

Invariance Principle for Random Walk in Dynamic Markovian Environment

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- 1 Introduction
- 2 Model Description
- 3 Assumptions and the Results
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- 5 Main Ideas in the Proofs
 - To Get a Renewal Structure
 - Construction of a "Regeneration Time"
 - Redefining the Processes
 - Quenched IP

The Basic Setup

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- **Two Stages of Randomness:**
 - **The Environment:** It is the transition laws which will tell us *how to take the next step* from the current position.
Note: These laws can be random!
 - **The Walk:** Given the environment we have an *walker* who moves on the lattice \mathbb{Z}^d starting from $\mathbf{0}$ according to the transition laws.
Note: The walker provides second stage of randomness.

Classical RWRE (Static Environment)

- **The Environment:** At the beginning of time, at every location $\mathbf{x} \in \mathbb{Z}^d$, we choose the transition kernels according to some probability distribution, and keep them fixed throughout the time evolution.

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A Classical Example of a Statics RWRE: Sinai Walk

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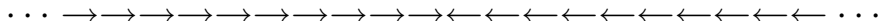
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- The walker starts at $\mathbf{0}$ and moves using the biased coin giving positive bias towards the direction of the arrow at his current location.
- Note that the average increment at each step is 0.

Sinai Walk Example Continued ...

In this case, there are “large traps”! For example, the following configuration of arrows appear with probability one.

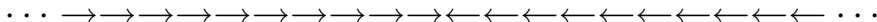
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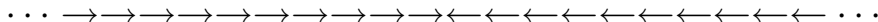
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In this case, there are “large traps”! For example, the following configuration of arrows appear with probability one.



- With high probability the walker will spend a “lot of time” in such a “trap”, this will then “slow down” the walk.
- In fact Sinai [1982] showed that in this case given a typical static environment with high probability the walker will be at $c(\log n)^2$ distance from the origin at time n .

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- Consider again the integer line \mathbb{Z} .
- The walker starts at $\mathbf{0}$ and carries both the coins.
- Before a move he first tosses the unbiased coin independently of the past, and then the biased coin again independently of the past. If the unbiased turns up a head, then he gives the bias to the right, else he gives the bias to the left of his current position.

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- Conceivably yes!
- We will expect if the environment has a dynamics which is “fast mixing” then the “traps” wouldn't exist to “slow down the walk”, and hence we can get a CLT.

Some Notations

- **The Environment:** At a site $\mathbf{x} \in \mathbb{Z}^d$ and at time $t \geq 0$ “environment” is a transition law, it will be denoted by $\omega_t(\mathbf{x}, \cdot)$.

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- **The Walk:** The position of the walker at time t will be denoted by X_t .

Quenched and Annealed Laws

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- **Annealed:** The marginal distribution of the walk, that is, integrating out the *quenched* law with respect to the environment distribution.

Note: The walk may not be a Markov chain under the annealed law.

Quenched Law

Definition of the Quenched Law

Given the entire environment

$$\omega := \left\{ (\omega_t(\mathbf{x}, \cdot))_{t \geq 0} \mid \mathbf{x} \in \mathbb{Z}^d \right\},$$

the *quenched law* of $(X_t)_{t \geq 0}$ starting from \mathbf{x} denoted by $\mathbf{P}_\omega^{\mathbf{x}}$, is the distribution of the time inhomogeneous Markov chain $(X_t)_{t \geq 0}$ on \mathbb{Z}^d , such that for every $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$

$$\mathbf{P}_\omega^{\mathbf{x}}(X_{t+1} = \mathbf{y} \mid X_t = \mathbf{x}) = \omega_t(\mathbf{x}, \mathbf{y}),$$

and

$$\mathbf{P}_\omega^{\mathbf{x}}(X_0 = \mathbf{x}) = 1.$$

Definition of the Annealed Law

The *annealed law* of $(X_t)_{t \geq 0}$ starting from \mathbf{x} denoted by $\mathbb{P}^{\mathbf{x}}$, is defined by

$$\mathbb{P}^{\mathbf{x}}(\cdot) := \int \mathbf{P}_{\omega}^{\mathbf{x}}(\cdot) \mathbf{P}(d\omega),$$

where $\omega \sim \mathbf{P}$.

Dynamic Markovian Environment

- We will assume that for every $\mathbf{x} \in \mathbb{Z}^d$ the environment chain at \mathbf{x} , given by

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- By \mathbf{P}^π we will denote the distribution of the entire environment ω .

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- Our example of RWDRE, falls under this model where the environment chains are just *i.i.d.* chains.

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(A2) There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon q(\mathbf{x}, \mathbf{y}) \quad \text{a.s. } [\mathbf{P}^\pi],$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

Discussion on the Assumptions

Assumption (A0)

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- This is for simplicity, the arguments can be easily generalized to transitions with bounded increment.

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- If the environment chains are i.i.d. chains then (A1) holds trivially with $\kappa = 1$.

Discussion on the Assumptions

Assumption (A2)

There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon q(\mathbf{x}, \mathbf{y}) \quad \text{a.s. } [\mathbf{P}^\pi].$$

- This condition essentially means that the random environment has a “deterministic” part q , which is non-degenerate. Presumably it is a “small” (random) perturbation of q .

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- Comparing with classical (static) RWRE literature, this condition can be termed as an *ellipticity* condition on the environment.
- This condition was also assumed in BMP [1997, 2000] and Stannat [2004].

Annealed SLLN and Invariance Principle

Theorem 1 (Annealed SLLN)

Suppose assumptions (A0), (A1) and (A2) hold. Then there exists a constant vector $\mathbf{v} \in \mathbb{R}^d$, such that

$$\frac{X_n}{n} \longrightarrow \mathbf{v} \text{ a.s. } \left[\mathbb{P}^0 \right], \text{ as } n \rightarrow \infty.$$

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Theorem 2 (Annealed Invariance Principle)

Suppose assumptions (A0), (A1) and (A2) hold. Then there exists a $(d \times d)$ (non-random) positive definite matrix Σ , s.t. under \mathbb{P}^0 ,

$$\left(\frac{X_{\lfloor nt \rfloor} - nt \mathbf{v}}{\sqrt{n}} \right)_{t \geq 0} \xrightarrow{d} BM_d(\Sigma), \text{ as } n \rightarrow \infty.$$

Quenched Invariance Principle

Theorem 3 (Quenched Invariance Principle)

Suppose assumptions (A0), (A1) and (A2) hold. Then for a.s. all ω with respect to \mathbf{P}^π , under the *quenched law* \mathbf{P}_ω^0 we have

$$\left(\frac{X_{\lfloor nt \rfloor} - nt \mathbf{v}}{\sqrt{n}} \right)_{t \geq 0} \xrightarrow{d} BM_d(\Sigma), \text{ as } n \rightarrow \infty.$$

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- Stannat [2004]
 - Gave a simpler but still analytic proof for any dimension $d \geq 1$.
- Rassoul-Agha and Seppäläinen [2005]
 - Proved invariance principle using probabilistic techniques as a special case of a more general result.

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- Boldrighini, Minlos and Pellegrinotti [2000]
 - For dimension $d \geq 3$.
 - Proofs are based on “hard” analytic techniques.

Our Earlier Contribution

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- For the quenched IP we needed more assumptions, namely

$$\kappa + \varepsilon^6 > 1 \text{ and } d > 7.$$

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- Here we prove the results without any non-intuitive/technical assumptions.
- For dimensions $d = 1$ and $d = 2$ the results are new. In fact earlier BMP has raised doubt on existence of quenched IP in $d = 1$, based on some simulations. So this work completely clears the matter for this model.

Our Main Strategy

- We will show that on an appropriate probability space there is a version of this process and an increasing sequence of *random times* $(\tau_n)_{n \geq 0}$ with $\tau_0 = 0$ such that the pairs $(\tau_n - \tau_{n-1}, X_{\tau_n} - X_{\tau_{n-1}})_{n \geq 1}$ form an *i.i.d.* sequence under \mathbb{P}^0

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- Moreover we will show that under our assumptions τ_1 has exponential tail.
- Because of nearest neighbor walk this will imply annealed SLLN and IP.
- For quenched IP we need to do some more work!

Construction of ε -Coins

Recall the Assumption (A2): There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon q(\mathbf{x}, \mathbf{y}) \quad \text{a.s. } [\mathbf{P}^\pi],$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

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- If $\epsilon_{t+1} = 1$ then it takes a move according to the fixed transition kernel q .
- If $\epsilon_{t+1} = 0$ then it takes a move according to the random transition kernel

$$\frac{\omega_t(\cdot, \cdot) - \varepsilon q(\cdot, \cdot)}{1 - \varepsilon}.$$

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- We will say a step taken by the walker is a "proper step" if and only if, it was taken when the ε -coin was 0.
- Note that only by taking a "proper" step the walker learns about the environment.

A "Regeneration Time"

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For every time duration $s \geq 0$ for every site \mathbf{z} which could possibly be reached from the current position of the walker and within time $t + s$, the environment chain at location \mathbf{z} have gone through a "*regeneration*" in the time interval $[t - L + 1, t + s]$.
Note: Here by "*regeneration*" of an environment chain, we mean that it starts afresh from its stationary distribution.

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- Note that such a time τ depends on the future of the environment chains at every location. So naturally it is NOT a *stopping time*. But it DOES NOT take into consideration any specifics of the future path of the walker.

Construction of κ -Coins

(an easy way to get environment "*regeneration*")

Recall the Assumption (A1): There exists $0 < \kappa \leq 1$ such that

$$K(w, \cdot) \geq \kappa \pi(\cdot), \quad \forall w \in \mathcal{S}.$$

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 - if $\alpha_{t+1}(\mathbf{x}) = 1$ then it moves to a state selected independently from the stationary distribution π , in other words, **goes through a regeneration**;
 - if $\alpha_{t+1}(\mathbf{x}) = 0$ then it moves to a state according to the kernel

$$\frac{K(\omega_t(\mathbf{x}, \cdot), \cdot) - \kappa\pi(\cdot)}{1 - \kappa}.$$

No regeneration in this case!

Precise Definition of the "Regeneration Time"

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- Fix $t \geq 0$ and $\mathbf{x} \in \mathbb{Z}^d$.
- Let $L_t := \sup \left\{ l > 0 \mid \varepsilon_{t-l} = 1 \right\}$ if $\varepsilon_t = 0$, otherwise put $L_t = 0$.

Thus L_t is the length of the "improper" steps before a "proper" step at time t .

Precise Definition of the "Regeneration Time"

- Consider the following event:

$$\bigcup_{l=1}^{\infty} \left([L_t = l] \bigcap \bigcap_{s=0}^{\infty} \bigcap_{\substack{z \in \mathbb{Z} \\ |x-z| \leq s}} \bigcup_{u=t-l+1}^{t+s} [\alpha_u(\mathbf{x}) = 1] \right).$$

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- Now define

$$\tau := \inf \left\{ t \geq 1 \mid \text{the event } S(t, X_t) \text{ has occurred} \right\}.$$

Properties of τ_1

Proposition 4

Let the assumptions (A0), (A1) and (A2) hold. Then for all $t \geq 0$

$$\mathbb{P}^0(\tau > t) \leq Ae^{-bt},$$

for some constants $A < \infty$ and $b > 0$.

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Remark

In particular this proposition proves that

$$\tau < \infty \text{ a.s. .}$$

A New Walk

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- Let $\mathbf{X}^{(1)}$ be the position of the walker moving in the first environment $(\omega^{(1)}, \alpha^{(1)})$ using $\varepsilon^{(1)}$.
- We then define

$$\tau_1 := \tau \left(\omega^{(1)}, \alpha^{(1)}; \mathbf{X}^{(1)}, \varepsilon^{(1)} \right)$$

A New Walk

- Having defined $\{(\mathbf{X}^{(i)}, \tau_i)\}$ for $i = 1, 2, \dots, (n-1)$, let $\mathbf{X}^{(n)}$ be the position of the walker in the n^{th} environment $(\omega^{(n)}, \alpha^{(n)})$ starting from $X_{\tau_{n-1}}^{(n-1)}$.

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- Define τ_n recursively as

$$\tau_n := \tau_{n-1} + \tau \left(\omega^{(n)}, \alpha^{(n)}; \mathbf{X}^{(n)}, \varepsilon^{(n)} \right)$$

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$$\tau_n := \tau_{n-1} + \tau\left(\omega^{(n)}, \alpha^{(n)}; \mathbf{X}^{(n)}, \varepsilon^{(n)}\right)$$

- Finally define a *new* walk $(Y_t)_{t \geq 0}$ by

$$Y_t := X_{t-\tau_{n-1}}^{(n)} \quad \text{if } \tau_{n-1} \leq t < \tau_n.$$

Properties of this New Walk

Proposition 5

Let $(\tau_n)_{n \geq 1}$ and $(Y_t)_{t \geq 0}$ be as defined before, then $(\tau_n - \tau_{n-1}, Y_{\tau_n} - Y_{\tau_{n-1}})_{n \geq 1}$ is an i.i.d. sequence, where $\tau_0 = 0$.

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Theorem 6

The (annealed) law of $(Y_t)_{t \geq 0}$ is same as that of $(X_t)_{t \geq 0}$.

Brief Sketch of the Proof for the Quenched IP

- We use a technique introduced by Bolthausen and Sznitman [2002].
- Let $B_t^n := (X_{\lfloor nt \rfloor} - nt \mathbf{v}) / \sqrt{n}$, and \mathcal{B}_t^n be the polygonal interpolation of $(k/n) \mapsto B_{k/n}^n$.
- Bolthausen and Sznitman technique says that if we have the *annealed* IP then the *quenched* IP will follow if we can show that for all $T > 0$,

$$\sum_{m=1}^{\infty} \text{Var}_{\mathbf{P}^{\pi}} \left(\mathbf{E}_{\omega}^0 \left[F \left(\mathcal{B}^{\lfloor b^m \rfloor} \right) \right] \right) < \infty,$$

for every Lipschitz function F on $C([0, T], \mathbb{R}^d)$ and $b \in (1, 2]$.

Brief Sketch of the Proof for the Quenched IP

- To check that the above sum of variances is finite, we work with two walkers which are independent given the environment, along with a martingale trick which uses the time as one more *extra dimension* and helps in getting the proper estimates for "low dimensions" (i.e. when $d \leq 2$).

Thank You