Tail of a Recursive Tree Process : Application to Frozen Percolation on Regular Trees

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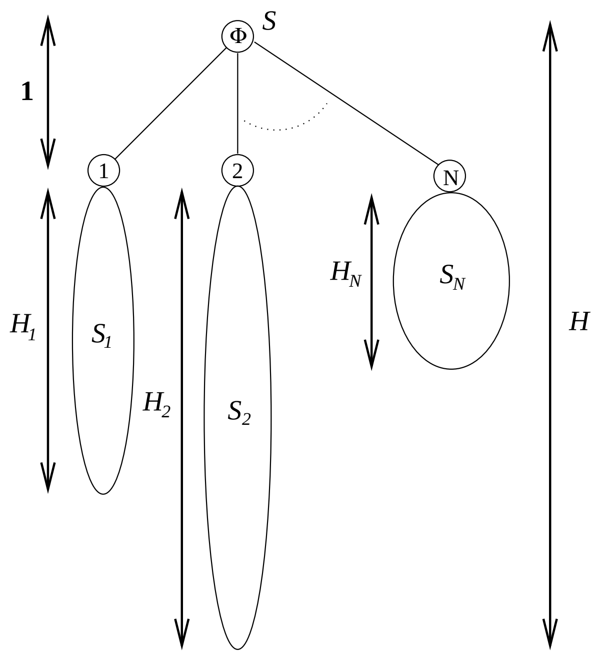
Three Examples

Examples 1 : Consider a *(sub)-critical* Galton-Watson branching process with the progeny distribution N, so $\mathbf{E}[N] \leq 1$; we assume $\mathbf{P}(N = 1) < 1$.

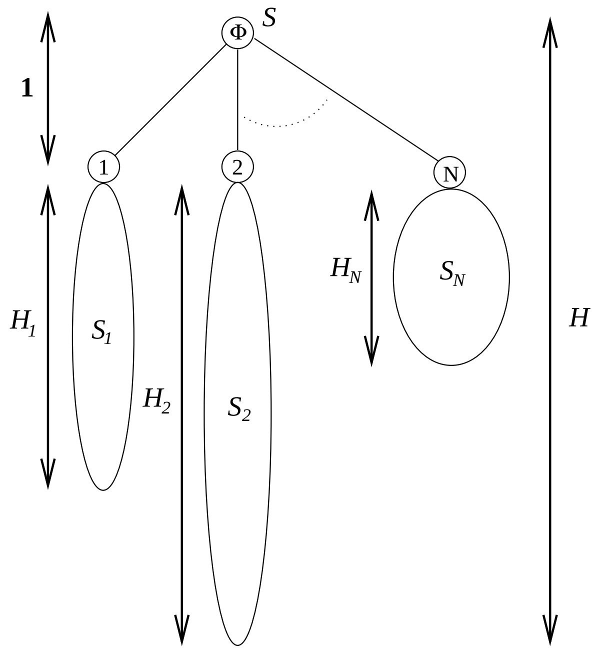


Height of the Tree : Let $H := 1+$ height of the G-W tree, then $H < \infty$ a.s. and

$$H \stackrel{d}{=} 1 + \max(H_1, H_2, \ldots, H_N) \quad \text{on } \mathbb{N},$$

where $(H_j)_{j \geq 1}$ are i.i.d. with same law as of H and are independent of N.

Examples 2 : Consider the same *(sub)-critical* Galton-Watson branching process.



Size of the Tree : Let $S :=$ total size of the tree. Once again $S < \infty$ a.s. since the process is (sub)-critical. Further

$$S \stackrel{d}{=} 1 + (S_1 + S_2 + \cdots + S_N) \quad \text{on } \mathbb{N},$$

where $(S_j)_{j \geq 1}$ are i.i.d. with same law as of S and are independent of N.

We will call such equations *Recursive Distributional Equations* (RDE).

Example 3 : Fix $0 < q < 1$ and consider the following process

$$X_i = \xi_i + X_{i+1} \pmod 2,$$

where $(\xi_i)_{i \geq 0}$ are i.i.d. Bernoulli(q) and X_{i+1} is independent of $(\xi_0, \xi_1, \ldots, \xi_i)$ for all $i \geq 0$.

Remarks :

- The process $(X_i)_{i \geq 0}$ exists provided the following RDE has a solution :

$$X \stackrel{d}{=} \xi + X_1 \pmod 2 \quad \text{on} \quad \{0, 1\},$$

  where $\xi \sim$ Bernoulli(q) and is independent of X_1 which has same distribution as of X.

- It is easy to see that the RDE has unique solution given by $X \sim$ Bernoulli $\left(\frac{1}{2}\right)$.

- Note that $(X_i)_{i \geq 0}$ is nothing but a stationary Markov chain when time is reversed.

Typical features of RDEs

Ex. 1 : $X \stackrel{d}{=} 1 + \max(X_1, X_2, \ldots, X_N)$ on $\mathbb{N}$

Ex. 2 : $X \stackrel{d}{=} 1 + (X_1 + X_2 + \cdots + X_N)$ on $\mathbb{N}$

Ex. 3 : $X \stackrel{d}{=} \xi + X_1 \pmod 2$ on $\{0, 1\}$

- **Unknown Quantity :** Distribution of X.
- **Known Quantities :**
  - $N \leq \infty$ which may or may not be random (e.g. $N \equiv 1$ in Ex. 3).
  - Possibly some more randomness whose distribution is known (e.g. ξ in the Ex. 3).
  - How we combine the known and unknown randomness (e.g. "$1 + \max$" operation in Ex. 1).
- **What is the RDE doing ?** To find a distribution μ such that when we take i.i.d. samples $(X_j)_{j \geq 1}$ from it and only use N many of them (where N is independent of the samples) and do the manipulation then we end up with another sample $X \sim \mu$.

Remark : In the case $N = 1$ a.s. it reduces to the question of finding a stationary distribution of a discrete time Markov chain.

Two main uses of RDEs

- **Direct use :** The RDE is used directly to define a distribution. Examples include,
  - ▶ The height (and also the size) of a (sub)-critical Galton-Watson tree (the first two examples).
  - ▶ The Quicksort distribution (not discussed here).
  - ▶ Discounted tree sums / inhomogeneous percolation on trees (not discussed here).
  - ▶ ... *and many others*.

- **Indirect use:** The RDE is used to define some auxiliary variables which help in defining/characterizing some other quantity of interest. Among others the following two type of applications are of special interest
  - ▶ 540° *argument* ! (will give an example).
  - ▶ Determining critical points and scaling laws (will not give an example).

General Setup

- Let $(S, \mathfrak{S})$ be a measurable space, and $\mathcal{P}$ be the collection of all probabilities on $(S, \mathfrak{S})$.

- Let (ξ, N) be a pair of random variables such that N takes values in $\{0, 1, 2, \ldots ; \infty\}$.

- Let $(X_j)_{j \geq 1}$ be **i.i.d** S-valued random variables, which are independent of (ξ, N).

- $g(\cdot)$ is a S-valued measurable function with appropriate domain.

Recursive Distributional Equation (RDE)

Definition 1 *The following* fixed-point equation *on* $\mathcal{P}$ *is called a* Recursive Distributional Equation (RDE)

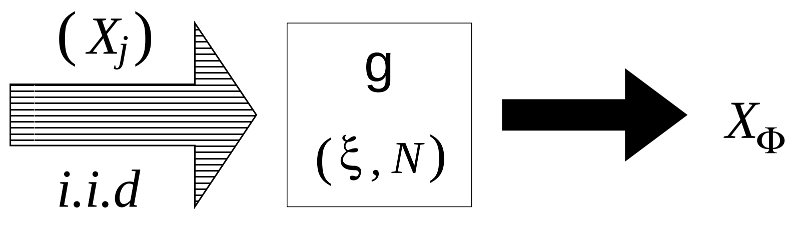
$$X \stackrel{d}{=} g\left(\xi; \left(X_j, 1 \leq j \leq^* N\right)\right) \quad \textit{on} \;\; S,$$

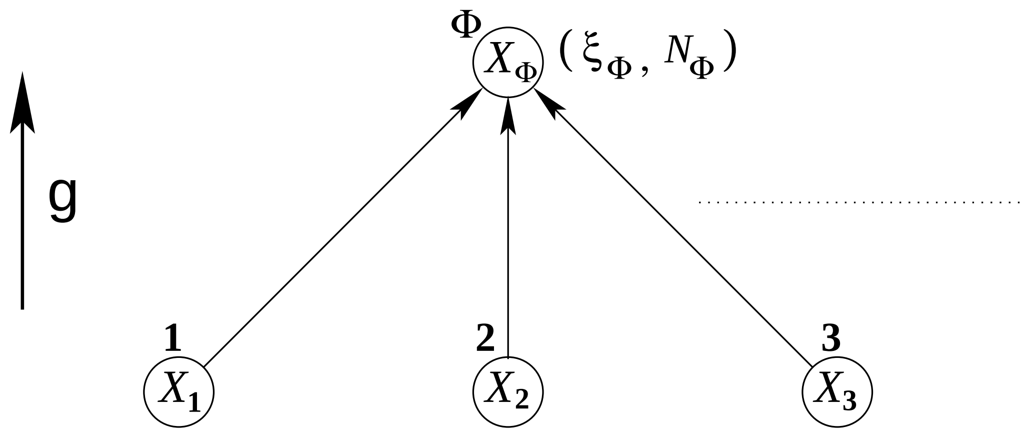
where $(X_j)_{j \geq 1}$ *are independent copies of* X *and are independent of* (ξ, N).

Remark : A more conventional (analysis) way of writing the equation would be

$$\mu = T(\mu)$$

where T is the operator associated with the above equation, which depends on the function g and the joint distribution of the pair (ξ, N), and μ is the (unknown) law of X.





Φ
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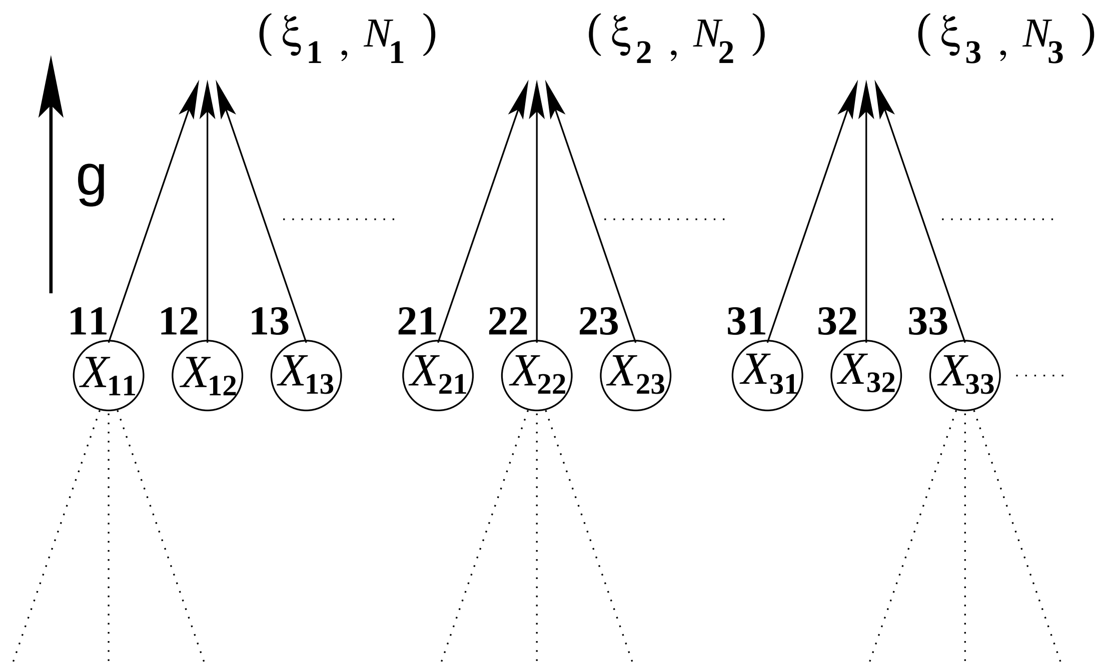
(ξ1 , N1)
(ξ2 , N2)
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g
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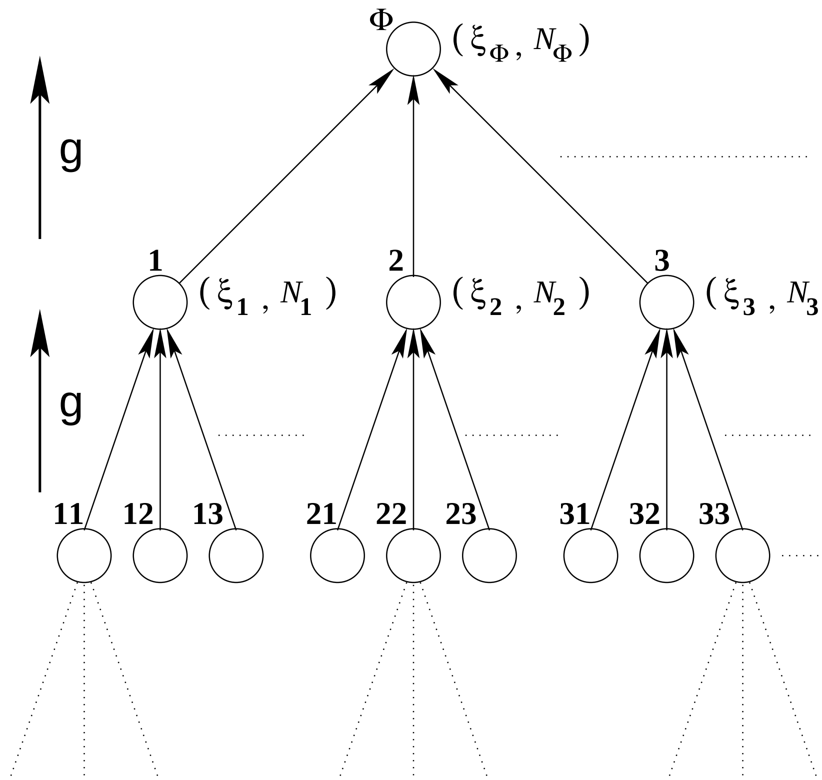
Recursive Tree Framework (RTF)

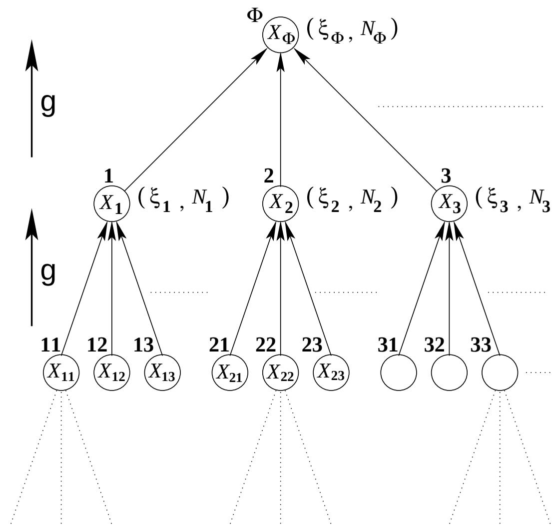


- **Skeleton :** $\mathbb{T}_\infty := (\mathcal{V}, \mathcal{E})$ is the canonical infinite tree with vertex set $\mathcal{V} := \left\{ \mathbf{i} \,|\, \mathbf{i} \in \mathbb{N}^d, \, d \geq 1 \right\} \cup \{\emptyset\}$, and edge set $\mathcal{E} := \{ e = (\mathbf{i}, \mathbf{i}j) \,|\, \mathbf{i} \in \mathcal{V}, j \in \mathbb{N} \}$, and root $\emptyset$.

- **Innovations :** Collection of **i.i.d** pairs $\{(\xi_\mathbf{i}, N_\mathbf{i}) \,|\, \mathbf{i} \in \mathcal{V}\}$.

- **Function :** The function $g(\cdot)$.

Recursive Tree Process (RTP)



Consider a **RTF** and let μ be a solution of the associated **RDE** . A collection of S-valued random variables $(X_{\mathbf{i}})_{\mathbf{i}\in\mathcal{V}}$ is called an invariant *Recursive Tree Process (RTP)* with marginal μ if

- $X_{\mathbf{i}} \sim \mu \ \forall \ \mathbf{i} \in \mathcal{V}$.
- Fix $d \geq 0$ then $(X_{\mathbf{i}})_{|\mathbf{i}|=d}$ are independent.
- $X_{\mathbf{i}} = g\left(\xi_{\mathbf{i}}; X_{\mathbf{i}j}, 1 \leq j \leq^* N_{\mathbf{i}}\right)$ a.s. $\forall \ \mathbf{i} \in \mathcal{V}$.
- $X_{\mathbf{i}}$ is independent of $\{(\xi_{\mathbf{i}'}, N_{\mathbf{i}'}) \mid |\mathbf{i}'| < |\mathbf{i}| \}$ $\forall \ \mathbf{i} \in \mathcal{V}$.

Remark : Using *Kolmogorov's consistency*, an invariant RTP with marginal μ exists if and only if μ is a solution of the associated RDE.

Influence of Infinite Boundary at the Root

Question : Is there a possible influence of the *boundary at infinity* on the root value $X_\emptyset$ of a RTP ?

Two Extreme Cases :

1. Recall the Example 1, the height of a (sub)-critical Galton-Watson tree.

$$H \stackrel{d}{=} 1 + \max\left(H_1, H_2, \ldots, H_N\right) \quad \text{on } \mathbb{N},$$

   Observation : The RTP lives a.s. on a finite tree.

   Intuition : There should not be any influence of infinity at the root.

2. Now consider the following example

$$X \stackrel{d}{=} \frac{X_1 + X_2}{\sqrt{2}} \text{ on } \mathbb{R}.$$

   Observation : The solution set is the Normal $\left(0, \sigma^2\right)$ family. But the associated RTF has no randomness, because the innovation process is non-random.

   Intuition : All the randomness must be coming from infinity !

Two Rigorous Notions

- **Endogeny :**

  Idea : If the root value $X_\emptyset$ only depends on the innovation process (the *data*), namely, $(\xi_{\mathbf{i}}, N_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$.

  Definition 2 *Let $\mathcal{G}$ be the σ-field generated by the* innovation process $\{\,(\xi_{\mathbf{i}}, N_{\mathbf{i}})\,|\,\mathbf{i} \in \mathcal{V}\,\}$. *We will say an invariant RTP is* endogenous *if $X_\emptyset$ is* almost surely *$\mathcal{G}$-measurable.*

- **Tail-Triviality :**

  Idea : If the tail σ-algebra of the RTP $(X_{\mathbf{i}})_{\mathbf{i} \in \mathcal{V}}$ is trivial.

  Definition 3 *Let*

  $$\mathcal{H}_n := \sigma\left(\{X_{\mathbf{i}}\,|\,|\mathbf{i}| \geq n\,\}\right),$$

  then the tail σ-algebra of the RTP is defined as

  $$\mathcal{H} = \bigcap_{n \geq 0} \mathcal{H}_n.$$

  An invariant RTP with marginal μ is called tail-trivial *is the σ-filed $\mathcal{H}$ is trivial.*

Two “not so difficult” Facts

- **Observation :** Associated with a RTF there is a Galton-Watson branching process tree rooted at $\emptyset$ defined only through $\{N_{\mathbf{i}} \mid \mathbf{i} \in \mathcal{V}\}$, call it $\mathcal{T}$. Essentially any associated invariant RTP lives on $\mathcal{T}$.

  Proposition 1 *If $\mathcal{T}$ is almost surely finite (equivalently $\mathbf{E}[N] \leq 1$ and $\mathbf{P}(N = 1) < 1$) then the associated RDE has unique solution and the RTP is endogenous.*

  Remark : The RDEs in the first two examples have unique solutions and are endogenous.

- **Proposition 2** *If an invariant RTP with marginal μ is endogenous then it must also have a trivial tail.*

What about the Converse of Proposition 2 ?

Answer : The converse is not true !

Counter Example :

- Recall the Example 3,

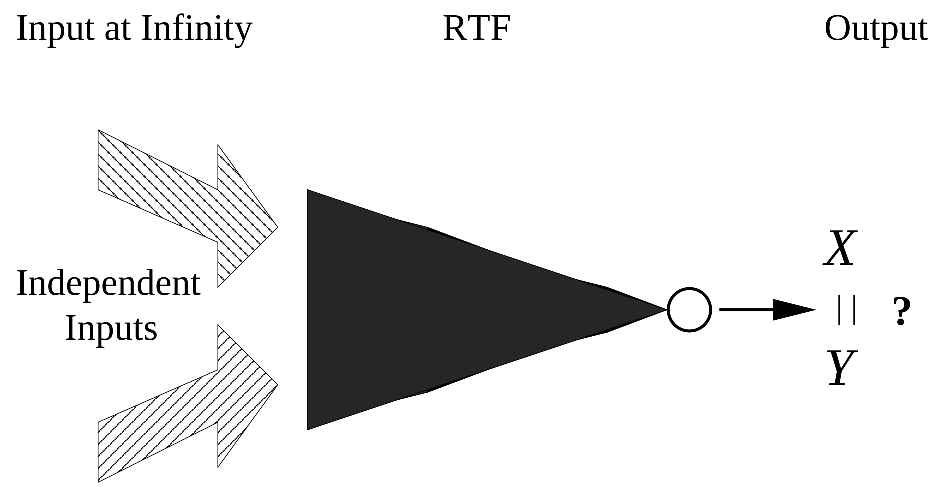
$$X_i = \xi_i + X_{i+1} \pmod 2\,,$$

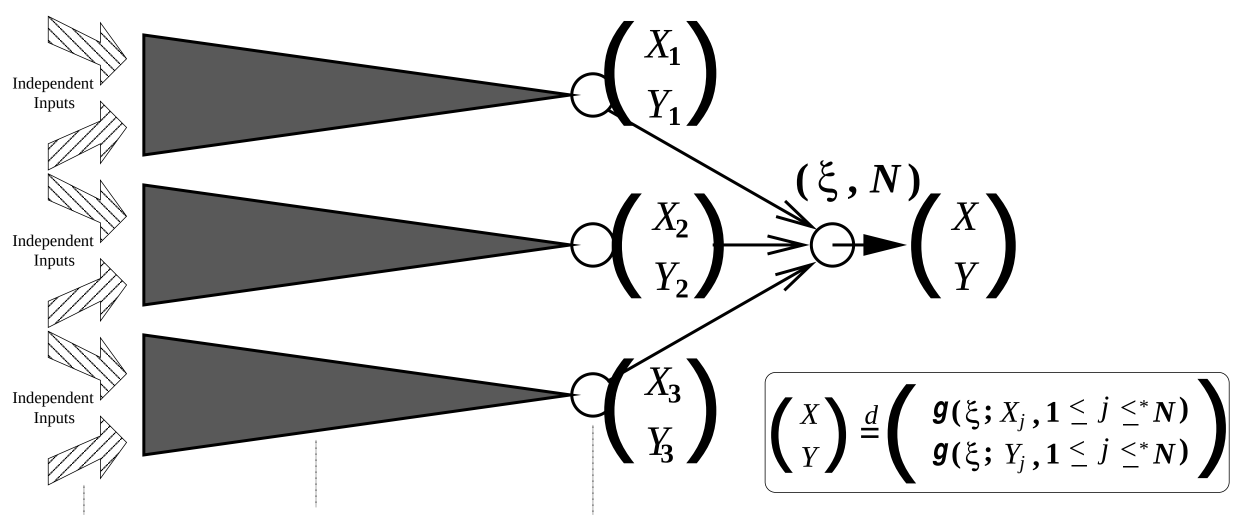
  where $(\xi_i)_{i\geq 0}$ are i.i.d. Bernoulli(q), and X_{i+1} is independent of $(\xi_0, \xi_1, \ldots, \xi_i)$ for all $i \geq 0$.

- It is easy to see that X_0 which is the root variable is independent of the innovation process $(\xi_i)_{i\geq 0}$. Thus it is not endogenous.

- On the other it is not difficult to show that it has a trivial tail !

One Possible Way to Determine Influence of Infinity





Bivariate Uniqueness of First Kind

Consider the following **bivariate RDE**,

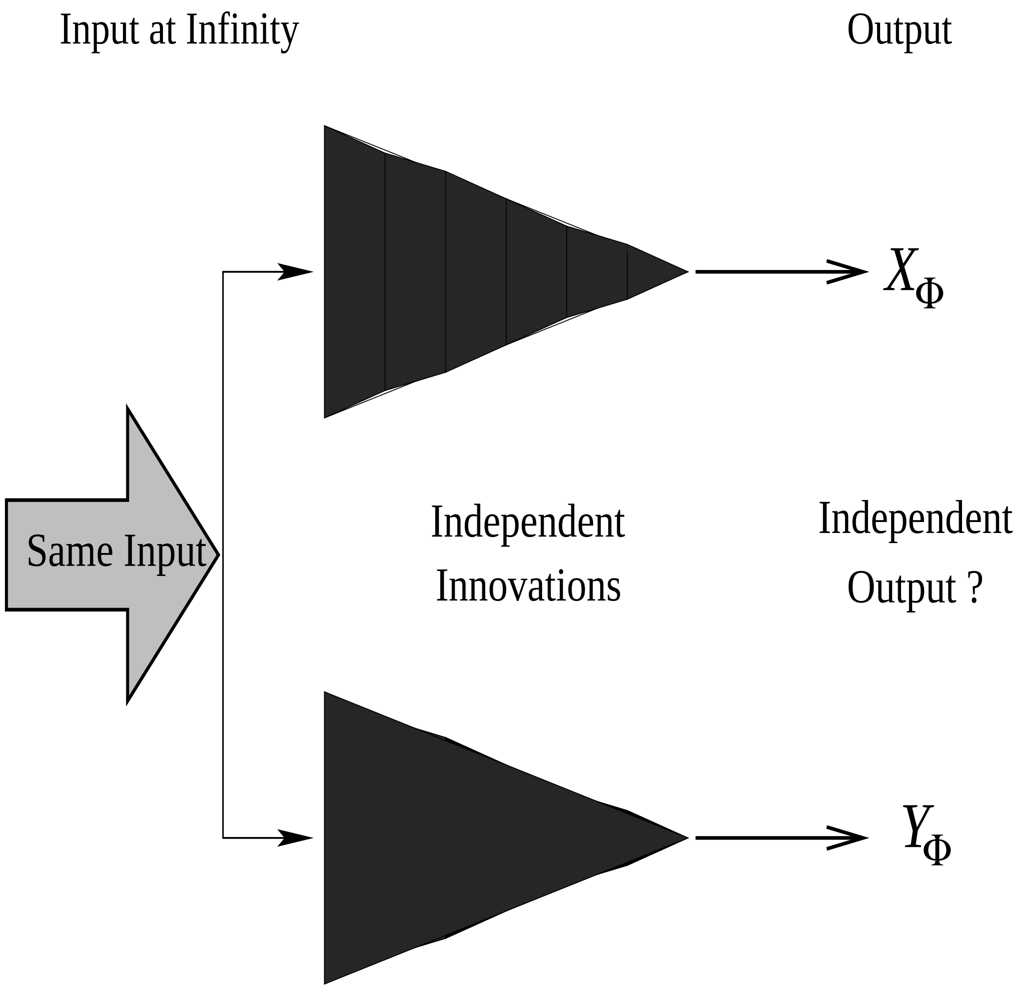
$$\begin{pmatrix} X \\ \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g\left(\xi; \left(X_j, 1 \leq j \leq^* N\right)\right) \\ \\ g\left(\xi; \left(Y_j, 1 \leq j \leq^* N\right)\right) \end{pmatrix}$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d and has the same law as of (X, Y), and are independent of the innovation (ξ, N).

Definition 4 *An invariant RTP with marginal μ has* **bivariate uniqueness** *property of the* **first kind** *if the above* bivariate RDE *has unique solution as $X = Y$ a.s on the space of joint probabilities with both marginals μ.*

Theorem 1 (Aldous and B. (2005)) *Suppose S is a* Polish *space. Consider an invariant RTP with marginal distribution μ. Then RTP is endogenous if and only if, the bivariate uniqueness property of the first kind holds.*

Another Possible Way to Determine Influence of Infinity



Bivariate Uniqueness of Second Kind

Now consider the following **bivariate RDE**,

$$\begin{pmatrix} X \\ \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} g\left(\xi; \left(X_j, 1 \leq j \leq^* N\right)\right) \\ \\ g\left(\eta; \left(Y_j, 1 \leq j \leq^* M\right)\right) \end{pmatrix}$$

where $(X_j, Y_j)_{j \geq 1}$ are i.i.d and has the same law as of (X, Y), and are independent of the innovations (ξ, N) and (η, M), which are i.i.d.

Definition 5 *An invariant RTP with marginal μ has* **bivariate uniqueness** *property of the* **second kind** *if the above* bivariate RDE *has unique solution $\mu \otimes \mu$, on the space of joint probabilities with both marginals μ.*

Second Equivalence Theorem

Theorem 2 (B. (2006)) *Suppose S is a* Polish *space. Consider an invariant RTP with marginal distribution μ.*

(a) If the RTP has a trivial tail then the bivariate uniqueness property of the second kind holds.

(b) Conversely, (under some technical conditions) if the bivariate uniqueness property of the second kind holds then the tail of the RTP is trivial.

(c) If $T \otimes T$ be the operator associated with the bivariate RDE *then the RTP has trivial tail if and only if*

$$(T \otimes T)^n \left(\mu^{\nearrow}\right) \stackrel{d}{\longrightarrow} \mu \otimes \mu,$$

where $\mu^{\nearrow}$ is the measure concentrated on the diagonal with both marginal μ.

Remark : This theorem parallels the first equivalence theorem.

Back to Example 3

- The RDE :

$$X \stackrel{d}{=} \xi + X_1 \pmod 2 \quad \text{on} \quad \{0,1\},$$

where X_1 has same distribution as of X and it is independent of $\xi \sim$ Bernoulli(q).

Solution : Unique solution $X \sim$ Bernoulli $\left(\frac{1}{2}\right)$.

- The Second Bivariate Version :

$$\begin{pmatrix} X \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \xi + X_1 \\ \eta + Y_1 \end{pmatrix} \pmod 2 \quad \text{on} \quad \{0,1\}^2,$$

where (X_1, Y_1) is an independent copy of the pair (X, Y) and it is independent of (ξ, η) which are i.i.d. Bernoulli(q).

Solution : The bivariate equation has unique solution given by the product measure

$$\text{Bernoulli}\left(\tfrac{1}{2}\right) \otimes \text{Bernoulli}\left(\tfrac{1}{2}\right).$$

- Thus the RTP has a trivial tail.

Frozen Percolation on Regular Binary Tree

The Setup :

- Let $\mathbb{T}_3 = (\mathbb{V}, \mathbb{E})$ be the infinite regular binary tree.

- Each edge $e \in \mathbb{E}$ is equipped with independent edge weight $U_e \sim \text{Uniform}[0, 1]$.

- Think of time moving from 0 to 1.

Frozen Percolation Process (informal description):

- For an edge $e \in \mathbb{E}$ at the time instance $t = U_e$ open the edge e if each of its end vertex is in a finite component; otherwise do not open e.

- Let $(\mathcal{A}_t)_{t \geq 0}$ be set process of open edges starting from $\mathcal{A}_0 = \emptyset$.

The Regular Percolation Process :

- For an edge $e \in \mathbb{E}$ at the time instance $t = U_e$ open the edge e.

- If $(\mathcal{B}_t)_{t \geq 0}$ be the set process of open edges the it can be described as

$$\mathcal{B}_t = \{ e \in \mathbb{E} \,|\, U_e \leq t \}$$

Remarks : Unlike the regular percolation process it is not clear whether the *frozen percolation process* exists and if so whether it admits a simpler description using only the edge weights.

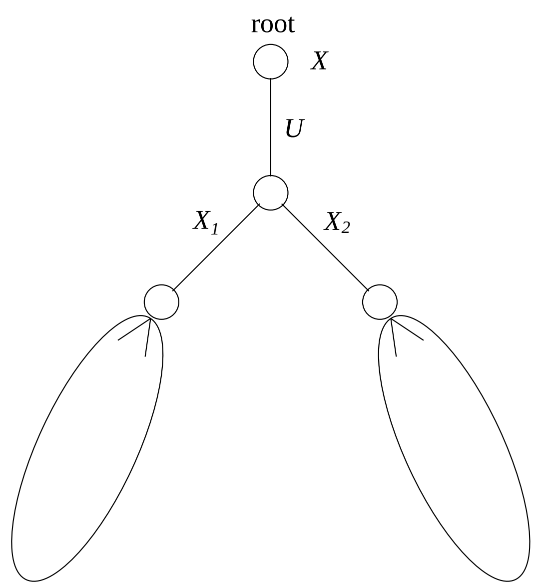
Two Easy Observations : If frozen percolation process exists then following must hold

- $\mathcal{A}_t \subseteq \mathcal{B}_t$ for all $t \in [0, 1]$.

- $\mathcal{A}_t = \mathcal{B}_t$ if $t \leq \frac{1}{2}$ (since the critical probability for infinite binary tree is $\frac{1}{2}$).

540° **Argument** [Aldous, 2000]

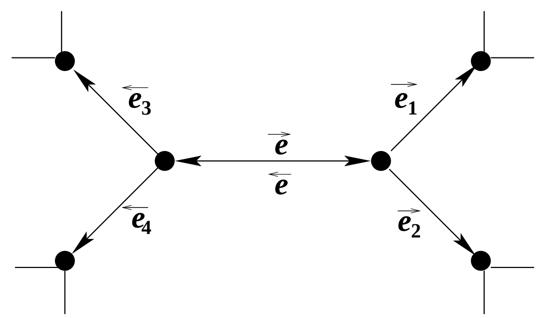
- **Stage 1 :** Suppose that the process exists on $\mathbb{T}_3$. Let $\widetilde{\mathbb{T}_3}$ be the *planted* binary tree which is a modification of $\mathbb{T}_3$ where we distinguish a vertex of degree 1 as the *root* and all other vertices have degree 3.



  - $X :=$ Time it takes for the root to join ∞ (will write $X = \infty$ if it never joins).
  - $X_j :=$ Time it takes for the root to join to ∞ in the j^{th} sub-tree for $j = 1, 2$.
  - X_1 and X_2 are independent copies of X.
  - It is easy to see that

$$X \stackrel{d}{=} \begin{cases} X_1 \wedge X_2 & \text{if } X_1 \wedge X_2 > U \\ \infty & \text{otherwise} \end{cases}$$

- **Stage 2 :**
  - The RDE has only one solution with full support given by
$$\nu(dy) = \frac{dy}{2y^2},\ \frac{1}{2} < y < 1,\ \nu(\{\infty\}) = \frac{1}{2}.$$
So using the general theory we can construct the invariant RTP with marginal ν.



  - Each edge $e \in \mathbb{E}$ defines two directed edges, and each directed edge $\overrightarrow{e}$ defines one *planted* tree, let $X_{\overrightarrow{e}}$ be the corresponding X variable.
  - Each directed edge $\overrightarrow{e}$ has two children say $\overrightarrow{e}_1$ and $\overrightarrow{e}_2$ then $\left\{X_{\overrightarrow{e}_1}, X_{\overrightarrow{e}_2}\right\}$ and $X_{\overrightarrow{e}}$ satisfies the equation with the edge weight U_e.
  - Each edge $e \in \mathbb{E}$ has a set of four *children* which are the four directed edges away from e. We denote it by $\partial\{e\}$.
  - Define $\mathcal{A}_1 := \left\{e \in \mathbb{E} \,|\, U_e < \min\left(X_f : f \in \partial\{e\}\right)\right\}$ and $\mathcal{A}_t := \{e \in \mathcal{A}_1 \,|\, U_e \le t\}$ for $0 \le t < 1$.

- **Stage 3 :** Using this *external* random variables $\left(X_{\overrightarrow{e}}\right)$ repeat the original computation to prove the existence of a frozen percolation process on $\mathbb{T}_3$. In fact it is easy to see that this construction gives an automorphism invariant version of the process.

Remarks :

- The construction of the process not only uses the edge weights (U_e) but also (possibly) *external* random variables, namely $\left(X_{\overrightarrow{e}}\right)$.

- For every $\overrightarrow{e}$ the variable $X_{\overrightarrow{e}}$ is a root value of a invariant RTP with marginal ν.

- Endogeny in this case will prove the measurability of the frozen percolation process on infinite regular binary tree.

- We can show using the second equivalence theorem that the associated RTP has a trivial tail.

- Endogeny remains as an *open problem* !

Frozen Percolation RDE

- Recall the RDE associated with the frozen percolation process,

$$X \stackrel{d}{=} \Phi\left(X_1 \wedge X_2; U\right)$$

where X_1, X_2 are independent copies of X and are independent of $U \sim$ Uniform$[0,1]$ and the function Φ is given by

$$\Phi(x;u) := \begin{cases} x & \text{if } x > u \\ \infty & \text{otherwise} \end{cases}.$$

- Also recall that it has *unique* solution with full support given by

$$\nu(dy) = \frac{dy}{2y^2},\ \frac{1}{2} < y < 1,\ \ \nu(\{\infty\}) = \frac{1}{2}.$$

Theorem 3 (B. (2006)) *The invariant RTP with marginal ν has bivariate uniqueness property of the second kind, that is, the following* bivariate RDE *has* unique *solution given by $\nu \otimes \nu$*

$$\begin{pmatrix} X \\ \\ Y \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \Phi\left(X_1 \wedge X_2; U\right) \\ \\ \Phi\left(Y_1 \wedge Y_2; V\right) \end{pmatrix}$$

where $(X_j, Y_j)_{j=1,2}$ are independent copies of (X, Y), and are independent of (U, V) which are i.id. Uniform$[0, 1]$.

Corollary 3.1 *The invariant RTP with marginal ν has trivial tail.*

Some Future Directions

- Find some more “interesting” and/or “natural” examples where we have trivial tail for the RTP but it is not endogenous.

- Can we characterize such RTPs ?

- How does the conditional distribution of $X_\emptyset$ given $\mathcal{G}$ look like for such a RTP ?