Problem Set for Extra Credit (Due: Friday, 6th December)

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Note: There are 8 problems each with 10 points. Solve as many as you can. Maximum extra credit you can get from this problem set is 8 points for your final grading.

1. Suppose \( X \) and \( Y \) are i.i.d. \( N(0, 1) \) random variables. Let \((R, \Theta)\) be the polar coordinate of the random point \((X,Y) \in \mathbb{R}^2\). We have seen in the class that \( R^2 \) has a distribution such that \( R^2 \sim \text{Exponential}(1/2) \).
   (a) Show that \( \Theta \sim \text{Unif}(0, 2\pi) \). [5 points]
   (b) Let \( Z = (X + Y)/\sqrt{2} \) and \( W = (X - Y)/\sqrt{2} \), show that \( Z \) and \( W \) are independent. Find the marginal distributions of \( Z \) and \( W \). [5 points]

2. Let \( X \) be a random variable with a continuous density function \( f \) such that \( f(x) > 0 \) for all \(-\infty < x < \infty\). Let \( F \) be the CDF of \( X \).
   (a) Show that \( F^{-1} \) exists as a function from \((0, 1)\) to \( \mathbb{R} \). [2 points]
   (b) Let \( U \sim \text{Unif}(0,1) \), find the CDF of \( Y = F^{-1}(U) \). [4 points]
   (c) Find the distribution of \( W = F(X) \). [4 points]

3. Suppose you have a computer routine which can generate i.i.d. \( \text{Unif}(0,1) \) variables, as many as you want. \( \lambda > 0 \) is a given number.
   (a) Use this routine to generate one \( \text{Exponential}(\lambda) \) random variable. [2 points]
   (b) Use the same routine to generate one \( \text{Poisson}(\lambda) \) random variable. [8 points]

Note: by “generate” I mean, you have to give a procedure whose end result will have the desired distribution. Such procedures are called simulations. (Hint for (b): Think of a Poisson arrival process of rate \( \lambda \).)

4. It is a math fact that if \( Z \) is a non-negative random variable and \( E[Z] = 0 \) then \( P(Z = 0) = 1. \) Use this fact to show that if \((X,Y)\) are two random variables such that
  \[
  E[X | Y] = Y, \quad \text{and} \quad E[Y | X] = X.
  \]
  Then \( P(X = Y) = 1. \) (Hint: Take \( Z = (X - Y)^2 \) and compute \( E[Z] \).) [10 points]

5. Suppose there are \( n \) balls labeled \( \{1, 2, \ldots, n\} \), and \( n \) boxes labeled \( \{1, 2, \ldots, n\} \). Balls are being placed at random in the boxes. Any ball can go into any box, and a box may contain more than one ball. Let \( X \) be the number of empty boxes. Find \( \text{Var}(X) \) and \( E[X^2] \). (Hint: Use indicators as we did for the midterm problem.) [8 + 2 points]

6. Problem # 6.3.2 of the text. [10 points]
7. Problem # 6.3.4 of the text. [10 points]
8. Problem # 6.3.6 of the text. [10 points]