Hard-Core Model on Random Graphs

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A Problem by David Aldous

- For $r \geq 2$ and $n \geq 3$, let $\mathbf{G}(n,r)$ be a random graph selected uniformly at random from the set of all r-regular graphs on n vertices.

- **Conjecture of Aldous [2003] :**

  Let I_n be a *maximum independent set* then

  $$\frac{\mathbf{E}[|I_n|]}{n} \to \kappa \quad \text{as} \quad n \to \infty,$$

  where $\kappa > 0$ is a constant which may depend on r.

- In combinatorics for a finite graph G the size of a maximum independent set is known as the *independence number* of G.

An Approach Towards Resolving the Conjecture

- We will consider a probability model on the set of all independent sets of the random graph $\mathbf{G}$ such that
$$\mathbb{P}_\lambda(I) \propto \lambda^{|I|},$$
where I is an independent set of $G(n,r)$.

- It is easy to see that given $\mathbf{G}(n,r)$ the probability measures $\mathbb{P}_\lambda$ concentrate on the *maximum* independent sets as $\lambda \to \infty$.

- So perhaps studying this model $\mathbb{P}_\lambda$ on random graphs may help to resolve Aldous' conjecture.

- We will see what we can do ... !

Hard-Core Model on a Finite Graph

Setup :

- Let $G := (V, E)$ be a finite graph.
- We say a subset $I \subseteq V$ is an *independent set* of G, if for any two vertices $u, v \in I$ there is no edge between u and v.
- Let $\mathcal{I}_G$ be the set of all independent sets of G.
- We would like to define a measure on $\mathcal{I}_G$.

Description 1 :

- Fix $\lambda > 0$.

- *Hard-core model on G with activity λ* is a probability distribution on $\mathcal{I}_G$ such that

$$\mathbb{P}_\lambda^G(I) \propto \lambda^{|I|}, \quad I \in \mathcal{I}_G.$$

- Thus

$$\mathbb{P}_\lambda^G(I) = \frac{\lambda^{|I|}}{Z_\lambda(G)}, \quad I \in \mathcal{I}_G$$

  where $Z_\lambda(G) := \sum_{I \in \mathcal{I}_G} \lambda^{|I|}$ is the proportionality constant, known as the *partition function*.

Observations :

- If $\lambda = 1$ then we get the uniform distribution on $\mathcal{I}_G$ and $Z_\lambda(G)$ is the size of $\mathcal{I}_G$.

- Also we have already noticed, $\lambda \to \infty$ the measures $\mathbb{P}_\lambda^G$ concentrate on maximal size independent sets.

Description 2 :

- Fix $\lambda > 0$ and let $p := \frac{\lambda}{1+\lambda} \in (0,1)$.
- Suppose $(C_v)_{v \in V}$ are i.i.d. Bernoulli$\,(p)$.
- Let $I := \{v \in V \,|\, C_v = 1\}$.
- The measure $\mathbf{P}\,(\cdot \,|\, I \in \mathcal{I}_G)$ on $\mathcal{I}_G$ is same as $\mathbb{P}_\lambda^G$.

Remark :

- This gives a way to get exact samples from $\mathbb{P}_\lambda^G$.

Hard-Core Model on an Infinite Graph

Problems with the Two Previous Descriptions :

- For Description 1, we note that $\mathcal{I}_G$ is infinite and hence the partition function $Z_\lambda(G) = \infty$!
- For Description 2, we end up with the (same type of) problem that the event $[I \in \mathcal{I}_G]$ has *zero* probability under the i.i.d. coin tossing measure.

An Observation on Finite Graph :

Fix any vertex $v \in V$ and let σ be an independent set for the graph with vertex set $V \setminus \{v\}$ then

$$\mathbb{P}_\lambda^G \left(v \in I \,|\, I \setminus \{v\} = \sigma \right) = \begin{cases} \frac{\lambda}{1+\lambda} & \text{if } \sigma \cup \{v\} \in \mathcal{I}_G \\ 0 & \text{otherwise} \end{cases}$$

where $I \in \mathcal{I}_G$.

Statistical Physics Definition :

Definition 1 *Given a finite or countably infinite, but locally finite graph* $G = (V, E)$ *and* $\lambda > 0$, *a probability measure* $\mathbb{P}_\lambda^G$ *on* $\{0,1\}^V$, *is said to be a* Gibbs measure for the hard-core model on G with activity λ, *if it admits conditional probabilities such that for all* $v \in V$ *and for any* $\sigma \in \{0,1\}^{V\setminus\{v\}}$,

$$\mathbb{P}_\lambda^G\left(I(v) = 1 \,|\, I(V \setminus \{v\}) = \sigma\right) = \begin{cases} \frac{\lambda}{1+\lambda} & \textit{if } \sigma \vee \mathbf{1}_v \in \mathcal{I}_G \\ \\ 0 & \textit{otherwise} \end{cases}$$

where I *is a* $\{0,1\}^V$*-valued random variable with distribution* $\mathbb{P}_\lambda^G$.

Remarks :

- This is what is known as Dobrushin-Lanford-Ruelle (DLR) definition of infinite-volume Gibbs measure.
- Similar definitions are used for defining Ising model and q-Potts model on infinite graphs.

Existence and Uniqueness

- In general a Gibbs measure exists by compactness argument.
- If G is finite then uniqueness holds trivially.
- It is not necessary that the uniqueness will hold when G is infinite.

Definition 2 *For a fixed graph G we say that a* phase transition *occurs for hard-core model with activity $\lambda > 0$, if there are more than one Gibbs measures of the form $\mathbb{P}_\lambda^G$.*

Note : There is no phase transition if G is finite.

What are Known ?

- First introduced by Dobrushin (1968) on $\mathbb{Z}^d$ for model of lattice gas.

- Phase transition is well studied for $\mathbb{Z}^d$.
  - ▶ No phase transition for $d = 1$.
  - ▶ For $d \geq 2$ no phase transition for small λ, but phase transition occurs for large λ.

  Not Known : Is phase transition *monotone* ? In other words is there a critical value in λ ?

- Arguably the most well studied case is the model on regular trees, $\mathbb{T}_r$ for $r \geq 2$. [Kelly, 1985]
  - ▶ For a r-regular tree $\mathbb{T}_r$, there exists a critical value $\lambda_c(r)$ such that, no phase transition when $\lambda \leq \lambda_c(r)$ and phase transition occurs when $\lambda > \lambda_c(r)$.
  - ▶ $\lambda_c(r) = \frac{(r-1)^{r-1}}{(r-2)^r}$.

- It is also known that there are infinite trees for which phase transition is not *monotone* ! [Brightwell, Häggström, Winkler, 1998]

Hard-Core Model on Random Graphs

Setup :

- $\mathcal{G}$ be a set of graphs which are finite or countably infinite and are locally finite.
- Suppose $\mathbf{P}$ is a probability on $\mathcal{G}$.
- Let $\mathbf{G} \sim \mathbf{P}$. We will write $\mathbf{G}(\omega)$ for a realization of the random graph $\mathbf{G}$.
- Given $\mathbf{G}(\omega)$ a hard-core model with activity $\lambda > 0$ on $\mathbf{G}(\omega)$ will be denoted by $\mathbb{P}_{\lambda}^{\omega}$.
- We will denote the joint measure as $\mathbf{P}_{\lambda}$.

Remark :

- Note that there are two stages of randomness and there are two parameters :
  - ▶ One is the probability distribution $\mathbf{P}$ on $\mathcal{G}$ governing the randomness of the underlying graphical structure.
  - ▶ The other is λ which is governing the hard-core model given the graph.

Phase Transition

Definition 3 *Given a random graph model* $(\mathcal{G}, \mathbf{P})$, *we say that there is a* phase transition *for the hard-core model with activity* $\lambda > 0$ *on a random graph* $\mathbf{G} \sim \mathbf{P}$ *if*

$$\mathbf{P}\left(\exists \textit{ multiple measures of the form } \mathbb{P}_\lambda^{\mathbf{G}}\right) > 0.$$

Remark :

- If the random graph model is such that $\mathbf{G}$ is finite a.s. then there will be no phase transition for any activity $\lambda > 0$.
- It is possible to construct an example of $(\mathcal{G}, \mathbf{P})$ such that phase transition occurs for every $\lambda > 0$.

An Example

- Let $\mathcal{G} := \{\mathbb{T}_r \,|\, r \geq 2\}$ and $\mathbf{P}$ be given by $\mathbf{P}(\mathbb{T}_r) = \frac{1}{2^{r-1}}$.

- Recall that from Kelly's work (1985) it is known that for hard-core model on r-regular tree $\mathbb{T}_r$, phase transition occurs if an only if

$$\lambda > \lambda_c(r) = \frac{(r-1)^{r-1}}{(r-2)^r}.$$

- But $\lambda_c(r) \to 0$ as $r \to \infty$.

- So for every $\lambda > 0$ for large enough r we must have $\lambda_c(r) < \lambda$ and thus a phase transition would occur for the random graph model $(\mathcal{G}, \mathbf{P})$.

Remark :

- It is important to note that for the model $(\mathcal{G}, \mathbf{P})$ we can have realizations having arbitrarily large degree with positive probability.

- It is known that for bounded degree (fixed) graphs there should be no phase transition for *small* values of λ. [van den Berg and Steif, 1994]

Random Graph Models

- **GW-Tree :** Galton-Watson branching process tree with a given progeny distribution denoted by N.
  - ▶ The parameter here is the distribution of N.

- **Sparse Random Graphs :**
  - ▶ **Erdös and Rényi Random Graph :** A random graph on $n \geq 1$ vertices labeled by $[n] := \{1, 2, \ldots, n\}$ where each pair of vertices are connected by an edge independently with probability $\frac{c}{n}$, where $c > 0$. This would be denoted by $\mathcal{G}\left(n, \frac{c}{n}\right)$.
    - ▶ The parameter here is $c > 0$.
  - ▶ **Random r-regular Graph :** This is to select one graph at random from the set of all r-regular graphs with vertex set $[n]$. We will denote this model by $\mathcal{G}_r(n)$.

    Note : In order for this model to make sense we will always assume that nr is even.
    - ▶ The parameter here is $r \geq 2$.

Motivations

- Aldous' conjecture for the scaling of the independent number of a sparse random graph.

- Interesting from Statistical Physics point of view, well studied for non-random graphs. [Dobrushin 1970, Kelley 1985, van den Berg & Steif 1994, Brightwell, Häggström & Winkler 1998, Brightwell & Winkler 1999]

- Has applications in engineering fields, like in *multicast networking* problems. [Ramanan et al, 2002]

Sparse Random Graphs and GW-Trees

- **Known :** If $\mathcal{G}_n$ be a model for sparse random graph then for "large" enough n from the "view point" of a fixed vertex "locally it looks like" a (possibly random) rooted tree.
  - For $\mathcal{G}\left(n, \frac{c}{n}\right)$ it is a rooted Galton-Watson tree with Poisson (c) offspring distribution.
  - For $\mathcal{G}(n, r)$ it is a rooted r-regular tree.
- **Conclusion :** So for computing "large" n limit of hard-core model on these kind of graphs we may need to consider the similar model on respective GW-trees.
- **Note :** For a r-regular tree, one slight annoyance is that it is not really a GW-tree ! But by removing one vertex (the root) it can be viewed as a collection of r GW-trees with progeny distribution $N \equiv r - 1$.

Hard-Core Model on GW-Trees

Proposition 1 *Fix $\lambda > 0$ then the followings hold for a GW-tree with progeny distribution N.*

(a) If $\mathbf{E}[N] \leq 1$ then there is no phase transition.

(b) If $\mathbf{E}[N] > 1$ then on the event of `non-extinction` *phase transition occurs with probability* 0 *or* 1*.*

Proof of Proposition 1 :

- Nothing to prove for part (a).

- For part (b) notice that the property that a (fixed) rooted tree $\mathcal{T}$ has no phase transition implies that if v is a child of the root, and $\mathcal{T}(v)$ is the sub-tree rooted at v consisting only of the descendants of v, then $\mathcal{T}(v)$ also has no phase transition.

- Let $\beta := \mathbf{P}_\lambda($ no phase transition in $\mathcal{T})$ where $\mathcal{T}$ is a GW-tree, and let $\{v_1, v_2, \ldots, v_N\}$ be the children of the root in $\mathcal{T}$. Then

$$\begin{aligned} \pi &\leq \mathbf{P}_\lambda(\text{ no phase transition in } \mathcal{T}(v_j),\ \forall\, j) \\ &= \sum_{n=0}^{\infty} \mathbf{P}\,(N = n)\,\pi^n = f(\pi) \end{aligned}$$

  where f is the generating function for N.

- Moreover $\beta \geq q :=$ extinction probability, because $[\texttt{extinction}] \subseteq [\texttt{no phase transition}]$

- Thus $\beta \in \{q, 1\}$ and this completes the proof.

Key Recursion on a Finite Tree

- Suppose $\mathcal{T}$ be a finite (fixed) rooted tree and we consider the hard-core model on it with activity $\lambda > 0$.

- Suppose $\emptyset$ be the root and it has $n(\emptyset)$ many children which are denoted by $1, 2, \ldots, n(\emptyset)$.

- Let I be a random independent set distributed according to the hard-core model with activity $\lambda > 0$. We define $\eta_{\emptyset}^{\mathcal{T}} := \mathbb{P}_{\lambda}^{\mathcal{T}}(\emptyset \in I)$.

- For a child j, let $\mathcal{T}_j$ be the sub-tree rooted at j obtained by removing $\emptyset$. Suppose $\eta_j^{\mathcal{T}_j}$ be defined similarly of $\eta_{\emptyset}^{\mathcal{T}}$.

- The following *key recursion* holds

$$\eta_{\emptyset}^{\mathcal{T}} = \frac{\lambda \prod_{j=1}^{n(\emptyset)} \left(1 - \eta_j^{\mathcal{T}_j}\right)}{1 + \lambda \prod_{j=1}^{n(\emptyset)} \left(1 - \eta_j^{\mathcal{T}_j}\right)}$$

“Superscript Dropping Principle” Recursive Distributional Equation (RDE)

We consider the following distributional identity :

$$\eta \stackrel{d}{=} \frac{\lambda \prod_{j=1}^{N} \left(1 - \eta_j\right)}{1 + \lambda \prod_{j=1}^{N} \left(1 - \eta_j\right)} \quad \text{on} \quad [0,1],$$

where (η_j) are i.i.d. copies of η and are independent of N.

- We also define an operator $T : \mathcal{P}([0,1]) \to \mathcal{P}([0,1])$ using the right-hand side of the above RDE, namely,

$$T(\mu) := \text{dist}\left(\frac{\lambda \prod_{j=1}^{N} \left(1 - \eta_j\right)}{1 + \lambda \prod_{j=1}^{N} \left(1 - \eta_j\right)}\right)$$

  where (η_j) are i.i.d. with distribution μ on $[0,1]$ and are independent of N.

- We put $S = T^2$.

RDE Continued ...

Properties of the RDE and the Operator T :

- $T(\delta_0) = \delta_{\lambda/(1+\lambda)}$.
- $\delta_0 \preccurlyeq T(\mu) \preccurlyeq \delta_{\lambda/(1+\lambda)}$, for any probability μ on $[0,1]$.
- T is anti-monotone $\Rightarrow$ S is monotone.
- T is continuous with respect to the weak convergence topology on $\mathcal{P}([0,1])$.
- So there exist $\mu_* \preccurlyeq \mu^*$ two fixed points of S such that $S^n(\delta_0) \uparrow \mu_*$ and $S^n\left(\delta_{\lambda/(1+\lambda)}\right) \downarrow \mu^*$.
- $T(\mu_*) = \mu^*$ and $T(\mu^*) = \mu_*$.
- S has unique fixed point if and only if $\mu_* = \mu^*$.
- T is a strict contraction with respect to the Wasserstine metric when $\lambda\,\mathbf{E}[N] < 1$.

Uniqueness Domain

Definition 4 *We will say that we are in the* uniqueness domain *if* $\mu_* = \mu^*$.

Characterization of Phase Transition for GW-Tree Model

Theorem 2 *For GW-tree with progeny distribution* N*, there is* no phase transition *for the hard-core model with activity* $\lambda > 0$*, if and only if, we are in uniqueness domain for the associated RDE.*

Specialization to r-regular Tree

- Notice that if $\mathbb{T}_r(\emptyset)$ denote a rooted r-regular tree, that is, a tree whose root $\emptyset$ has degree $r-1$ and all other vertices have degree r, then it is a GW-tree with progeny distribution $N \equiv r-1$.

- So for this model N is non random, that is the operator T has no random part in its definition.

- This then implies both μ_* and μ^* are degenerate measures.

- So basically we need to consider fixed point of a deterministic function $s = t^2$ where $t\colon [0,1] \to [0,1]$ given by

$$t(p) = \frac{\lambda\,(1-p)^{r-1}}{1+\lambda\,(1-p)^{r-1}}, \;\; p \in [0,1].$$

- This is exactly what Kelly did in his 1985 paper and this leads to the critical value $\lambda_c(r)$.

When Does Uniqueness Domain Hold ?

Corollary 3 *For a GW-tree with progeny distribution* N*, there is no phase transition for the hard-core model with activity* $\lambda > 0$ *if*

(a) $\mathbf{E}[N] \leq 1$ *or,*

(b) $\lambda \mathbf{E}[N] < 1$*.*

Remarks :

- In particular it shows that for any GW-tree (with $\mathbf{E}[N] < \infty$) at least for sufficiently small λ there is no phase transition. Such result is expected. But note that we do not assume that the progeny distribution is bounded.

- In fact a better bound holds using *Van den Berg-Steif inequality*, namely $\lambda (\mathbf{E}[N] - 1) < 1$.

Main Results for hard-Core Model on Sparse Random Graphs

Theorem 4 *Suppose $X_\lambda^\omega(n,c)$ be the size of a random independent set distributed according to the hard-core model with activity $\lambda > 0$ on a Erdös-Rényi random graph $\mathcal{G}\left(n, \frac{c}{n}\right)$. If the GW-tree with Poisson(c) progeny distribution has no phase transition then*

$$\lim_{n\to\infty} \frac{\mathbf{E}_\lambda\left[X_\lambda^\omega(n,c)\right]}{n} = \gamma_\lambda(c)$$

where $\gamma_\lambda(c) := \mathbf{E}[\eta]$ and η is the unique solution of the RDE.

Theorem 5 *Suppose $X_\lambda^\omega(n,r)$ be the size of a random independent set distributed according to the hard-core model with activity $\lambda > 0$ on a random regular graph $\mathcal{G}_r(n)$. If the r-regular tree has no phase transition, that is, if $\lambda < \lambda_c(r) = (r-1)^{(r-1)}/(r-2)^r$, then*

$$\lim_{n\to\infty} \frac{\mathbf{E}_\lambda\left[X_\lambda^\omega(n,r)\right]}{n} = \alpha_\lambda(r)$$

where $\alpha_\lambda(r) = w/(1+2w)$ with w is the unique positive solution of the equation $\lambda = w(1+w)^{r-1}$.

Back to Aldous' Conjecture

Conjecture [Aldous, 2003] : For a sparse random graph if I_n is a maximum independent set then

$$\lim_{n\to\infty} \frac{\mathbf{E}[|I_n|]}{n} = \kappa$$

for some constant $\kappa > 0$ (explicitly computable ?).

- Our method fails ! This is because it seems (for the general GW-tree case) that the uniqueness domain does not hold for *large* λ.
- For example it is the case with r-regular trees and hence for the sparse random graph model $\mathcal{G}_r(n)$.
- In fact for $\mathcal{G}_r(n)$ model it has been postulated (proved using non-rigorous methods) in physics literature that such asymptotic limit exists and has the same answer as Theorem 5 when λ is smaller than the so called "*extremality threshold*" (which is bigger than the "*uniqueness threshold*").
- Our Theorems 4 and 5 provides rigorous argument when λ is in the uniqueness domain (that is, under the *uniqueness threshold* for the $\mathcal{G}_r(n)$ model).

Background : Recursive Tree Process (RTP)

Consider the RDE

$$\eta \stackrel{d}{=} \frac{\lambda \prod_{j=1}^{N} (1-\eta_j)}{1+\lambda \prod_{j=1}^{N} (1-\eta_j)} \quad \text{on} \quad [0,1],$$

where (η_j) are i.i.d. copies of η and are independent of N.

Notations :

- Let μ be a solution of the RDE.
- Let $\mathbb{T}_\infty = (\mathcal{V}, \mathcal{E})$ be the canonical infinite tree with vertex set $\mathcal{V} := \left\{ \mathbf{i} \,|\, \mathbf{i} \in \mathbb{N}^d,\, d \geq 1 \right\} \cup \{\emptyset\}$. We will consider it as rooted at $\emptyset$.
- Suppose $(N_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ be i.i.d. copies of the progeny distribution N.

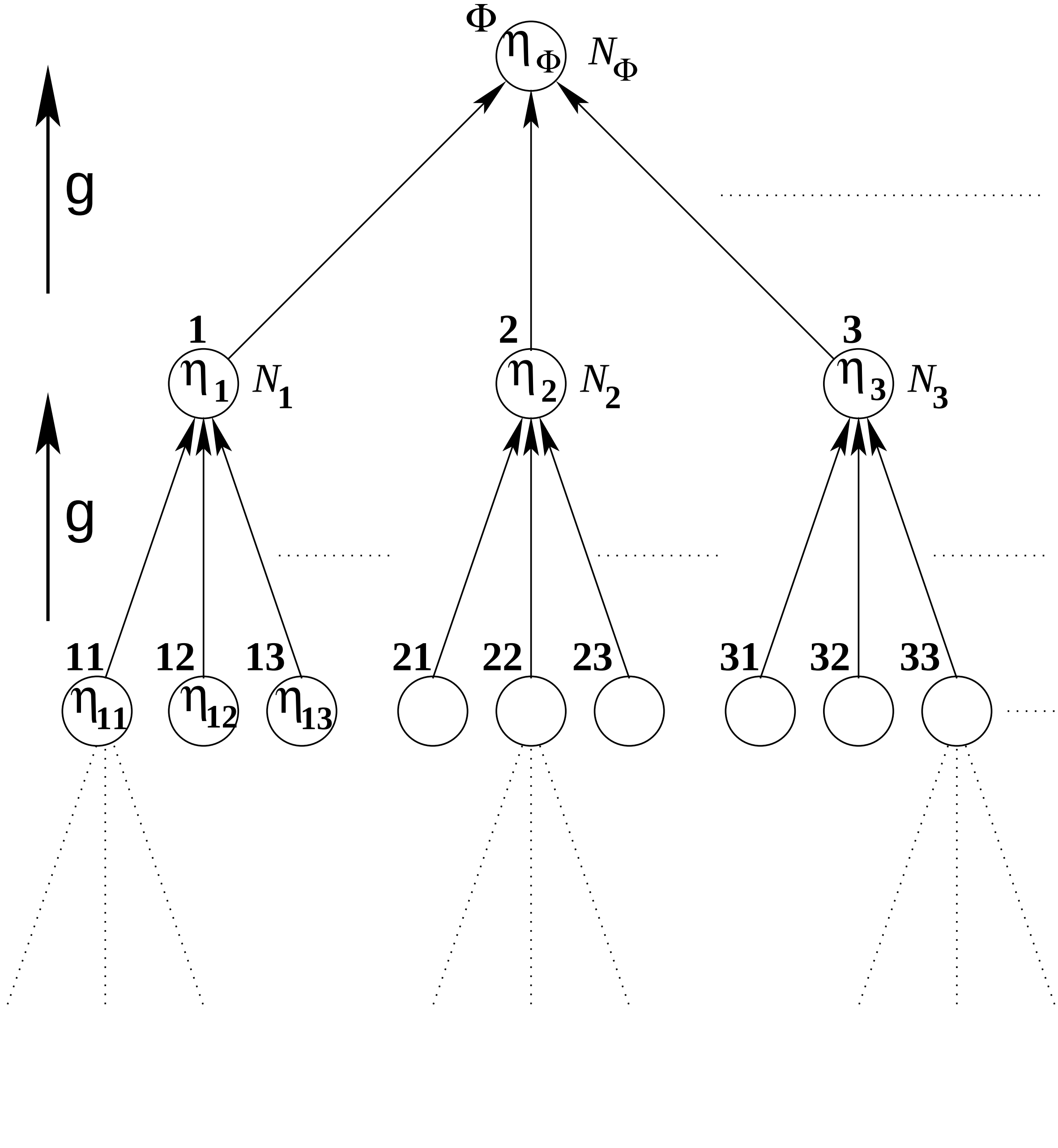
Recursive Tree Process (RTP)

A collection of $[0,1]$-valued random variables $(\eta_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ is called an invariant *Recursive Tree Process (RTP)* with marginal μ if

- $\eta_{\mathbf{i}} \sim \mu \;\; \forall\, \mathbf{i} \in \mathcal{V}$.
- Fix $d \geq 0$ then $(\eta_{\mathbf{i}})_{|\mathbf{i}|=d}$ are independent.
- $\eta_{\mathbf{i}} = \dfrac{\lambda \prod_{j=1}^{N_{\mathbf{i}}} (1-\eta_{\mathbf{i}j})}{1+\lambda \prod_{j=1}^{N_{\mathbf{i}}} (1-\eta_{\mathbf{i}j})}$ a.s. $\forall\, \mathbf{i} \in \mathcal{V}$.
- $\eta_{\mathbf{i}}$ is independent of $\{N_{\mathbf{i}'} \,|\, |\mathbf{i}'| < |\mathbf{i}|\,\} \;\; \forall\, \mathbf{i} \in \mathcal{V}$.

Remark : Using *Kolmogorov's consistency*, an invariant RTP with marginal μ exists if and only if μ is a solution of the RDE.

Recursive Tree Process (RTP)



Towards Proving Theorem 2

Long Range Independence Property

- Fix $d \geq 0$.
- Write $\mathbf{x}_d$ for a vector $(x_{\mathbf{i}})_{|\mathbf{i}|=d}$ where each $x_{\mathbf{i}} \in [0, 1]$.
- Let $\mathcal{T}$ be the realization of the GW-tree rooted at $\emptyset$ obtained from the realizations of $(N_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$.
- Let $\left(\eta_{\mathbf{i}}^{(d)}(\mathbf{x}_d)\right)_{|\mathbf{i}| \leq d}$ be the d-depth RTP with values at level d given by $\mathbf{x}_d$.

Lemma 6 (Long range independence) *Suppose we are in the uniqueness domain, that is $\mu_* = \mu^*$, then*

$$\lim_{d \to \infty} \sup_{\mathbf{x}_d} |\eta_{\emptyset}^{(d)}(\mathbf{x}_d) - \eta_{\emptyset}| = 0 \quad a.s.$$

Remark :

- The proof of Theorem 2 follows from this Lemma.

Proof of Lemma 6 :

- For the vector $\mathbf{x}_d$ if all the components are same as c the we will write the vector itself as $\mathbf{c}$.

- $\eta_\emptyset^{(2d)}(\mathbf{0}) \uparrow \eta_\emptyset$, a.s. and also $\eta_\emptyset^{(2d+1)}(\mathbf{0}) \downarrow \eta_\emptyset$ a.s.

- $\eta_\emptyset^{(d)}\left(\frac{\lambda}{1+\lambda}\right) = \eta_\emptyset^{(d+1)}(\mathbf{0})$, so $\eta_\emptyset^{(d)}\left(\frac{\lambda}{1+\lambda}\right) \to \eta_\emptyset$ a.s.

- If $0 \le x_{\mathbf{i}} \le \frac{\lambda}{1+\lambda}$ for all $\mathbf{i} \in \mathcal{V}$ then
$$\eta_\emptyset^{(2d)}(\mathbf{0}) \le \eta_\emptyset^{(2d)}(\mathbf{x}_{2d}) \le \eta_\emptyset^{(2d)}\left(\frac{\lambda}{1+\lambda}\right), \quad \text{and}$$
$$\eta_\emptyset^{(2d+1)}\left(\frac{\lambda}{1+\lambda}\right) \le \eta_\emptyset^{(2d+1)}(\mathbf{x}_{2d+1}) \le \eta_\emptyset^{(2d+1)}(\mathbf{0}).$$
So $\eta_\emptyset^{(d)}(\mathbf{x}_d) \to \eta_\emptyset$ a.s. as $d \to \infty$.

- Now notice that $\eta_\emptyset^{(d)}(\mathbf{1}) = \eta_\emptyset^{(d-1)}(\mathbf{x}_{d-1})$ where each $x_{\mathbf{i}} \in \{0, \lambda/(1+\lambda)\}$. So $\eta_\emptyset^{(d)}(\mathbf{1}) \to \eta_\emptyset$ a.s.

- Finally, if $0 \le x_{\mathbf{i}} \le 1$ for all $\mathbf{i} \in \mathcal{V}$ then
$$\eta_\emptyset^{(2d)}(\mathbf{0}) \le \eta_\emptyset^{(2d)}(\mathbf{x}_{2d}) \le \eta_\emptyset^{(2d)}(\mathbf{1}), \quad \text{and}$$
$$\eta_\emptyset^{(2d+1)}(\mathbf{1}) \le \eta_\emptyset^{(2d+1)}(\mathbf{x}_{2d+1}) \le \eta_\emptyset^{(2d+1)}(\mathbf{0}).$$
So $\eta_\emptyset^{(d)}(\mathbf{x}_d) \to \eta_\emptyset$ uniformly a.s. as $d \to \infty$, proving the lemma.

Remarks on the Proofs of Theorem 4 and Theorem 5

- First thing to note is

  $$\frac{1}{n}\mathbf{E}_\lambda\left[X_\lambda^\omega\right] = \mathbf{P}_\lambda\left(v_0 \in I_\lambda^\omega\right),$$

  where I_λ^ω is the random independent set selected according to the (random) distribution $\mathbb{P}_\lambda^\omega$, and v_0 is a fixed vertex.

- For any fixed $d > 0$ the distribution of the d-depth neighborhood of v_0 converges to the distribution of a d-depth Poisson(c) GW-tree for $\mathcal{G}\left(n, \frac{c}{n}\right)$ model, and to a d-depth r-regular tree for $\mathcal{G}_r\left(n\right)$ model.

- We can then apply the *local weak convergence* technique of Aldous and Steele (2004) using the (strong) *long range independence* property which holds under the uniqueness domain. These will give the stated results after a little more careful probability computations !

Open Problems/Questions

- We know that for r-regular tree the phase transition is a *monotone* property in λ, and the critical value $\lambda_c(r)$ is explicitly know.

  Is phase transition monotone (in λ) for a general GW-tree ?

  Comment : Most possibly not ! But is it at least the case for GW-tree with Poisson progeny distribution ?

- For GW-tree with Poisson(c) progeny distribution is phase transition monotone in c ? That is for every fixed $\lambda > 0$ if $c > c'$ and we have no phase transition for Poisson(c) GW-tree then can we say that we have no phase transition for Poisson(c') GW-tree ?

  Comment : We know this is true if $\lambda \times c < 1$.

- If the answer to above question is yes (which is most possibly the case) can we also get the critical value for c ? (explicitly or bounds ?)