset} = \min_{j \geq 1} (\xi_j - X_j) \quad \text{on } \mathbb{R},$$

  where $(X_j)_{j\geq 1}$ are i.i.d. with same law as of $X_{\emptyset}$, and are independent of $(\xi_j)_{j\geq 1}$ which are points of a Poisson point process of rate 1 on $(0, \infty)$.
- **Step 3 :** One can show [Aldous, 2001]
  - The RDE is well defined and has unique solution as the *Logistic distribution*. We will call this RDE the *Logistic RDE*.
  - Now we can reconstruct (rigorously) the optimal matching on **PWIT** using the variables $X_{\mathbf{i}}$. For example, match root $\emptyset$ with $\arg\min_{j\geq 1} (\xi_j - X_j)$.

Role of Endogeny

- Endogeny will show that the optimal solution is a measurable function of the data (innovations), in the infinite-size problem. Since a measurable function is a.s. continuous, we can pull back to define almost-feasible solution of the finite size-n problem with almost equal cost.

- It will then remain to show that in the finite size-n problem one can patch an almost-feasible solution into a feasible solution for asymptotically negligible cost.

Theorem 3 (B. (2002)) *The unique solution of the* Logistic RDE *is endogenous.*

Remark : Proof uses the *equivalence theorem* and involves analytic argument.

My Apology for Not Covering ...

- The **linear** RDEs which have been extensively studied by various authors, mainly in the context of probabilistic analysis of random algorithms.
- RDEs related to **branching processes** and **branching random walks**.
- Relation of certain max-type RDEs (appear in context of BRW) with linear RDEs.
- ... *and many more some of which I may not even be aware of !*

Thank You