Invariance Principle for Random Walk in Dynamic Markovian Environment

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Outline

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  - **The Environment:** It is the transition laws which will tell us *how to take the next step* from the current position.

    Note: These laws can be random!
  - **The Walk:** Given the environment we have an *walker* who moves on the lattice $\mathbb{Z}^d$ starting from $\mathbf{0}$ according to the transition laws.

    Note: The walker provides second stage of randomness.

Classical RWRE (Static Environment)

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- **The Walk:** Given the transition laws the walker then moves according to a *time homogeneous* Markov chain, starting from $\mathbf{0}$.

A Classical Example of a Statics RWRE: Sinai Walk

Suppose there are two coins, one is unbiased $(1/2, 1/2)$ and one is biased $(3/4, 1/4)$.

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- The walker starts at $\mathbf{0}$ and moves using the biased coin giving positive bias towards the direction of the arrow at his current location.
- Note that the average increment at each step is 0.

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- With high probability the walker will spend a "lot of time" in such a "trap", this will then "slows down" the walk.
- In fact Sinai [1982] showed that in this case given a typical static environment with high probability the walker will be at $c\,(\log n)^2$ distance from the origin at time n.

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- Consider again the integer line $\mathbb{Z}$.
- The walker starts at $\mathbf{0}$ and carries both the coins.
- Before a move he first tosses the unbiased coin independently of the past, and then the biased coin again independently of the past. If the unbiased turns up a head, then he gives the bias to the right, else he gives the bias to the left of his current position.

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Question: Do the "traps" disappear in this case ?

- Conceivably yes!
- We will expect if the environment has a dynamics which is "fast mixing" then the "traps" wouldn't exists to "slow down the walk", and hence we can get a CLT.

Some Notations

- **The Environment:** At a site $\mathbf{x} \in \mathbb{Z}^d$ and at time $t \geq 0$ "environment" is a transition law, it will be denoted by $\omega_t(\mathbf{x}, \cdot)$.

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- **The Walk:** The position of the walker at time t will be denoted by X_t.

Quenched and Annealed Laws

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- **Annealed:** The marginal distribution of the walk, that is, integrating out the *quenched* law with respect to the environment distribution.

  Note: The walk may not be a Markov chain under the annealed law.

Quenched Law

Definition of the Quenched Law

Given the entire environment

$$\omega := \left\{ \left(\omega_t\left(\mathbf{x}, \cdot\right)\right)_{t \geq 0} \,\middle|\, \mathbf{x} \in \mathbb{Z}^d \right\},$$

the *quenched law* of $\left(X_t\right)_{t \geq 0}$ starting from $\mathbf{x}$ denoted by $\mathbf{P}_\omega^{\mathbf{x}}$, is the distribution of the time inhomogeneous Markov chain $\left(X_t\right)_{t \geq 0}$ on $\mathbb{Z}^d$, such that for every $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$

$$\mathbf{P}_\omega^{\mathbf{x}} \left(X_{t+1} = \mathbf{y} \,\middle|\, X_t = \mathbf{x} \right) = \omega_t\left(\mathbf{x}, \mathbf{y}\right),$$

and

$$\mathbf{P}_\omega^{\mathbf{x}} \left(X_0 = \mathbf{x} \right) = 1.$$

Definition of the Annealed Law

The *annealed law* of $(X_t)_{t \geq 0}$ starting from $\mathbf{x}$ denoted by $\mathbb{P}^{\mathbf{x}}$, is defined by

$$\mathbb{P}^{\mathbf{x}}(\cdot) := \int \mathbf{P}^{\mathbf{x}}_{\omega}(\cdot)\; \mathbf{P}(d\omega),$$

where $\omega \sim \mathbf{P}$.

Dynamic Markovian Environment

- We will assume that for every $\mathbf{x} \in \mathbb{Z}^d$ the environment chain at $\mathbf{x}$, given by
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- By $\mathbf{P}^{\pi}$ we will denote the distribution of the entire environment ω.

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- Our example of RWDRE, falls under this model where the environment chains are just *i.i.d.* chains.

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$$K(w, \cdot) \geq \kappa \, \pi(\cdot), \quad \forall \; w \in \mathcal{S}.$$

(A2) There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon \, q(\mathbf{x}, \mathbf{y}) \quad \text{a.s.} \quad [\mathbf{P}^{\pi}],$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

Discussion on the Assumptions

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- This is for simplicity, the arguments can be easily generalized to transitions with bounded increment.

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- If the state space for the environment chains $\mathcal{S}$ is finite and K is irreducible and a periodic (assumption made by BMP [2000]), then assumption (A1) may fail, but it does hold if K is replaced by K^r for some fixed integer $r \geq 1$. A slight modification of our arguments applies to that case, too.

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- If the environment chains are i.i.d. chains then (A1) holds trivially with $\kappa = 1$.

Discussion on the Assumptions

Assumption (A2)

There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon \, q(\mathbf{x}, \mathbf{y}) \quad \text{a.s.} \quad [\mathbf{P}^{\pi}] \, .$$

- This condition essentially means that the random environment has a "deterministic" part q, which is non-degenerate. Presumably it is a "small" (random) perturbation of q.

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- This condition essentially means that the random environment has a "deterministic" part q, which is non-degenerate. Presumably it is a "small" (random) perturbation of q.
- Comparing with classical (static) RWRE literature, this condition can be termed as an *ellipticity* condition on the environment.
- This condition was also assumed in BMP [1997, 2000] and Stannat [2004].

Annealed SLLN and Invariance Principle

Theorem 1 (Annealed SLLN)

Suppose assumptions (A0), (A1) and (A2) hold. Then there exists a constant vector $\mathbf{v} \in \mathbb{R}^d$, such that

$$\frac{X_n}{n} \longrightarrow \mathbf{v} \;\text{ a.s. }\; \left[\mathbb{P}^{\mathbf{0}}\right], \;\text{ as }\; n \to \infty\,.$$

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Theorem 2 (Annealed Invariance Principle)

Suppose assumptions (A0), (A1) and (A2) hold. Then there exists a $(d \times d)$ (non-random) positive definite matrix Σ, s.t. under $\mathbb{P}^{\mathbf{0}}$,

$$\left(\frac{X_{\lfloor nt \rfloor} - nt\,\mathbf{v}}{\sqrt{n}} \right)_{t \geq 0} \xrightarrow{d} BM_d\left(\Sigma\right), \ \text{ as } \ n \to \infty .$$

Quenched Invariance Principle

Theorem 3 (Quenched Invariance Principle)

Suppose assumptions (A0), (A1) and (A2) hold. Then for a.s. all ω with respect to $\mathbf{P}^{\pi}$, under the *quenched law* $\mathbf{P}_{\omega}^{\mathbf{0}}$ we have

$$\left(\frac{X_{\lfloor nt \rfloor} - nt\,\mathbf{v}}{\sqrt{n}} \right)_{t \geq 0} \xrightarrow{d} BM_d\,(\Sigma)\,, \quad \text{as} \quad n \to \infty\,.$$

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  - They used non-trivial analytic methods, including a specific type of *cluster expansion* technique.
- Stannat [2004]
  - Gave a simpler but still analytic proof for any dimension $d \geq 1$.
- Rassoul-Agha and Seppäläinen [2005]
  - Proved invariance principle using probabilistic techniques as a special case of a more general result.

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For dynamic Markovian environment model, exactly similar to ours, but with some more (technical) assumptions the *quenched CLT* was proved by

- Boldrighini, Minlos and Pellegrinotti [2000]
  - For dimension $d \geq 3$.
  - Proofs are based on “hard” analytic techniques.

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- For the annealed results we had to assume

$$\kappa + \varepsilon^2 > 1\,.$$

- For the quenched IP we needed more assumptions, namely

$$\kappa + \varepsilon^6 > 1 \text{ and } d > 7\,.$$

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- Here we prove the results without any non-intuitive/technical assumptions.
- For dimensions $d = 1$ and $d = 2$ the results are new. In fact earlier BMP has raised doubt on existence of quenched IP in $d = 1$, based on some simulations. So this work completely clears the matter for this model.

Our Main Strategy

- We will show that on an appropriate probability space there is a version of this process and an increasing sequence of *random times* $(\tau_n)_{n\geq 0}$ with $\tau_0 = 0$ such that the pairs $\left(\tau_n - \tau_{n-1}, X_{\tau_n} - X_{\tau_{n-1}}\right)_{n\geq 1}$ form an *i.i.d.* sequence under $\mathbb{P}^{\mathbf{0}}$

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- These times will be called *"regeneration times"*.
- Moreover we will show that under our assumptions τ_1 has exponential tail.
- Because of nearest neighbor walk this will imply annealed SLLN and IP.
- For quenched IP we need to do some more work!

Construction of ε-Coins

Recall the Assumption (A2): There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon\, q(\mathbf{x}, \mathbf{y}) \quad \text{a.s.} \quad [\mathbf{P}^\pi],$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

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- If $\varepsilon_{t+1} = 1$ then it takes a move according to the fixed transition kernel q.
- If $\varepsilon_{t+1} = 0$ then it takes a move according to the random transition kernel
$$\frac{\omega_t(\cdot,\cdot) - \varepsilon q(\cdot,\cdot)}{1-\varepsilon}.$$

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- By taking a step when the ε-coin is 1 the walker do not collect any information about the environment.
- We will say a step taken by the walker is a "proper step" if and only if, it was taken when the ε-coin was 0.
- Note that only by taking a "proper" step the walker learns about the environment.

A "Regeneration Time"

We would like to define a random time, say τ, which will be the first time t such that,

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  Note: Here by *"regeneration"* of an environment chain, we mean that it starts afresh from its stationary distribution.

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- Note that such a time τ depends on the future of the environment chains at every location. So naturally it is NOT a *stopping time*. But it DOES NOT take into consideration any specifics of the future path of the walker.

Construction of κ-Coins
(an easy way to get environment *"regeneration"*)

Recall the Assumption (A1): There exists $0 < \kappa \leq 1$ such that

$$K(w, \cdot) \geq \kappa \, \pi(\cdot), \quad \forall \; w \in \mathcal{S}.$$

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  - if $\alpha_{t+1}(\mathbf{x}) = 1$ then it moves to a state selected independently from the stationary distribution π, in other words, **goes through a regeneration**;
  - if $\alpha_{t+1}(\mathbf{x}) = 0$ then it moves to a state according to the kernel
    $$\frac{K(\omega_t(\mathbf{x}, \cdot), \cdot) - \kappa\pi(\cdot)}{1-\kappa}.$$
    No regeneration in this case!

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- Fix $t \geq 0$ and $\mathbf{x} \in \mathbb{Z}^d$.
- Let $L_t := \sup \left\{ l > 0 \,\middle|\, \varepsilon_{t-l} = 1 \right\}$ if $\varepsilon_t = 0$, otherwise put $L_t = 0$.

  Thus L_t is the *length of the "improper" steps before a "proper" step at time t.*

Precise Definition of the "Regeneration Time"

- Consider the following event:

$$\bigcup_{l=1}^{\infty}\left([L_t = l] \bigcap \bigcap_{s=0}^{\infty} \bigcap_{\substack{\mathbf{z}\in\mathbb{Z} \\ |\mathbf{x}-\mathbf{z}|\leq s}} \bigcup_{u=t-l+1}^{t+s} [\alpha_u(\mathbf{x}) = 1]\right).$$

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  We denote it by $S(t,\mathbf{x})$.

- Now define

$$\tau := \inf\left\{t \geq 1 \,\middle|\, \text{the event } S(t, X_t) \text{ has occurred}\right\}.$$

Properties of τ_1

Proposition 4

Let the assumptions (A0), (A1) and (A2) hold. Then for all $t \geq 0$

$$\mathbb{P}^{\mathbf{0}} \left(\tau > t \right) \leq A e^{-bt} \, ,$$

for some constants $A < \infty$ and $b > 0$.

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Remark

In particular this proposition proves that

$$\tau < \infty \text{ a.s.}.$$

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- Let $\left(\omega^{(i)}, \alpha^{(i)}\right)_{i \geq 1}$ be i.i.d. copies of the environment driven by κ coins, as defined earlier.

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- We the define

$$\tau_1 := \tau\left(\omega^{(1)}, \alpha^{(1)}; \mathbf{X}^{(1)}, \varepsilon^{(1)}\right)$$

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- Having defined $\left\{\left(\mathbf{X}^{(i)}, \tau_i\right)\right\}$ for $i = 1, 2, \ldots, (n-1)$, let $\mathbf{X}^{(n)}$ be the position of the walker in the n^{th} environment $\left(\omega^{(n)}, \alpha^{(n)}\right)$ starting from $X^{(n-1)}_{\tau_{n-1}}$.

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- Define τ_n recursively as

$$\tau_n := \tau_{n-1} + \tau \left(\omega^{(n)}, \alpha^{(n)}; \mathbf{X}^{(n)}, \varepsilon^{(n)} \right)$$

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- Finally define a *new* walk $(Y_t)_{t \geq 0}$ by

$$Y_t := X^{(n)}_{t-\tau_{n-1}} \text{ if } \tau_{n-1} \leq t < \tau_n\,.$$

Properties of this New Walk

Proposition 5

Let $(\tau_n)_{n\geq 1}$ and $(Y_t)_{t\geq 0}$ be as defined before, then $\left(\tau_n - \tau_{n-1}, Y_{\tau_n} - Y_{\tau_{n-1}}\right)_{n\geq 1}$ is an i.i.d. sequence, where $\tau_0 = 0$.

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Theorem 6

The (annealed) law of $(Y_t)_{t\geq 0}$ is same as that of $(X_t)_{t\geq 0}$.

Brief Sketch of the Proof for the Quenched IP

- We use a technique introduced by Bolthausen and Sznitman [2002].
- Let $B_t^n := \left(X_{\lfloor nt \rfloor} - nt\,\mathbf{v}\right)/\sqrt{n}$, and $\mathcal{B}_t^n$ be the polygonal interpolation of $(k/n) \mapsto B_{k/n}^n$.
- Bolthausen and Sznitman technique says that if we have the *annealed* IP then the *quenched* IP will follow if we can show that for all $T > 0$,

$$\sum_{m=1}^{\infty} \mathrm{Var}_{\mathbf{P}^\pi}\left(\mathbf{E}_\omega^{\mathbf{0}}\left[F\left(\mathcal{B}^{\lfloor b^m \rfloor}\right)\right]\right) < \infty,$$

for every Lipschitz function F on $C\left([0,T],\mathbb{R}^d\right)$ and $b \in (1,2]$.

Brief Sketch of the Proof for the Quenched IP

- To check that the above sum of variances is finite, we work with two walkers which are independent given the environment, along with a martingale trick which uses the time as one more *extra dimension* and helps in getting the proper estimates for "low dimensions" (i.e. when $d \leq 2$).

Thank You