Random Walk in Dynamic Markovian Random Environment

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The Basic Setup

- **Graph :** $\mathbb{Z}^d$, $d \geq 1$ with nearest neighbor links.
- **Two Stages of Randomness :**
  - **The Environment :** It is the transition laws which will tell us *how to take the next step* from the current position.

    Note : These laws can be random !

  - **The Walk :** Given the environment we have an *walker* who moves on the lattice $\mathbb{Z}^d$ starting from $\mathbf{0}$ according to the transition laws.

    Note : The walker provides second stage of randomness.

Three Types of Models

1. Classical Markovian Walk :

- **The Environment :** At every site $\mathbf{x} \in \mathbb{Z}^d$ we have a Markov transition kernel (fixed).
- **The Walk :** Usual Markov chain starting from $\mathbf{0}$.

Remarks :

- The transition kernel(s) is(are) non-random.
- They may or may not depend on the time (so called *time homogeneous* or *time inhomogeneous* chains).

Example 1 : Simple symmetric random walk on $\mathbb{Z}$.

Remark : We mostly understand what happens in this example.

2. Classical RWRE (Static Environment) :

- **The Environment :** At the beginning of time, at every location $\mathbf{x} \in \mathbb{Z}^d$, we choose the transition kernels according to some probability distribution, and keep them fixed through out the time evolution.

- **The Walk :** Given the transition laws the walker then moves according to a *time homogeneous* Markov chain, starting from $\mathbf{0}$.

Example 2 [Sinai Walk] : We have two coins, one unbiased $(1/2, 1/2)$ and one biased $(3/4, 1/4)$.

- Consider the integer line $\mathbb{Z}$.

- At every location give value 0 or 1 according to independent tosses of the unbiased coin.

- The walker starts at 0 and moves using the biased coin, with bias to right if the value of the current position is 1, and bias to left if it is 0.

Remark : There are "traps" !

$$\ldots 11111111110000000000 \ldots$$

The walker would have to spend "lot of time" in such "traps" which will "slow down" the walk.

3. Random Walk in Dynamic Random Environment :

- **The Environment :** At every location the transition laws evolve over time.

- **The Walk :** Given (all) the transition kernels, the walker moves according to a *time inhomogeneous* Markov chain, starting from $\mathbf{0}$.

Example 3 : Again say we have two coins, one unbiased $(1/2, 1/2)$ and one biased $(3/4, 1/4)$.

- Consider again the integer line $\mathbb{Z}$.

- The walker starts at 0 and carries both the coins.

- Before a move he first tosses the unbiased coin independently of the past, and then the biased coin again independently of the past. If the unbiased end up in 1, then he puts the bias to right, else put bias to left.

Question : What happens for this walk ?

More Precise Viewpoint
Quenched and Annealed Laws

- **Quenched :** The conditional law of the walk given realization of the environment.

  Note : The walk is a (possibly) time inhomogeneous Markov chain under this law.

- **Annealed :** The marginal distribution of the walk, that is, integrating out the *quenched* law with respect to the environment distribution.

  Note : The walk may not be a Markov chain under the annealed law.

Some Notations

- **The Environment :** At a site $\mathbf{x} \in \mathbb{Z}^d$ and at time $t \geq 0$ "environment" is a transition law, it will be denoted by $\omega_t\left(\mathbf{x}, \cdot\right)$.

- **The Walk :** The position of the walker at time t will be denoted by X_t.

Definition of the Quenched Law

- The *quenched law* of $(X_t)_{t \geq 0}$ starting from $\mathbf{x}$ will be denoted by $\mathbf{P}_\omega^{\mathbf{x}}$.

- Given the entire environment
$$\omega := \left\{ \left(\omega_t\left(\mathbf{x}, \cdot\right)\right)_{t \geq 0} \,\middle|\, \mathbf{x} \in \mathbb{Z}^d \right\},$$
$\mathbf{P}_\omega^{\mathbf{x}}$ is the law of the time inhomogeneous Markov chain $(X_t)_{t \geq 0}$ on $\mathbb{Z}^d$, such that
$$\mathbf{P}_\omega^{\mathbf{x}}\left(X_{t+1} = \mathbf{y} \,\middle|\, X_t = \mathbf{x}\right) = \omega_t\left(\mathbf{x}, \mathbf{y}\right),$$
and
$$\mathbf{P}_\omega^{\mathbf{x}}\left(X_0 = \mathbf{x}\right) = 1.$$

Definition of the Annealed Law

- The *annealed law* of $(X_t)_{t\geq 0}$ starting from $\mathbf{x}$ will be denoted by $\mathbb{P}^{\mathbf{x}}$.
- It is defined by

$$\mathbb{P}^{\mathbf{x}}(\cdot) := \int \mathbf{P}_{\omega}^{\mathbf{x}}(\cdot)\ \mathbf{P}(d\omega),$$

  where $\omega \sim \mathbf{P}$.

Dynamic Markovian Environment

- We will assume that for every $\mathbf{x} \in \mathbb{Z}^d$ the environment chain at $\mathbf{x}$, given by
$$(\omega_t(\mathbf{x}, \cdot))_{t \geq 0}$$
is a stationary Markov chain with some state space $\mathcal{S}$.

- We will also assume that the chains $(\omega_t(\mathbf{x}, \cdot))_{t \geq 0}$ are i.i.d. as $\mathbf{x}$ varies.

- Let K be the transition kernel for the environment chain and π be a stationary distribution.

- By $\mathbf{P}^\pi$ we will denote the distribution of the entire environment ω.

Remarks :

- This particular model was first introduced by Boldrighini, Minlos and Pellegrinotti [2000].

- The Example 3, falls under this model where the environment chains are just *i.i.d.* chains.

Assumptions

(A0) We have only nearest neighbor transitions.

(A1) There exists $0 < \kappa \leq 1$ such that

$$K\,(w,\cdot) \geq \kappa\,\pi\,(\cdot)\,, \quad \forall\ w \in \mathcal{S}.$$

(A2) There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t\,(\mathbf{x},\mathbf{y}) \geq \varepsilon\, q\,(\mathbf{x},\mathbf{y}) \quad \text{a.s.} \quad [\mathbf{P}^{\pi}]\,,$$

for all $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

(A3) $\kappa + \varepsilon^2 > 1$.

Discussion on the Assumptions

(A0) We have only nearest neighbor transitions.

- This is for simplicity, the arguments can be easily generalized to transitions with bounded increment.

(A1) There exists $0 < \kappa \leq 1$ such that

$$K(w, \cdot) \geq \kappa \, \pi(\cdot), \quad \forall \ w \in \mathcal{S}.$$

- This condition provides a uniform "fast mixing" rate for the environment chains.
- If the state space for the environment chains $\mathcal{S}$ is finite and K is irreducible and a periodic (assumption made by BMP [2000]), then assumption (A1) may fail, but it does hold if K is replaced by K^r for some fixed integer $r \geq 1$. A slight modification of our arguments applies to that case, too.
- If the environment chains are i.i.d. chains (like in Example 3) then (A1) holds trivially with $\kappa = 1$.

(A2) There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t(\mathbf{x}, \mathbf{y}) \geq \varepsilon\, q(\mathbf{x}, \mathbf{y}) \quad \text{a.s. } [\mathbf{P}^\pi],$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

- This condition essentially means that the random environment has a "deterministic" part q, which is non-degenerate.
- Comparing with classical (static) RWRE literature, this condition can be referred as an *ellipticity* condition.
- This condition was also assumed in BMP [1997, 2000] and Stannat [2004].

(A3) $\kappa + \varepsilon^2 > 1$.

- Technical but absolutely crucial for our arguments !
- This conditions exhibits a trade off between the environment chain being fast mixing (κ close to 1) and the fluctuation in the environment being "small" (ε close to 1).
- ε close to 1 is also an assumption made by BMP [1997, 2000].
- This trade off is perhaps artificial !

Annealed SLLN and Invariance Principle

Theorem 1 (Annealed SLLN) *Suppose assumptions (A0) – (A3) hold. Then there exists a constant* $\mathbf{v} \in \mathbb{R}^d$*, such that*

$$\frac{X_n}{n} \longrightarrow \mathbf{v} \ \text{ a.s. } \ \left[\mathbb{P}^{\mathbf{0}}\right],$$

as $n \to \infty$*.*

Theorem 2 (Annealed Invariance Principle) *Suppose assumptions (A0) – (A3) hold. Then there exists a* $(d \times d)$ *positive definite matrix* Σ*, such that under* $\mathbb{P}^{\mathbf{0}}$*,*

$$\left(\frac{X_{\lfloor nt \rfloor} - nt\,\mathbf{v}}{\sqrt{n}}\right)_{t \geq 0} \stackrel{d}{\longrightarrow} \mathrm{BM}_d\left(\Sigma\right),$$

as $n \to \infty$*.*

Quenched Invariance Principle

Further Assumption (A3)' : We assume $\kappa + \varepsilon^6 > 1$.

Theorem 3 (Quenched Invariance Principle) *Suppose assumptions (A0) – (A3)' hold, and also assume that the dimension d is "large" ($d > 6$ would do). Then a.s. for all $\omega \sim \mathbf{P}^\pi$, under the* quenched law $\mathbf{P}_\omega^{\mathbf{0}}$,

$$\left(\frac{X_{\lfloor nt \rfloor} - nt\,\mathbf{v}}{\sqrt{n}} \right)_{t \geq 0} \underset{d}{\longrightarrow} \mathrm{BM}_d\,(\Sigma)\,,$$

as $n \to \infty$.

Bit of History

- If the environment chains are i.i.d. chains then *quenched CLT* has been proved by
  - ▶ Boldrighini, Minlos and Pellegrinotti [1997]
    - ∗ In this work they proved for $d \geq 2$ which was extended to $d = 1$ in a later work [1998].
    - ∗ They used non-trivial analytic methods, including a specific type of *cluster expansion* technique.
  - ▶ Stannat [2004]
    - ∗ Gives a simpler but still analytic proof which works for any dimension $d \geq 1$.
  - ▶ Rassoul-Agha and Seppäläinen [2005]
    - ∗ Proved invariance principle using probabilistic techniques (a special case of a more general result).
- For dynamic Markovian environment model, exactly similar to ours, the *quenched CLT* has been proved by
  - ▶ Boldrighini, Minlos and Pellegrinotti [2000]
    - ∗ For dimension $d \geq 3$.
    - ∗ Proofs are based on "hard" analytic techniques.

Then What is New ?

- *Nothing* new for the results !
- Our goal was to provide *simple* proofs using probabilistic techniques.
- We achieved doing so, for the *annealed* results.
- For *quenched* result ... well well ... "simple" is a relative term ! But the proof is probabilistic.

Some More Comments on the Quenched Invariance Principle

- It is not known if the *quenched IP* holds for dimension $d \leq 2$.

- Our assumption $d > 6$ is clearly not optimal, for example, when the environment chains are i.i.d. chains, then it is known [Stannat 2004] that *quenched* CLT holds for any dimension $d \geq 1$.

- We (and also BMP) believe that it should be true for any $d \geq 1$.

- This believe is in no contradiction with Sinai Walk, as the classical (static) case is not included in our model. In fact the static environment is in some sense the "hardest" case to analyze.

Our Main Strategy

- We will show that there is an increasing sequence of *stopping times* $(\tau_n)_{n\geq 0}$ with $\tau_0 = 0$ such that under $\mathbb{P}^{\mathbf{0}}$ the pairs $(\tau_n - \tau_{n-1}, X_{\tau_n} - X_{\tau_{n-1}})_{n\geq 1}$ are *i.i.d.*

- Moreover we will show that under our assumptions τ_1 has finite second moment.

- Because of nearest neighbor walk this will imply annealed SLLN and IP.

- For quenched IP we need to do more !

Construction of ε-Coins

Recall the Assumption (A2) : There exist $0 < \varepsilon \leq 1$ and a fixed Markov kernel q with only nearest neighbor transition which is non-degenerate, such that

$$\omega_t\left(\mathbf{x}, \mathbf{y}\right) \geq \varepsilon\, q\left(\mathbf{x}, \mathbf{y}\right) \quad \text{a.s.} \quad \left[\mathbf{P}^{\pi}\right],$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$, and $t \geq 0$.

So we can construct (on an extended probability space) a sequence of i.i.d. coin tosses, say $\left\{\epsilon_t\right\}_{t \geq 1}$ such that

- $\mathbf{P}\left(\epsilon_1 = 1\right) = \varepsilon$.
- $\left\{\epsilon_t\right\}_{t \geq 1}$ are independent of the environment chains.
- At a time $t \geq 0$ before taking the step for time $t+1$, the walker observes the outcome of the ε-coin ε_{t+1}.
- If $\varepsilon_{t+1} = 1$ then it takes a move according to the fixed transition kernel q.
- If $\varepsilon_{t+1} = 0$ then it takes a move according to the random transition kernel

$$\frac{\omega_t\left(\cdot, \cdot\right) - \varepsilon q\left(\cdot, \cdot\right)}{1 - \varepsilon}.$$

Remarks :

- By taking a step when the ε-coin is 1 the walker do not collect any information about the environment.
- We will say a step taken by the walker is a “proper step” if and only if, it was taken when the ε-coin was 0.

  So by taking a “proper” step the walker learns about the environment.

The Regeneration Time

- We would like to define a random time τ_1 such that,

  for every site $\mathbf{x} \in \mathbb{Z}^d$ which has been visited by the walker by taking a "proper" step before time τ_1, we would demand the following to hold :

  from the time of last "proper" visit to $\mathbf{x}$ by the walker and before time τ_1, the environment chain at $\mathbf{x}$ goes through a *regeneration*, in the sense that, at some time before time τ_1 it should start afresh from the stationary distribution.

- If we can do this, then at time τ_1 the walker would have forgotten every information it may have learned about the environment while doing the walk.

Construction of κ-Coins

Recall the Assumption (A1) : There exists $0 < \kappa \leq 1$ such that

$$K(w, \cdot) \geq \kappa \, \pi(\cdot), \quad \forall \; w \in \mathcal{S}.$$

So at every site $\mathbf{x} \in \mathbb{Z}^d$, we can construct (another extension of the probability space) a sequence of i.i.d. coin tosses, say $\{\alpha_t(\mathbf{x})\}_{t \geq 1}$, such that

- $\mathbf{P}(\alpha_1(\mathbf{x}) = 1) = \kappa$.
- $\{\alpha(\mathbf{x})_t\}_{t \geq 1}$ are independent as $\mathbf{x}$ varies.
- At a site $\mathbf{x}$, the environment chain moves from time t to time $t+1$ in the following way :
  - if $\alpha_{t+1}(\mathbf{x}) = 1$ then it moves to a state selected independently from the stationary distribution π,
  - if $\alpha_{t+1}(\mathbf{x}) = 0$ then it moves to a state according to the kernel

$$\frac{K(\omega_t(\mathbf{x}, \cdot), \cdot) - \kappa \pi(\cdot)}{1 - \kappa}.$$

Precise Definition of τ_1

- Fix $t \geq 0$ and $\mathbf{x} \in \mathbb{Z}^d$.
- **Number of "proper" visits to $\mathbf{x}$ before time t :**
$$I_t(\mathbf{x}) := \sum_{s=0}^{t} \mathbf{1}\left(X_s = \mathbf{x}, \epsilon_s = 0\right).$$
- **Time of Last "proper" visit to $\mathbf{x}$ before time t :**
$$\gamma_t(\mathbf{x}) := \sup\left\{ s \leq t \,\middle|\, X_s = \mathbf{x}, \epsilon_s = 0 \right\}.$$
- **Time of "regeneration" after last visit to $\mathbf{x}$:**
$$\eta_t(\mathbf{x}) := \inf\left\{ s \geq 0 \,\middle|\, \alpha_{\gamma_t(\mathbf{x})+s}(\mathbf{x}) = 1 \right\}.$$
- If $I_t(\mathbf{x}) = 0$ then $\gamma_t(\mathbf{x}) = \eta_t(\mathbf{x}) = 0$.
- **The Regeneration Time :**
$$\tau_1 := \inf\left\{ t > 0 \,\middle|\, \gamma_t(\mathbf{x}) + \eta_t(\mathbf{x}) < t \;\; \forall\, \mathbf{x} \in \mathbb{Z}^d \right\}.$$

Properties of τ_1

Proposition 4 *Let the assumptions (A0) – (A3) hold. Then* $\mathbb{E}^{\mathbf{0}}\left[\tau_1^2\right] < \infty$.

Lemma 5 *Let* $\mathcal{H}_1 := \sigma\left(\tau_1; \{X_t\}_{0 \le t \le \tau_1}; \left\{(\alpha_t(\cdot))_{t \le \tau_1}\right\}\right)$, *then under* $\mathbb{P}^{\mathbf{0}}$ *the conditional distribution of*

$$\left(\{X_{\tau_1+t} - X_{\tau_1}\}_{t \ge 0}, \left\{(\omega_{\tau_1+t}(\cdot,\cdot))_{t \ge 0}\right\}, \left\{(\alpha_t(\cdot))_{t \ge \tau_1+1}\right\}\right)$$

given $\mathcal{H}_1$ *is same as the distribution of*

$$\left(\{X_t\}_{t \ge 0}, \left\{(\omega_t(\cdot,\cdot))_{t \ge 0}\right\}, \left\{(\alpha_t(\cdot))_{t \ge 1}\right\}\right).$$

Main Idea for Proving the Annealed Results

- The sequence of time $(\tau_n)_{n\geq 1}$ are defined inductively by

  $$\tau_{n+1} := \tau_n + \tau_1\left((X_{\tau_n+t})_{t\geq 0}\,;\left\{(\omega_{\tau_n+t}(\cdot,\cdot))_{t\geq 0}\right\};\left\{(\alpha_{\tau_n+t}(\cdot))_{t\geq 1}\right\}\right),$$

  starting from $\tau_0 = 0$.

- Rerun of Lemma 5 will show that the pairs

  $$(\tau_n - \tau_{n-1}, X_{\tau_n} - X_{\tau_{n-1}})_{n\geq 1}$$

  are i.i.d. under $\mathbb{P}^{\mathbf{0}}$.

- Rest follows from standard argument !

Proof of Proposition 4

- Let $\{L(t)\}_{t\geq 1}$ be a fixed sequence of integers increasing to ∞ and $L(t) < t$. We will choose appropriate value of $L(t)$ later.

- Let β_t be the first time there is a run of length $L(t)$ on non-zero ε-coins ending at it, formally,

$$\beta_t := \inf \left\{ s \geq L(t) \;\middle|\; \epsilon_s = \epsilon_{s-1} = \cdots = \epsilon_{s-L(t)+1} = 1 \right\}.$$

- From definition of τ_1 we get

$$\begin{aligned} & \overline{\mathbb{P}}^{\mathbf{0}} (\tau_1 > t) \\ \leq\ & \overline{\mathbb{P}}^{\mathbf{0}} (\beta_t > t) + \\ & \overline{\mathbb{P}}^{\mathbf{0}} \left(\beta_t \leq t,\ \exists\ \mathbf{x} \in \mathbb{Z}^d \text{ s.t. } \eta_{\beta_t}(\mathbf{x}) \geq \beta_t - \gamma_{\beta_t}(\mathbf{x}) \right) \end{aligned}$$

Continuing the proof ...

- Simple calculation shows

$$
\begin{aligned}
&\overline{\mathbb{P}}^{\mathbf{0}}\left(\beta_t > t\right) \\
\le\ & \overline{\mathbb{P}}^{\mathbf{0}}\left(\bigcap_{j=0}^{\lfloor\frac{t-L(t)+1}{L(t)}\rfloor}\left[\epsilon_{jL(t)+1} = \cdots = \epsilon_{(j+1)L(t)} = 1\right]^c\right) \\
\le\ & \left(1 - \varepsilon^{L(t)}\right)^{\frac{t-L(t)+1}{L(t)}-1}
\end{aligned}
$$

- For the other probability we observe

$$
\begin{aligned}
&\overline{\mathbb{P}}^{\mathbf{0}}\left(\beta_t \le t,\ \exists\ \mathbf{x} \in \mathbb{Z}^d \text{ s.t. } \eta_{\beta_t}(\mathbf{x}) \ge \beta_t - \gamma_{\beta_t}(\mathbf{x})\right) \\
\le\ & C_0 \sum_{r=0}^{\infty} r^{d-1} e^{-\lambda\,(L(t)\vee r)} \\
=\ & C_0 \left(\left(\sum_{r\le L(t)} r^{d-1}\right) e^{-\lambda L(t)} + \sum_{r>L(t)} r^{d-1} e^{-\lambda r}\right)
\end{aligned}
$$

where $\lambda = -\log(1-\kappa)$ and the inequality follows from the observation that

$$
\beta_t - \gamma_{\beta_t}(\mathbf{x}) \ge L(t) \vee |\mathbf{x} - X_{\beta_t - L(t)}|.
$$

Continuing the proof ...

- Recall the assumption (A3), $\kappa + \varepsilon^2 > 1$, thus we can find $0 < \delta < 1$ such that $\varepsilon^2 > (1-\kappa)^\delta$. Taking

$$L(t) = \left\lfloor -\frac{\delta \log t}{\log \varepsilon} \right\rfloor$$

  and using above estimates, we conclude,

$$\mathbb{P}^{\mathbf{0}}(\tau_1 > t) \le \frac{C_1}{t^{2+\zeta}}$$

  for some constants $C_1, \zeta > 0$.

Remarks on the Proof of the Quenched Invariance Principle

- We use a technique introduced by Bolthausen and Sznitman [2002].

- Let $B_t^n := \left(X_{\lfloor nt \rfloor} - nt\,\mathbf{v}\right)/\sqrt{n}$, and $\mathcal{B}_t^n$ be the polygonal interpolation of $(k/n) \mapsto B_{k/n}^n$.

- Bolthausen and Sznitman technique says, that if we have the *annealed* IP then the *quenched* IP will follow if we can show that for all $T > 0$,

$$\sum_{m=1}^{\infty} \text{Var}_{\mathbf{P}^\pi}\left(\mathbf{E}_\omega^{\mathbf{0}}\left[F\left(\mathcal{B}^{\lfloor b^m \rfloor}\right)\right]\right) < \infty,$$

  for every Lipschitz function F on $C\left([0,T], \mathbb{R}^d\right)$ and $b \in (1, 2]$.

- To check that the above sum of variances is finite, we work with two walkers which are independent given the environment, and then one needs to "control" the probability of intersection of paths. It is precisely here while doing the estimates, we need the "large" dimension assumption.