Recursive Distributional Equations and Recursive Tree Processes

Antar Bandyopadhyay
(Some parts are joint work with David J. Aldous)

Statistics and Mathematics Colloquium

Theoretical Statistics and Mathematics Unit
Indian Statistical Institute
New Delhi, India

Department of Mathematics
Chalmers University of Technology
Göteborg, Sweden

<http://www.math.chalmers.se/∼antar>

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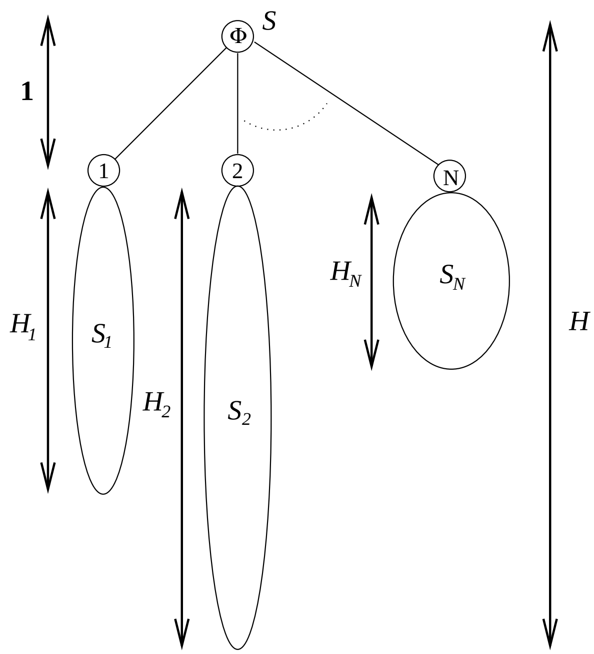
Brief Outline of the Talk

- Some examples of *Recursive Distributional Equations* (RDE).
- Indicate some basic general theory :
  - ▶ A mathematically natural structure : *Recursive Tree Process* (RTP).
  - ▶ Discuss the possible influence of infinite boundary.
  - ▶ Define two mathematically natural notions : *Endogeny* and *Tail-Triviality* of a RTP.
  - ▶ Discuss how to determine endogeny/tail-triviality of a RTP : two *equivalence theorems*.
- Discuss some *non-trivial* application(s).

Three *not so difficult* Examples

Example 1 : Consider a *(sub)-critical* Galton-Watson branching process with the progeny distribution N, so $\mathbf{E}[N] \leq 1$; we assume $\mathbf{P}(N = 1) < 1$.

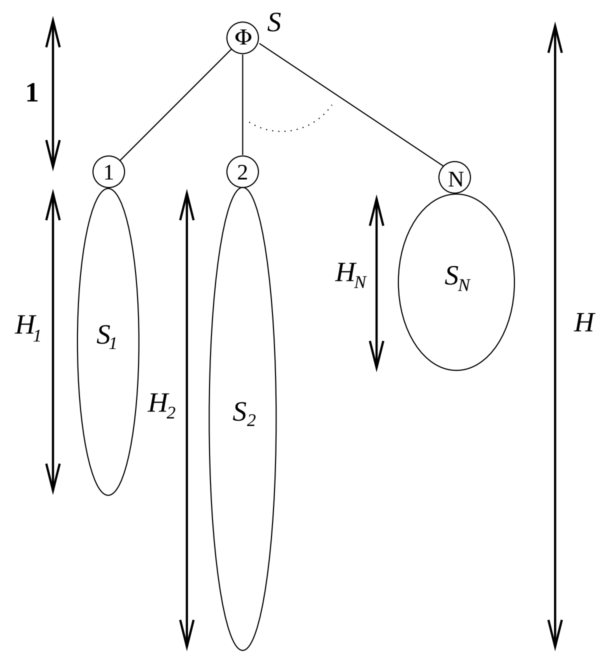


Height of the Tree : Let $H := 1+$ height of the G-W tree, then $H < \infty$ a.s. and

$$H \stackrel{d}{=} 1 + \max(H_1, H_2, \ldots, H_N) \quad \text{on } \mathbb{N},$$

where $(H_j)_{j \geq 1}$ are i.i.d. with same law as of H and are independent of N.

Example 2 : Consider the same *(sub)-critical* Galton-Watson branching process.



Size of the Tree : Let $S :=$ total size of the tree. Once again $S < \infty$ a.s. since the process is (sub)-critical. Further

$$S \stackrel{d}{=} 1 + (S_1 + S_2 + \cdots + S_N) \quad \text{on} \quad \mathbb{N},$$

where $(S_j)_{j \geq 1}$ are i.i.d. with same law as of S and are independent of N.

We will call such equations *Recursive Distributional Equations* (RDE).

Example 3 : Fix $0 < q < 1$ and consider the following process

$$X_i = \xi_i + X_{i+1} \pmod{2},$$

where $(\xi_i)_{i \geq 0}$ are i.i.d. Bernoulli(q) and X_{i+1} is independent of $(\xi_0, \xi_1, \ldots, \xi_i)$ for all $i \geq 0$.

Remarks :

- The process $(X_i)_{i \geq 0}$ exists provided the following RDE has a solution :

  $$X \stackrel{d}{=} \xi + X_1 \pmod{2} \quad \text{on} \quad \{0, 1\},$$

  where $\xi \sim$ Bernoulli(q) and is independent of X_1 which has same distribution as of X.

- It is easy to see that the RDE has unique solution given by $X \sim$ Bernoulli $\left(\frac{1}{2}\right)$.

- Note that $(X_i)_{i \geq 0}$ is nothing but a stationary Markov chain when time is reversed.

Three *non-trivial* Examples

Example 4 (Quicksort RDE) :

- Consider n numbers in a random order.
- Divide the last $(n-1)$ numbers into two piles, according to *less* than or *greater* than the first number.
- Recursively sort the two piles (which are now smaller in size).
- $X(n) := \#$ of comparisons, then

$$X(n) \stackrel{d}{=} X_1(U_n) + X_2(n-1-U_n) + (n-1),$$

where $X_1(\cdot)$ and $X_2(\cdot)$ are i.i.d. with same law as of $X(\cdot)$ and are independent of U_n which is uniform on $\{0, 1, 2, \ldots, n-1\}$.

- Rösler (1990) showed $\mathbf{E}[X(n)] \sim 2n \log n$. Moreover

$$\frac{X(n) - 2n \log n}{n} \stackrel{d}{\longrightarrow} Y,$$

with the distribution of Y satisfying the RDE

$$Y \stackrel{d}{=} UY_1 + (1-U)Y_2 + c(U) \quad \text{on } \mathbb{R},$$

where Y_1 and Y_2 are i.i.d. with same law as of Y and are independent of $U \sim$ Uniform$[0,1]$, and $c(u) := 1 + 2u \log u + 2(1-u)\log(1-u)$.

Example 5 (Logistic RDE) : Consider the following RDE

$$X \stackrel{d}{=} \min_{j \geq 1} \left(\xi_j - X_j \right) \quad \text{on } \mathbb{R},$$

where $(X_j)_{j \geq 1}$ are i.i.d. with same distribution as X and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson point process of rate 1 on $(0, \infty)$.

- This RDE appears in the study of the asymptotic limit of the *mean-field random assignment problem*. [Aldous 2001]

- It is not so difficult (but not obvious either) to see that this RDE has a unique solution, given by the *Logistic distribution*,

$$\mathbf{P}(X \leq x) = \frac{1}{1 + e^{-x}}, \quad x \in \mathbb{R}.$$

Example 6 (Frozen Percolation RDE) : Consider the following RDE

$$X \stackrel{d}{=} \Phi\left(X_1 \wedge X_2; U\right) \quad \text{on } I := \left[\tfrac{1}{2}, 1\right] \cup \{\infty\},$$

where (X_1, X_2) are independent copies of X and are independent of $U \sim$ Uniform$[0,1]$ and the function Φ is given by

$$\Phi(x; u) := \begin{cases} x & \text{if } x > u \\ \infty & \text{otherwise} \end{cases}.$$

- This RDE plays a central role in rigorous construction of a *frozen percolation process* on the infinite 3-regular tree. [Aldous 2000]

- Again it is not difficult (but not so obvious either) to show that this RDE has a *unique* solution with full support I, which is given by

$$\nu(dy) = \frac{dy}{2y^2},\ \tfrac{1}{2} < y < 1,\ \ \nu(\{\infty\}) = \tfrac{1}{2}.$$

Typical features of RDEs

$$\text{Ex. } 1: \quad X \stackrel{d}{=} 1 + \max(X_1, X_2, \ldots, X_N) \quad \text{on } \mathbb{N}$$
$$\text{Ex. } 2: \quad X \stackrel{d}{=} 1 + (X_1 + X_2 + \cdots + X_N) \quad \text{on } \mathbb{N}$$
$$\text{Ex. } 4: \quad X \stackrel{d}{=} UX_1 + (1-U)X_2 + c(U) \quad \text{on } \mathbb{R}$$

- **Unknown Quantity :** Distribution of X.
- **Known Quantities :**
  - $N \leq \infty$ which may or may not be random (e.g. $N \equiv 2$ in Ex. 4).
  - Possibly some more randomness whose distribution is known (e.g. U in the Ex. 4).
  - How we combine the known and unknown randomness (e.g. "$1 + \max$" operation in Ex. 1).
- **What is the RDE doing ?** To find a distribution μ such that when we take i.i.d. samples $(X_j)_{j \geq 1}$ from it and only use N many of them (where N is independent of the samples) and do the manipulation then we end up with another sample $X \sim \mu$.

Remark : When $N = 1$ a.s. (e.g. Ex. 3) then solving the RDE basically means finding a stationary distribution of a discrete time Markov chain.

General Setup

- Let $(S, \mathfrak{S})$ be a measurable space, and $\mathcal{P}$ be the collection of all probabilities on $(S, \mathfrak{S})$.

- Let (ξ, N) be a pair of random variables such that N takes values in $\{0, 1, 2, \ldots; \infty\}$.

- Let $(X_j)_{j \geq 1}$ be **i.i.d** S-valued random variables, which are independent of (ξ, N).

- $g(\cdot)$ is a S-valued measurable function with appropriate domain.

Recursive Distributional Equation (RDE)

Definition 1 *The following* fixed-point equation *on* $\mathcal{P}$ *is called a* Recursive Distributional Equation (RDE)

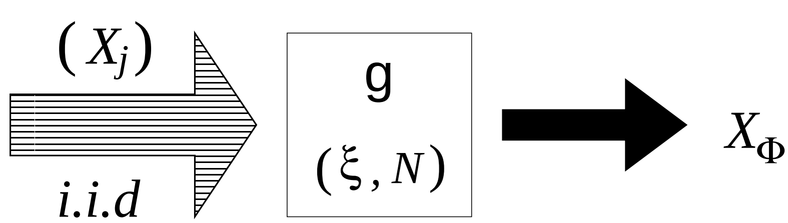
$$X \stackrel{d}{=} g\left(\xi; \left(X_j, 1 \leq j \leq^* N\right)\right) \quad \textit{on} \;\; S,$$

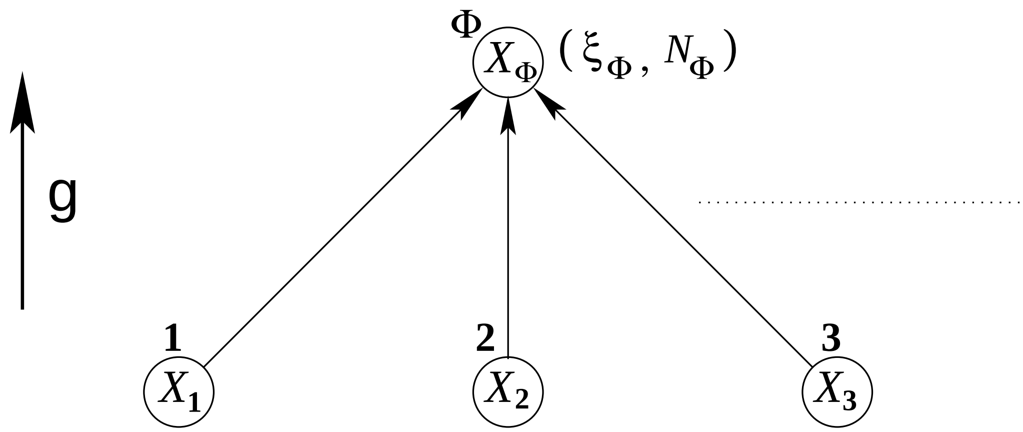
where $(X_j)_{j \geq 1}$ *are independent copies of* X *and are independent of* (ξ, N)*.*

Remark : A more conventional (analysis) way of writing the equation would be

$$\mu = T(\mu)$$

where T is the operator associated with the above equation, which depends on the function g and the joint distribution of the pair (ξ, N), and μ is the (unknown) law of X.




Φ
X_Φ
(ξ_Φ, N_Φ)
g
1
X_1
2
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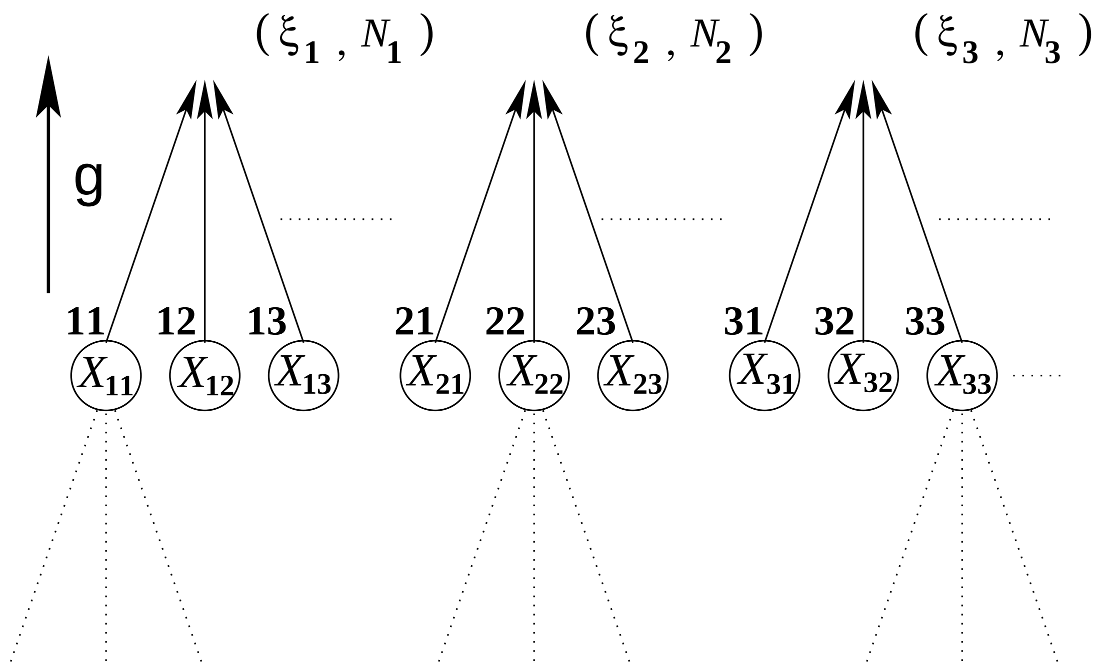
3
X_3



(ξ1 , N1)
(ξ2 , N2)
(ξ3 , N3)
g
11
12
13
21
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X11
X12
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X31
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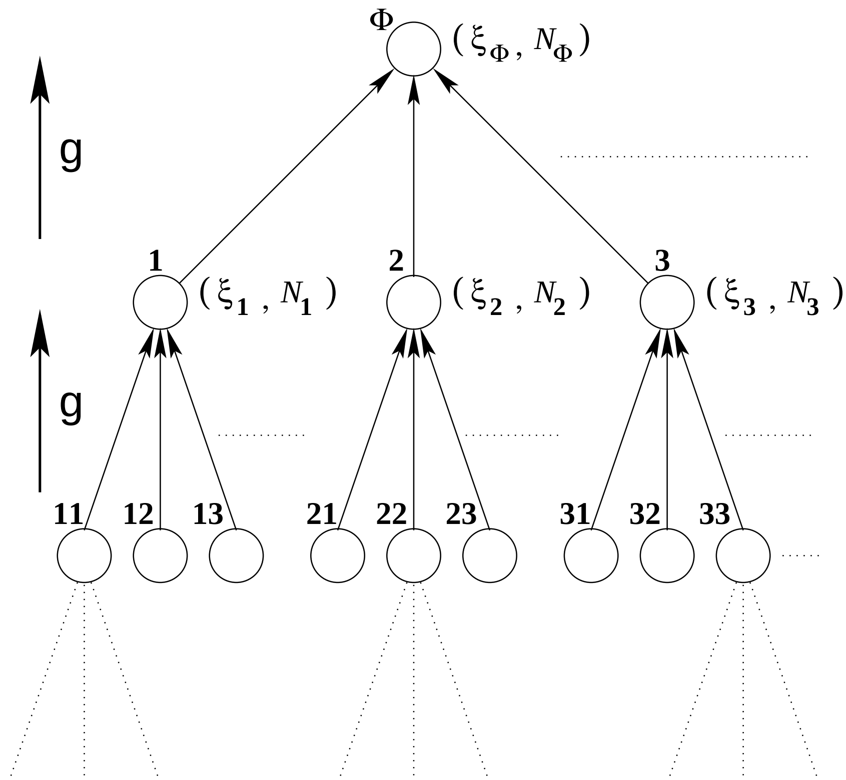
X33

Recursive Tree Framework (RTF)

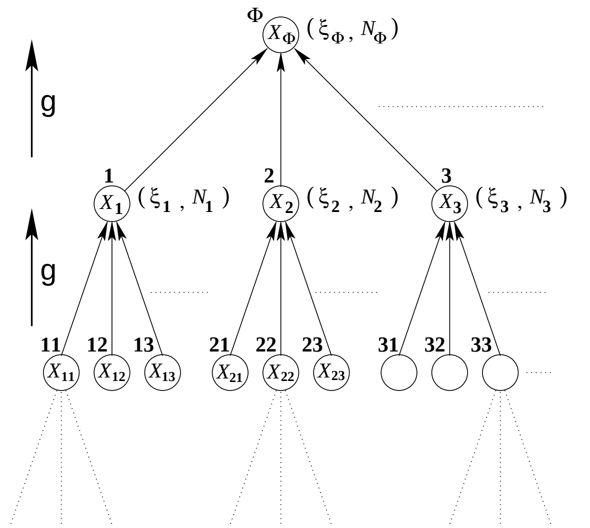


- **Skeleton :** $\mathbb{T}_\infty := (\mathcal{V}, \mathcal{E})$ is the canonical infinite tree with vertex set $\mathcal{V} := \left\{ \mathbf{i} \,|\, \mathbf{i} \in \mathbb{N}^d,\ d \geq 1 \right\} \cup \{\emptyset\}$, and edge set $\mathcal{E} := \{ e = (\mathbf{i}, \mathbf{i}j) \,|\, \mathbf{i} \in \mathcal{V}, j \in \mathbb{N} \}$, and root $\emptyset$.

- **Innovations :** Collection of **i.i.d** pairs $\{(\xi_\mathbf{i}, N_\mathbf{i}) \,|\, \mathbf{i} \in \mathcal{V}\}$.

- **Function :** The function $g(\cdot)$.

Recursive Tree Process (RTP)



Consider a **RTF** and let μ be a solution of the associated **RDE** . A collection of S-valued random variables $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ is called an invariant *Recursive Tree Process (RTP)* with marginal μ if

- $X_{\mathbf{i}} \sim \mu \ \forall\ \mathbf{i} \in \mathcal{V}$.
- Fix $d \geq 0$ then $(X_{\mathbf{i}})_{|\mathbf{i}|=d}$ are independent.
- $X_{\mathbf{i}} = g\left(\xi_{\mathbf{i}}; X_{\mathbf{i}j}, 1 \leq j \leq^* N_{\mathbf{i}}\right)$ a.s. $\forall\ \mathbf{i} \in \mathcal{V}$.
- $X_{\mathbf{i}}$ is independent of $\{(\xi_{\mathbf{i}'}, N_{\mathbf{i}'}) \mid |\mathbf{i}'| < |\mathbf{i}|\,\}\ \ \forall\ \mathbf{i} \in \mathcal{V}$.

Remark : Using *Kolmogorov's consistency*, an invariant RTP with marginal μ exists if and only if μ is a solution of the associated RDE.

Influence of Infinite Boundary at the Root

A Mathematically Natural Question : Is there a possible influence of the *boundary at infinity* on the root value $X_\emptyset$ of a RTP ?

Two Extreme Cases :

1. Recall the Example 1, the height of a (sub)-critical Galton-Watson tree.

$$H \stackrel{d}{=} 1 + \max\left(H_1, H_2, \ldots, H_N\right) \quad \text{on} \;\; \mathbb{N},$$

   Observation : The RTP lives a.s. on a finite tree.

   Intuition : There should not be any influence of infinity at the root.

2. Now consider the following example

$$X \stackrel{d}{=} \frac{X_1 + X_2}{\sqrt{2}} \;\; \text{on} \;\; \mathbb{R}.$$

   Observation : The solution set is the Normal $\left(0, \sigma^2\right)$ family. But the associated RTF has no randomness, because the innovation process is non-random.

   Intuition : All the randomness must be coming from infinity !

Two Rigorous Notions

- **Endogeny :**

  Idea : If the root value $X_\emptyset$ only depends on the innovation process (the *data*), namely, $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$.

  Definition 2 *Let $\mathcal{G}$ be the σ-field generated by the* innovation process $\{ (\xi_{\mathbf{i}}, N_{\mathbf{i}}) \mid \mathbf{i} \in \mathcal{V} \}$. *We will say an invariant RTP is* endogenous *if $X_\emptyset$ is* almost surely *$\mathcal{G}$-measurable.*

- **Tail-Triviality :**

  Idea : If the tail σ-algebra of the RTP $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ is trivial.

  Definition 3 *Let*

$$\mathcal{H}_n := \sigma\left(\{X_{\mathbf{i}} \mid |\mathbf{i}| \geq n\}\right),$$

  then the tail σ-algebra of the RTP is defined as

$$\mathcal{H} = \bigcap_{n\geq 0} \mathcal{H}_n.$$

  An invariant RTP with marginal μ is called tail-trivial *is the σ-filed $\mathcal{H}$ is trivial.*

Two *not so difficult* **Facts**

- **Observation :** Associated with a RTF there is a Galton-Watson branching process tree rooted at $\emptyset$ defined only through $\{N_{\mathbf{i}} \,|\, \mathbf{i} \in \mathcal{V}\}$, call it $\mathcal{T}$. Essentially any associated invariant RTP lives on $\mathcal{T}$.

  Proposition 1 *If $\mathcal{T}$ is almost surely finite (equivalently $\mathbf{E}[N] \leq 1$ and $\mathbf{P}(N = 1) < 1$) then the associated RDE has unique solution and the RTP is endogenous.*

  Remark : The RDEs in the first two examples have unique solutions and are endogenous.

- **Proposition 2** *If an invariant RTP is endogenous then it must also have a trivial tail.*

  Remark : Thus tail-triviality of an invariant RTP is weaker than endogeny, but it can be useful to prove *non-endogeny*.

What about the Converse of Proposition 2 ?

Answer : The converse is not true !

Counter Example :

- Recall the Example 3,

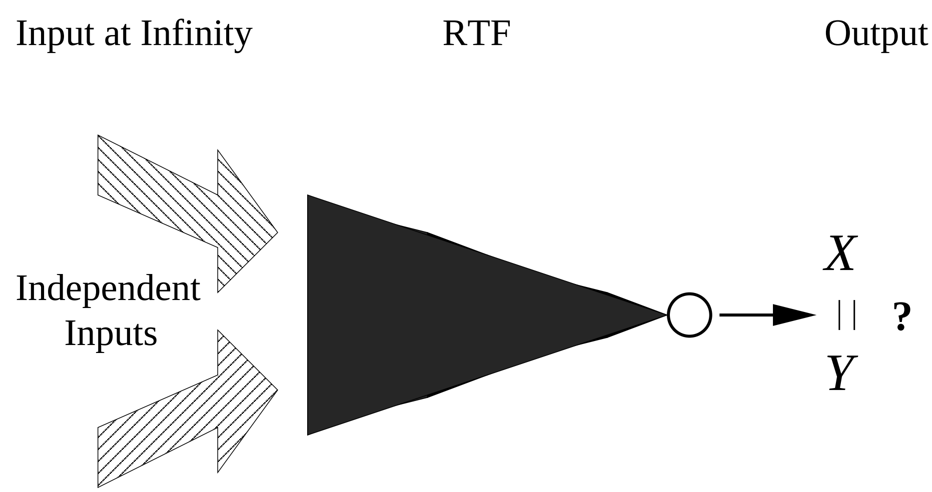
$$X_i = \xi_i + X_{i+1} \pmod 2\,,$$

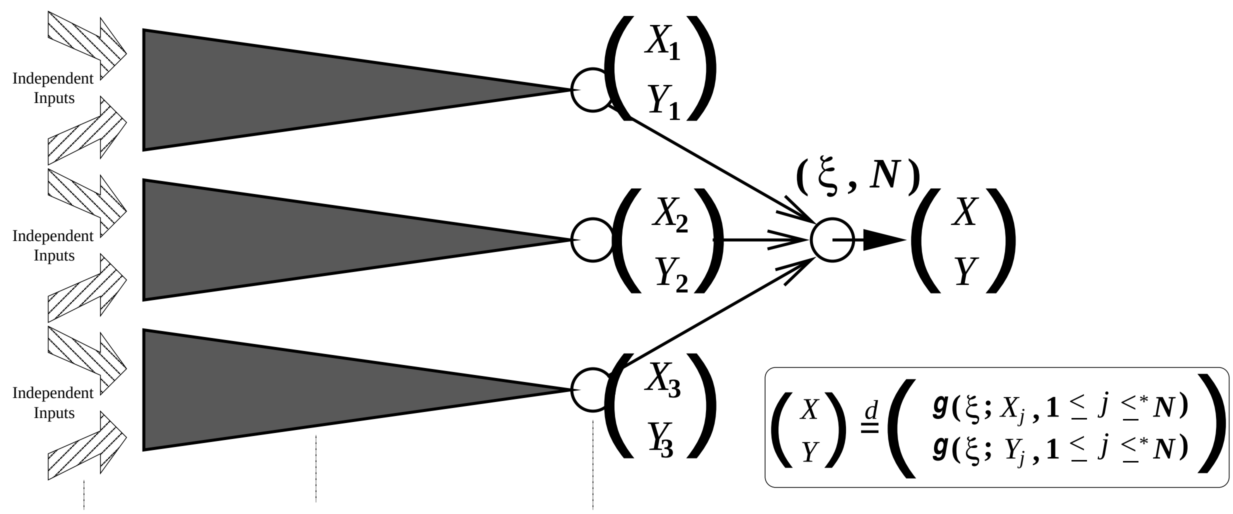
  where $(\xi_i)_{i \geq 0}$ are i.i.d. Bernoulli(q), and X_{i+1} is independent of $(\xi_0, \xi_1, \ldots, \xi_i)$ for all $i \geq 0$.

- It is easy to see that X_0 which is the root variable is independent of the innovation process $(\xi_i)_{i \geq 0}$. Thus it is not endogenous.

- On the other it is not difficult to show that it has a trivial tail !

One Possible Way to Determine Influence of Infinity





Bivariate Uniqueness of the First Kind

Consider the following **bivariate RDE**,

$$\begin{pmatrix} X \\ \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g\left(\xi; \left(X_j, 1 \le j \le^* N\right)\right) \\ \\ g\left(\xi; \left(Y_j, 1 \le j \le^* N\right)\right) \end{pmatrix}$$

where $(X_j, Y_j)_{j \ge 1}$ are i.i.d and have the same joint law as of (X, Y), and are independent of the innovation (ξ, N).

Definition 4 *An invariant RTP with marginal* μ *has* **bivariate uniqueness** *property of the* **first kind** *if the above* bivariate RDE *has unique solution as* $X = Y$ *a.s on the space of joint probabilities with both marginals* μ*.*

The First Equivalence Theorem

Theorem 1 *Suppose the S is a* Polish *space. Consider an invariant RTP with marginal distribution μ.*

(a) If the endogenous property holds then the bivariate uniqueness property of the first kind holds.

(b) Conversely, (under some technical condition) if the bivariate uniqueness property of the first kind holds then the endogenous property holds.

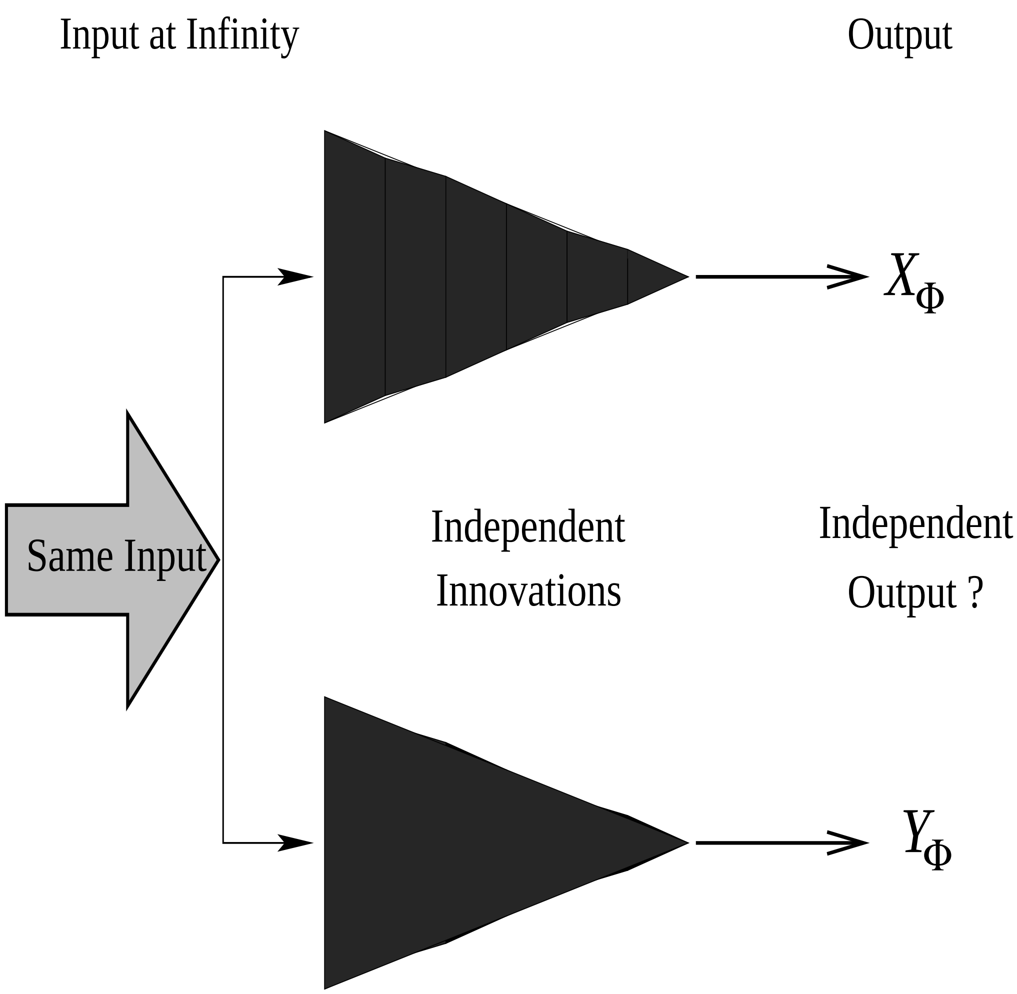
(c) If $T^{(2)}$ be the operator associated with the bivariate RDE *then endogenous property holds if and only if*

$$T^{(2)^n}(\mu \otimes \mu) \stackrel{d}{\longrightarrow} \mu^{\nearrow},$$

where $\mu \otimes \mu$ is the product measure, and $\mu^{\nearrow}$ is the measure concentrated on the diagonal with both marginals μ.

Remark : Recently Christophe Leuridan and Jean Brossard communicated to us that the technical condition in part (b) can be removed.

Another Possible Way to Determine Influence of Infinity



Bivariate Uniqueness of the Second Kind

Now consider the following **bivariate RDE**,

$$\begin{pmatrix} X \\ \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g\left(\xi; \left(X_j, 1 \leq j \leq^* N\right)\right) \\ \\ g\left(\eta; \left(Y_j, 1 \leq j \leq^* M\right)\right) \end{pmatrix}$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d and have the same joint law as of (X, Y), and are independent of the innovations (ξ, N) and (η, M), which are i.i.d.

Definition 5 *An invariant RTP with marginal* μ *has* **bivariate uniqueness** *property of the* **second kind** *if the above* bivariate RDE *has unique solution* $\mu \otimes \mu$*, on the space of joint probabilities with both marginals* μ*.*

The Second Equivalence Theorem

Theorem 2 (B. (2006)) *Suppose S is a* Polish *space. Consider an invariant RTP with marginal distribution μ.*

(a) If the RTP has a trivial tail then the bivariate uniqueness property of the second kind holds.

(b) Conversely, (under some technical condition) if the bivariate uniqueness property of the second kind holds then the tail of the RTP is trivial.

(c) If $T \otimes T$ be the operator associated with the bivariate RDE *then the RTP has trivial tail if and only if*

$$(T \otimes T)^n \left(\mu^{\nearrow}\right) \stackrel{d}{\longrightarrow} \mu \otimes \mu,$$

where $\mu^{\nearrow}$ is the measure concentrated on the diagonal with both marginals μ.

Applications of Endogeny/Tail-Triviality

- **Characterization :** Some time the RDE may have many solutions but only one of them (*the fundamental solution*) is endogenous.

  ▶ In case of the *Quicksort RDE* (Example 4) only the limiting *Quicksort distribution* is endogenous.

  [We will not discuss any details of this example.]

- **In proving limit theorems :** In certain combinatorial optimization problem over random data, where the limiting structure is a (random) tree, endogeny is technically helpful in deriving limit results.

  ▶ *Mean-field random assignment problem* $\leftrightarrow$ Logistic RDE (Example 5).

  [We will briefly discuss this example.]

- **To prove measurability of a process :** If a process is constructed using the consistency theorem then endogeny basically helps to resolve the measurability question.

  ▶ Frozen percolation process on infinite regular trees (Example 6).

  [We will discuss this example and see what we can achieve.]

Application of Endogeny
Back to the Logistic RDE

$$X \stackrel{d}{=} \min_{j \geq 1} \left(\xi_j - X_j \right) \quad \text{on } \mathbb{R},$$

where $(X_j)_{j \geq 1}$ are i.i.d. with same distribution as X and are independent of $(\xi_j)_{j \geq 1}$ which are points of a Poisson point process of rate 1 on $(0, \infty)$.

Remarks :

- This RDE is the key to derive the asymptotic limit for the *mean-field random assignment problem*.
- In fact the RTP associated with this RDE helps to construct the limiting optimal solution.
- For this example we can successfully use the first equivalence theorem.

Theorem 3 (B. (2002)) *The bivariate uniqueness property of the first kind holds for the Logistic RDE, thus the associated invariant RTP is endogenous.*

Brief Digression to Frozen Percolation Process on the Infinite 3-Regular Tree

The Setup :

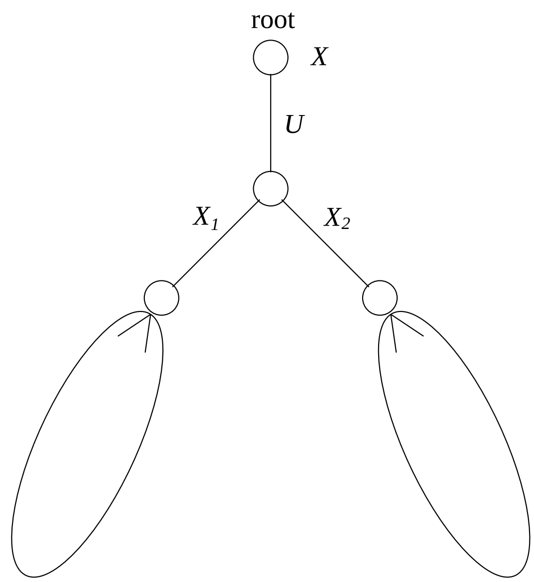
- Let $\mathbb{T}_3 = (\mathbb{V}, \mathbb{E})$ be the infinite regular binary tree.
- Each edge $e \in \mathbb{E}$ is equipped with independent edge weight $U_e \sim$ Uniform$[0, 1]$.
- Think of time moving from 0 to 1.

Frozen Percolation Process (informal description):

- For an edge $e \in \mathbb{E}$ at the time instance $t = U_e$ open the edge e if each of its end vertex is in a finite component; otherwise do not open e.
- Let $(\mathcal{A}_t)_{t \geq 0}$ be set process of open edges starting from $\mathcal{A}_0 = \emptyset$.

A 540° **Argument** [Aldous, 2000]

- **Stage 1 :** Suppose that the process exists on $\mathbb{T}_3$.



- $X :=$ Time it takes for the root to join ∞ (will write $X = \infty$ if it never joins).
- $X_j :=$ Time it takes for the root to join to ∞ in the j^{th} sub-tree for $j = 1, 2$.
- X_1 and X_2 are independent copies of X.
- It is easy to see that

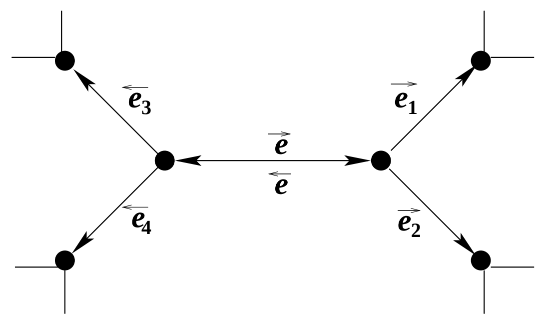
$$X \stackrel{d}{=} \begin{cases} X_1 \wedge X_2 & \text{if } X_1 \wedge X_2 > U \\ \infty & \text{otherwise} \end{cases}$$

- **Stage 2 :**

  - ▶ The RDE has only one solution with full support given by

$$\nu(dy) = \frac{dy}{2y^2}, \ \frac{1}{2} < y < 1, \ \nu(\{\infty\}) = \frac{1}{2}.$$

    So using the general theory we can construct the invariant RTP with marginal ν.



  - ▶ Each edge $e \in \mathbb{E}$ defines two directed edges, and each directed edge $\overrightarrow{e}$ defines one *planted* tree, let $X_{\overrightarrow{e}}$ be the corresponding root value of the RTP.

- **Stage 3 :** Using this *external* random variables $\left(X_{\overrightarrow{e}}\right)$ repeat the original computation to prove the existence of a frozen percolation process on $\mathbb{T}_3$. In fact this gives an automorphism invariant version of the process.

Remarks :

- The construction of the process not only uses the edge weights (U_e) but also (possibly) *external* random variables from the RTPs, namely $\left(X_{\overrightarrow{e}}\right)$.

- Endogeny in this case will prove the measurability of the frozen percolation process with respect to the i.i.d. Uniform$[0, 1]$ edge weights.

What we can do :

Theorem 4 (B. (2006)) *The bivariate uniqueness property of the second kind holds for the solution ν of the frozen percolation RDE, thus the associated invariant RTP with marginal ν has trivial tail.*

What we have not been able to do :

- Above result does not resolve the question of endogeny.

- The analysis seems to be too hard for resolving the endogeny question using the first equivalence theorem.

- Simulations strongly suggest *non-endogeny* !

Some Related Future Directions

- Find some more "interesting" and/or "natural" examples where we have trivial tail for the RTP but it is not endogenous.
- Can we characterize such RTPs ?
- How does the conditional distribution of $X_\emptyset$ given $\mathcal{G}$ look like for such a RTP ?