Recursive Distributional Equations and More Applications of the *Objective Method*

Antar Bandyopadhyay
(Some parts are joint work with David Aldous and David Gamarnik)

Second Lectures on Probability and Stochastic Processes
Indian Statistical Institute, Bangalore Centre

Theoretical Statistics and Mathematics Unit
Indian Statistical Institute, New Delhi Centre

http://www.isid.ac.in/∼antar

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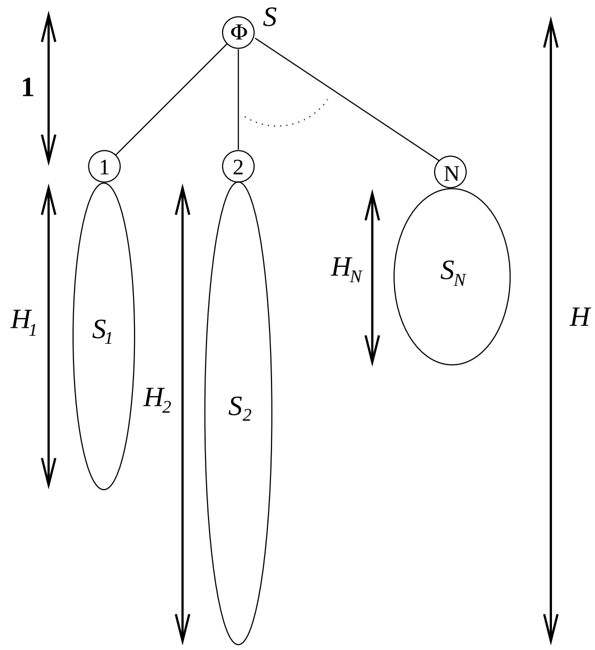
Recursive Distributional Equations

Three More Examples

Examples 1: Consider a *(sub)-critical* Galton-Watson branching process with the progeny distribution N, so $\mathbf{E}[N] \leq 1$; we assume $\mathbf{P}(N = 1) < 1$.

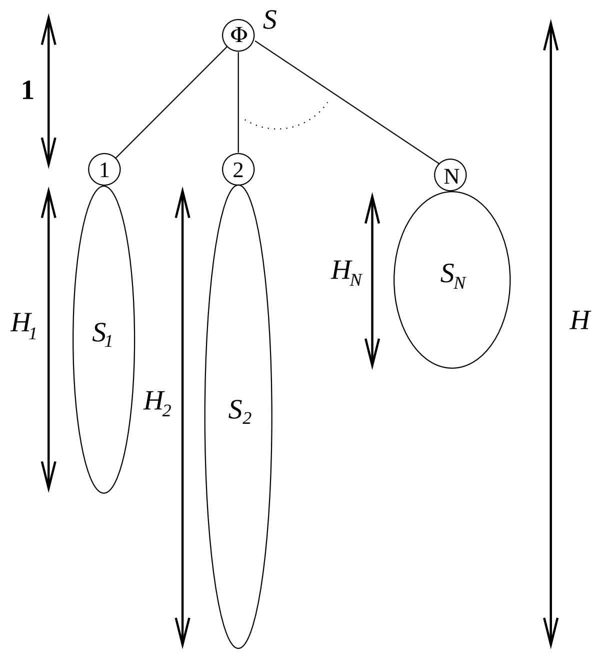


Height of the Tree: Let $H := 1+$ height of the G-W tree, then $H < \infty$ a.s. and

$$H \stackrel{d}{=} 1 + \max(H_1, H_2, \ldots, H_N) \quad \text{on} \;\; \mathbb{N},$$

where $(H_j)_{j \geq 1}$ are i.i.d. with same law as of H and are independent of N.

Examples 2: Consider the same *(sub)-critical* Galton-Watson branching process.



Size of the Tree: Let $S :=$ total size of the tree. Once again $S < \infty$ a.s. since the process is (sub)-critical. Further

$$S \stackrel{d}{=} 1 + (S_1 + S_2 + \cdots S_N) \quad \text{on} \quad \mathbb{N},$$

where $(S_j)_{j \geq 1}$ are i.i.d. with same law as of S and are independent of N.

We will call such equations *Recursive Distributional Equations* (RDE).

Example 3 (Quicksort Algorithm/Distribution):

- Select the first number from a pile of n numbers and divide the other $(n-1)$ numbers into two piles, according to *less* than or *bigger* than the first number.

- Recursively sort the two piles (which are now smaller in size).

- $X(n) :=$ # comparisons needed to sort n numbers starting from a uniform random permutation of $[n]$. Then

$$X(n) \stackrel{d}{=} X_1(U_n) + X_2(n-1-U_n) + (n-1),$$

  where $X_1(\cdot)$ and $X_2(\cdot)$ are i.i.d. with same law as of $X(\cdot)$ and are independent of U_n which is uniform on $\{0, 1, 2, \ldots, n-1\}$.

- Rösler (1990) showed $\mathbf{E}[X(n)] \sim 2n \log n$ and moreover

$$\frac{X(n) - 2n \log n}{n} \xrightarrow{d} Y,$$

  where distribution of Y satisfies the RDE

$$Y \stackrel{d}{=} UY_1 + (1-U)Y_2 + C(U) \quad \text{on } \mathbb{R},$$

  where Y_1 and Y_2 are i.i.d. with same law as of Y and are independent of $U \sim$ Uniform$[0,1]$, and $c(u) := 1 + 2u \log u + 2(1-u) \log(1-u)$.

Typical features of RDEs

Ex. 1: $X \stackrel{d}{=} 1 + \max(X_1, X_2, \ldots, X_N)$ on $\mathbb{N}$

Ex. 2: $X \stackrel{d}{=} 1 + (X_1 + X_2 + \cdots + X_N)$ on $\mathbb{N}$

Ex. 3: $X \stackrel{d}{=} UX_1 + (1 - U)X_2 + C(U)$ on $\mathbb{R}$

- **Unknown Quantity:** Distribution of X.
- **Known Quantities:**
  - $N \leq \infty$ which may or may not be random (e.g. $N \equiv 2$ in Ex. 3).
  - Possibly some more randomness whose distribution is known (e.g. U in the Ex. 3).
  - How we combine the known and unknown randomness (e.g. "$1 + \max$" operation in Ex. 1).
- **What is the RDE doing ?** To find a distribution μ such that when we take i.i.d. samples $(X_j)_{j \geq 1}$ from it and only use N many of them (where N is independent of the samples) and do the manipulation then we end up with another sample $X \sim \mu$.

Remark: In the case $N = 1$ a.s. it reduces to the question of finding a stationary distribution of a discrete time Markov chain.

Two main uses of RDEs

- **Direct use:** The RDE is used directly to define a distribution. Examples include,
  - The height (and also the size) of a (sub)-critical Galton-Watson tree (the first two examples).
  - The Quicksort distribution (Example 3).
  - Discounted tree sums / inhomogeneous percolation on trees. [Aldous and B. 2005]
  - ... *and many others*.

- **Indirect use:** The RDE is used to define some auxiliary variables which help in defining/characterizing some other quantity of interest. Among others the following two type of applications are of special interest
  - 540° *argument* ! (we have seen one).
  - Determining critical points and scaling laws (will not give an example).

General Setup

- Let $(S, \mathfrak{S})$ be a measurable space, and $\mathcal{P}$ be the collection of all probabilities on $(S, \mathfrak{S})$.

- Let (ξ, N) be a pair of random variables such that N takes values in $\{0, 1, 2, \ldots; \infty\}$.

- Let $(X_j)_{j \geq 1}$ be **i.i.d** S-valued random variables, which are independent of (ξ, N).

- $g(\cdot)$ is a S-valued measurable function with appropriate domain.

Recursive Distributional Equation (RDE)

Definition 1 *The following* fixed-point equation *on* $\mathcal{P}$ *is called a* Recursive Distributional Equation (RDE)

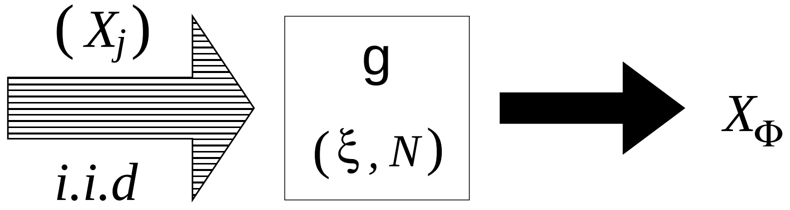
$$X \stackrel{d}{=} g\left(\xi; \left(X_j, 1 \leq j \leq^* N\right)\right) \quad \textit{on} \;\; S,$$

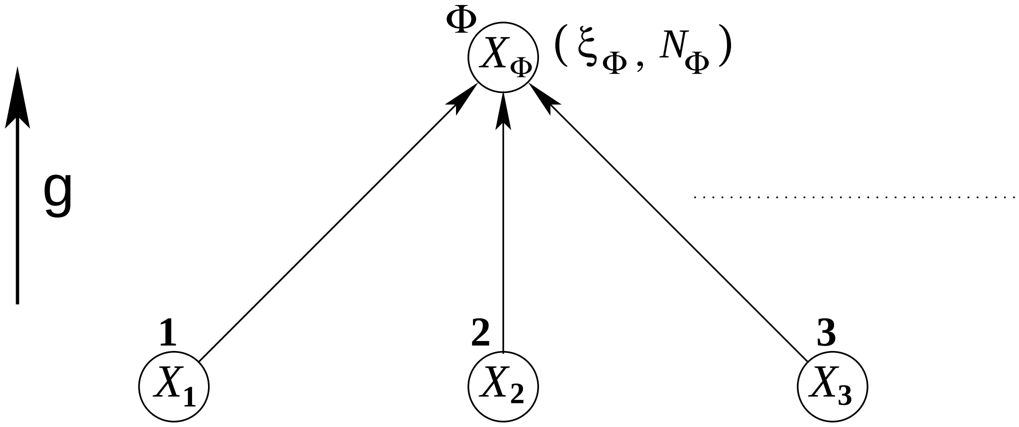
where $(X_j)_{j \geq 1}$ *are independent copies of* X *and are independent of* (ξ, N).

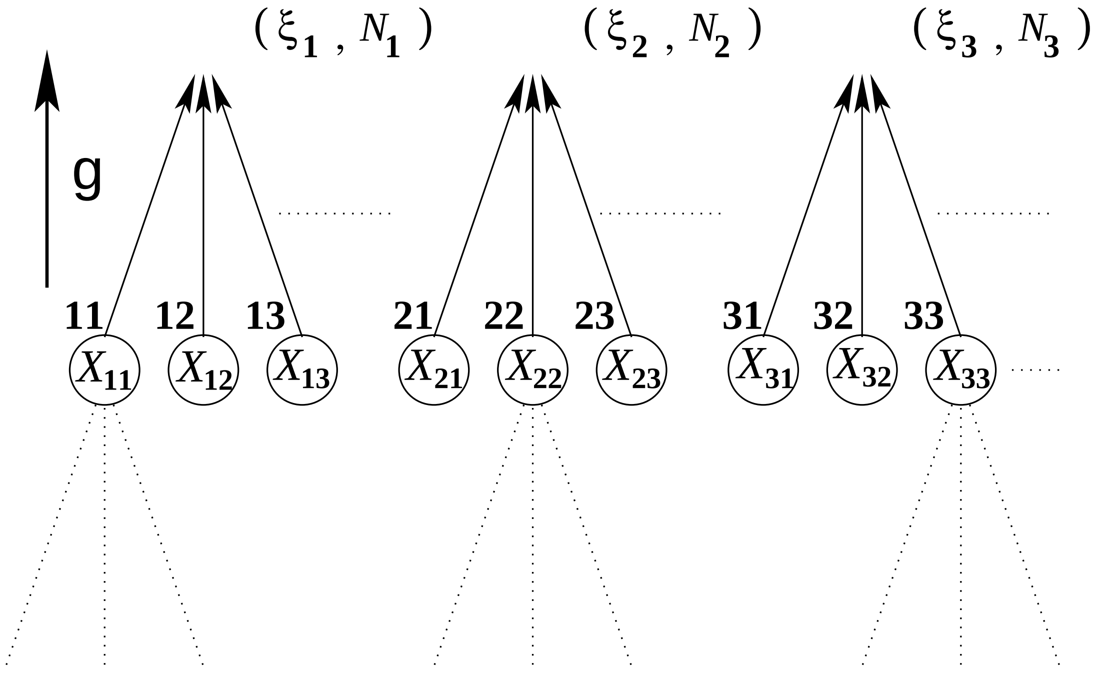
Remark: A more conventional (analysis) way of writing the equation would be

$$\mu = T(\mu)$$

where T is the operator associated with the above equation, which depends on the function g and the joint distribution of the pair (ξ, N), and μ is the (unknown) law of X.




g
Φ
X_Φ
(ξ_Φ, N_Φ)
1
2
3
X_1
X_2
X_3



(ξ1 , N1)
(ξ2 , N2)
(ξ3 , N3)
g
11
12
13
21
22
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X11
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X31
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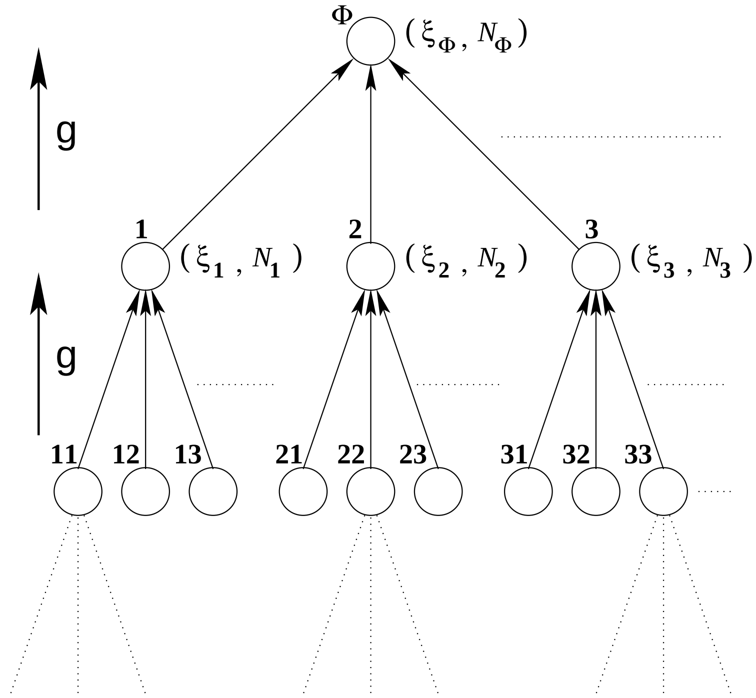
X33

Recursive Tree Framework (RTF)

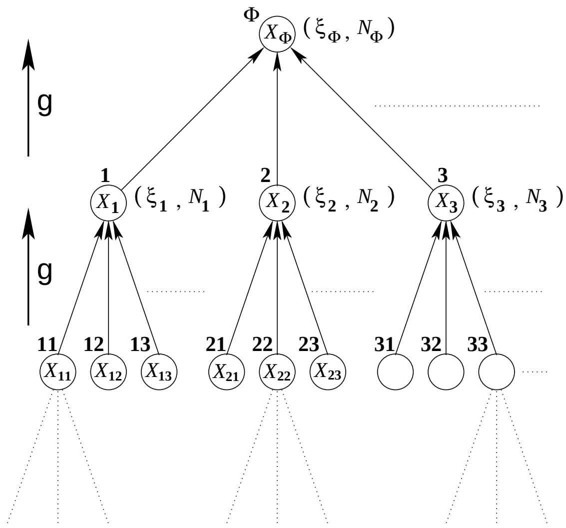


- **Skeleton:** $\mathcal{T} := (\mathcal{V}, \mathcal{E})$ is the canonical infinite tree with vertex set $\mathcal{V} := \left\{ \mathbf{i} \,\middle|\, \mathbf{i} \in \mathbb{N}^d,\, d \geq 1 \right\} \cup \{\emptyset\}$, and edge set $\mathcal{E} := \left\{ e = (\mathbf{i}, \mathbf{i}j) \,\middle|\, \mathbf{i} \in \mathcal{V}, j \in \mathbb{N} \right\}$, and root $\emptyset$.

- **Innovations:** Collection of **i.i.d** pairs $\left\{ (\xi_{\mathbf{i}}, N_{\mathbf{i}}) \,\middle|\, \mathbf{i} \in \mathcal{V} \right\}$.

- **Function:** The function $g(\cdot)$.

Recursive Tree Process (RTP)



Consider a **RTF** and let μ be a solution of the associated **RDE** . A collection of S-valued random variables $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ is called an invariant *Recursive Tree Process (RTP)* with marginal μ if

- $X_{\mathbf{i}} \sim \mu \ \forall\ \mathbf{i} \in \mathcal{V}$.
- Fix $d \geq 0$ then $(X_{\mathbf{i}})_{|\mathbf{i}|=d}$ are independent.
- $X_{\mathbf{i}} = g\left(\xi_{\mathbf{i}}; X_{\mathbf{i}j}, 1 \leq j \leq^* N_{\mathbf{i}}\right)$ a.s. $\forall\ \mathbf{i} \in \mathcal{V}$.
- $X_{\mathbf{i}}$ is independent of $\left\{(\xi_{\mathbf{i}'}, N_{\mathbf{i}'}) \,\middle|\, |\mathbf{i}'| < |\mathbf{i}|\right\}$ $\forall\ \mathbf{i} \in \mathcal{V}$.

Remark: Using *Kolmogorov's consistency*, an invariant RTP with marginal μ exists if and only if μ is a solution of the associated RDE.

A Fact for "essentially" finite RTF

Remark: Associated with a RTF there is a Galton-Watson branching process tree rooted at $\emptyset$ defined only through $\left\{ N_{\mathbf{i}} \,\middle|\, \mathbf{i} \in \mathcal{V} \right\}$, call it $\mathcal{T}$. Essentially any associated invariant RTP lives on $\mathcal{T}$.

Proposition 1 *If $\mathcal{T}$ is almost surely finite (equivalently $\mathbf{E}[N] \leq 1$ and $\mathbf{P}(N = 1) < 1$) then the associated RDE has unique solution full* domain of attaraction.

Remark: The RDEs in the first two examples have unique solutions.

Domain of the Function g

- The innovation ξ takes values in some measurable space $(\Theta, \mathfrak{F})$.
- Recall our sample space is S.
- The function g which takes values in S, is defined on the space

$$\Theta^* := \Theta \times \bigcup_{0 \le d \le \infty} S^d.$$

Here S^∞ is the usual infinite product space and $S^0 := \{\Delta\}$ where Δ is some "known object" !

Tree-Structured Coupling From the Past: Proof of the Proposition 1

- Let $\mathfrak{I}$ be the set of all finite rooted trees with vertex weights.

- We define a function $h : \mathfrak{I} \to S$ as follows
  - Let $\mathcal{T} \in \mathfrak{I}$ with weights $(w_{\mathbf{i}})$.
  - If a vertex $\mathbf{i}$ is a leaf then define
  $$x_{\mathbf{i}} := g\left(w_{\mathbf{i}}; \Delta\right).$$
  - For an internal vertex $\mathbf{i}$ with $n_{\mathbf{i}} \geq 1$ children recursively define
  $$x_{\mathbf{i}} := g\left(w_{\mathbf{i}}; \left(x_{\mathbf{i}j}, 1 \leq j \leq n_{\mathbf{i}}\right)\right)$$
  - Take $h(\mathcal{T}) = x_{\emptyset}$, where $\emptyset$ is the root of $\mathcal{T}$.

Continuing ...

- GW tree $\mathcal{T}$ with the node weights $(\xi_{\mathbf{i}})$ is an element of $\mathfrak{I}$, let $X_{\emptyset}$ be the h value of it.

- For a vertex $\mathbf{i}$ of $\mathcal{T}$ let $\mathcal{T}_{\mathbf{i}}$ be the family tree generated by $\mathbf{i}$. Then $\mathcal{T}_{\mathbf{i}}$ with the node weights is also an element of $\mathfrak{I}$, let $X_{\mathbf{i}}$ be it's h value.

- It follows from definition of h that $(X_{\mathbf{i}})$ is a RTP with some marginal. Thus the RDE has a solution.

- Finally if μ is a solution of the RDE, let $(Y_{\mathbf{i}})$ be invariant RTP with marginal μ. From definition for a leaf $\mathbf{i}$ we must have $Y_{\mathbf{i}} = X_{\mathbf{i}}$ a.s. Now since the tree is a.s. finite so by recursion we get

$$Y_{\emptyset} = X_{\emptyset} \text{ a.s.}$$

  This proves the uniqueness.

Hard-Core Model on Random Graphs

A Problem by David Aldous

- For $r \geq 2$ and $n \geq 3$, let $\mathbf{G}(n, r)$ be a random graph selected uniformly at random from the set of all r-regular graphs on n vertices.

- **Conjecture of Aldous [2003]:**

  Let I_n be a *maximum independent set* then

$$\frac{\mathbf{E}[|I_n|]}{n} \to \kappa \quad \text{as} \quad n \to \infty,$$

  where $\kappa > 0$ is a constant which may depend on r.

- In combinatorics for a finite graph G the size of a maximum independent set is known as the *independence number* of G.

An Approach Towards Resolving the Conjecture

- We will consider a probability model on the set of all independent sets of the random graph $\mathbf{G}$ such that
$$\mathbb{P}_\lambda(I) \propto \lambda^{|I|},$$
where I is an independent set of $G(n,r)$.

- It is easy to see that given $\mathbf{G}(n,r)$ the probability measures $\mathbb{P}_\lambda$ concentrate on the *maximum* independent sets as $\lambda \to \infty$.

- So perhaps studying this model $\mathbb{P}_\lambda$ on random graphs may help to resolve Aldous' conjecture.

- We will see what we can do ... !

Hard-Core Model on a Finite Graph

Setup:

- Let $G := (V, E)$ be a finite graph.
- We say a subset $I \subseteq V$ is an *independent set* of G, if for any two vertices $u, v \in I$ there is no edge between u and v.
- Let $\mathcal{I}_G$ be the set of all independent sets of G.
- We would like to define a measure on $\mathcal{I}_G$.

Description 1:

- Fix $\lambda > 0$.

- *Hard-core model on G with activity λ* is a probability distribution on $\mathcal{I}_G$ such that

$$\mathbb{P}_\lambda^G(I) \propto \lambda^{|I|}, \quad I \in \mathcal{I}_G.$$

- Thus

$$\mathbb{P}_\lambda^G(I) = \frac{\lambda^{|I|}}{Z_\lambda(G)}, \quad I \in \mathcal{I}_G$$

  where $Z_\lambda(G) := \sum_{I \in \mathcal{I}_G} \lambda^{|I|}$ is the proportionality constant, known as the *partition function*.

Observations:

- If $\lambda = 1$ then we get the uniform distribution on $\mathcal{I}_G$ and $Z_\lambda(G)$ is the size of $\mathcal{I}_G$.

- Also we have already noticed, $\lambda \to \infty$ the measures $\mathbb{P}_\lambda^G$ concentrate on maximal size independent sets.

Description 2:

- Fix $\lambda > 0$ and let $p := \frac{\lambda}{1+\lambda} \in (0, 1)$.
- Suppose $(C_v)_{v \in V}$ are i.i.d. Bernoulli$\,(p)$.
- Let $I := \left\{ v \in V \,\middle|\, C_v = 1 \right\}$.
- The measure $\mathbf{P}\left(\cdot \,\middle|\, I \in \mathcal{I}_G \right)$ on $\mathcal{I}_G$ is same as $\mathbb{P}_\lambda^G$.

Remark: This gives a way to get exact samples from $\mathbb{P}_\lambda^G$.

Hard-Core Model on an Infinite Graph

Problems with the Two Previous Descriptions:

- For Description 1, we note that $\mathcal{I}_G$ is infinite and hence the partition function $Z_\lambda(G) = \infty$!
- For Description 2, we end up with the (same type of) problem that the event $[I \in \mathcal{I}_G]$ has *zero* probability under the i.i.d. coin tossing measure.

An Observation on Finite Graph:

Fix any vertex $v \in V$ and let σ be an independent set for the graph with vertex set $V \setminus \{v\}$ then

$$\mathbb{P}_\lambda^G\left(v \in I \,\middle|\, I \setminus \{v\} = \sigma\right) = \begin{cases} \frac{\lambda}{1+\lambda} & \text{if } \sigma \cup \{v\} \in \mathcal{I}_G \\ 0 & \text{otherwise} \end{cases}$$

where $I \in \mathcal{I}_G$.

Statistical Physics Definition:

Definition 2 *Given a finite or countably infinite, but locally finite graph $G = (V, E)$ and $\lambda > 0$, a probability measure $\mathbb{P}_\lambda^G$ on $\{0,1\}^V$, is said to be a* Gibbs measure for the hard-core model on G with activity λ, *if it admits conditional probabilities such that for all $v \in V$ and for any $\sigma \in \{0,1\}^{V \setminus \{v\}}$,*

$$\mathbb{P}_\lambda^G \left(I(v) = 1 \,\middle|\, I(V \setminus \{v\}) = \sigma \right) = \begin{cases} \frac{\lambda}{1+\lambda} & \text{if } \sigma \vee \mathbf{1}_v \in \mathcal{I}_G \\ 0 & \text{otherwise} \end{cases}$$

where I is a $\{0,1\}^V$-valued random variable with distribution $\mathbb{P}_\lambda^G$.

Remarks:

- This is what is known as Dobrushin-Lanford-Ruelle (DLR) definition of infinite-volume Gibbs measure.
- Similar definitions are used for defining Ising model and q-Potts model on infinite graphs.

Existence and Uniqueness

- In general a Gibbs measure exists by compactness argument.
- If G is finite then uniqueness holds trivially.
- It is not necessary that the uniqueness will hold when G is infinite.

Definition 3 *For a fixed graph G we say that a* phase transition *occurs for hard-core model with activity $\lambda > 0$, if there are more than one Gibbs measures of the form $\mathbb{P}_\lambda^G$.*

Note: There is no phase transition if G is finite.

What are Known ?

- First introduced by Dobrushin (1968) on $\mathbb{Z}^d$ for model of lattice gas.

- Phase transition is well studied for $\mathbb{Z}^d$.
  - ▶ No phase transition for $d = 1$.
  - ▶ For $d \geq 2$ no phase transition for small λ, but phase transition occurs for large λ.

  Not Known: Is phase transition *monotone* ? In other words is there a critical value in λ ?

- Arguably the most well studied case is the model on regular trees, $\mathbb{T}_r$ for $r \geq 2$. [Kelly, 1985]
  - ▶ For a r-regular tree $\mathbb{T}_r$, there exists a critical value $\lambda_c(r)$ such that, no phase transition when $\lambda \leq \lambda_c(r)$ and phase transition occurs when $\lambda > \lambda_c(r)$.
  - ▶ $\lambda_c(r) = \frac{(r-1)^{r-1}}{(r-2)^r}$.

- It is also known that there are infinite trees for which phase transition is not *monotone* ! [Brightwell, Häggström, Winkler, 1998]

Hard-Core Model on Random Graphs

Setup:

- $\mathcal{G}$ be a set of graphs which are finite or countably infinite and are locally finite.
- Suppose $\mathbf{P}$ is a probability on $\mathcal{G}$.
- Let $\mathbf{G} \sim \mathbf{P}$. We will write $\mathbf{G}(\omega)$ for a realization of the random graph $\mathbf{G}$.
- Given $\mathbf{G}(\omega)$ a hard-core model with activity $\lambda > 0$ on $\mathbf{G}(\omega)$ will be denoted by $\mathbb{P}_\lambda^\omega$.
- We will denote the joint measure as $\mathbf{P}_\lambda$.

Remark: Note that there are two stages of randomness and there are two parameters:

- One is the probability distribution $\mathbf{P}$ on $\mathcal{G}$ governing the randomness of the underlying graphical structure.
- The other is λ which is governing the hard-core model given the graph.

Phase Transition

Definition 4 *Given a random graph model* $(\mathcal{G}, \mathbf{P})$*, we say that there is a* phase transition *for the hard-core model with activity* $\lambda > 0$ *on a random graph* $\mathbf{G} \sim \mathbf{P}$ *if*

$$\mathbf{P}\left(\exists \textit{ multiple measures of the form } \mathbb{P}_\lambda^{\mathbf{G}}\right) > 0.$$

Remarks:

- If the random graph model is such that $\mathbf{G}$ is finite a.s. then there will be no phase transition for any activity $\lambda > 0$.

- It is possible to construct an example of $(\mathcal{G}, \mathbf{P})$ such that phase transition occurs for every $\lambda > 0$.

An Example

- Let $\mathcal{G} := \left\{ \mathbb{T}_r \,\middle|\, r \geq 2 \right\}$ and $\mathbf{P}$ be given by $\mathbf{P}(\mathbb{T}_r) = \frac{1}{2^{r-1}}$.

- Recall that from Kelly's work (1985) it is known that for hard-core model on r-regular tree $\mathbb{T}_r$, phase transition occurs if an only if

$$\lambda > \lambda_c(r) = \frac{(r-1)^{r-1}}{(r-2)^r}.$$

- But $\lambda_c(r) \to 0$ as $r \to \infty$.

- So for every $\lambda > 0$ for large enough r we must have $\lambda_c(r) < \lambda$ and thus a phase transition would occur for the random graph model $(\mathcal{G}, \mathbf{P})$.

Remarks:

- It is important to note that for the model $(\mathcal{G}, \mathbf{P})$ we can have realizations having arbitrarily large degree with positive probability.

- It is known that for bounded degree (fixed) graphs there should be no phase transition for *small* values of λ. [van den Berg and Steif, 1994]

Random Graph Models

- **GW-Tree:** Galton-Watson branching process tree with a given progeny distribution denoted by N.
  - ▶ The parameter here is the distribution of N.

- **Sparse Random Graphs:**
  - ▶ **Erdös and Rényi Random Graph:** A random graph on $n \geq 1$ vertices labeled by $[n] := \{1, 2, \ldots, n\}$ where each pair of vertices are connected by an edge independently with probability $\frac{c}{n}$, where $c > 0$. This would be denoted by $\mathcal{G}\left(n, \frac{c}{n}\right)$.
    - $*$ The parameter here is $c > 0$.
  - ▶ **Random r-regular Graph:** This is to select one graph at random from the set of all r-regular graphs with vertex set $[n]$. We will denote this model by $\mathcal{G}_r(n)$.

    Note: In order for this model to make sense we will always assume that nr is even.
    - $*$ The parameter here is $r \geq 2$.

Motivations

- Aldous' conjecture for the scaling of the independent number of a sparse random graph.

- Interesting from Statistical Physics point of view, well studied for non-random graphs. [Dobrushin 1970, Kelley 1985, van den Berg & Steif 1994, Brightwell, Häggström & Winkler 1998, Brightwell & Winkler 1999]

- Has applications in engineering fields, like in *multicast networking* problems. [Ramanan et al, 2002]

Sparse Random Graphs and GW-Trees

- **Known:** If $\mathcal{G}_n$ be a model for sparse random graph then for "large" enough n from the "view point" of a fixed vertex "locally it looks like" a (possibly random) rooted tree.
  - For $\mathcal{G}\left(n, \frac{c}{n}\right)$ it is a rooted Galton-Watson tree with Poisson (c) offspring distribution.
  - For $\mathcal{G}(n, r)$ it is a rooted r-regular tree.
- **Conclusion:** So for computing "large" n limit of hard-core model on these kind of graphs we may need to consider the similar model on respective GW-trees.
- **Note:** For a r-regular tree, one slight annoyance is that it is not really a GW-tree ! But by removing one vertex (the root) it can be viewed as a collection of r GW-trees with progeny distribution $N \equiv r - 1$.

Hard-Core Model on GW-Trees

Proposition 2 *Fix $\lambda > 0$ then the followings hold for a GW-tree with progeny distribution N.*

(a) If $\mathbf{E}[N] \le 1$ then there is no phase transition.

(b) If $\mathbf{E}[N] > 1$ then on the event of `non-extinction` *phase transition occurs with probability* 0 *or* 1*.*

Proof of Proposition 2:

- Nothing to prove for part (a).

- For part (b) notice that the property that a (fixed) rooted tree $\mathcal{T}$ has no phase transition implies that if v is a child of the root, and $\mathcal{T}(v)$ is the sub-tree rooted at v consisting only of the descendants of v, then $\mathcal{T}(v)$ also has no phase transition.

- Let $\beta := \mathbf{P}_\lambda($ no phase transition in $\mathcal{T})$ where $\mathcal{T}$ is a GW-tree, and let $\{v_1, v_2, \ldots, v_N\}$ be the children of the root in $\mathcal{T}$. Then

$$\begin{aligned} \pi &\leq \mathbf{P}_\lambda(\text{ no phase transition in } \mathcal{T}(v_j),\ \forall\, j) \\ &= \sum_{n=0}^{\infty} \mathbf{P}\,(N = n)\, \pi^n = f(\pi) \end{aligned}$$

  where f is the generating function for N.

- Moreover $\beta \geq q :=$ extinction probability, because $[\texttt{extinction}] \subseteq [\texttt{no phase transition}]$

- Thus $\beta \in \{q, 1\}$ and this completes the proof.

Key Recursion on a Finite Tree

- Suppose $\mathcal{T}$ be a finite (fixed) rooted tree and we consider the hard-core model on it with activity $\lambda > 0$.

- Suppose $\emptyset$ be the root and it has $n(\emptyset)$ many children which are denoted by $1, 2, \ldots, n(\emptyset)$.

- Let I be a random independent set distributed according to the hard-core model with activity $\lambda > 0$. We define $\eta_{\emptyset}^{\mathcal{T}} := \mathbb{P}_{\lambda}^{\mathcal{T}}(\emptyset \in I)$.

- For a child j, let $\mathcal{T}_j$ be the sub-tree rooted at j obtained by removing $\emptyset$. Suppose $\eta_j^{\mathcal{T}_j}$ be defined similarly of $\eta_{\emptyset}^{\mathcal{T}}$.

- The following *key recursion* holds

$$\eta_{\emptyset}^{\mathcal{T}} = \frac{\lambda \prod_{j=1}^{n(\emptyset)} \left(1 - \eta_j^{\mathcal{T}_j}\right)}{1 + \lambda \prod_{j=1}^{n(\emptyset)} \left(1 - \eta_j^{\mathcal{T}_j}\right)}$$

“Superscript Dropping Principle” Recursive Distributional Equation (RDE)

We consider the following distributional identity:

$$\eta \stackrel{d}{=} \frac{\lambda \prod_{j=1}^{N} \left(1 - \eta_j\right)}{1 + \lambda \prod_{j=1}^{N} \left(1 - \eta_j\right)} \quad \text{on} \quad [0, 1],$$

where (η_j) are i.i.d. copies of η and are independent of N.

- We also define an operator $T : \mathcal{P}([0,1]) \to \mathcal{P}([0,1])$ using the right-hand side of the above RDE, namely,

$$T(\mu) := \text{dist} \left(\frac{\lambda \prod_{j=1}^{N} \left(1 - \eta_j\right)}{1 + \lambda \prod_{j=1}^{N} \left(1 - \eta_j\right)} \right)$$

  where (η_j) are i.i.d. with distribution μ on $[0, 1]$ and are independent of N.

- We put $S = T^2$.

RDE Continued ...

Properties of the RDE and the Operator T:

- $T(\delta_0) = \delta_{\lambda/(1+\lambda)}$.
- $\delta_0 \preccurlyeq T(\mu) \preccurlyeq \delta_{\lambda/(1+\lambda)}$, for any probability μ on $[0,1]$.
- T is anti-monotone $\Rightarrow$ S is monotone.
- T is continuous with respect to the weak convergence topology on $\mathcal{P}([0,1])$.
- So there exist $\mu_* \preccurlyeq \mu^*$ two fixed points of S such that $S^n(\delta_0) \uparrow \mu_*$ and $S^n\left(\delta_{\lambda/(1+\lambda)}\right) \downarrow \mu^*$.
- $T(\mu_*) = \mu^*$ and $T(\mu^*) = \mu_*$.
- S has unique fixed point if and only if $\mu_* = \mu^*$.
- T is a strict contraction with respect to the Wasserstine metric when $\lambda\,\mathbf{E}[N] < 1$.

Uniqueness Domain

Definition 5 *We will say that we are in the* uniqueness domain *if* $\mu_* = \mu^*$*.*

Characterization of Phase Transition for GW-Tree Model

Theorem 3 *For GW-tree with progeny distribution* N*, there is* no phase transition *for the hard-core model with activity* $\lambda > 0$*, if and only if, we are in uniqueness domain for the associated RDE.*

Specialization to r-regular Tree

- Notice that if $\mathbb{T}_r(\emptyset)$ denote a rooted r-regular tree, that is, a tree whose root $\emptyset$ has degree $r-1$ and all other vertices have degree r, then it is a GW-tree with progeny distribution $N \equiv r-1$.

- So for this model N is non random, that is the operator T has no random part in its definition.

- This then implies both μ_* and μ^* are degenerate measures.

- So basically we need to consider fixed point of a deterministic function $s = t^2$ where $t \colon [0,1] \to [0,1]$ given by

$$t(p) = \frac{\lambda\,(1-p)^{r-1}}{1+\lambda\,(1-p)^{r-1}}, \;\; p \in [0,1].$$

- This is exactly what Kelly did in his 1985 paper and this leads to the critical value $\lambda_c(r)$.

When Does Uniqueness Domain Hold ?

Corollary 4 *For a GW-tree with progeny distribution* N*, there is no phase transition for the hard-core model with activity* $\lambda > 0$ *if*

(a) $\mathbf{E}[N] \leq 1$ *or,*

(b) $\lambda \mathbf{E}[N] < 1$*.*

Remarks:

- In particular it shows that for any GW-tree (with $\mathbf{E}[N] < \infty$) at least for sufficiently small λ there is no phase transition. Such result is expected. But note that we do not assume that the progeny distribution is bounded.

- In fact a better bound holds using *Van den Berg-Steif inequality*, namely $\lambda (\mathbf{E}[N] - 1) < 1$.

Main Results for hard-Core Model on Sparse Random Graphs

Theorem 5 *Suppose $X_\lambda^\omega(n,c)$ be the size of a random independent set distributed according to the hard-core model with activity $\lambda > 0$ on a Erdös-Rényi random graph $\mathcal{G}\left(n, \frac{c}{n}\right)$. If the GW-tree with Poisson(c) progeny distribution has no phase transition then*

$$\lim_{n\to\infty} \frac{\mathbf{E}_\lambda\left[X_\lambda^\omega(n,c)\right]}{n} = \gamma_\lambda(c)$$

where $\gamma_\lambda(c) := \mathbf{E}[\eta]$ and η is the unique solution of the RDE.

Theorem 6 *Suppose $X_\lambda^\omega(n,r)$ be the size of a random independent set distributed according to the hard-core model with activity $\lambda > 0$ on a random regular graph $\mathcal{G}_r(n)$. If the r-regular tree has no phase transition, that is, if $\lambda < \lambda_c(r) = (r-1)^{(r-1)}/(r-2)^r$, then*

$$\lim_{n\to\infty} \frac{\mathbf{E}_\lambda\left[X_\lambda^\omega(n,r)\right]}{n} = \alpha_\lambda(r)$$

where $\alpha_\lambda(r) = w/(1+2w)$ with w is the unique positive solution of the equation $\lambda = w(1+w)^{r-1}$.

Back to Aldous' Conjecture

Conjecture [Aldous, 2003]: For a sparse random graph if I_n is a maximum independent set then

$$\lim_{n\to\infty} \frac{\mathbf{E}\left[|I_n|\right]}{n} = \kappa$$

for some constant $\kappa > 0$ (explicitly computable ?).

- Our method fails ! This is because it seems (for the general GW-tree case) that the uniqueness domain does not hold for *large* λ.

- For example it is the case with r-regular trees and hence for the sparse random graph model $\mathcal{G}_r(n)$.

- In fact for $\mathcal{G}_r(n)$ model it has been postulated (proved using non-rigorous methods) in physics literature that such asymptotic limit exists and has the same answer as Theorem 6 when λ is smaller than the so called "*extremality threshold*" (which is bigger than the "*uniqueness threshold*").

- Our Theorems 5 and 6 provides rigorous argument when λ is in the uniqueness domain (that is, under the *uniqueness threshold* for the $\mathcal{G}_r(n)$ model).

Towards Proving Theorem 3

Long Range Independence Property

- Fix $d \geq 0$.
- Write $\mathbf{x}_d$ for a vector $(x_{\mathbf{i}})_{|\mathbf{i}|=d}$ where each $x_{\mathbf{i}} \in [0,1]$.
- Let $\mathcal{T}$ be the realization of the GW-tree rooted at $\emptyset$ obtained from the realizations of $(N_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$.
- Let $\left(\eta_{\mathbf{i}}^{(d)}(\mathbf{x}_d)\right)_{|\mathbf{i}| \leq d}$ be the d-depth RTP with values at level d given by $\mathbf{x}_d$.

Lemma 7 (Long range independence) *Suppose we are in the uniqueness domain, that is $\mu_* = \mu^*$, then*

$$\lim_{d \to \infty} \sup_{\mathbf{x}_d} \left| \eta_{\emptyset}^{(d)}(\mathbf{x}_d) - \eta_{\emptyset} \right| = 0 \quad a.s.$$

Remark: The proof of Theorem 3 follows from this Lemma.

Proof of Lemma 7:

- For the vector $\mathbf{x}_d$ if all the components are same as c the we will write the vector itself as $\mathbf{c}$.

- $\eta_\emptyset^{(2d)}(\mathbf{0}) \uparrow \eta_\emptyset$, a.s. and also $\eta_\emptyset^{(2d+1)}(\mathbf{0}) \downarrow \eta_\emptyset$ a.s.

- $\eta_\emptyset^{(d)}\left(\frac{\lambda}{1+\lambda}\right) = \eta_\emptyset^{(d+1)}(\mathbf{0})$, so $\eta_\emptyset^{(d)}\left(\frac{\lambda}{1+\lambda}\right) \to \eta_\emptyset$ a.s.

- If $0 \le x_{\mathbf{i}} \le \frac{\lambda}{1+\lambda}$ for all $\mathbf{i} \in \mathcal{V}$ then

$$\eta_\emptyset^{(2d)}(\mathbf{0}) \le \eta_\emptyset^{(2d)}(\mathbf{x}_{2d}) \le \eta_\emptyset^{(2d)}\left(\frac{\lambda}{1+\lambda}\right), \quad \text{and}$$

$$\eta_\emptyset^{(2d+1)}\left(\frac{\lambda}{1+\lambda}\right) \le \eta_\emptyset^{(2d+1)}(\mathbf{x}_{2d+1}) \le \eta_\emptyset^{(2d+1)}(\mathbf{0}).$$

  So $\eta_\emptyset^{(d)}(\mathbf{x}_d) \to \eta_\emptyset$ a.s. as $d \to \infty$.

- Now notice that $\eta_\emptyset^{(d)}(\mathbf{1}) = \eta_\emptyset^{(d-1)}(\mathbf{x}_{d-1})$ where each $x_{\mathbf{i}} \in \{0, \lambda/(1+\lambda)\}$. So $\eta_\emptyset^{(d)}(\mathbf{1}) \to \eta_\emptyset$ a.s.

- Finally, if $0 \le x_{\mathbf{i}} \le 1$ for all $\mathbf{i} \in \mathcal{V}$ then

$$\eta_\emptyset^{(2d)}(\mathbf{0}) \le \eta_\emptyset^{(2d)}(\mathbf{x}_{2d}) \le \eta_\emptyset^{(2d)}(\mathbf{1}), \quad \text{and}$$

$$\eta_\emptyset^{(2d+1)}(\mathbf{1}) \le \eta_\emptyset^{(2d+1)}(\mathbf{x}_{2d+1}) \le \eta_\emptyset^{(2d+1)}(\mathbf{0}).$$

  So $\eta_\emptyset^{(d)}(\mathbf{x}_d) \to \eta_\emptyset$ uniformly a.s. as $d \to \infty$, proving the lemma.

Remarks on the Proofs of Theorem 5 and Theorem 6

- First thing to note is

$$\frac{1}{n}\mathbf{E}_\lambda\left[X_\lambda^\omega\right] = \mathbf{P}_\lambda\left(v_0 \in I_\lambda^\omega\right),$$

  where I_λ^ω is the random independent set selected according to the (random) distribution $\mathbb{P}_\lambda^\omega$, and v_0 is a fixed vertex.

- For any fixed $d > 0$ the distribution of the d-depth neighborhood of v_0 converges to the distribution of a d-depth Poisson(c) GW-tree for $\mathcal{G}\left(n, \frac{c}{n}\right)$ model, and to a d-depth r-regular tree for $\mathcal{G}_r\left(n\right)$ model.

- We can then apply the *local weak convergence* technique of Aldous and Steele (2004) using the (strong) *long range independence* property which holds under the uniqueness domain. These will give the stated results after a little more careful probability computations !

Open Problems/Questions

- We know that for r-regular tree the phase transition is a *monotone* property in λ, and the critical value $\lambda_c(r)$ is explicitly know.

  Is phase transition monotone (in λ) for a general GW-tree ?

  Comment: Most possibly not ! But is it at least the case for GW-tree with Poisson progeny distribution ?

- For GW-tree with Poisson(c) progeny distribution is phase transition monotone in c ? That is for every fixed $\lambda > 0$ if $c > c'$ and we have no phase transition for Poisson(c) GW-tree then can we say that we have no phase transition for Poisson(c') GW-tree ?

  Comment: We know this is true if $\lambda \times c < 1$.

- If the answer to above question is yes (which is most possibly the case) can we also get the critical value for c ? (explicitly or bounds ?)

Counting without Sampling: Asymptotics of the log-Partition Function

Two Counting Problems

Definition 6 (Independent Set) *Suppose* $G := (V, E)$ *be a finite graph. We will say a subset* $I \subseteq V$ *is an* independent set *of* G*, if for any two vertices* $u, v \in I$ *there is no edge between* u *and* v*.*

We will denote by $\mathcal{I}_G$*, the set of all independent sets of* G*.*

Problem 1: Given a finite graph G, count the *number of independent sets* of G.

Definition 7 (Proper q-Coloring) *Fix* $q \geq 2$ *an integer, and suppose* $G := (V, E)$ *be a finite graph. A map* $C : V \to \{1, 2, \ldots, q\}$ *is called a* proper q-coloring *of* G*, if no two vertices of same color share an edge.*

We will denote by $\mathcal{C}_G(q)$*, the set of all proper* q*-colorings of* G*.*

Problem 2: Given a finite graph G, and $q \geq 2$ an integer, count the *number of proper* q*-colorings* of G.

Exact/Approximate Counting

Q: Can we do exact counting ?

A:
- Perhaps not !
- The sets are typically exponentially large.
- No polynomial time algorithm [Valiant 1979].

Q: So what do we do ?

A: We can try "approximate" counting.

Q:
- How do we approximate ?
- What kind of approximation ?

A:
- Typical approach is to use a *Markov chain Monte Carlo* (MCMC) sampling scheme.
- One need to prove *rapid mixing* for the chain.

Some Success Stories for Problems Similar to Ours (using MCMC techniques)

- Computing the permanent:
  - Jerrum and Sinclair (1989, 1997).
  - Jerrum, Sinclair and Vigoda (2004).
- Computing the volume of a convex body:
  - Dyer, Frieze and Kannan (1991).
  - Kannan, Lovasz and Simonovits (1997).
  - Lovasz and Vempala (2003).
- Counting independent set:
  - Luby and Vigoda (1997).

Remark: Such MCMC techniques typically provide a randomized ε-approximation to the counting problem, such that the running time is a polynomial in the size of the problem (e.g. the size of V), and also in the error ε.

What Do We Propose to Do ?

- We will give *deterministic* approximation schemes, which will not use sampling.

- But we will provide ε-approximation to $\log |\mathcal{I}_G|$ and $\log |\mathcal{C}_G(q)|$. (Unfortunately, this is *obviously* less efficient !)

- Moreover, we will need restrictions on our graphs ! For example, we will need *low degree* graphs, and a "*large girth*" assumption (will be more specific later).

What is the use !!!!

We are *obviously* doing *less* than what is known !

Then why did we work on this ?

- Well well ... I like this work !
- But there are more reasons than just that !

Motivation and Achievements

- Our motivation comes from *statistical physics*.

- Computation of log $|\mathcal{I}_G|$ or log $|\mathcal{C}_G(q)|$ are interesting, because they correspond to the *free energy* for certain models in statistical physics (the models will be given later).

- We can achieve (new) explicit results for regular graphs, which are not possible to derive using the MCMC methods. To give some example:
  - ▶ We can show that for every 4-regular graph of n vertices and large girth, the number of independent sets is approximately $(1.494\ldots)^n$.
  - ▶ We can also show that if $q \geq r+1$ then for every r-regular graph with large girth, the number of proper q-coloring is approximately
$$\left[q\left(1-\frac{1}{q}\right)^{\frac{r}{2}}\right]^n .$$

- We can drop the "large girth" assumption and work with random regular graphs to get concentration results.

Two Statistical Physics Models

(1) Hard-Core Model: Given a finite graph G and a real number $\lambda > 0$, consider a (discrete) probability distribution on $\mathcal{I}_G$ given by

$$\mathbb{P}(I) \propto \lambda^{|I|} \;\Leftrightarrow\; \mathbb{P}(I) = \frac{\lambda^{|I|}}{Z(\lambda, G)}, \quad I \in \mathcal{I}_G\,,$$

where

$$Z(\lambda, G) := \sum_{I \in \mathcal{I}_G} \lambda^{|I|}\,.$$

Remarks:

- $\mathbb{P}$ is called the *Gibbs distribution* on $\mathcal{I}_G$.
- $Z(\lambda, G)$ is called the *partition function*.
- λ is called the *activity parameter*.
- Observe $Z(\lambda, G) = |\mathcal{I}_G|$ when $\lambda = 1$, then we are back to the original counting problem.

(2) Model on Proper q-Colorings: Given $q \geq 2$ an integer, and a finite graph G, let $\lambda_k > 0$, for $1 \leq k \leq q$. Consider a (discrete) probability distribution on $\mathcal{C}_G(q)$ given by

$$\mathbb{P}(C) = \frac{\prod_{1 \leq k \leq q} \lambda_k^{|C^{-1}(\{k\})|}}{Z(\lambda, q, G)}, \quad C \in \mathcal{C}_G(q)$$

where

$$Z(\lambda, q, G) := \sum_{C \in \mathcal{C}_G(q)} \prod_{1 \leq k \leq q} \lambda_k^{|C^{-1}(\{k\})|}.$$

Remarks:

- $\mathbb{P}$ is called the *Gibbs distribution* on $\mathcal{C}_G(q)$.
- $Z(\lambda, q, G)$ is the *partition function* and λ_k's are called the *activity parameters*.
- If all the λ_k's are equal then $Z(\lambda, q, G) = |\mathcal{C}_G(q)|$ and we are back to the original counting problem. For this case we will denote the partition function by $Z(q, G)$.

Some Families of Graphs

- **Large girth:** An infinite family of graphs $\mathcal{G}$ is defined to have *large girth*, if there exists an increasing function $f : \mathbb{N} \to \mathbb{N}$ with $\lim_{s\to\infty} f(s) = \infty$, such that for every $G \in \mathcal{G}$ with n vertices, we have

$$\text{girth}\,(G) \geq f(n)\,.$$

Recall: girth (G) := size of the smallest cycle in G.

- **[Low degree]:** Let $\mathcal{G}\,(n, r, g)$ be the family of graphs on n vertices, such that the maximum degree of any vertex is bounded by r and each graph has girth at least g.

- **Regular:** Let $\mathcal{G}_{\texttt{reg}}\,(n, r, g)$ be the family of r-regular graphs on n vertices, such that each graph has girth at least g.

The Main Results

Algorithm Result:

Theorem 8 (Independent Sets) *For every family of graphs $\mathcal{G}$ with maximum degree at most 4 and large girth, there is an algorithm $\mathcal{A}$, such that for any $\varepsilon > 0$ and $G \in \mathcal{G}$, $\mathcal{A}$ produces a quantity $\mathfrak{Z}$ in time polynomial in $n := |V|$, such that*

$$(1-\varepsilon)\frac{\log|\mathcal{I}_G|}{n} \le \mathfrak{Z} \le (1+\varepsilon)\frac{\log|\mathcal{I}_G|}{n}.$$

Theorem 9 (Colorings) *Fix $q \geq r+1$ be two integers then*

$$\lim_{g\to\infty} \sup_{G\in\mathcal{G}(n,r,g)} \left| \frac{\log|\mathcal{C}_G(q)|}{n} - \frac{1}{n} \sum_{1\le k\le n} \log\left[q\left(1-\frac{1}{q}\right)^{r_{G_{k-1}}(v_k)} \right] \right| = 0.$$

where $V := \{v_1, v_2, \ldots, v_n\}$ and $G_k := G \setminus \{v_1, v_2, \ldots, v_k\}$, and by $r_G(v)$ we mean the degree of vertex v in graph G.

In particular, we can get an algorithm result for counting the number of proper q-colorings, which is similar to the previous theorem.

Results for the Regular Graphs with Large Girth:

Theorem 10 (Independent Sets) *Suppose $\lambda < \lambda_c(r)$ where $\lambda_c(r) := (r-1)^{r-1}/(r-2)^r$, then*

$$\lim_{g\to\infty} \sup_{G\in\mathcal{G}_{\mathrm{reg}}(n,r,g)} \left| \frac{\log Z(\lambda, G)}{n} - \log\left(x^{-\frac{r}{2}}(2-x)^{-\frac{r-2}{2}}\right)\right| = 0\,,$$

where x is the unique positive solution of

$$x = 1/(1+\lambda x^{r-1})\,.$$

In particular, if $r = 2,3,4,5$ and $\lambda = 1$, then the corresponding limits for $\frac{\log|\mathcal{I}_G|}{n}$ are respectively, $\log 1.618\ldots$, $\log 1.545\ldots$, $\log 1.494\ldots$ *and* $\log 1.453\ldots$.

Theorem 11 (Colorings) *For every $q \geq r+1$, the number of q-colorings of graphs $G \in \mathcal{G}_{\mathtt{reg}}(n,r,g)$ satisfies*

$$\lim_{g\to\infty} \sup_{G\in\mathcal{G}_{\mathrm{reg}}(n,r,g)} \left| \frac{\log Z(q, G)}{n} - \log\left[q\left(1-\frac{1}{q}\right)^{\frac{r}{2}}\right]\right| = 0\,.$$

Results for the Random Regular Graphs:

Theorem 12 (Independent Sets) *For every $r \geq 2$ and every $\lambda < \lambda_c(r)$, the (random) partition function $Z(\lambda, G_r(n))$, of a random r-regular graph $G_r(n)$ satisfies*

$$\frac{\log Z(\lambda, G_r(n))}{n} \to \log\left[x^{-\frac{r}{2}}(2-x)^{-\frac{r-2}{2}}\right],$$

with high probability (w.h.p.), as $n \to \infty$, where x is the unique positive solution of $x = 1/(1 + \lambda x^{r-1})$.

Theorem 13 (Colorings) *For every $r \geq 2$ and $q \geq r+1$, the (random) partition function $Z(q, G_r(n))$ of a random r-regular graph $G_r(n)$ corresponding to the uniform distribution on proper q-colorings satisfies*

$$\frac{\log Z(q, G_r(n))}{n} \to \log\left[q\left(1 - \frac{1}{q}\right)^{\frac{r}{2}}\right].$$

w.h.p. as $n \to \infty$.

Remark: Theorem 13 was proved earlier by Achlioptas and Moore (2004) using second moment method.

Two Main Steps of the Algorithm (Illustrated only for the Independent Sets)

STEP 1 (The Cavity Equation):

- In this step we relate the computation of the partition function to the computation of the marginal probabilities.

- This is done by creating a *cavity* in the original graph.

- Let G_1 be the original graph G with one vertex, say v_1, removed.

- By definition

$$Z(\lambda, G_1) = \sum_{I \in \mathcal{I}_{G_1}} \lambda^{|I|} = \sum_{\substack{I \in \mathcal{I}_G \\ v_1 \notin I}} \lambda^{|I|}.$$

- *Cavity Equation:*

$$\frac{Z(\lambda, G_1)}{Z(\lambda, G)} = \mathbb{P}_G(v_1 \notin \mathbf{I}) \ .$$

Cavity Equation Continued ...

Proposition 14 *Let* $V := \{v_1, v_2, \ldots, v_n\}$*, and for* $1 \leq k \leq (n-1)$ *we define* $G_k := G \setminus \{v_1, v_2, \ldots, v_k\}$ *as the graph obtained from* G *after creating* k cavities*. Put* $G_0 = G$*. Then the following relation holds*

$$\frac{Z(\lambda, G_1)}{Z(\lambda, G_0)} = \mathbb{P}_{G_0}(v_1 \notin \mathbf{I}) \ ,$$

where $\mathbf{I}$ *is a random independent set distributed according the Gibbs measure* $\mathbb{P}$*. As a result we get*

$$Z(\lambda, G) = \prod_{k=1}^{n} \left(\mathbb{P}_{G_{k-1}}(v_k \notin \mathbf{I})\right)^{-1} .$$

Remark: This proposition is well known in Physics literature and also in the Markov chain based approximation algorithms for counting.

STEP 2 (Computation on Trees):

- Note our *large girth* assumption makes our graphs "*locally*" tree like !

- So in this step we only make computation for the marginal probabilities when the graph is a finite tree.

- This can be done easily by a recursive method, essentially the same cavity trick works.

Computation on Trees Continued ...

- T be a finite tree with root v_0, and let $\{v_1, v_2, \ldots, v_k\}$ be the children of v_0.

- By the cavity equation we get:

$$\begin{aligned}
\mathbb{P}_T\left(v_0 \notin \mathbf{I}\right) &= \frac{Z\left(\lambda, T \setminus \{v_0\}\right)}{Z\left(\lambda, T\right)} \\
&= \frac{1}{1 + \frac{\sum_{I \in \mathcal{I}_T, v_0 \in I} \lambda^{|I|}}{Z(\lambda, T \setminus \{v_0\})}} \\
&= \frac{1}{1 + \lambda \frac{\sum_{I \in \mathcal{I}_{T \setminus \{v_0\}},\, v_j \notin I \ \forall\ 1 \le j \le k} \lambda^{|I|}}{Z(\lambda, T \setminus \{v_0\})}} \\
&= \frac{1}{1 + \lambda \prod_{1 \le j \le k} \mathbb{P}_{T(v_j)}\left(v_j \notin \mathbf{I}\right)}
\end{aligned}$$

where $T(v_j)$ is the tree rooted at the child v_j.

Computation on Trees Continued ...

Proposition 15 *Suppose* T *be a finite rooted tree with root* v_0*, and let* $\{v_1, v_2, \ldots, v_k\}$ *be* $k \geq 0$ *children of* v_0*. For each* $1 \leq j \leq k$*, let* $T(v_j)$ *denote the tree rooted at* v_j *consisting only the descendants of* v_j *(if any). Then the following recursion holds*

$$\mathbb{P}_T\left(v_0 \notin \mathbf{I}\right) = \frac{1}{1 + \lambda \prod_{1 \leq j \leq k} \mathbb{P}_{T(v_j)}\left(v_j \notin \mathbf{I}\right)}.$$

The Algorithm

INPUT: A graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$, and a number $\varepsilon > 0$.

BEGIN

1. Compute the girth $g = g(G)$.

2. If $(0.9)^{\frac{g}{2}-2} \geq \varepsilon$ then find $|\mathcal{I}_G|$ by enumeration and STOP.

If not then

3. Set $Z \leftarrow 1$, $t \leftarrow \lfloor g/2 \rfloor$ and $k \leftarrow 1$.

4. Find the t-depth neighborhood $T(v_k)$ of v_k.

5. Compute the marginal probability $p = \mathbb{P}_{T(v_k)}(v_k \notin \mathbf{I})$ for the finite tree $T(v_k)$.

6. Set $Z \leftarrow Z/p$, $G \leftarrow G \setminus \{v_k\}$, $k \leftarrow k + 1$.

7. If $k \leq n$ then goto Step 4, otherwise STOP.

END

OUTPUT: Z

Why Does It Works ?

⋆ **Strong Correlation Decay:**

- We prove that under certain assumptions, for example,
  - ▶ $\lambda < \lambda_c(r) = (r-1)^{r-1}/(r-2)^r$ (for the r-regular case),
  - ▶ or $r \leq 4$ (for deriving the algorithm),

  the *influence* on the root of the boundary at a distance d decreases exponentially fast as d increases.

- A statistical physics consequence of this is the Gibbs measure on the "limiting infinite graph" (if any !) is unique, that is there is no *phase transition*.

- For the infinite r-regular trees it was shown by Kelly (1985), that there is no *phase transition* for the hard-core model if and only if $\lambda \leq \lambda_c(r)$.

- For counting independent sets we extend this result to the class of finite trees with maximum degree at most 4, which is the most crucial result for our algorithm to succeed.

Strong Correlation Decay Continued ...

- Suppose T be a finite tree with *large* depth.

- If the maximum degree of T is at most 4, then for any two boundary conditions b_1 and b_2 we show that

$$\mathbb{P}_T^{\lambda=1}\left(v_0 \notin \mathbf{I} \,\middle|\, b_1\right) \approx \mathbb{P}_T^{\lambda=1}\left(v_0 \notin \mathbf{I} \,\middle|\, b_2\right) .$$

- Moreover the error in approximation is exponentially small in the depth of the tree.

Strong Correlation Decay Continued ...

- Further, if T is a tree such that every vertex has degree r except the root, which has degree $(r-1)$ and the vertices at the last generation, which have degree 1, then for $\lambda < \lambda_c(r)$ it is know (Kelly, 1985) that

$$\mathbb{P}_T\left(v_0 \notin \mathbf{I} \,\middle|\, b\right) \approx x\,,$$

  for any boundary condition b, where x is the unique solution of the *deterministic* fixed point equation

$$x = 1/(1 + \lambda x^{r-1})\,.$$

- If T is a tree with all internal vertices having degree r then under the same assumption

$$\mathbb{P}_T\left(v_0 \notin \mathbf{I} \,\middle|\, b\right) \approx \frac{1}{2-x}\,.$$

Lemma 16 *The following bounds holds for every rooted tree T with depth $t \geq 2$ and degree of any vertex at most 4*

$$\frac{1}{2} \leq \mathbb{P}_T^{\lambda=1}\left(v_0 \notin \mathbf{I} \,\middle|\, b\right) \leq \frac{8}{9},$$

and

$$\left|\mathbb{P}_T^{\lambda=1}\left(v_0 \notin \mathbf{I} \,\middle|\, b_1\right) - \mathbb{P}_T^{\lambda=1}\left(v_0 \notin \mathbf{I} \,\middle|\, b_2\right)\right| \leq (.9)^{t-2},$$

where b, b_1, b_2 are boundary conditions.

Moreover, when $\lambda < \lambda_c(r)\,(r-1)^{r-1}/(r-2)^r$, let x be the unique non-negative solution of the fixed point equation $x = 1/(1+\lambda x^{r-1})$. Suppose all the nodes of T except for leaves and the root have degree r, and suppose the root has degree $r-1$. Then for all boundary conditions b

$$\left|\mathbb{P}_T\left(v_0 \notin \mathbf{I} \,\middle|\, b\right) - x\right| \leq \alpha^t,$$

for some constant $\alpha = \alpha(\lambda) < 1$. If on the other hand, all the nodes except for leaves, have degree r (including the root), then

$$\left|\mathbb{P}_T\left(v_0 \notin \mathbf{I} \,\middle|\, b\right) - \frac{1}{2-x}\right| \leq \alpha^t,$$

with the same constant $\alpha < 1$.

Strong Correlation Decay Continued ...

Remarks:

- The proof involves only elementary math ! But at some point we had to take help of *computer* (MATLAB) [not me] !!

- The correlation decay for the counting of proper q-colorings was proved by Jonasson (2002) for finite depth r-regular tree, but his result extends to any finite tree with bounded degree, which we use for counting proper-q coloring problem.

⋆ **From the Tree to the Original Graph:**

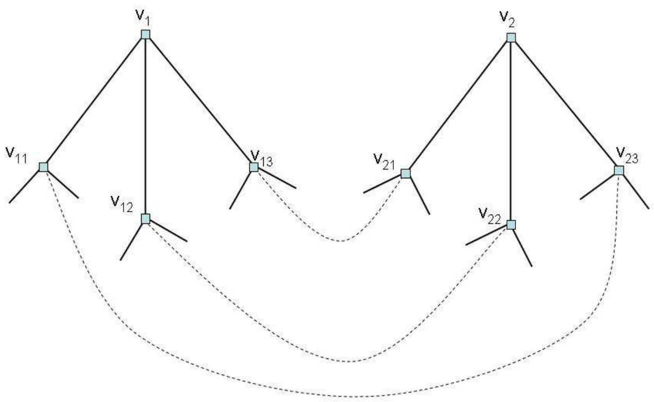
- In this step we show that the error we make by taking a *local tree* around a vertex is *small*.

  Note: The *local tree* comes from the *large girth* assumption.

- This is again done by using the strong correlation decay property and the (spacial) Markovian nature of the Gibbs distribution.

Special Tricks for the Regular Graphs

- For regular graphs creating a *cavity* destroy the regularity !
- Instead we do the following which we call the *rewiring*. Similar idea has been used in Physics literature [Mezard and Parisi, 2005].



Note: v_1 and v_2 are not neighbors and their neighbors are not neighbors !

New "Cavity" Equation for Regular Graphs

Proposition 17 *Given an r-regular graph G, and $\lambda > 0$, the graph G^o obtained from G by rewiring on nodes $v_1, v_2 \in G$, the following relation holds*

$$\frac{Z(\lambda, G^o)}{Z(\lambda, G)} = \mathbb{P}_G(v_1, v_2 \notin \mathbf{I})\mathbb{P}_{G\setminus\{v_1,v_2\}}\left(\bigcap_{1\le j\le r} [v_{1j} \notin \mathbf{I} \text{ or } v_{2j} \notin \mathbf{I}]\right)$$

where $v_{ij}, j = 1, \ldots, r$ are the neighbors of $v_i, i = 1, 2$ in G.

Strong Correlation Decay Result for Regular Graphs

Lemma 18 *Given* $r \geq 3, \lambda < (r-1)^{r-1}/(r-2)^r$ *and* $\epsilon > 0$, *there exists a sufficiently large constant* $g = g(r, \epsilon, \lambda)$ *such that for every* r*-regular graph* G *with girth* $g(G) \geq g$, *and for every pair of nodes* $v_1, v_2 \in G$ *at distance at least* $2g + 1$

$$\left| \mathbb{P}_G(v_1, v_2 \notin \mathbf{I}) - \frac{1}{(2-x)^2} \right| < \epsilon,$$

and

$$\left| \mathbb{P}_{G \setminus \{v_1, v_2\}} \left(\bigcap_{1 \leq j \leq r} [v_{1j} \notin \mathbf{I} \text{ or } v_{2j} \notin \mathbf{I}] \right) - (2x - x^2)^r \right| < \epsilon,$$

where $v_{ij}, j = 1, \ldots, r$ *is the set of neighbors of* v_i *in* G, $i = 1, 2$, *and* x *is the unique positive solution of* $x = 1/(1 + \lambda x^{r-1})$.

A Technical Result needed for Regular Graphs

Lemma 19 *Given an n-node r-regular graph G, consider any integer $4 \leq g \leq g(G)$. The rewiring operation can be performed for at least $(n/2)-(2g+1)r^{2g}$ steps on pairs of nodes which are at least $2g+1$ distance apart. In every step the resulting graph is r-regular with girth at least g.*

Some Final Remarks

- A recent work of Weitz (2006) provides a *fully polynomial approximation scheme* for any finite graph with low degree (maximum degree at most 5) for the problem of counting the independent sets, but it does not give explicit limit results such as ours for the regular graphs.

- Gamarnik and Katz (2006) (personal communication) have extended the work of Weitz (2006) for other counting problems, e.g. counting colorings, and counting matchings on general finite graphs.

- It seems to me that each of this is a "success story" for making a rigorous argument for a very powerful method of statistical physics, called the **cavity method** ! But the full math picture is yet to be discovered.

Thank You