Hard-Core Model on Random Graphs

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Hard-Core Model on a Finite Graph

- Let $G := (V, E)$ be a finite graph.

- We say a subset $I \subseteq V$ is an *independent set* of G, if for any two $u, v \in I$ there is no edge between u and v.

- Let $\mathcal{I}_G$ be the set of all independent sets of G.

- Fix $0 < p < 1$ and let $\lambda = \frac{p}{1-p}$.

- Suppose $(C_v)_{v \in V}$ be i.i.d. Bernoulli (p).

- Define $I := \{v \in V \mid C_v = 1\}$.

- The measure $\mathbf{P}(\cdot \mid I \in \mathcal{I}_G)$ on $\mathcal{I}_G$ is called the *hard-core model* or *random independent set model* with *activity* λ. We will denote it by $\mathbb{P}$.

- It is easy to see that $\mathbb{P}$ is the probability on $\mathcal{I}_G$ which puts mass proportional to $\lambda^{|I|}$ for $I \in \mathcal{I}_G$.

Sparse Random Graphs

- Two types of *sparse random graphs* :
  - $\mathcal{G}\left(n, \frac{\mu}{n}\right)$: A random graph with n vertices and each edge is present with probability $\frac{\mu}{n}$ independently, where $\mu > 0$. [Erdös & Rényi 1959 - 1968]
  - $\mathcal{G}(n, r+1)$: Pick a graph uniformly at random from the set of all $(r+1)$-regular graphs with n vertices.
- Given a particular realization G_ω of a sparse random graph, we will consider the *hard-core model* with activity $\lambda > 0$ on that finite graph as described before.
- Note there are two stages of randomness and there are two parameters,
  - $\mu > 0$ dealing with the randomness of the graph configuration.
  - $\lambda > 0$ dealing with the randomness of the hard-core model given a configuration.

Motivations

- Interesting from Statistical Physics point of view, well studied for non-random graphs. [Kelley 1985, van den Berg & Steif 1994, Brightwell, Häggström & Winkler 1998, Brightwell & Winkler 1999]

- Has applications in engineering fields, like in *multi-cast networking* problems. [Ramanan et al 2002]

- Conjecture of Aldous [2003] :

  For a sparse random graph if I_n be the *maximal independent set* then

  $$\frac{\mathbf{E}\left[|I_n|\right]}{n} \to c \quad \text{as} \quad n \to \infty,$$

  where c is a constant which depends on the model for the sparse random graph.

Remark : For a hard-core model on a finite graph if we take $\lambda \to \infty$ limit then it concentrate on the maximal independent set(s).

Sparse Random Graphs and GW-Trees

- **Known :** If $\mathcal{G}_n$ be a model for sparse random graph then for "large" enough n "locally it looks like" a (possibly random) rooted tree.
  - For $\mathcal{G}\left(n, \frac{\mu}{n}\right)$ it is rooted Galton-Watson tree with Poisson (μ) offspring distribution.
  - For $\mathcal{G}\left(n, r+1\right)$ it is rooted $(r+1)$-regular tree.
- **Conclusion :** So for computing "large" n limit of hard-core model on these kind graphs we need to consider the similar model on respective GW-trees.
- **Problem :** The trees we get may be infinite with positive probability.
- **Solution :** In that case we need to consider Gibbs measure with activity $\lambda > 0$ which has appropriate conditional laws ("DLR condition").
- **Warning :** It is then no longer true that there is only one such measure and we will say that a *phase transition* occurs if there are multiple Gibbs measures for a given activity $\lambda > 0$.

Key Recursion on a Finite Tree

- Suppose $\mathcal{T}$ be a finite rooted tree and we consider the hard-core model on it with activity $\lambda > 0$.

- Suppose $\emptyset$ be the root and it has $n(\emptyset)$ many children which are denoted by $1, 2, \ldots, n(\emptyset)$.

- Let I be a random independent set distributed according to the hard-core model with activity $\lambda > 0$. Then we define $\eta_{\mathcal{T}}^{\emptyset} := \mathbb{P}(\emptyset \in I)$.

- For a child j, let $\mathcal{T}_j$ be the sub-tree rooted at j obtained by removing $\emptyset$. Suppose $\eta_{\mathcal{T}_j}^{j}$ be defined similarly of $\eta_{\mathcal{T}}^{\emptyset}$.

- The following *key recursion* holds

$$\eta_{\mathcal{T}}^{\emptyset} = \frac{\lambda \prod_{j=1}^{n(\emptyset)} \left(1 - \eta_{\mathcal{T}_j}^{j}\right)}{1 + \lambda \prod_{j=1}^{n(\emptyset)} \left(1 - \eta_{\mathcal{T}_j}^{j}\right)}$$

Related RDE

$$\eta \stackrel{d}{=} \frac{\lambda \prod_{j=1}^{N} \left(1-\eta_j\right)}{1+\lambda \prod_{j=1}^{N} \left(1-\eta_j\right)} \quad \text{on} \quad [0,1],$$

where (η_j) are i.i.d. copies of η and are independent of N.

Properties : Let T be the associated operator and $S = T^2$ then

- $T(\delta_0) = \delta_{\lambda/(1+\lambda)}$.
- $\delta_0 \preccurlyeq T(m) \preccurlyeq \delta_{\lambda/(1+\lambda)}$, for any probability m on $[0,1]$.
- T is anti-monotone $\Rightarrow$ S is monotone.
- So there exist $m_* \preccurlyeq m^*$ two fixed points of S such that $S^n(\delta_0) \uparrow m_*$ and $S^n\left(\delta_{\lambda/(1+\lambda)}\right) \downarrow m^*$.
- $T(m_*) = m^*$.
- S has unique fixed point if and only if $m_* = m^*$.

Uniqueness Domain

Definition 1 *We will say that we are in* uniqueness domain *if* $m_* = m^*$.

Results

- **Theorem 1** *For a GW-Tree with progeny distribution N and for activity $\lambda > 0$ we are in uniqueness domain if and only if, there is a unique Gibbs measure with activity λ a.s. with respect to the randomness in the configuration of the tree.*

  Note : The *phase transition* is characterize by the uniqueness of solution of a RDE.

- **Theorem 2** *For $\mathcal{G}\left(n, \frac{\mu}{n}\right)$ suppose we are in the uniqueness domain for $\lambda > 0$ and with $N \sim$ Poisson (μ) and let I_n be a random independent set with hard-core distribution with activity λ, then*

$$\frac{\mathbf{E}\left[|I_n|\right]}{n} \to \mathbf{E}\left[\eta\right]$$

  where $\eta \sim m_* = m^*$.

- **Theorem 3** *A similar statement for $\mathcal{G}\left(n, r+1\right)$.*

When Uniqueness Domain Holds ?

- **Small μ :** If $\mu \leq 1$ then the graphical structure is in the (sub)-critical domain and hence it will be finite and so uniqueness domain holds for any $\lambda > 0$. This is not the interesting case !

- **Small λ :** If $\lambda \times \mu < 1$ then T is a contraction and hence uniqueness domain holds. Thus for any $\mu > 0$ for activity $\lambda < \frac{1}{\mu}$ we are in the uniqueness domain.

Remarks :

- I believe (do not have complete proofs yet) that uniqueness domain will not hold for large μ or large λ (and $\mu > 1$).

- So it seems that we may not be able to resolve Aldous' conjecture by this method. But perhaps we can ... that is yet another story !

- At least we do get a nice example of *phase transition* phenomenon which is characterize by uniqueness of solution of a RDE.