Stat-134 (Section 02), Fall 2002

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Solution of Practice Final

1. (a) $|X|$ takes values in $(0, \infty)$. Thus $F_{|X|}(x) = 0$ if $x \leq 0$. Fix $x > 0$, then

$$\begin{aligned} F_{|X|}(x) &= \mathbf{P}(|X| \leq x) \\ &= \int_{-x}^{x} \frac{dt}{\pi(1+t^2)} \\ &= \frac{2}{\pi} \tan^{-1} x. \end{aligned}$$

So the CDF of $|X|$ is

$$F_{|X|}(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \frac{2}{\pi} \tan^{-1} x & \text{if } x > 0. \end{cases}$$

(b) X^2 also takes only positive values, so the density $f_{X^2}(y) = 0$ if $y \leq 0$. Further, $F_{X^2}(y) = \mathbf{P}\left(X^2 \leq y\right) = \mathbf{P}\left(|X| \leq \sqrt{y}\right) = \frac{2}{\pi} \tan^{-1} \sqrt{y}$. So by differentiating we get

$$f_{X^2}(y) = \begin{cases} \frac{1}{\pi\sqrt{y}(1+y)} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

2. Let $W_1, W_2, W_3, \ldots$, be the inter-arrival times, which are **i.i.d.** Exponential(λ). From definition $T_1 = W_1$ and $T_2 = W_1 + W_2$.

Fix $0 < t < s < \infty$, then

$$\begin{aligned} \mathbf{P}(T_1 \in dt, T_2 \in ds) &= \mathbf{P}(N((0,t]) = 0, \text{one arrival in } (t, t+dt], N((t,s]) = 0, \text{one arrival in } (s, s+ds]) \\ &= e^{-\lambda t} \times \lambda\, dt \times e^{-\lambda(s-t)} \times \lambda\, ds \\ &= \lambda^2 e^{-\lambda s}\, dt\, ds. \end{aligned}$$

So the joint density of (T_1, T_2) is given by

$$f(t,s) = \begin{cases} \lambda^2 e^{-\lambda s} & \text{if } 0 < t < s < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Further we know that $T_2 \sim \text{Gamma}(2, \lambda)$, so the marginal density of T_2 is

$$f_{T_2}(s) = \begin{cases} \lambda^2 s e^{-\lambda s} & \text{if } s > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) So the conditional density of T_1 given $T_2 = 10$ is

$$f_{T_1 \big| T_2}\left(t \big| s = 10\right) = \frac{f(t,10)}{f_{T_2}(10)} = \begin{cases} \dfrac{1}{10} & \text{if } 0 < t < 10, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, given $T_2 = 10$, $T_1 \sim \text{Unif}(0,10)$, and hence $\mathbf{E}\left[T_1 \big| T_2 = 10\right] = 5$.

(b) We know that $T_1 = W_1$ and $T_2 = W_1 + W_2$. Hence,

$$\mathbf{E}\left[T_1 T_2\right] = \mathbf{E}\left[W_1\left(W_1 + W_2\right)\right] = \mathbf{E}\left[W_1^2\right] + \mathbf{E}\left[W_1 W_2\right] = \frac{3}{\lambda^2}.$$

3. In this problem $n := 90$ is the number of trials, and $p := \mathbf{P}\,(\text{a student gets 2 or more aces})$ is the success probability of an individual trial. Using *Hypergeometric* distribution we get that

$$p = \frac{\binom{4}{2}\binom{48}{11}}{\binom{52}{13}} + \frac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}} + \frac{\binom{4}{4}\binom{48}{9}}{\binom{52}{13}} \approx 0.2573$$

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Let X be the number of students who got 2 or more aces, then clearly $X \sim \text{Binomial}(n,p)$.
So $\mu = np \approx 23.2573$ and $\sigma = \sqrt{np(1-p)} \approx 4.1473$, so using Normal approximation to Binomial probabilities we get that

$$\mathbf{P}\,(\text{at least 50 students get 2 or more aces}) \approx 1 - \Phi\left(\frac{50.5 - 23.1601}{4.1473}\right) \approx 1 - \Phi\,(6.5922) \approx 0.0000.$$

4. Let X be the number of times I have to toss my coin before getting a head, and Y be the number of times you have to toss your coin before getting a head. So X and Y are **i.i.d.** Geometric($\frac{1}{2}$) variables.

(a)

$$\begin{aligned} \mathbf{P}\,(\text{we stop simultaneously}) &= \mathbf{P}\,(X = Y) \\ &= \sum_{k=1}^{\infty} \mathbf{P}\,(X = k, Y = k) \\ &= \sum_{k=1}^{\infty} \mathbf{P}\,(X = k)\,\mathbf{P}\,(Y = k) \\ &= \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3}. \end{aligned}$$

(b) Notice that given the event $[X = Y]$ the number of coin tosses is well defined and it is X (or Y). So for any $k \geq 1$,

$$\mathbf{P}\left(X = k \big| X = Y\right) = \frac{\mathbf{P}\,(X = k, Y = k)}{\mathbf{P}\,(X = Y)} = \frac{1/4^k}{1/3} = \frac{3}{4}\left(\frac{1}{4}\right)^{k-1}.$$

Thus given $[X = Y]$, the number of tosses follows Geometric($\frac{3}{4}$) distribution.

5. (a) Clearly, X only takes values in $(-1,1)$. So $f_X(x) = 0$ if $|x| \geq 1$. Let $-1 < x < 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-(1-|x|)}^{(1-|x|)} \frac{dy}{2} = 1 - |x|.$$

(b) The conditional density of Y given $X = \dfrac{1}{2}$ is then given by

$$f_{Y|X}\left(y \middle| x = \frac{1}{2}\right) = \frac{f\left(\frac{1}{2}, y\right)}{f_X\left(\frac{1}{2}\right)} = \begin{cases} 1 & \text{if } -\frac{1}{2} < y < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus given $\left[Y = \dfrac{1}{2}\right]$, Y follows Unif $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$.

6. First of of all from definition we note that the marginal distributions of X and Y are same, and it is Binomial $\left(10, \dfrac{1}{3}\right)$.

Let I_i be the indicator of the event that in i^{th} draw we got a green ball, and J_j be the event that in j^{th} draw we got a black ball. Trivially, I_i and J_j are independent if $1 \leq i \neq j \leq 10$, and $I_i \times J_i = 0$ for all $1 \leq i \leq 10$. Also $X = I_1 + I_2 + \cdot + I_{10}$ and $Y = J_1 + J_2 + \cdots + J_{10}$.

(a) $XY = \sum\sum_{1 \leq i \neq j \leq 10} I_i \times J_j$. Hence $\mathbf{E}[XY] = 10 \times (10 - 1) \times \dfrac{1}{3} \times \dfrac{1}{3} = 10$.

(b) $\mathbf{E}[X] = \mathbf{E}[Y] = \dfrac{10}{3}$. So $\mathbf{E}[XY] \neq \mathbf{E}[X]\,\mathbf{E}[Y]$, thus X and Y are not independent.

7. Suppose that Julia plans to arrive at the air-port at time t (in the standard unit of hours and minutes). Let X be the time when she actually arrives, so $X \sim \text{Unif}(t, t + 15 \text{ minutes})$. Also let Y be the time when the flight actually be leaving the air-port. Thus $Y \sim \text{Unif}(10:30 \text{ AM}, 10:45 \text{ AM})$. We will assume that X and Y are independent.

$$\begin{aligned} \mathbf{P}(\text{Julia will not be able to catch the flight}) &= \mathbf{P}(X > Y) \\ &= \frac{1}{2}(t + 15 - 10:30 \text{ AM})^2 \times \frac{1}{15^2}. \end{aligned}$$

So for Julia to have 90% chance of catching the flight, we need to make the above probability exactly 10%, that is we need $t = 10:30 \text{ AM} - 15 + 15 \times \sqrt{0.2}$ minutes $\approx 10:22\text{AM}$.

8. (a) $Z = \min(X, Y)$, so Z takes values in $\mathbb{R}$. Fix $-\infty < z < \infty$, then

$$\begin{aligned} F_Z(z) &= \mathbf{P}(Z \leq z) \\ &= 1 - \mathbf{P}(\min(X, Y) > z) \\ &= 1 - \mathbf{P}(X > z, Y > z) \\ &= 1 - \mathbf{P}(X > z)\,\mathbf{P}(Y > z) \\ &= 1 - (1 - \Phi(z - \mu))(1 - \Phi(z)). \end{aligned}$$

So the density of Z is given by

$$f_Z(z) = (1 - \Phi(z - \mu))\,\phi(z) + (1 - \Phi(z))\,\phi(z - \mu).$$

(b) Consider the following two cases

Case-1 : $t \leq 0$, then $\mathbf{P}(\max(X, Y) - \min(X, Y) > t) = 1$.

Case-2 : $t > 0$, then

$$\begin{aligned} \mathbf{P}(\max(X, Y) - \min(X, Y) > t) &= \mathbf{P}(|X - Y| > t) \\ &= 1 - \Phi\left(\frac{t - \mu}{\sqrt{2}}\right) + \Phi\left(\frac{-t - \mu}{\sqrt{2}}\right). \end{aligned}$$

Note that $X - Y \sim \text{Normal}(\mu, 2)$.

9. Let A_i be the event that there is a match at i^{th} position, so $\mathbf{P}(A_i) = \dfrac{(n-1)!}{n!} = \dfrac{1}{n}$. From definition $X = I_{A_1} + I_{A_2} + \cdots + I_{A_n}$. Hence $\mathbf{E}[X] = n \times \dfrac{1}{n} = 1$. Now,

$$\begin{aligned} X^2 &= \sum_{i=1}^{n} I_{A_i}^2 + \sum\sum_{1\le i\ne j\le n} I_{A_i} I_{A_j} \\ &= \sum_{i=1}^{n} I_{A_i} + \sum\sum_{1\le i\ne j\le n} I_{A_i\cap A_j}. \end{aligned}$$

Further, $\mathbf{P}(A_i \cap A_j) = \dfrac{(n-2)!}{n!} = \dfrac{1}{n(n-1)}$ for $1 \le i \ne j \le n$. So we get that,

$$\mathbf{E}\left[X^2\right] = n \times \frac{1}{n} + n(n-1) \times \frac{1}{n(n-1)} = 2.$$

Finally, $\mathbf{Var}(X) = 2 - 1^2 = 1$.

10. The joint density of (X, Y) is given by

$$f(x,y) = \begin{cases} \dfrac{\alpha-1}{y^{\alpha+1}} & \text{if } 0 < x < y,\ y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Observe that X only takes positive values, thus $f_X(x) = 0$ if $x \le 0$. Fix $x > 0$, then

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{\max(x,1)}^{\infty} \frac{\alpha-1}{y^{\alpha+1}}\,dy = \frac{\alpha-1}{\alpha\left(\max(x,1)\right)^{\alpha}}.$$

So the marginal density of X is

$$f_X(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \dfrac{\alpha-1}{\alpha} & \text{if } 0 < x < 1, \\ \dfrac{\alpha-1}{\alpha x^{\alpha}} & \text{if } x \ge 1. \end{cases}$$

(b) So the conditional density of Y given $X = x$ is well defined when $x > 0$.

Case-1 : $0 < x < 1$,

$$f_{Y|X}\left(y\middle|x\right) = \begin{cases} \dfrac{\alpha}{y^{\alpha+1}} & \text{if } y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

So $\mathbf{E}\left[Y\middle|X=x\right] = \int_1^\infty \dfrac{y\alpha}{y^{\alpha+1}}\,dy = \alpha \displaystyle\int_1^\infty \frac{dy}{y^\alpha} = \frac{\alpha}{\alpha-1}$.

Case-2 : $x \ge 1$,

$$f_{Y|X}\left(y\middle|x\right) = \begin{cases} \dfrac{\alpha x^\alpha}{y^{\alpha+1}} & \text{if } y > x, \\ 0 & \text{otherwise.} \end{cases}$$

So $\mathbf{E}\left[Y\middle|X=x\right] = \int_x^\infty \dfrac{y\alpha x^\alpha}{y^{\alpha+1}}\,dy = \alpha x^\alpha \displaystyle\int_x^\infty \frac{dy}{y^\alpha} = \frac{\alpha x}{\alpha-1}$.