

Annealed and Quenched IP for Random Walk in Dynamic Markovian Environment

Antar Bandyopadhyay
(Joint work with Ofer Zeitouni)

Theoretical Statistics and Mathematics Unit
Indian Statistical Institute, Delhi Centre
<http://www.isid.ac.in/~antar>

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Outline

- 1 Introduction
- 2 Model Description
- 3 Assumptions and the Results
- 4 History and Achievements
- 5 Main Ideas in the Proofs
 - To Get a Renewal Structure
 - Construction of a "Regeneration Time"
 - Redefining the Processes
 - Quenched IP

The Basic Setup

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- **Two Stages of Randomness:**
 - **The Environment:** It is the transition laws which will tell us *how to take the next step* from the current position.
Note: These laws can be random!
 - **The Walk:** Given the environment we have an *walker* who moves on the lattice \mathbb{Z}^d starting from $\mathbf{0}$ according to the transition laws.
Note: The walker provides second stage of randomness.

Classical RWRE (Static Environment)

- **The Environment:** At the beginning of time, at every location $\mathbf{x} \in \mathbb{Z}^d$, we choose the random transition kernels according to some probability distribution, and keep them fixed through out the time evolution.

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- **The Walk:** Given the transition laws the walker then moves according to a *time homogeneous* Markov chain, starting from $\mathbf{0}$.

A Classical Example of a Statics RWRE: Sinai Walk

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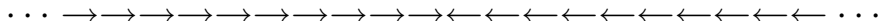
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- The walker starts at $\mathbf{0}$ and moves using the biased coin giving positive bias towards the direction of the arrow at his current location.
- Note that the average increment at each step is 0.

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In this case, there are “large traps”! For example, the following configuration of arrows appear with probability one.

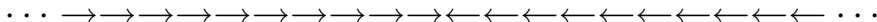
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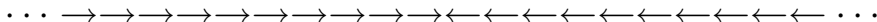
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In this case, there are “large traps”! For example, the following configuration of arrows appear with probability one.



- With high probability the walker will spend a “lot of time” in such a “trap”, this will then “slow down” the walk.
- In fact Sinai [1982] showed that in this case given a typical static environment with high probability the walker will be at $c(\log n)^2$ distance from the origin at time n .

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- Consider again the integer line \mathbb{Z} .
- The walker starts at $\mathbf{0}$ and carries both the coins.
- Before a move he first tosses the unbiased coin independently of the past, and then the biased coin again independently of the past. If the unbiased turns up a head, then he gives the bias to the right, else he gives the bias to the left of his current position.

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- In this case hopefully here we will have a CLT.

Some Notations

- **The Environment:** At a site $\mathbf{x} \in \mathbb{Z}^d$ and at time $t \geq 0$ “environment” is a transition law, it will be denoted by $\omega_t(\mathbf{x}, \cdot)$.

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- **The Walk:** The position of the walker at time t will be denoted by X_t .

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- **Annealed:** The marginal distribution of the walk, that is, integrating out the *quenched* law with respect to the environment distribution.

Note: The walk may not be a Markov chain under the annealed law.

Quenched Law

Definition of the Quenched Law

Given the entire environment

$$\omega := \left\{ (\omega_t(\mathbf{x}, \cdot))_{t \geq 0} \mid \mathbf{x} \in \mathbb{Z}^d \right\},$$

the *quenched law* of $(X_t)_{t \geq 0}$ starting from \mathbf{x} denoted by $\mathbf{P}_\omega^{\mathbf{x}}$, is the distribution of the time inhomogeneous Markov chain $(X_t)_{t \geq 0}$ on \mathbb{Z}^d , such that for every $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$

$$\mathbf{P}_\omega^{\mathbf{x}}(X_{t+1} = \mathbf{y} \mid X_t = \mathbf{x}) = \omega_t(\mathbf{x}, \mathbf{y}),$$

and

$$\mathbf{P}_\omega^{\mathbf{x}}(X_0 = \mathbf{x}) = 1.$$

Annealed Law

Definition of the Annealed Law

The *annealed law* of $(X_t)_{t \geq 0}$ starting from \mathbf{x} denoted by $\mathbb{P}^{\mathbf{x}}$, is defined by

$$\mathbb{P}^{\mathbf{x}}(\cdot) := \int \mathbf{P}_{\omega}^{\mathbf{x}}(\cdot) \mathbf{P}(d\omega),$$

where $\omega \sim \mathbf{P}$.

Dynamic Markovian Environment

- We will assume that for every $\mathbf{x} \in \mathbb{Z}^d$ the environment chain at \mathbf{x} , given by

$$(\omega_t(\mathbf{x}, \cdot))_{t \geq 0}$$

is a *stationary Markov chain* with some state space \mathcal{S} and transition kernel K .

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- By \mathbf{P}^π we will denote the distribution of the entire environment ω .

Remarks

- This particular model was first introduced by Boldrighini, Minlos and Pellegrinotti [2000].

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- The example of RWDRE which we discussed earlier falls under this model where the environment chains are just *i.i.d.* chains.

Assumptions

(A0) We have only nearest neighbor transitions.

(A1) There exists $0 < \kappa \leq 1$ such that

$$K(w, \cdot) \geq \kappa \pi(\cdot), \quad \forall w \in \mathcal{S}.$$

(A2) There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon q(\mathbf{x}, \mathbf{y}) \quad \text{a.s. } [\mathbf{P}^\pi],$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

Discussion on the Assumptions

Assumption (A0)

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- This is only for simplicity. The arguments can be easily generalized to transitions with bounded increment.

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- If the environment chains are i.i.d. chains then (A1) holds trivially with $\kappa = 1$.

Discussion on the Assumptions

Assumption (A2)

There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transitions which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon q(\mathbf{x}, \mathbf{y}) \quad \text{a.s. } [\mathbf{P}^\pi].$$

- This condition essentially means that the random environment has a “deterministic” part q , which is non-degenerate. In fact under this assumption the environment is nothing but a (random) perturbation of q .

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- Comparing with classical (static) RWRE literature, this condition can be termed as an *ellipticity* condition on the environment.
- This condition was also assumed in BMP [1997, 2000] and Stannat [2004].

Annealed SLLN and Invariance Principle

Theorem 1 (Annealed SLLN)

Suppose assumptions (A0), (A1) and (A2) hold. Then there exists a constant vector $\mathbf{v} \in \mathbb{R}^d$, such that

$$\frac{X_n}{n} \longrightarrow \mathbf{v} \text{ a.s. } \left[\mathbb{P}^0 \right], \text{ as } n \rightarrow \infty.$$

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Theorem 2 (Annealed Invariance Principle)

Suppose assumptions (A0), (A1) and (A2) hold. Then there exists a $(d \times d)$ (non-random) positive definite matrix Σ , s.t. under \mathbb{P}^0 ,

$$\left(\frac{X_{\lfloor nt \rfloor} - nt \mathbf{v}}{\sqrt{n}} \right)_{t \geq 0} \xrightarrow{d} BM_d(\Sigma), \text{ as } n \rightarrow \infty.$$

Quenched Invariance Principle

Theorem 3 (Quenched Invariance Principle)

Suppose assumptions (A0), (A1) and (A2) hold. Then for a.s. all ω with respect to \mathbf{P}^π , under the *quenched law* \mathbf{P}_ω^0 we have

$$\left(\frac{X_{\lfloor nt \rfloor} - nt \mathbf{v}}{\sqrt{n}} \right)_{t \geq 0} \xrightarrow{d} BM_d(\Sigma), \text{ as } n \rightarrow \infty.$$

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- Stannat [2004]
 - Gave a simpler but still analytic proof for any dimension $d \geq 1$.
- Rassoul-Agha and Seppäläinen [2005]
 - Proved invariance principle using probabilistic techniques as a special case of a more general result.

History Continued ...

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For dynamic Markovian environment model, exactly similar to ours, but with some more (technical) assumptions the *quenched CLT* was proved by

- Boldrighini, Minlos and Pellegrinotti [2000]
 - For dimension $d \geq 3$.
 - Proofs are based on “hard” analytic techniques.

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- For the quenched IP we needed more assumptions, namely

$$\kappa + \varepsilon^6 > 1 \text{ and } d > 7.$$

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- Here we prove the results without any non-intuitive/technical assumptions.
- For dimensions $d = 1$ and $d = 2$ the results are new. In fact earlier BMP has raised doubt on existence of quenched IP in $d = 1$, based on some simulations. So this work completely clears the matter for this model.

Our Main Strategy

- We will show that on an appropriate probability space there is a version of this process and an increasing sequence of *random times* $(\tau_n)_{n \geq 0}$ with $\tau_0 = 0$ such that the pairs $(\tau_n - \tau_{n-1}, X_{\tau_n} - X_{\tau_{n-1}})_{n \geq 1}$ form an *i.i.d.* sequence under \mathbb{P}^0

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- Moreover we will show that under our assumptions τ_1 has exponential tail.
- Because of nearest neighbor walk this will imply annealed SLLN and IP.
- For quenched IP we need to do some more work!

Construction of ε -Coins

Recall the Assumption (A2): There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon q(\mathbf{x}, \mathbf{y}) \quad \text{a.s. } [\mathbf{P}^\pi],$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

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- If $\varepsilon_{t+1} = 1$ then it takes a move according to the fixed transition kernel q .
- If $\varepsilon_{t+1} = 0$ then it takes a move according to the random transition kernel

$$\frac{\omega_t(\cdot, \cdot) - \varepsilon q(\cdot, \cdot)}{1 - \varepsilon}.$$

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- We will say a step taken by the walker is a "proper step" if and only if, it was taken when the ε -coin was 0.

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Remarks:

- By taking a step when the ε -coin is 1 the walker does not collect any information about the environment.
- We will say a step taken by the walker is a "proper step" if and only if, it was taken when the ε -coin was 0.
- Note that only by taking a "proper" step the walker learns about the environment.

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For every time duration $s \geq 0$, for every site \mathbf{z} which could possibly be reached from the current position of the walker and within time $t + s$, the environment chain at location \mathbf{z} have gone through a "*regeneration*" in the time interval $[t - l + 1, t + s]$.

Note: "*Regeneration*" of an environment chain means that it "starts afresh" from its stationary distribution.

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- If we can do this, then the random time τ will have the desired property that after this time wherever the walker goes, by the time it will reach a location it will have "no information" about its environment!

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- If we can do this, then the random time τ will have the desired property that after this time wherever the walker goes, by the time it will reach a location it will have "no information" about its environment!
- Note that such a time τ depends on the future of the environment chains at every location. So naturally it is NOT a *stopping time*. But also note it DOES NOT take into consideration any specifics of the future path of the walker.

Construction of κ -Coins

(an easy way to get environment "*regeneration*")

Recall the Assumption (A1): There exists $0 < \kappa \leq 1$ such that

$$K(w, \cdot) \geq \kappa \pi(\cdot), \quad \forall w \in \mathcal{S}.$$

Construction of κ -Coins

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 - if $\alpha_{t+1}(\mathbf{x}) = 1$ then it moves to a state selected independently from the stationary distribution π , in other words, **goes through a "regeneration"**;
 - if $\alpha_{t+1}(\mathbf{x}) = 0$ then it moves to a state according to the kernel

$$\frac{K(\omega_t(\mathbf{x}, \cdot), \cdot) - \kappa\pi(\cdot)}{1 - \kappa}.$$

No regeneration in this case!

Precise Definition of the "Regeneration Time"

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- Fix $t \geq 0$ and $\mathbf{x} \in \mathbb{Z}^d$.
- Let $L_t := \sup \left\{ l > 0 \mid \varepsilon_{t-l} = 1 \right\}$ if $\varepsilon_t = 0$, otherwise put $L_t = 0$.

Thus L_t is the length of the "improper" steps before a "proper" step at time t .

Precise Definition of the "Regeneration Time"

- Consider the following event:

$$\bigcup_{l=1}^{\infty} \left([L_t = l] \bigcap \bigcap_{s=0}^{\infty} \bigcap_{\substack{z \in \mathbb{Z} \\ |x-z| \leq s}} \bigcup_{u=t-l+1}^{t+s} [\alpha_u(\mathbf{x}) = 1] \right).$$

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- Now define

$$\tau := \inf \left\{ t \geq 1 \mid \text{the event } S(t, X_t) \text{ has occurred} \right\}.$$

Properties of τ_1

Proposition 4

Let the assumptions (A0), (A1) and (A2) hold. Then for all $t \geq 0$

$$\mathbb{P}^0(\tau > t) \leq Ae^{-bt},$$

for some constants $A < \infty$ and $b > 0$.

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Remark

In particular this proposition proves that

$$\tau < \infty \text{ a.s. .}$$

Moreover it has all moments finite.

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- We then define

$$\tau_1 := \tau \left(\omega^{(1)}, \alpha^{(1)}; \mathbf{X}^{(1)}, \varepsilon^{(1)} \right)$$

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- Having defined $\{(\mathbf{X}^{(i)}, \tau_i)\}$ for $i = 1, 2, \dots, (n-1)$, let $\mathbf{X}^{(n)}$ be the position of the walker in the n^{th} environment $(\omega^{(n)}, \alpha^{(n)})$ starting from $X_{\tau_{n-1}}^{(n-1)}$.

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- Finally define a *new* walk $(Y_t)_{t \geq 0}$ by

$$Y_t := X_{t-\tau_{n-1}}^{(n)} \quad \text{if } \tau_{n-1} \leq t < \tau_n.$$

Properties of this New Walk

Proposition 5

Let $(\tau_n)_{n \geq 1}$ and $(Y_t)_{t \geq 0}$ be as defined before, then $(\tau_n - \tau_{n-1}, Y_{\tau_n} - Y_{\tau_{n-1}})_{n \geq 1}$ is an i.i.d. sequence, where $\tau_0 = 0$.

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Theorem 6

The (annealed) law of $(Y_t)_{t \geq 0}$ is same as that of $(X_t)_{t \geq 0}$.

Brief Sketch of the Proof for the Quenched IP

- We use a technique introduced by Bolthausen and Sznitman [2002].
- Let $B_t^n := (X_{\lfloor nt \rfloor} - nt \mathbf{v}) / \sqrt{n}$, and \mathcal{B}_t^n be the polygonal interpolation of $(k/n) \mapsto B_{k/n}^n$.
- Bolthausen and Sznitman technique says that if we have the *annealed* IP then the *quenched* IP will follow if we can show that for all $T > 0$,

$$\sum_{m=1}^{\infty} \text{Var}_{\mathbf{P}^{\pi}} \left(\mathbf{E}_{\omega}^0 \left[F \left(\mathcal{B}^{\lfloor b^m \rfloor} \right) \right] \right) < \infty,$$

for every Lipschitz function F on $C([0, T], \mathbb{R}^d)$ and $b \in (1, 2]$.

Brief Sketch of the Proof for the Quenched IP

- To check that the above sum of variances is finite, we work with two walkers which are independent given the environment, along with a martingale trick which uses the time as one more *extra dimension* and helps in getting the proper estimates for "low dimensions" (i.e. when $d \leq 2$).

Thank You