Hard-Core Model on Random Graphs

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A Problem by David Aldous

• For $r \geq 2$ and $n \geq 3$, let $G(n, r)$ be a random graph selected uniformly at random from the set of all $r$-regular graphs on $n$ vertices.

• Conjecture of Aldous [2003] :
  Let $I_n$ be a maximum independent set then
  \[
  \frac{E[|I_n|]}{n} \to \kappa \quad \text{as} \quad n \to \infty,
  \]
  where $\kappa > 0$ is a constant which may depend on $r$.

• In combinatorics for a finite graph $G$ the size of a maximum independent set is known as the independence number of $G$. 
An Approach Towards Resolving the Conjecture

- We will consider a probability model on the set of all independent sets of the random graph $G$ such that
  \[ P_\lambda(I) \propto \lambda^{|I|}, \]
  where $I$ is an independent set of $G(n,r)$.

- It is easy to see that given $G(n,r)$ the probability measures $P_\lambda$ concentrate on the maximum independent sets as $\lambda \to \infty$.

- So perhaps studying this model $P_\lambda$ on random graphs may help to resolve Aldous’ conjecture.

- We will see what we can do ... !
Hard-Core Model on a Finite Graph

Setup:

- Let $G := (V, E)$ be a finite graph.
- We say a subset $I \subseteq V$ is an independent set of $G$, if for any two vertices $u, v \in I$ there is no edge between $u$ and $v$.
- Let $\mathcal{I}_G$ be the set of all independent sets of $G$.
- We would like to define a measure on $\mathcal{I}_G$. 
Description 1 :

- Fix $\lambda > 0$.

- *Hard-core model on $G$ with activity $\lambda$* is a probability distribution on $\mathcal{I}_G$ such that

$$
\mathbb{P}^G_\lambda(I) \propto \lambda^{|I|}, \quad I \in \mathcal{I}_G.
$$

- Thus

$$
\mathbb{P}^G_\lambda(I) = \frac{\lambda^{|I|}}{Z_\lambda(G)}, \quad I \in \mathcal{I}_G
$$

where $Z_\lambda(G) := \sum_{I \in \mathcal{I}_G} \lambda^{|I|}$ is the proportionality constant, known as the *partition function*.

Observations :

- If $\lambda = 1$ then we get the uniform distribution on $\mathcal{I}_G$ and $Z_\lambda(G)$ is the size of $\mathcal{I}_G$.

- Also we have already noticed, $\lambda \to \infty$ the measures $\mathbb{P}^G_\lambda$ concentrate on maximal size independent sets.
Description 2:

- Fix $\lambda > 0$ and let $p := \frac{\lambda}{1+\lambda} \in (0, 1)$.
- Suppose $(C_v)_{v \in V}$ are i.i.d. Bernoulli $(p)$.
- Let $I := \{v \in V \mid C_v = 1\}$.
- The measure $P(\cdot \mid I \in I_G)$ on $I_G$ is same as $P^G_\lambda$.

Remark:

- This gives a way to get exact samples from $P^G_\lambda$. 
Hard-Core Model on an Infinite Graph

Problems with the Two Previous Descriptions:

- For Description 1, we note that $\mathcal{I}_G$ is infinite and hence the partition function $Z_\lambda(G) = \infty$!

- For Description 2, we end up with the (same type of) problem that the event $[I \in \mathcal{I}_G]$ has zero probability under the i.i.d. coin tossing measure.

An Observation on Finite Graph:

Fix any vertex $v \in V$ and let $\sigma$ be an independent set for the graph with vertex set $V \setminus \{v\}$ then

$$P^G_\lambda (v \in I \mid I \setminus \{v\} = \sigma) = \begin{cases} \frac{\lambda}{1+\lambda} & \text{if } \sigma \cup \{v\} \in \mathcal{I}_G \\ 0 & \text{otherwise} \end{cases}$$

where $I \in \mathcal{I}_G$. 
Statistical Physics Definition :

**Definition 1** Given a finite or countably infinite, but locally finite graph $G = (V, E)$ and $\lambda > 0$, a probability measure $\mathbb{P}^G_\lambda$ on $\{0, 1\}^V$, is said to be a Gibbs measure for the hard-core model on $G$ with activity $\lambda$, if it admits conditional probabilities such that for all $v \in V$ and for any $\sigma \in \{0, 1\}^{V \setminus \{v\}}$,

$$
\mathbb{P}^G_\lambda(I(v) = 1 | I(V \setminus \{v\}) = \sigma) = \begin{cases} 
\frac{\lambda}{1+\lambda} & \text{if } \sigma \lor 1_v \in \mathbb{I}_G \\
0 & \text{otherwise}
\end{cases}
$$

where $I$ is a $\{0, 1\}^V$-valued random variable with distribution $\mathbb{P}^G_\lambda$.

**Remarks :**

- This is what is known as Dobrushin-Lanford-Ruelle (DLR) definition of infinite-volume Gibbs measure.
- Similar definitions are used for defining Ising model and $q$-Potts model on infinite graphs.
Existence and Uniqueness

- In general a Gibbs measure exists by compactness argument.

- If $G$ is finite then uniqueness holds trivially.

- It is not necessary that the uniqueness will hold when $G$ is infinite.

**Definition 2** For a fixed graph $G$ we say that a phase transition occurs for hard-core model with activity $\lambda > 0$, if there are more than one Gibbs measures of the form $\mathbb{P}^G_\lambda$.

**Note**: There is no phase transition if $G$ is finite.
What are Known?

• First introduced by Dobrushin (1968) on $\mathbb{Z}^d$ for model of lattice gas.

• Phase transition is well studied for $\mathbb{Z}^d$.
  - No phase transition for $d = 1$.
  - For $d \geq 2$ no phase transition for small $\lambda$, but phase transition occurs for large $\lambda$.

Not Known: Is phase transition monotone? In other words is there a critical value in $\lambda$?

• Arguably the most well studied case is the model on regular trees, $\mathbb{T}_r$ for $r \geq 2$. [Kelly, 1985]
  - For a $r$-regular tree $\mathbb{T}_r$, there exists a critical value $\lambda_c(r)$ such that, no phase transition when $\lambda \leq \lambda_c(r)$ and phase transition occurs when $\lambda > \lambda_c(r)$.
  - $\lambda_c(r) = \frac{(r-1)^{r-1}}{(r-2)^r}$.

• It is also known that there are infinite trees for which phase transition is not monotone! [Brightwell, Häggström, Winkler, 1998]
Hard-Core Model on Random Graphs

Setup :

- $\mathcal{G}$ be a set of graphs which are finite or countably infinite and are locally finite.
- Suppose $\mathbf{P}$ is a probability on $\mathcal{G}$.
- Let $G \sim \mathbf{P}$. We will write $G(\omega)$ for a realization of the random graph $G$.
- Given $G(\omega)$ a hard-core model with activity $\lambda > 0$ on $G(\omega)$ will be denoted by $\mathbb{P}_\lambda^{\omega}$.
- We will denote the joint measure as $\mathbf{P}_\lambda$.

Remark :

- Note that there are two stages of randomness and there are two parameters :
  - One is the probability distribution $\mathbf{P}$ on $\mathcal{G}$ governing the randomness of the underlying graphical structure.
  - The other is $\lambda$ which is governing the hard-core model given the graph.
**Phase Transition**

**Definition 3** Given a random graph model \((G, P)\), we say that there is a phase transition for the hard-core model with activity \(\lambda > 0\) on a random graph \(G \sim P\) if

\[
P\left(\exists \text{ multiple measures of the form } P^G_\lambda \right) > 0.
\]

**Remark :**

- If the random graph model is such that \(G\) is finite a.s. then there will be no phase transition for any activity \(\lambda > 0\).

- It is possible to construct an example of \((G, P)\) such that phase transition occurs for every \(\lambda > 0\).
An Example

• Let \( \mathcal{G} := \{ \mathbb{T}_r | r \geq 2 \} \) and \( P \) be given by \( P(\mathbb{T}_r) = \frac{1}{2^{r-1}}. \)

• Recall that from Kelly’s work (1985) it is known that for hard-core model on \( r \)-regular tree \( \mathbb{T}_r \), phase transition occurs if and only if

\[
\lambda > \lambda_c(r) = \frac{(r - 1)^{r-1}}{(r - 2)^r}.
\]

• But \( \lambda_c(r) \to 0 \) as \( r \to \infty. \)

• So for every \( \lambda > 0 \) for large enough \( r \) we must have \( \lambda_c(r) < \lambda \) and thus a phase transition would occur for the random graph model \( (\mathcal{G}, P) \).

Remark :

• It is important to note that for the model \( (\mathcal{G}, P) \) we can have realizations having arbitrarily large degree with positive probability.

• It is known that for bounded degree (fixed) graphs there should be no phase transition for small values of \( \lambda. \) [van den Berg and Steif, 1994]
Random Graph Models

- **GW-Tree**: Galton-Watson branching process tree with a given progeny distribution denoted by $N$.
  - The parameter here is the distribution of $N$.

- **Sparse Random Graphs**:
  - **Erdős and Rényi Random Graph**: A random graph on $n \geq 1$ vertices labeled by $[n] := \{1, 2, \ldots, n\}$ where each pair of vertices are connected by an edge independently with probability $\frac{c}{n}$, where $c > 0$. This would be denoted by $G(n, \frac{c}{n})$.
  - The parameter here is $c > 0$.
  - **Random $r$-regular Graph**: This is to select one graph at random from the set of all $r$-regular graphs with vertex set $[n]$. We will denote this model by $G_r(n)$.
    - **Note**: In order for this model to make sense we will always assume that $nr$ is even.
  - The parameter here is $r \geq 2$. 

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Motivations

• Aldous’ conjecture for the scaling of the independent number of a sparse random graph.


• Has applications in engineering fields, like in multicast networking problems. [Ramanan et al, 2002]
Sparse Random Graphs and GW-Trees

- **Known**: If $\mathcal{G}_n$ be a model for sparse random graph then for “large” enough $n$ from the “view point” of a fixed vertex “locally it looks like” a (possibly random) rooted tree.
  
  ▶ For $\mathcal{G}(n, \frac{c}{n})$ it is a rooted Galton-Watson tree with Poisson $(c)$ offspring distribution.
  
  ▶ For $\mathcal{G}(n, r)$ it is a rooted $r$-regular tree.

- **Conclusion**: So for computing “large” $n$ limit of hard-core model on these kind of graphs we may need to consider the similar model on respective GW-trees.

- **Note**: For a $r$-regular tree, one slight annoyance is that it is not really a GW-tree! But by removing one vertex (the root) it can be viewed as a collection of $r$ GW-trees with progeny distribution $N \equiv r - 1$. 

Hard-Core Model on GW-Trees

**Proposition 1** Fix $\lambda > 0$ then the followings hold for a GW-tree with progeny distribution $N$.

(a) If $E[N] \leq 1$ then there is no phase transition.

(b) If $E[N] > 1$ then on the event of non-extinction phase transition occurs with probability 0 or 1.
Proof of Proposition 1 :

- Nothing to prove for part (a).

- For part (b) notice that the property that a (fixed) rooted tree $T$ has no phase transition implies that if $v$ is a child of the root, and $T(v)$ is the sub-tree rooted at $v$ consisting only of the descendants of $v$, then $T(v)$ also has no phase transition.

- Let $\beta := P_\lambda(\text{no phase transition in } T)$ where $T$ is a GW-tree, and let $\{v_1, v_2, \ldots, v_N\}$ be the children of the root in $T$. Then
  \[
  \pi \leq P_\lambda(\text{no phase transition in } T(v_j), \forall j) = \sum_{n=0}^{\infty} P(N = n) \pi^n = f(\pi)
  \]
  where $f$ is the generating function for $N$.

- Moreover $\beta \geq q := \text{extinction probability, because } \text{[extinction]} \subseteq \text{[no phase transition]}$

- Thus $\beta \in \{q, 1\}$ and this completes the proof.
Key Recursion on a Finite Tree

- Suppose $\mathcal{T}$ be a finite (fixed) rooted tree and we consider the hard-core model on it with activity $\lambda > 0$.

- Suppose $\emptyset$ be the root and it has $n(\emptyset)$ many children which are denoted by $1, 2, \ldots, n(\emptyset)$.

- Let $I$ be a random independent set distributed according to the hard-core model with activity $\lambda > 0$. We define $\eta_{\emptyset}^{\mathcal{T}} := \mathbb{P}_{\lambda}^{\mathcal{T}} (\emptyset \in I)$.

- For a child $j$, let $\mathcal{T}_j$ be the sub-tree rooted at $j$ obtained by removing $\emptyset$. Suppose $\eta_{j}^{\mathcal{T}_j}$ be defined similarly of $\eta_{\emptyset}^{\mathcal{T}}$.

- The following key recursion holds

\[
\eta_{\emptyset}^{\mathcal{T}} = \frac{\lambda \prod_{j=1}^{n(\emptyset)} (1 - \eta_{j}^{\mathcal{T}_j})}{1 + \lambda \prod_{j=1}^{n(\emptyset)} (1 - \eta_{j}^{\mathcal{T}_j})}
\]
“Superscript Dropping Principle”
Recursive Distributional Equation (RDE)

We consider the following distributional identity:

\[ \eta \overset{d}{=} \frac{\lambda \prod_{j=1}^{N} (1 - \eta_j)}{1 + \lambda \prod_{j=1}^{N} (1 - \eta_j)} \quad \text{on } [0, 1], \]

where \((\eta_j)\) are i.i.d. copies of \(\eta\) and are independent of \(N\).

- We also define an operator \(T : \mathcal{P}([0, 1]) \rightarrow \mathcal{P}([0, 1])\) using the right-hand side of the above RDE, namely,

\[ T(\mu) := \text{dist} \left( \frac{\lambda \prod_{j=1}^{N} (1 - \eta_j)}{1 + \lambda \prod_{j=1}^{N} (1 - \eta_j)} \right) \]

where \((\eta_j)\) are i.i.d. with distribution \(\mu\) on \([0, 1]\) and are independent of \(N\).

- We put \(S = T^2\).
RDE Continued ...

Properties of the RDE and the Operator $T$:

- $T(\delta_0) = \delta_{\lambda/(1+\lambda)}$.
- $\delta_0 \preceq T(\mu) \preceq \delta_{\lambda/(1+\lambda)}$, for any probability $\mu$ on $[0,1]$.
- $T$ is anti-monotone $\Rightarrow$ $S$ is monotone.
- $T$ is continuous with respect to the weak convergence topology on $\mathcal{P}([0,1])$.
- So there exist $\mu_* \preceq \mu^*$ two fixed points of $S$ such that $S^n(\delta_0) \uparrow \mu_*$ and $S^n(\delta_{\lambda/(1+\lambda)}) \downarrow \mu^*$.
- $T(\mu_*) = \mu^*$ and $T(\mu^*) = \mu_*$.
- $S$ has unique fixed point if and only if $\mu_* = \mu^*$.
- $T$ is a strict contraction with respect to the Wasserstein metric when $\lambda E[N] < 1$. 
**Uniqueness Domain**

**Definition 4** We will say that we are in the uniqueness domain if $\mu_* = \mu^*$. 

**Characterization of Phase Transition for GW-Tree Model**

**Theorem 2** For GW-tree with progeny distribution $N$, there is no phase transition for the hard-core model with activity $\lambda > 0$, if and only if, we are in uniqueness domain for the associated RDE.
Specialization to $r$-regular Tree

- Notice that if $T_r(\emptyset)$ denote a rooted $r$-regular tree, that is, a tree whose root $\emptyset$ has degree $r - 1$ and all other vertices have degree $r$, then it is a GW-tree with progeny distribution $N \equiv r - 1$.

- So for this model $N$ is non random, that is the operator $T$ has no random part in its definition.

- This then implies both $\mu_*$ and $\mu^*$ are degenerate measures.

- So basically we need to consider fixed point of a deterministic function $s = t^2$ where $t : [0, 1] \to [0, 1]$ given by

$$t(p) = \frac{\lambda (1 - p)^{r-1}}{1 + \lambda (1 - p)^{r-1}}, \quad p \in [0, 1].$$

- This is exactly what Kelly did in his 1985 paper and this leads to the critical value $\lambda_c(r)$.
When Does Uniqueness Domain Hold?

**Corollary 3** For a GW-tree with progeny distribution $N$, there is no phase transition for the hard-core model with activity $\lambda > 0$ if

(a) $\mathbb{E}[N] \leq 1$ or,

(b) $\lambda \mathbb{E}[N] < 1$.

Remarks:

- In particular it shows that for any GW-tree (with $\mathbb{E}[N] < \infty$) at least for sufficiently small $\lambda$ there is no phase transition. Such result is expected. But note that we do not assume that the progeny distribution is bounded.

- In fact a better bound holds using Van den Berg-Steif inequality, namely $\lambda (\mathbb{E}[N] - 1) < 1$. 

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Main Results for hard-Core Model on Sparse Random Graphs

**Theorem 4** Suppose $X^\omega_{\lambda}(n, c)$ be the size of a random independent set distributed according to the hard-core model with activity $\lambda > 0$ on a Erdös-Rényi random graph $G(n, \frac{c}{n})$. If the GW-tree with Poisson$(c)$ progeny distribution has no phase transition then

$$
\lim_{n \to \infty} \frac{\mathbb{E}_\lambda [X^\omega_{\lambda}(n, c)]}{n} = \gamma_\lambda(c)
$$

where $\gamma_\lambda(c) := \mathbb{E}[\eta]$ and $\eta$ is the unique solution of the RDE.

**Theorem 5** Suppose $X^\omega_{\lambda}(n, r)$ be the size of a random independent set distributed according to the hard-core model with activity $\lambda > 0$ on a random regular graph $G_r(n)$. If the $r$-regular tree has no phase transition, that is, if $\lambda < \lambda_c(r) = \frac{(r-1)^{r-1}}{(r-2)r}$, then

$$
\lim_{n \to \infty} \frac{\mathbb{E}_\lambda [X^\omega_{\lambda}(n, r)]}{n} = \alpha_\lambda(r)
$$

where $\alpha_\lambda(r) = \frac{w}{1 + 2w}$ with $w$ is the unique positive solution of the equation $\lambda = w(1 + w)^{r-1}$.
Back to Aldous’ Conjecture

Conjecture [Aldous, 2003]: For a sparse random graph if $I_n$ is a maximum independent set then

$$\lim_{n \to \infty} \frac{\mathbb{E}[|I_n|]}{n} = \kappa$$

for some constant $\kappa > 0$ (explicitly computable?).

- Our method fails! This is because it seems (for the general GW-tree case) that the uniqueness domain does not hold for large $\lambda$.

- For example it is the case with $r$-regular trees and hence for the sparse random graph model $G_r(n)$.

- In fact for $G_r(n)$ model it has been postulated (proved using non-rigorous methods) in physics literature that such asymptotic limit exists and has the same answer as Theorem 5 when $\lambda$ is smaller than the so called “extremality threshold” (which is bigger than the “uniqueness threshold”).

- Our Theorems 4 and 5 provides rigorous argument when $\lambda$ is in the uniqueness domain (that is, under the uniqueness threshold for the $G_r(n)$ model).
Background : Recursive Tree Process (RTP)

Consider the RDE

\[
\eta \overset{d}{=} \frac{\lambda \prod_{j=1}^{N} (1 - \eta_j)}{1 + \lambda \prod_{j=1}^{N} (1 - \eta_j)} \quad \text{on } [0, 1],
\]

where \((\eta_j)\) are i.i.d. copies of \(\eta\) and are independent of \(N\).

Notations :

- Let \(\mu\) be a solution of the RDE.

- Let \(T_\infty = (V, E)\) be the canonical infinite tree with vertex set \(V := \{i \mid i \in \mathbb{N}^d, d \geq 1\} \cup \{\emptyset\}\). We will consider it as rooted at \(\emptyset\).

- Suppose \((N_i)_{i \in V}\) be i.i.d. copies of the progeny distribution \(N\).
Recursive Tree Process (RTP)

A collection of $[0, 1]$-valued random variables $(\eta_i)_{i \in \mathcal{V}}$ is called an invariant Recursive Tree Process (RTP) with marginal $\mu$ if

- $\eta_i \sim \mu$ $\forall$ $i \in \mathcal{V}$.
- Fix $d \geq 0$ then $(\eta_i)_{|i|=d}$ are independent.
- $\eta_i = \frac{\lambda \prod_{j=1}^{N_i} (1-\eta_j)}{1+\lambda \prod_{j=1}^{N_i} (1-\eta_j)}$ a.s. $\forall$ $i \in \mathcal{V}$.
- $\eta_i$ is independent of $\{N_{i'} : |i'| < |i|\}$ $\forall$ $i \in \mathcal{V}$.

Remark: Using Kolmogorov's consistency, an invariant RTP with marginal $\mu$ exists if and only if $\mu$ is a solution of the RDE.
Recursive Tree Process (RTP)
Towards Proving Theorem 2

Long Range Independence Property

• Fix $d \geq 0$.

• Write $x_d$ for a vector $(x_i)_{|i|=d}$ where each $x_i \in [0,1]$.

• Let $T$ be the realization of the GW-tree rooted at $\emptyset$ obtained from the realizations of $(N_i)_{i \in \mathcal{V}}$.

• Let $(\eta^{(d)}_i (x_d))_{|i| \leq d}$ be the $d$-depth RTP with values at level $d$ given by $x_d$.

Lemma 6 (Long range independence) Suppose we are in the uniqueness domain, that is $\mu_* = \mu^*$, then

$$\lim_{d \to \infty} \sup_{x_d} |\eta^{(d)}_\emptyset (x_d) - \eta_\emptyset| = 0 \text{ a.s.}$$

Remark:

• The proof of Theorem 2 follows from this Lemma.
Proof of Lemma 6:

- For the vector \( x_d \) if all the components are same as \( c \) the we will write the vector itself as \( c \).
- \( \eta^{(2d)}(0) \uparrow \eta_0 \), a.s. and also \( \eta^{(2d+1)}(0) \downarrow \eta_0 \) a.s.
- \( \eta^{(d)}\left(\frac{\lambda}{1+\lambda}\right) = \eta^{(d+1)}(0) \), so \( \eta^{(d)}\left(\frac{\lambda}{1+\lambda}\right) \rightarrow \eta_0 \) a.s.
- If \( 0 \leq x_i \leq \frac{\lambda}{1+\lambda} \) for all \( i \in \mathcal{V} \) then
  \[
  \eta^{(2d)}(0) \leq \eta^{(2d)}(x_{2d}) \leq \eta^{(2d)}\left(\frac{\lambda}{1+\lambda}\right), \quad \text{and} \quad \eta^{(2d+1)}\left(\frac{\lambda}{1+\lambda}\right) \leq \eta^{(2d+1)}(x_{2d+1}) \leq \eta^{(2d+1)}(0).
  \]
  So \( \eta^{(d)}(x_d) \rightarrow \eta_0 \) a.s. as \( d \rightarrow \infty \).

- Now notice that \( \eta^{(d)}(1) = \eta^{(d-1)}(x_{d-1}) \) where each \( x_i \in \{0, \lambda/(1 + \lambda)\} \). So \( \eta^{(d)}(1) \rightarrow \eta_0 \) a.s.

- Finally, if \( 0 \leq x_i \leq 1 \) for all \( i \in \mathcal{V} \) then
  \[
  \eta^{(2d)}(0) \leq \eta^{(2d)}(x_{2d}) \leq \eta^{(2d)}(1), \quad \text{and} \quad \eta^{(2d+1)}(1) \leq \eta^{(2d+1)}(x_{2d+1}) \leq \eta^{(2d+1)}(0).
  \]
  So \( \eta^{(d)}(x_d) \rightarrow \eta_0 \) uniformly a.s. as \( d \rightarrow \infty \), proving the lemma.
Remarks on the Proofs of Theorem 4 and Theorem 5

- First thing to note is
  \[ \frac{1}{n} \mathbb{E}_\lambda \left[ X_\lambda^\omega \right] = P_\lambda \left( v_0 \in I_\lambda^\omega \right), \]
  where \( I_\lambda^\omega \) is the random independent set selected according to the (random) distribution \( P_\lambda^\omega \), and \( v_0 \) is a fixed vertex.

- For any fixed \( d > 0 \) the distribution of the \( d \)-depth neighborhood of \( v_0 \) converges to the distribution of a \( d \)-depth Poisson(\( c \)) GW-tree for \( G(n, \frac{c}{n}) \) model, and to a \( d \)-depth \( r \)-regular tree for \( G_r(n) \) model.

- We can then apply the local weak convergence technique of Aldous and Steele (2004) using the (strong) long range independence property which holds under the uniqueness domain. These will give the stated results after a little more careful probability computations!
Open Problems/Questions

• We know that for \( r \)-regular tree the phase transition is a \textit{monotone} property in \( \lambda \), and the critical value \( \lambda_c(r) \) is explicitly known.

Is phase transition monotone (in \( \lambda \)) for a general GW-tree?

\textbf{Comment} : Most possibly not ! But is it at least the case for GW-tree with Poisson progeny distribution ?

• For GW-tree with Poisson(\( c \)) progeny distribution is phase transition monotone in \( c \) ? That is for every fixed \( \lambda > 0 \) if \( c > c' \) and we have no phase transition for Poisson(\( c \)) GW-tree then can we say that we have no phase transition for Poisson(\( c' \)) GW-tree ?

\textbf{Comment} : We know this is true if \( \lambda \times c < 1 \).

• If the answer to above question is yes (which is most possibly the case) can we also get the critical value for \( c \) ? (explicitly or bounds ?)