Recursive Distributional Equations : Application to Hard-Core Model on Random Graphs

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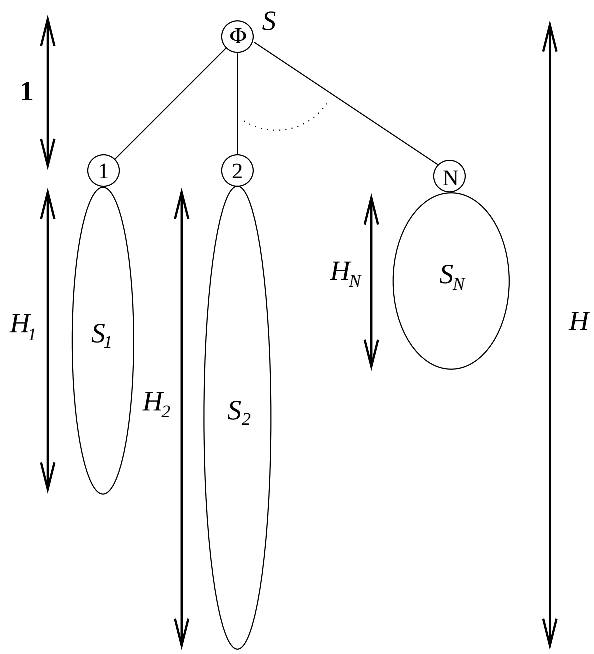
Two Examples

Examples 1 : Consider a *(sub)-critical* Galton-Watson branching process with the progeny distribution N, so $\mathbf{E}[N] \leq 1$; we assume $\mathbf{P}(N = 1) < 1$.



Height of the Tree : Let $H := 1+$ height of the G-W tree, then $H < \infty$ a.s. and

$$H \stackrel{d}{=} 1 + \max(H_1, H_2, \ldots, H_N) \quad \text{on } \mathbb{N},$$

where $(H_j)_{j \geq 1}$ are i.i.d. with same law as of H and are independent of N.

Example 2 (Perhaps the best known !) : Consider the following fixed point equation

$$Z \stackrel{d}{=} \frac{Z_1 + Z_2}{\sqrt{2}} \quad \text{on} \quad \mathbb{R},$$

where (Z_1, Z_2) are i.i.d. copies of Z.

- The set of all solutions is given by the Normal $\left(0, \sigma^2\right)$, $\sigma^2 \geq 0$ family.
- This example also extends to give characterizations of stable laws.

We will call such an equation a *recursive distributional equation* (RDE).

Recursive Distributional Equation (RDE)

Definition 1 *The following* fixed-point equation *on* $\mathcal{P}$ *is called a* Recursive Distributional Equation (RDE)

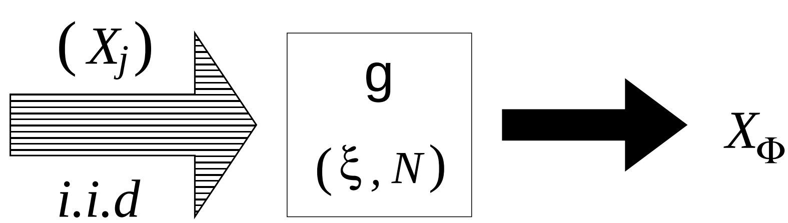
$$X \stackrel{d}{=} g\left(\xi; X_j, 1 \leq j \leq^* N\right), \quad \textit{on} \;\; S$$

where $(X_j)_{j \geq 1}$ *are independent copies of* X *and are independent of* (ξ, N).

Remark : A more conventional (analysis) way of writing the equation would be

$$\mu = T(\mu)$$

where T is the operator associated with the above equation, which depends on the function g and the joint distribution of the pair (ξ, N), and μ is the (unknown) law of X.



Hard-Core Model

Setup :

- Let $G := (V, E)$ be a graph.
- We say a subset $I \subseteq V$ is an *independent set* of G, if for any two vertices $u, v \in I$ there is no edge between u and v.
- Let $\mathcal{I}_G$ be the set of all independent sets of G.
- We would like to define a measure on $\mathcal{I}_G$.

For Finite Graphs :

- Fix $\lambda > 0$.
- *Hard-core model on G with activity λ* is a probability distribution on $\mathcal{I}_G$ such that

$$\mathbb{P}_\lambda(I) \propto \lambda^{|I|}, \quad I \in \mathcal{I}_G.$$

- Thus

$$\mathbb{P}_\lambda(I) = \frac{\lambda^{|I|}}{Z_\lambda(G)}, \quad I \in \mathcal{I}_G$$

where $Z_\lambda(G) := \sum_{I \in \mathcal{I}_G} \lambda^{|I|}$ is the proportionality constant, known as the *partition function*.

For Infinite Graph :

- Use a Statistical Physics definition of *infinite volume Gibbs measure* on $\mathcal{I}_G$ (similar to that of Ising model and q-Potts model).
- It always exists but may not be unique ! If uniqueness fails then we will say that a *phase transition* occurs.

Random Graph Models

- **GW-Tree :** Galton-Watson branching process tree with a given progeny distribution denoted by N.
  - ▶ The parameter here is the distribution of N.

- **Erdös and Rényi Random Graph :** A random graph on $n \geq 1$ vertices labeled by $[n] := \{1, 2, \ldots, n\}$ where each pair of vertices are connected by an edge independently with probability $\frac{c}{n}$, where $c > 0$. This would be denoted by $\mathcal{G}\left(n, \frac{c}{n}\right)$.
  - ▶ The parameter here is $c > 0$.

- Can also work with **"Random r-regular Graph"** model (this we will not discuss) !

The Recursive Distributional Equation (RDE)

$$\eta \stackrel{d}{=} \frac{\lambda \prod_{j=1}^{N} \Big(1 - \eta_j\Big)}{1 + \lambda \prod_{j=1}^{N} \Big(1 - \eta_j\Big)} \quad \text{on} \quad [0, 1],$$

where $\Big(\eta_j\Big)$ are i.i.d. copies of η and are independent of N.

Characterization of Phase Transition for GW-Tree Model

Theorem 1 *For GW-tree with progeny distribution* N*, there is* no phase transition *for the hard-core model with activity* $\lambda > 0$*, if and only if, for the associated RDE, the operator* T^2 *has unique fixed-point.*

Main Results for Hard-Core Model on Sparse Random Graphs

Theorem 2 *Suppose* $X_\lambda^\omega(n,c)$ *be the size of a random independent set distributed according to the hard-core model with activity* $\lambda > 0$ *on a Erdös-Rényi random graph* $\mathcal{G}\left(n, \frac{c}{n}\right)$*. If the GW-tree with Poisson(*c*) progeny distribution has no phase transition then*

$$\lim_{n \to \infty} \frac{\mathbf{E}_\lambda\left[X_\lambda^\omega(n,c)\right]}{n} = \gamma_\lambda(c)$$

where $\gamma_\lambda(c) := \mathbf{E}[\eta]$ *and* η *is the unique solution of the RDE.*

Theorem 3 *A similar statement for the random* r*-regular graph model.*