Statistics - 134 (Lecture - 2), Fall 2002

Solutions of the Final Exam Problems

1. (a) **TRUE**.

$$\mathbf{P}\left(A \middle| B\right) = \frac{\mathbf{P}\left(A \cap B\right)}{\mathbf{P}\left(B\right)} = \frac{\mathbf{P}\left(A\right)}{\mathbf{P}\left(B\right)} \geq \mathbf{P}\left(A\right).$$

   (b) **FALSE**. Notice that $\mathbf{E}\left[X - Y\right] = 0$ and $\mathbf{Var}\left(X - Y\right) = 2$ so by Chybecyeb inequality we get

$$\mathbf{P}\left(|X - Y| > 2\right) \leq \frac{1}{2}$$

   (c) **FALSE**. Certainly, $\mathbf{E}\left[\frac{X}{Y}\right] = \mathbf{E}\left[X\right]\mathbf{E}\left[\frac{1}{Y}\right]$, because X and Y are independent. But in general, $\mathbf{E}\left[\frac{1}{Y}\right] \neq \frac{1}{\mathbf{E}[Y]}$.

   (d) **TRUE**. Observe that $\mathbf{Var}\left(X\right) = \mathbf{E}\left[X^2\right] - \left(\mathbf{E}\left[X\right]\right)^2 \geq 0$.

   (e) **TRUE**. By convolution formula, $X - Y$ has a density and hence it is a continuous random variable. So $\mathbf{P}\left(X = Y\right) = \mathbf{P}\left(X - Y = 0\right) = 0$.

2. Apply the change of variable formula to get (note that the transformation is $y = e^x$, which is one-to-one, with differentiable inverse)

$$f_Y\left(y\right) = \begin{cases} \frac{1}{\sqrt{2\pi}y} e^{-\frac{1}{2}(\log y)^2} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

3. (a)

$$\begin{aligned} \mathbf{P}\left(T_3 > 3\right) &= \mathbf{P}\left(N_3 \leq 2\right) \\ &= e^{-9}\left(1 + 9 + \frac{9^2}{2!}\right) \\ &\approx 0.0062 \end{aligned}$$

   (b)

$$\begin{aligned} \mathbf{P}\left(T_6 - T_3 \leq 2\right) &= \mathbf{P}\left(W_4 + W_5 + W_6 \leq 2\right) \\ &= \mathbf{P}\left(T_3 \leq 2\right) \\ &= \mathbf{P}\left(N_2 \geq 3\right) \\ &= 1 - e^{-6}\left(1 + 6 + \frac{6^2}{2!}\right) \\ &\approx 0.9380 \end{aligned}$$

4. (a) The joint density of (X, Y) is given by

$$f\left(x, y\right) = \frac{1}{2\pi} e^{-\frac{1}{2}\left[x^2 + (x-y)^2\right]} \quad -\infty < x, y < \infty.$$

   So to compute the marginal density of Y we need to integrate $f(x, y)$ over x. Now, notice that it is then nothing but a convolution of two Normal$(0, 1)$ variables, and hence $Y \sim$ Normal$(0, 2)$.

   (b) Simple calculation will show that given $[Y = y]$ the random variable $X \sim$ Normal$(\frac{y}{2}, \frac{1}{2})$. Thus, $\mathbf{E}\left[X \middle| Y = y\right] = \frac{y}{2}$.

5. Let Z be the total profit after 100 draws, then $Z = X_1 + X_2 + \cdots + X_{100}$, where $X_1, X_2, \ldots, X_{100}$ are **i.i.d.** such that $\mathbf{P}(X_1 = 2) = \mathbf{P}(X_1 = -1) = 3/10$, $\mathbf{P}(X_1 = 0) = 4/10$. Use **Normal approximation** method to conclude that $\mathbf{P}(Z > 45) \approx 0.1038$.

6. (a) Note that X takes values in $(0,1)$. Fix $0 < x < 1$, then

$$f_X(x) = \int_0^1 f(x,y)\,dy = \frac{1}{5}\left(3x^2 + 4x + 2\right).$$

(b)

$$\begin{aligned} \mathbf{P}(X \le Y) &= \iint_{x \le y} f(x,y)\,dx\,dy \\ &= \frac{1}{5}\int_0^1 \left(\int_0^y \left(3x^2 + 4xy + 6y^2 + 2x\right)\,dx\right)\,dy \\ &= \frac{1}{5}\int_0^1 \left(9y^3 + y^2\right)\,dy \\ &= \frac{31}{60} \end{aligned}$$

7. (a) The values of T are $\{1, 2, 3, \ldots,\}$. Fix $k \ge 1$, then

$$\mathbf{P}(T = k) = \frac{2}{3}\left(\frac{1}{2}\right)^k + \frac{1}{3}\left(\frac{1}{2} + \theta\right)\left(\frac{1}{2} - \theta\right)^{k-1}.$$

(b)

$$\mathbf{P}\left(\text{I got the biased coin} \middle| T = 10\right) = \frac{\frac{1}{3}\left(\frac{1}{2} + \theta\right)\left(\frac{1}{2} - \theta\right)^9}{\frac{2}{3}\left(\frac{1}{2}\right)^{10} + \frac{1}{3}\left(\frac{1}{2} + \theta\right)\left(\frac{1}{2} - \theta\right)^9}$$

8. (a) The values of Y are $\{0, 1, 2, \ldots\}$.

(b) Fix $k \ge 0$, then

$$\begin{aligned} \mathbf{P}(Y = k) &= \mathbf{P}(k \le X < k+1) \\ &= e^{-k} - e^{-(k+1)} \\ &= \left(1 - e^{-1}\right) e^{-k}. \end{aligned}$$

(c) So Y follows Geometric distribution supported on $\{0, 1, 2, \ldots\}$, and with success probability $\left(1 - e^{-1}\right)$.

9. Let X be the time when Hermione arrives and Y be the time when Harry arrives. From the given information we get that X and Y are independent normal random variables with $X - 12:00$ noon $\sim$ Normal$(-5, 9)$ and $Y - 12:00$ noon $\sim$ Normal$(5, 9)$.

(a) $\mathbf{P}(Y < X) = \mathbf{P}(X - Y > 0) = 1 - \Phi\left(\frac{10}{\sqrt{18}}\right) \approx 0.0091$.

(b) $\mathbf{P}(Y - X > 10) = 0.5$.

10. Notice that $X \sim$ Exponential(λ).

(a) $\mathbf{E}[X^n]$ can now be computed by looking at the form of Gamma$(n+1, \lambda)$ density, and it is $n!/\lambda^n$.

(b) By geometric series formula the required sum is $\lambda/(\lambda - z)$.