Random Walks in Dynamic Markovian Random Environment

Antar Bandyopadhyay
(Joint work with Professor Ofer Zeitouni)

Mini-Workshop on Motion in Random Media
University of Minnesota
Minneapolis, USA

Department of Mathematics
Chalmers University of Technology
Göteborg, Sweden

http://www.math.chalmers.se/~antar

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The Setup

- **Graph :** $\mathbb{Z}^d$, $d \geq 1$ with nearest neighbor links is the basic underlying graph structure.

- **Environment :** At a site $\mathbf{x} \in \mathbb{Z}^d$ and at time $t \geq 0$ "environment" is a transition law $\omega_t(\mathbf{x}, \cdot)$ (we will assume nearest neighbor transition only).

- **Walk :** Given environment

$$\omega := \left\{ (\omega_t(\mathbf{x}, \cdot))_{t \geq 0} \,\middle|\, \mathbf{x} \in \mathbb{Z}^d \right\}$$

(note both in space and time), the walk $\{X_t\}_{t \geq 0}$ is a time inhomogeneous Markov chain on $\mathbb{Z}^d$.

Dynamic Markovian Environment

- For each site $\mathbf{x} \in \mathbb{Z}^d$, the environment at $\mathbf{x}$, namely $(\omega_t(\mathbf{x}, \cdot))_{t \geq 0}$ is a stationary Markov chain with some state space $\mathcal{S}$.

- The chains $(\omega_t(\mathbf{x}, \cdot))_{t \geq 0}$ are i.i.d. as $\mathbf{x}$ varies.

Quenched and Annealed Laws

- **Quenched law** starting at $\mathbf{x}$: $\mathbf{P}_\omega^{\mathbf{x}}$

  Note : $\{X_t\}_{t\geq 0}$ is a Markov chain under this law.

- **Annealed law** starting at $\mathbf{x}$: $\mathbb{P}^{\mathbf{x}}$

  Note : $\{X_t\}_{t\geq 0}$ is not necessarily a Markov chain under this law.

Assumptions on the Environment

(A0) **Finite State Space :** $\mathcal{S}$ is finite.

(A1) **Unique Stationary Distribution :** At any site $\mathbf{x}$, the environment chain $(\omega_t(\mathbf{x}, \cdot))_{t \geq 0}$ is irreducible and aperiodic, with (unique) stationary distribution π.

(A2) **Uniform Ellipticity :** The environment at every site and time is *uniformly elliptic*, that is, there exists an $\varepsilon > 0$ such that

$$\mathbf{P}^{\pi}\left(\omega_t\left(\mathbf{x}, \mathbf{x} \pm \mathbf{e}_i\right) \geq \varepsilon\right) = 1,$$

for all $1 \leq i \leq d$, $t \geq 0$ and $\mathbf{x} \in \mathbb{Z}^d$.

(A3) **Fast Mixing :** For the environment chain if T_w be the hitting time of a state $w \in \mathcal{S}$ then there exist constants $C_0 > 0$ and $\lambda > 2\left|\log(2d\varepsilon)\right|$, such that for any pair of states $w, w' \in \mathcal{S}$

$$P^{w'}\left(T_w \geq t\right) \leq C_0 \exp(-\lambda t) \quad \forall \; t \geq 0.$$

Annealed SLLN and CLT

Theorem 1 (Annealed SLLN) *There exists a constant* $\mathbf{v} \in \mathbb{R}^d$*, such that*

$$\frac{X_t}{t} \longrightarrow \mathbf{v} \quad \textit{a.s.} \quad \left[\mathbb{P}^{\mathbf{0}}\right].$$

Theorem 2 (Annealed CLT) *There exists a* $(d \times d)$ *positive definite matrix* Σ*, such that under* $\mathbb{P}^{\mathbf{0}}$*,*

$$\frac{X_t - t\,\mathbf{v}}{\sqrt{t}} \underset{d}{\longrightarrow} \mathrm{Normal}_d\left(\mathbf{0}, \Sigma\right).$$

What Is Known :

- For i.i.d. space-time environment *quenched CLT* (and more) has been proved by
  - Boldrighini, Minlos and Pellegrinotti [1997, 1998];
  - Stannat [2004] (simpler proof);
  - Rassoul-Agha and Seppäläinen [2005].
- For dynamic Markovian environment model (similar to ours) Boldrighini, Minlos and Pellegrinotti [2000] have proved *quenched CLT* !

Then What Is New ?

- BMP [2000] proof is hard and involves non-trivial analytic techniques.
- Our goal is to get a simple proof with probabilistic argument.

Remark : So far we have only achieved only *annealed* results which of course is weaker than known results !

Our Main Strategy

- We will show that there is an increasing sequence of *stopping times* $(\tau_n)_{n\geq 0}$ with $\tau_0 = 0$ such that under $\mathbb{P}^{\mathbf{0}}$ the pairs $(\tau_n - \tau_{n-1}, X_{\tau_n} - X_{\tau_{n-1}})_{n\geq 1}$ are *i.i.d.*

- Moreover we will show that under our assumptions τ_1 has finite second moment.

- Because of nearest neighbor walk this will imply annealed SLLN and CLT.

Construction of Some ε-Moves

- We can construct (on an extended probability space) a sequence of i.i.d. random variables, say $\{\epsilon_t\}_{t\geq 1}$, which are independent of the environment, such that
  - $\mathbf{P}(\epsilon_1 = \pm \mathbf{e}_i) = \varepsilon$ and $\mathbf{P}(\epsilon_1 = 0) = 1 - 2d\varepsilon$;
  - at time $t \geq 1$, the walker first observes the outcome of the ε-move ϵ_t;
  - if $\epsilon_t \neq 0$ then it moves in the direction given by ϵ_t irrespective of the environment;
  - otherwise the walker moves to a direction $\mathbf{e}$ with probability

$$\omega_t(X_{t-1}, X_{t-1} + \mathbf{e}) - \varepsilon$$

.

Remarks :

- By taking a step using the ε-move the walker do not collect any information about the environment.
- We will say a step taken by the walker is a "proper step" if and only if it is not taken using a ε-move.

What is the Time τ_1 ?

- We want τ_1 to be a (random) time such that, for every site $\mathbf{x} \in \mathbb{Z}^d$ which has been visited by the walker by taking a "proper" step before time τ_1, we want
  - τ_1 to occur before the walker takes another "proper" visit to the site $\mathbf{x}$; and
  - during the time interval of the last "proper" visit to $\mathbf{x}$ before τ_1 and till time τ_1, the environment chain at $\mathbf{x}$ goes through a *regeneration*, in the sense that, at time τ_1 it should look like as if it has started afresh from a stationary distribution.

Precise Description

- Let $\left\{ \left(\alpha_i\left(\mathbf{x}\right)\right)_{i \geq 1} \,\middle|\, \mathbf{x} \in \mathbb{Z}^d \right\}$ be i.i.d. $\mathcal{S}$-valued random variables with distribution π which are independent of the environment and the walk.

- Fix $t \geq 0$ and $\mathbf{x} \in \mathbb{Z}^d$.

- Let
$$I_t\left(\mathbf{x}\right) := \sum_{s=0}^{t} \mathbf{1}\left(X_s = \mathbf{x}, \epsilon_s = 0\right),$$
be the number of "proper" visits to the site $\mathbf{x}$.

- If $I_t\left(\mathbf{x}\right) > 0$ then we define
$$\gamma_t\left(\mathbf{x}\right) := \sup\left\{ s \leq t \,\middle|\, X_s = x, \epsilon_s = 0 \right\},$$
and
$$\eta_t\left(\mathbf{x}\right) := \inf\left\{ s \geq 0 \,\middle|\, \omega_{\gamma_t(\mathbf{x})+s}\left(\mathbf{x}, \cdot\right) = \alpha_{I_t(\mathbf{x})}\left(\mathbf{x}\right) \right\}.$$
We will take $\gamma_t\left(\mathbf{x}\right) = \eta_t\left(\mathbf{x}\right) = 0$ if $I_t\left(\mathbf{x}\right) = 0$.

- **The Time τ_1 :**
$$\tau_1 := \inf\left\{ t > 0 \,\middle|\, \gamma_t\left(\mathbf{x}\right) + \eta_t\left(\mathbf{x}\right) < t \;\; \forall\, \mathbf{x} \in \mathbb{Z}^d \right\}.$$

Properties of τ_1

Proposition 3 *Under our assumptions* $\mathbb{E}^{\mathbf{0}}\left[\tau_1^2\right] < \infty$.

Lemma 4 *Let* $\mathcal{H}_1 := \sigma\left(\tau_1; \{X_t\}_{0\le t\le\tau_1}; \left\{(\alpha_i(\cdot))_{i\le M_{\tau_1}}\right\}\right)$, *where* $M_t := \sup\left\{I_t(\mathbf{x}) \,\middle|\, \mathbf{x}\in\mathbb{Z}^d\right\}$, *then under* $\mathbb{P}^{\mathbf{0}}$ *the conditional distribution of*

$$\left(\{X_{\tau_1+t} - X_{\tau_1}\}_{t\ge 0}, \left\{(\omega_{\tau_1+t}(\cdot,\cdot))_{t\ge 0}\right\}, \left\{(\alpha_i(\cdot))_{i\ge M_{\tau_1}+1}\right\}\right)$$

given $\mathcal{H}_1$ *is same as the distribution of*

$$\left(\{X_t\}_{t\ge 0}, \left\{(\omega_t(\cdot,\cdot))_{t\ge 0}\right\}, \left\{(\alpha_i(\cdot))_{i\ge 1}\right\}\right).$$

Idea of the Proofs of the Main Results

- The sequence of time $(\tau_n)_{n\geq 1}$ are defined inductively by

$$\tau_{n+1} := \\ \tau_n + \tau_1 \left((X_{\tau_n+t})_{t\geq 0} ; \left\{ (\omega_{\tau_n+t}(\cdot,\cdot))_{t\geq 0} \right\} ; \left\{ (\alpha_i(\cdot))_{i\geq M_{\tau_n}+1} \right\} \right) .$$

- Rerun of Lemma 4 will show that the pairs

$$\left(\tau_n - \tau_{n-1}, X_{\tau_n} - X_{\tau_{n-1}}\right)_{n\geq 1}$$

are i.i.d.

- Rest follows from standard argument !

Proof of Proposition 3

- Let $\{L(t)\}_{t\geq 1}$ be a fixed sequence of integers increasing to ∞ and $L(t) < t$. We will choose appropriate value of $L(t)$ later.

- Let β_t be the first time there is a run of length $L(t)$ on non-zero ε-moves ending at it, formally,

$$\beta_t := \inf \left\{ s \geq L(t) \;\middle|\; \epsilon_s \neq 0, \epsilon_{s-1} \neq 0, \cdots, \epsilon_{s-L(t)+1} \neq 0 \right\}.$$

- From definition of τ_1 we get

$$\begin{aligned} \overline{\mathbb{P}}^{\mathbf{0}} (\tau_1 > t) \;\; \leq \;\; & \overline{\mathbb{P}}^{\mathbf{0}} (\beta_t > t) + \\ & \overline{\mathbb{P}}^{\mathbf{0}} \left(\beta_t \leq t, \; \exists \, \mathbf{x} \in \mathbb{Z}^d \text{ s.t. } \eta_{\beta_t}(\mathbf{x}) \geq \beta_t - \gamma_{\beta_t}(\mathbf{x}) \right) \end{aligned}$$

Continuing the proof ...

- Simple calculation shows

$$
\begin{aligned}
& \overline{\mathbb{P}}^{\mathbf{0}}\left(\beta_t > t\right) \\
\leq\ & \overline{\mathbb{P}}^{\mathbf{0}}\left(\bigcap_{j=0}^{\lfloor\frac{t-L(t)+1}{L(t)}\rfloor}\left[\epsilon_{jL(t)+1} \neq 0, \cdots, \epsilon_{(j+1)L(t)} \neq 0\right]^c\right) \\
\leq\ & \left(1-(2d\varepsilon)^{L(t)}\right)^{\frac{t-L(t)+1}{L(t)}-1}
\end{aligned}
$$

Continuing the proof ...

- For the other probability we observe

$$\begin{aligned}
& \overline{\mathbb{P}}^{\mathbf{0}}\left(\beta_t \leq t,\ \exists\ \mathbf{x} \in \mathbb{Z}^d \text{ s.t. } \eta_{\beta_t}(\mathbf{x}) \geq \beta_t - \gamma_{\beta_t}(\mathbf{x})\right) \\
\leq\ & C_0 \sum_{r=0}^{\infty} r^{d-1} e^{-\lambda\,(L(t) \vee r)} \\
=\ & C_0 \left(\left(\sum_{r \leq L(t)} r^{d-1} \right) e^{-\lambda L(t)} + \sum_{r > L(t)} r^{d-1} e^{-\lambda r} \right)
\end{aligned}$$

where the first inequality follows from the observation that

$$\beta_t - \gamma_t(\mathbf{x}) \geq L(t) \vee |\mathbf{x} - X_{\beta_t - L(t)}|.$$

- Now from assumption (A3) we can find $0 < \delta < 1$ such that $\lambda > \frac{2}{\delta}\Big| \log(2d\varepsilon) \Big|$, thus taking

$$L(t) = \lfloor \frac{\delta \log t}{|\log(2d\varepsilon)|} \rfloor$$

we get the correct tail estimate, that is,

$$\mathbb{P}^{\mathbf{0}}(\tau_1 > t) \leq \frac{C_1}{t^{2+\zeta}}$$

for some constants $C_1, \zeta > 0$.