Recursive Distributional Equation and Its Applications

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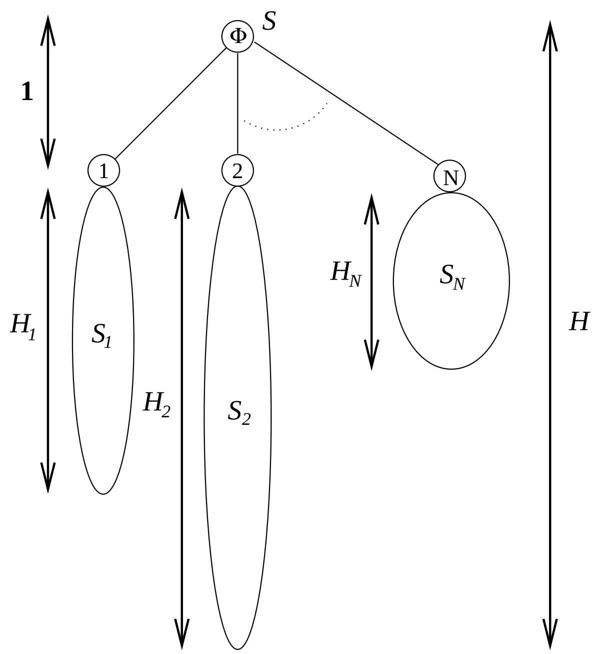
Two Examples

Examples 1 : Consider a *(sub)-critical* Galton-Watson branching process with the progeny distribution N, so $\mathbf{E}[N] \leq 1$; we assume $\mathbf{P}(N = 1) < 1$.



Height of the Tree : Let $H := 1+$ height of the G-W tree, then $H < \infty$ a.s. and

$$H \stackrel{d}{=} 1 + \max(H_1, H_2, \ldots, H_N) \quad \text{on } \mathbb{N},$$

where $(H_j)_{j \geq 1}$ are i.i.d. with same law as of H and are independent of N.

Example 2 (Perhaps the best known !) : Consider the following fixed point equation

$$Z \stackrel{d}{=} \frac{Z_1 + Z_2}{\sqrt{2}} \quad \text{on} \quad \mathbb{R},$$

where (Z_1, Z_2) are i.i.d. copies of Z.

- The set of all solutions is given by the Normal $\left(0, \sigma^2\right)$, $\sigma^2 \geq 0$ family.
- This example also extends to give characterizations of stable laws.

We will call such an equation a *recursive distributional equation* (RDE).

Typical features of RDEs

$$\text{Ex. } 1: \quad X \stackrel{d}{=} 1 + \max(X_1, X_2, \ldots, X_N) \quad \text{on } \mathbb{N}$$

$$\text{Ex. } 2: \quad X \stackrel{d}{=} (Z_1 + Z_2)/\sqrt{2} \quad \text{on } \mathbb{R}$$

- **Unknown Quantity :** Distribution of X.
- **Known Quantities :**
  - $N \leq \infty$ which may or may not be random (e.g. $N \equiv 2$ in Ex. 2).
  - Possibly some more randomness whose distribution is known (not present in both examples above).
  - How we combine the known and unknown randomness (e.g. "$1 + \max$" operation in Ex. 1).
- **What is the RDE doing ?** To find a distribution μ such that when we take i.i.d. samples $(X_j)_{j \geq 1}$ from it and only use N many of them (where N is independent of the samples) and do the manipulation then we end up with another sample $X \sim \mu$.

Remark : In the case $N = 1$ a.s. it reduces to the question of finding a stationary distribution of a discrete time Markov chain.

Two main uses of RDEs

- **Direct use :** The RDE is used directly to define a distribution. Examples include,
  - The height (and also the size) of a (sub)-critical Galton-Watson tree (Example 1).
  - The Quicksort distribution (from random algorithm literature).
  - Discounted tree sums / inhomogeneous percolation on trees.
  - ... *and many others*.

- **Indirect use:** The RDE is used to define some auxiliary variables which help in defining/characterizing some other quantity of interest. Among others the following two type of applications are of special interest
  - Characterizing *phase transition* or determining critical points and scaling laws. (will see an example.)
  - 540° *argument* ! (will not give an example.) [Aldous 2000, 2001 and Aldous & B. 2004]

General Setup

- Let $(S, \mathfrak{S})$ be a measurable space, and $\mathcal{P}$ be the collection of all probabilities on $(S, \mathfrak{S})$.

- Let (ξ, N) be a pair of random variables such that N takes values in $\{0, 1, 2, \ldots; \infty\}$.

- Let $(X_j)_{j \geq 1}$ be **i.i.d** S-valued random variables, which are independent of (ξ, N).

- $g(\cdot)$ is a S-valued measurable function with appropriate domain.

Recursive Distributional Equation (RDE)

Definition 1 *The following* fixed-point equation *on* $\mathcal{P}$ *is called a* Recursive Distributional Equation (RDE)

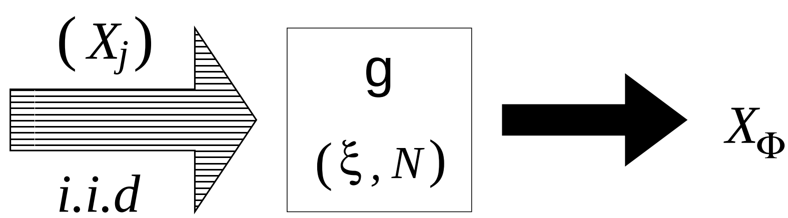
$$X \stackrel{d}{=} g\left(\xi; X_j, 1 \leq j \leq^* N\right), \quad \textit{on} \;\; S$$

where $(X_j)_{j \geq 1}$ *are independent copies of* X *and are independent of* (ξ, N)*.*

Remark : A more conventional (analysis) way of writing the equation would be

$$\mu = T(\mu)$$

where T is the operator associated with the above equation, which depends on the function g and the joint distribution of the pair (ξ, N), and μ is the (unknown) law of X.



Hard-Core Model on a Finite Graph

- Let $G := (V, E)$ be a finite graph.
- We say a subset $I \subseteq V$ is an *independent set* of G, if for any two $u, v \in I$ there is no edge between u and v.
- Let $\mathcal{I}_G$ be the set of all independent sets of G.
- Fix $0 < p < 1$ and let $\lambda = \frac{p}{1-p}$.
- Suppose $(C_v)_{v \in V}$ be i.i.d. Bernoulli$\,(p)$.
- Define $I := \{v \in V \,|\, C_v = 1\}$.
- The measure $\mathbf{P}\left(\cdot \,|\, I \in \mathcal{I}_G\right)$ on $\mathcal{I}_G$ is called the *hard-core model* or *random independent set model* with *activity* λ. We will denote it by $\mathbb{P}$.
- It is easy to see that $\mathbb{P}$ is the probability on $\mathcal{I}_G$ which puts mass proportional to $\lambda^{|I|}$ for $I \in \mathcal{I}_G$.

Sparse Random Graphs

- Two types of *sparse random graphs* :
  - $\mathcal{G}\left(n, \frac{\mu}{n}\right)$: A random graph with n vertices and each edge is present with probability $\frac{\mu}{n}$ independently, where $\mu > 0$. [Erdös & Rényi 1959 - 1968]
  - $\mathcal{G}(n, r+1)$: Pick a graph uniformly at random from the set of all $(r+1)$-regular graphs with n vertices.
- Given a particular realization G_ω of a sparse random graph, we will consider the *hard-core model* with activity $\lambda > 0$ on that finite graph as described before.
- Note there are two stages of randomness and there are two parameters,
  - $\mu > 0$ dealing with the randomness of the graph configuration.
  - $\lambda > 0$ dealing with the randomness of the hard-core model given a configuration.

Motivations

- Interesting from Statistical Physics point of view, well studied for non-random graphs. [Kelley 1985, van den Berg & Steif 1994, Brightwell, Häggström & Winkler 1998, Brightwell & Winkler 1999]

- Has applications in engineering fields, like in *multi-cast networking* problems. [Ramanan et al 2002]

- Conjecture of Aldous [2003] :

  For a sparse random graph if I_n be the *maximal independent set* then

  $$\frac{\mathbf{E}\left[|I_n|\right]}{n} \to c \quad \text{as} \quad n \to \infty,$$

  where c is a constant which depends on the model for the sparse random graph.

Remark : For a hard-core model on a finite graph if we take $\lambda \to \infty$ limit then it concentrate on the maximal independent set(s).

Sparse Random Graphs and GW-Trees

- **Known :** If $\mathcal{G}_n$ be a model for sparse random graph then for "large" enough n "locally it looks like" a (possibly random) rooted tree.
  - ▶ For $\mathcal{G}\left(n, \frac{\mu}{n}\right)$ it is rooted Galton-Watson tree with Poisson (μ) offspring distribution.
  - ▶ For $\mathcal{G}\left(n, r+1\right)$ it is rooted $(r+1)$-regular tree.

- **Conclusion :** So for computing "large" n limit of hard-core model on these kind graphs we need to consider the similar model on respective GW-trees.

- **Problem :** The trees we get may be infinite with positive probability.

- **Solution :** In that case we need to consider Gibbs measure with activity $\lambda > 0$ which has appropriate conditional laws ("DLR condition").

- **Warning :** It is then no longer true that there is only one such measure and we will say that a *phase transition* occurs if there are multiple Gibbs measures for a given activity $\lambda > 0$.

Key Recursion on a Finite Tree

- Suppose $\mathcal{T}$ be a finite rooted tree and we consider the hard-core model on it with activity $\lambda > 0$.

- Suppose $\emptyset$ be the root and it has $n(\emptyset)$ many children which are denoted by $1, 2, \ldots, n(\emptyset)$.

- Let I be a random independent set distributed according to the hard-core model with activity $\lambda > 0$. Then we define $\eta_{\mathcal{T}}^{\emptyset} := \mathbb{P}(\emptyset \in I)$.

- For a child j, let $\mathcal{T}_j$ be the sub-tree rooted at j obtained by removing $\emptyset$. Suppose $\eta_{\mathcal{T}_j}^{j}$ be defined similarly of $\eta_{\mathcal{T}}^{\emptyset}$.

- The following *key recursion* holds

$$\eta_{\mathcal{T}}^{\emptyset} = \frac{\lambda \prod_{j=1}^{n(\emptyset)} \left(1 - \eta_{\mathcal{T}_j}^{j}\right)}{1 + \lambda \prod_{j=1}^{n(\emptyset)} \left(1 - \eta_{\mathcal{T}_j}^{j}\right)}$$

Related RDE

$$\eta \stackrel{d}{=} \frac{\lambda \prod_{j=1}^{N} \left(1 - \eta_j\right)}{1 + \lambda \prod_{j=1}^{N} \left(1 - \eta_j\right)} \quad \text{on} \quad [0,1],$$

where (η_j) are i.i.d. copies of η and are independent of N.

Properties : Let T be the associated operator and $S = T^2$ then

- $T\left(\delta_0\right) = \delta_{\lambda/(1+\lambda)}$.
- $\delta_0 \preccurlyeq T\left(m\right) \preccurlyeq \delta_{\lambda/(1+\lambda)}$, for any probability m on $[0,1]$.
- T is anti-monotone $\Rightarrow$ S is monotone.
- So there exist $m_* \preccurlyeq m^*$ two fixed points of S such that $S^n\left(\delta_0\right) \uparrow m_*$ and $S^n\left(\delta_{\lambda/(1+\lambda)}\right) \downarrow m^*$.
- $T\left(m_*\right) = m^*$.
- S has unique fixed point if and only if $m_* = m^*$.

Uniqueness Domain

Definition 2 *We will say that we are in* uniqueness domain *if* $m_* = m^*$.

Results

- **Theorem 1** *For a GW-Tree with progeny distribution* N *and for activity* $\lambda > 0$ *we are in uniqueness domain if and only if, there is a unique Gibbs measure with activity* λ *a.s. with respect to the randomness in the configuration of the tree.*

  Note : The *phase transition* is characterize by the uniqueness of solution of a RDE.

- **Theorem 2** *For* $\mathcal{G}\left(n, \frac{\mu}{n}\right)$ *suppose we are in the uniqueness domain for* $\lambda > 0$ *and with* $N \sim$ *Poisson* (μ) *and let* I_n *be a random independent set with hard-core distribution with activity* λ*, then*

$$\frac{\mathbf{E}\left[|I_n|\right]}{n} \to \mathbf{E}\left[\eta\right]$$

  where $\eta \sim m_* = m^*$.

- **Theorem 3** *A similar statement for* $\mathcal{G}\left(n, r+1\right)$.

When Uniqueness Domain Holds ?

- **Small μ :** If $\mu \leq 1$ then the graphical structure is in the (sub)-critical domain and hence it will be finite and so uniqueness domain holds for any $\lambda > 0$. This is not the interesting case !

- **Small λ :** If $\lambda \times \mu < 1$ then T is a contraction and hence uniqueness domain holds. Thus for any $\mu > 0$ for activity $\lambda < \frac{1}{\mu}$ we are in the uniqueness domain.

Remarks :

- I believe (do not have complete proofs yet) that uniqueness domain will not hold for large μ or large λ (and $\mu > 1$).

- So it seems that we may not be able to resolve Aldous' conjecture by this method. But perhaps we can ... that is yet another story !

- At least we do get a nice example of *phase transition* phenomenon which is characterize by uniqueness of solution of a RDE.