Random Walk in Dynamic Markovian Random Environment

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The Basic Setup

- **Graph :** $\mathbb{Z}^d$, $d \geq 1$ with nearest neighbor links.

- **Two Stages of Randomness :**

  ▶ **The Environment :** It is the transition laws which will tell us *how to take the next step* from the current position.

  Notation : At a site $\mathbf{x} \in \mathbb{Z}^d$ and at time $t \geq 0$ the environment will be denoted by $\omega_t(\mathbf{x}, \cdot)$.

  Note : These laws can be random !

  ▶ **The Walk :** Given the environment we have an *walker* who moves on the lattice $\mathbb{Z}^d$ starting from $\mathbf{0}$ according to the transition laws.

  Notation : X_t is the position of the walker at time t.

  Note : The walker provides second stage of randomness.

More Precise Viewpoint
Quenched and Annealed Laws

- **Quenched :** The conditional law of the walk given realization of the environment.

  Note : The walk is a (possibly) time inhomogeneous Markov chain under this law.

  Notation : Given the entire environment

  $$\omega := \left\{ \left(\omega_t\left(\mathbf{x},\cdot\right)\right)_{t\geq 0} \,\middle|\, \mathbf{x} \in \mathbb{Z}^d \right\},$$

  the *quenched law* of $(X_t)_{t\geq 0}$ starting from $\mathbf{x}$ will be denoted by $\mathbf{P}_\omega^{\mathbf{x}}$.

- **Annealed :** The marginal distribution of the walk, that is, integrating out the *quenched* law with respect to the environment distribution.

  Note : The walk may not be a Markov chain under the annealed law.

  Notation : The *annealed law* of $(X_t)_{t\geq 0}$ starting from $\mathbf{x}$ will be denoted by $\mathbb{P}^{\mathbf{x}}$.

Dynamic Markovian Environment

- We will assume that for every $\mathbf{x} \in \mathbb{Z}^d$ the environment chain at $\mathbf{x}$, given by
$$\left(\omega_t\left(\mathbf{x}, \cdot\right)\right)_{t \geq 0}$$
is a stationary Markov chain with some state space $\mathcal{S}$.

- We will also assume that the chains $\left(\omega_t\left(\mathbf{x}, \cdot\right)\right)_{t \geq 0}$ are i.i.d. as $\mathbf{x}$ varies.

- Let K be the transition kernel for the environment chain and π be a stationary distribution.

- By $\mathbf{P}^{\pi}$ we will denote the distribution of the entire environment ω.

Remark : This particular model was first introduced by Boldrighini, Minlos and Pellegrinotti [2000].

Assumptions

(A0) We have only nearest neighbor transitions.

(A1) There exists $0 < \kappa \leq 1$ such that

$$K(w, \cdot) \geq \kappa \, \pi(\cdot), \quad \forall \; w \in \mathcal{S}.$$

(A2) There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon \, q(\mathbf{x}, \mathbf{y}) \quad \text{a.s.} \quad [\mathbf{P}^{\pi}],$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

(A3) $\kappa + \varepsilon^2 > 1$.

Annealed SLLN and Invariance Principle

Theorem 1 (Annealed SLLN) *Suppose assumptions (A0) – (A3) hold. Then there exists a constant* $\mathbf{v} \in \mathbb{R}^d$*, such that*

$$\frac{X_n}{n} \longrightarrow \mathbf{v} \ \ \textit{a.s.} \quad \left[\mathbb{P}^{\mathbf{0}}\right],$$

as $n \to \infty$*.*

Theorem 2 (Annealed Invariance Principle) *Suppose assumptions (A0) – (A3) hold. Then there exists a* $(d \times d)$ *positive definite matrix* Σ*, such that under* $\mathbb{P}^{\mathbf{0}}$*,*

$$\left(\frac{X_{\lfloor nt \rfloor} - nt\,\mathbf{v}}{\sqrt{n}}\right)_{t \geq 0} \xrightarrow{d} \mathrm{BM}_d\,(\Sigma)\,,$$

as $n \to \infty$*.*

Quenched Invariance Principle

Further Assumption (A3)' : We assume $\kappa + \varepsilon^6 > 1$.

Theorem 3 (Quenched Invariance Principle) *Suppose assumptions (A0) – (A3)' hold, and also assume that the dimension d is "large" ($d > 6$ would do). Then a.s. for all $\omega \sim \mathbf{P}^\pi$, under the* quenched law $\mathbf{P}_\omega^{\mathbf{0}}$,

$$\left(\frac{X_{\lfloor nt \rfloor} - nt\,\mathbf{v}}{\sqrt{n}} \right)_{t \geq 0} \stackrel{d}{\longrightarrow} \mathrm{BM}_d\,(\Sigma)\,,$$

as $n \to \infty$.

Bit of History

- If the environment chains are i.i.d. chains then *quenched CLT* has been proved by
  - ▶ Boldrighini, Minlos and Pellegrinotti [1997]
    - ∗ In this work they proved for $d \geq 2$ which was extended to $d = 1$ in a later work [1998].
    - ∗ They used non-trivial analytic methods, including a specific type of *cluster expansion* technique.
  - ▶ Stannat [2004]
    - ∗ Gives a simpler but still analytic proof which works for any dimension $d \geq 1$.
  - ▶ Rassoul-Agha and Seppäläinen [2005]
    - ∗ Proved invariance principle using probabilistic techniques (a special case of a more general result).
- For dynamic Markovian environment model, exactly similar to ours, the *quenched CLT* has been proved by
  - ▶ Boldrighini, Minlos and Pellegrinotti [2000]
    - ∗ For dimension $d \geq 3$.
    - ∗ Proofs are based on "hard" analytic techniques.

Then What is New ?

- *Nothing* new for the results !
- Our goal was to provide *simple* proofs using probabilistic techniques.
- We achieved doing so, for the *annealed* results.
- For *quenched* result ... well well ... "simple" is a relative term ! But the proof is probabilistic.

Our Main Strategy

- We will show that there is an increasing sequence of *stopping times* $(\tau_n)_{n\geq 0}$ with $\tau_0 = 0$ such that under $\mathbb{P}^{\mathbf{0}}$ the pairs $(\tau_n - \tau_{n-1}, X_{\tau_n} - X_{\tau_{n-1}})_{n\geq 1}$ are *i.i.d.*

- Moreover we will show that under our assumptions τ_1 has finite second moment.

- Because of nearest neighbor walk this will imply annealed SLLN and IP.

- For quenched IP we need to do more !

A Regeneration Time

- Recall that by assumption (A2) for every time step $t \geq 0$ and for any two states $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ we have

$$\omega_t (\mathbf{x}, \mathbf{y}) \geq \varepsilon \, q (\mathbf{x}, \mathbf{y}) \quad \text{a.s.} \quad [\mathbf{P}^\pi] \, .$$

  Thus at every time step there is a probability $\varepsilon > 0$ when the walker take a step according to the deterministic transition kernel q, and hence in that step does not "learn" any thing about the environment.

- Recall that by assumption (A1) the transition kernel for the environment chains is of the form

$$K (\omega, \cdot) \geq \kappa \, \pi (\cdot) \, , \quad \forall \ \omega \in \mathcal{S}.$$

  Thus at every time step $t \geq 0$, an environment chain at a specific location has a positive probability, namely $\kappa > 0$, to start afresh from the stationary distribution, this would be called a *regeneration*.

- Let τ_1 to be the first time such that the environment chains at all the locations visited by the walkers before time τ_1 have gone through a regeneration.

- At time τ_1 the walker "knows" nothing about the environments any more, so it will look like we have started afresh from time 0.

Properties of τ_1

Proposition 4 *Let the assumptions (A0) – (A3) hold. Then* $\mathbb{E}^{\mathbf{0}}\left[\tau_1^2\right] < \infty$.

Lemma 5 *We can construct the* regeneration time τ_1 *in such a way that there is a filtration* $(\mathcal{H}_t)_{t\geq 0}$ *such that* $(X_t, \{\omega_t(\cdot,\cdot)\})$ *is* $\mathcal{H}_t$*-measurable, and* τ_1 *is a stop-time with respect to it. Moreover under* $\mathbb{P}^{\mathbf{0}}$ *the conditional distribution of*

$$\left(\{X_{\tau_1+t} - X_{\tau_1}\}_{t\geq 0}, \left\{(\omega_{\tau_1+t}(\cdot,\cdot))_{t\geq 0}\right\}\right)$$

given $\mathcal{H}_{\tau_1}$ *is same as the distribution of*

$$\left(\{X_t\}_{t\geq 0}, \left\{(\omega_t(\cdot,\cdot))_{t\geq 0}\right\}\right).$$

Main Idea for Proving the Annealed Results

- The sequence of time $(\tau_n)_{n\geq 1}$ are defined inductively starting from $\tau_0 \equiv 0$.

- Rerun of Lemma 5 will show that the pairs

$$(\tau_n - \tau_{n-1}, X_{\tau_n} - X_{\tau_{n-1}})_{n\geq 1}$$

  are i.i.d. under $\mathbb{P}^{\mathbf{0}}$.

- Rest follows from standard argument !

Construction of ε-Coins

Recall the Assumption (A2) : There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x},\mathbf{y}) \geq \varepsilon\, q(\mathbf{x},\mathbf{y}) \quad \text{a.s.} \quad [\mathbf{P}^{\pi}],$$

for all $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

So we can construct (on an extended probability space) a sequence of i.i.d. coin tosses, say $\{\epsilon_t\}_{t\geq 1}$ such that

- $\mathbf{P}(\epsilon_1 = 1) = \varepsilon$.
- $\{\epsilon_t\}_{t\geq 1}$ are independent of the environment chains.
- At a time $t \geq 0$ before taking the step for time $t+1$, the walker observes the outcome of the ε-coin ε_{t+1}.
- If $\varepsilon_{t+1} = 1$ then it takes a move according to the fixed transition kernel q.
- If $\varepsilon_{t+1} = 0$ then it takes a move according to the random transition kernel
$$\frac{\omega_t(\cdot,\cdot) - \varepsilon q(\cdot,\cdot)}{1-\varepsilon}.$$

Remarks :

- By taking a step when the ε-coin is 1 the walker do not collect any information about the environment.
- We will say a step taken by the walker is a "proper step" if and only if, it was taken when the ε-coin was 0.

  So by taking a "proper" step the walker learns about the environment.

The Regeneration Time

- We would like to define a random time τ_1 such that,

  for every site $\mathbf{x} \in \mathbb{Z}^d$ which has been visited by the walker by taking a "proper" step before time τ_1, we would demand the following to hold :

  from the time of last "proper" visit to $\mathbf{x}$ by the walker and before time τ_1, the environment chain at $\mathbf{x}$ goes through a *regeneration*, in the sense that, at some time before time τ_1 it should start afresh from the stationary distribution.

- If we can do this, then at time τ_1 the walker would have forgotten every information it may have learned about the environment while doing the walk.

Construction of κ-Coins

Recall the Assumption (A1) : There exists $0 < \kappa \leq 1$ such that

$$K(w, \cdot) \geq \kappa \, \pi(\cdot), \quad \forall \; w \in \mathcal{S}.$$

So at every site $\mathbf{x} \in \mathbb{Z}^d$, we can construct (another extension of the probability space) a sequence of i.i.d. coin tosses, say $\{\alpha_t(\mathbf{x})\}_{t \geq 1}$, such that

- $\mathbf{P}(\alpha_1(\mathbf{x}) = 1) = \kappa$.
- $\{\alpha(\mathbf{x})_t\}_{t \geq 1}$ are independent as $\mathbf{x}$ varies.
- At a site $\mathbf{x}$, the environment chain moves from time t to time $t+1$ in the following way :
  - if $\alpha_{t+1}(\mathbf{x}) = 1$ then it moves to a state selected independently from the stationary distribution π,
  - if $\alpha_{t+1}(\mathbf{x}) = 0$ then it moves to a state according to the kernel
  $$\frac{K(\omega_t(\mathbf{x}, \cdot), \cdot) - \kappa\pi(\cdot)}{1 - \kappa}.$$

Precise Definition of τ_1

- Fix $t \geq 0$ and $\mathbf{x} \in \mathbb{Z}^d$.
- **Number of "proper" visits to x before time t :**

$$I_t(\mathbf{x}) := \sum_{s=0}^{t} \mathbf{1}\left(X_s = \mathbf{x}, \epsilon_s = 0\right).$$

- **Time of Last "proper" visit to x before time t :**

$$\gamma_t(\mathbf{x}) := \sup\left\{ s \leq t \,\middle|\, X_s = \mathbf{x}, \epsilon_s = 0 \right\}.$$

- **Time of "regeneration" after last visit to x :**

$$\eta_t(\mathbf{x}) := \inf\left\{ s \geq 0 \,\middle|\, \alpha_{\gamma_t(\mathbf{x})+s}(\mathbf{x}) = 1 \right\}.$$

- If $I_t(\mathbf{x}) = 0$ then $\gamma_t(\mathbf{x}) = \eta_t(\mathbf{x}) = 0$.
- **The Regeneration Time :**

$$\tau_1 := \inf\left\{ t > 0 \,\middle|\, \gamma_t(\mathbf{x}) + \eta_t(\mathbf{x}) < t \;\; \forall\, \mathbf{x} \in \mathbb{Z}^d \right\}.$$

Remarks on the Proof of the Quenched Invariance Principle

- We use a technique introduced by Bolthausen and Sznitman [2002].

- Let $B_t^n := \left(X_{\lfloor nt \rfloor} - nt\,\mathbf{v}\right)/\sqrt{n}$, and $\mathcal{B}_t^n$ be the polygonal interpolation of $(k/n) \mapsto B_{k/n}^n$.

- Bolthausen and Sznitman technique says, that if we have the *annealed* IP then the *quenched* IP will follow if we can show that for all $T > 0$,

$$\sum_{m=1}^{\infty} \text{Var}_{\mathbf{P}^\pi}\left(\mathbf{E}_\omega^{\mathbf{0}}\left[F\left(\mathcal{B}^{\lfloor b^m \rfloor}\right)\right]\right) < \infty,$$

  for every Lipschitz function F on $C\left([0,T],\mathbb{R}^d\right)$ and $b \in (1,2]$.

- To check that the above sum of variances is finite, we work with two walkers which are independent given the environment, and then one needs to "control" the probability of intersection of paths. It is precisely here while doing the estimates, we need the "large" dimension assumption.