Stat-134, Section 02
Section 02

Problem Set for Extra Credit (Due : Friday, 6th December)

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Note : There are 8 problems each with 10 points. Solve as many as you can. Maximum extra credit you can get from this problem set is 8 points for your final grading.

1. Suppose X and Y are **i.i.d** $N(0,1)$ random variables. Let (R, Θ) be the polar coordinate of the random point $(X, Y) \in \mathbb{R}^2$. We have seen in the class that R has a distribution such that $R^2 \sim$ Exponential(1/2).
   (a) Show that $\Theta \sim \text{Unif}(0, 2\pi)$. [5 points]
   (b) Let $Z = (X+Y)/\sqrt{2}$ and $W = (X-Y)/\sqrt{2}$, show that Z and W are independent. Find the marginal distributions of Z and W. [5 points]

2. Let X be a random variable with a continuous density function f such that $f(x) > 0$ for all $-\infty < x < \infty$. Let F be the CDF of X.
   (a) Show that F^{-1} exists as a function from $(0,1)$ to $\mathbb{R}$. [2 points]
   (b) Let $U \sim \text{Unif}(0,1)$, find the CDF of $Y = F^{-1}(U)$. [4 points]
   (c) Find the distribution of $W = F(X)$. [4 points]

3. Suppose you have a computer routine which can generate **i.i.d.** Unif$(0,1)$ variables, as many as you want. $\lambda > 0$ is a given number.
   (a) Use this routine to generate one Exponential(λ) random variable. [2 points]
   (b) Use the same routine to generate one Poisson(λ) random variable. [8 points]

   Note : by "generate" I mean, you have to give a procedure whose end result will have the desired distribution. Such procedures are called *simulations*. (Hint for (b) : Think of a Poisson arrival process of rate λ.)

4. It is a math fact that if Z is a non-negative random variable and $\mathbf{E}[Z] = 0$ then $\mathbf{P}(Z = 0) = 1$. Use this fact to show that if (X, Y) are two random variables such that

$$\mathbf{E}\left[X \,\middle|\, Y\right] = Y, \text{ and}$$

$$\mathbf{E}\left[Y \,\middle|\, X\right] = X.$$

   Then $\mathbf{P}(X = Y) = 1$. (Hint : Take $Z = (X-Y)^2$ and compute $\mathbf{E}[Z]$.) [10 points]

5. Suppose there are n balls labeled $\{1, 2, \ldots, n\}$, and n boxes labeled $\{1, 2, \ldots, n\}$. Balls are being placed at random in the boxes. Any ball can go into any box, and a box may contain more than one ball. Let X be the number of empty boxes. Find Var(X) and $\mathbf{E}\left[X^2\right]$. (Hint : Use indicators as we did for the midterm problem.) [$8+2$ points]

6. Problem # 6.3.2 of the text. [10 points]

7. Problem # 6.3.4 of the text. [10 points]

8. Problem # 6.3.6 of the text. [10 points]